Eidgenössische Technische Hochschule Zürich
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Quasi-Monte Carlo integration and applications

Semester paper

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Chapter 1

Introduction

In this project, we focus on a particular method to approximate integrals of the form
$$\int_{[0,1]^s} f(\mathbf{x})d\mathbf{x}$$
for a certain function $f \in L^1([0,1]^s)$ and $s \in \mathbb{N}$.
In Monte-Carlo integration, we look at this integral as an expectation of the random variable f on the probability space $([0,1]^s, \mathcal{B}, \lambda_s)$, where $\mathcal{B}$ is the Borel σ-algebra and λ_s the Lebesgue measure. We can then approximate the integral by the following estimator
$$\frac{1}{N}\sum_{n=0}^{N-1} f(X_n)$$
where $X_0, ..., X_{N-1}$ are i.i.d. uniformly distributed random variables on $[0,1]^s$ for $N \in \mathbb{N}$.
By the linearity of the expectation, this estimator is unbiased and by the strong law of large numbers, we then have
$$\mathbb{P}\left[\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(X_n) = \int_{[0,1]^s} f(\mathbf{x})d\mathbf{x}\right] = 1$$
where $\mathbb{P}$ is the probability measure of an arbitrary probability space supporting a sequence $(X_n)_{n\in\mathbb{N}_0}$ of i.i.d. uniformly distributed random variables on $[0,1]^s$.
In this project, we focus on quasi-Monte Carlo integration. That is, we want to find deterministic sequences $(\mathbf{x}_n)_{n\in\mathbb{N}_0} \subseteq [0,1]^s$ such that the property
$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n) = \int_{[0,1]^s} f(\mathbf{x})d\mathbf{x}$$

still holds.
At this point, it is not yet clear if such sequences exist. If they do, what kind of distribution properties do they have? How can we define a measure to compare such sequences?
In this project, we follow [1] to find some answers to these questions.

Chapter 2

Uniform distribution modulo 1

In this section, we consider a deterministic sequence $\mathcal{S} = (\mathbf{x}_n)_{n\in\mathbb{N}_0} \subseteq [0,1)^s$, for a certain dimension $s \in \mathbb{N}$. We want to define a notion of uniformity for $\mathcal{S}$. Intuitively, every region should contain a number of points which is proportional to the size of this region.
To formalize this idea, we will use the following notation

$$A([\mathbf{a},\mathbf{b}),\mathcal{S},N) = |\{n \in \mathbb{N}_0 \mid 0 \leq n \leq N-1 \text{ and } \mathbf{x}_n \in [\mathbf{a},\mathbf{b})\}|$$

where $[\mathbf{a},\mathbf{b}) \subseteq [0,1)^s$, $\mathbf{a} = (a_1,...,a_s), \mathbf{b} = (b_1,...,b_s)$ and $N \in \mathbb{N}$.
In words, $A([\mathbf{a},\mathbf{b}),\mathcal{S},N)$ is the number of points from the N first elements of the sequence $\mathcal{S}$ that lie in the interval $[\mathbf{a},\mathbf{b})$.
We will denote by λ_s, for $s \in \mathbb{N}$, the s-dimensional Lebesgue measure.

2.1 Definition and basic properties

Following the intuition given above, we consider the following notion of uniformity.

Definition 2.1 (Uniformly distributed modulo one) *An infinite sequence $\mathcal{S} \subseteq [0,1)^s$ is said to be uniformly distributed modulo 1, if for every interval $[\mathbf{a},\mathbf{b}) \subseteq [0,1)^s$ we have*

$$\lim_{N\to\infty} \frac{A([\mathbf{a},\mathbf{b}),\mathcal{S},N)}{N} = \lambda_s([\mathbf{a},\mathbf{b})).$$

We will see examples of such sequences later on.
Let's see how this definition connects to Monte-Carlo integration.

Theorem 2.2 *A sequence $\mathcal{S} = (\mathbf{x}_n)_{n\in\mathbb{N}_0} \subseteq [0,1)^s$ is uniformly distributed modulo one if and only if for every Riemann integrable function $f : [0,1]^s \to \mathbb{R}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) = \int_{[0,1]^s} f(\mathbf{x})d\mathbf{x}. \tag{2.1}$$

Proof $'' \Leftarrow ''$ Assume (2.1) holds for all Riemann integrable functions. Consider an interval $I = [\mathbf{a}, \mathbf{b}) \in [0,1)^s$, and its characteristic function $\mathbb{1}_I$. Then (2.1) holds for $\mathbb{1}_I$ and we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{1}_I(\mathbf{x}_n) = \int_{[0,1]^s} \mathbb{1}_I(\mathbf{x}) d\mathbf{x} = \lambda_s(I).$$

Now note that, by definition, we have $A(I, \mathcal{S}, N) = \sum_{n=0}^{N-1} \mathbb{1}_I(\mathbf{x}_n)$.
We deduce that $\lim_{N\to\infty} \frac{1}{N} A(I, \mathcal{S}, N) = \lambda_s(I)$ and, since I was arbitrary, $\mathcal{S}$ is uniformly distributed modulo one.

$'' \Rightarrow ''$ Now assume $\mathcal{S} = (\mathbf{x}_n)_{n\in\mathbb{N}_0}$ is uniformly distributed modulo one. We first show that (2.1) holds for step functions and then we conclude via approximation.
Let the intervals $E_1, ..., E_m$ form a partition of $[0,1]^s$ and consider the step function

$$g = \sum_{i=1}^{m} d_i \mathbb{1}_{E_i}$$

where $d_i \in \mathbb{R}$ for $i = 1, ..., m$. Then we have

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} g(\mathbf{x}_n) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{i=1}^{m} d_i \mathbb{1}_{E_i}(\mathbf{x}_n) \\
&= \sum_{i=1}^{m} d_i \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{1}_{E_i}(\mathbf{x}_n) \\
&= \sum_{i=1}^{m} d_i \lambda_s(E_i) = \int_{[0,1]^s} g(\mathbf{x}) d\mathbf{x}
\end{aligned}$$

where in the third equality we used that $(\mathbf{x}_n)_{n\in\mathbb{N}_0}$ is uniformly distributed modulo 1.
This shows that (2.1) holds for every step functions. Consider now a Riemann integrable function $f : [0,1]^s \to \mathbb{R}$. By definition of the Riemann integral, for all $\varepsilon > 0$ there exists two step functions $g_1, g_2 : [0,1]^s \to \mathbb{R}$ such that $g_1 \leq f \leq g_2$ and $\int_{[0,1]^s} (g_2(\mathbf{x}) - g_1(\mathbf{x}))\, d\mathbf{x} < \varepsilon$.

We deduce that

$$\begin{aligned}
\int_{[0,1]^s} f(\mathbf{x})d\mathbf{x} - \varepsilon \leq \int_{[0,1]^s} g_2(\mathbf{x})d\mathbf{x} - \varepsilon \leq \int_{[0,1]^s} g_1(\mathbf{x})d\mathbf{x} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} g_1(\mathbf{x}_n) \\
&= \liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} g_1(\mathbf{x}_n) \\
&\leq \liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) \\
&\leq \liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} g_2(\mathbf{x}_n) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} g_2(\mathbf{x}_n) \\
&= \int_{[0,1]^s} g_2(\mathbf{x})d\mathbf{x} \leq \int_{[0,1]^s} f(\mathbf{x})d\mathbf{x} + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we deduce that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) = \int_{[0,1]^s} f(\mathbf{x})d\mathbf{x}$. This concludes the proof. $\square$

The next theorem gives another characterization of sequences uniformly distributed modulo 1, and will be useful to prove Theorem 2.4 (Weyl's criterion).

Theorem 2.3 *A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}_0} \subseteq [0,1)^s$ is uniformly distributed modulo 1 if and only if* (2.1) *holds for every continuous one-periodic and complex valued function $f : [0,1]^s \to \mathbb{C}$.*

Proof $'' \Rightarrow ''$ We can write $f = f_1 + if_2$ where f_1 is the real part of f and f_2 the imaginary part of f.
Since f is continuous, also f_1 and f_2 are continuous and in particular Riemann integrable on the compact $[0,1]^s$.
Applying Theorem 2.2 to f_1 and f_2 yields

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (f_1(\mathbf{x}_n) + if_2(\mathbf{x}_n)) \\
&= \int_{[0,1]^s} f_1(\mathbf{x})d\mathbf{x} + i\int_{[0,1]^s} f_2(\mathbf{x})d\mathbf{x} \\
&= \int_{[0,1]^s} f(\mathbf{x})d\mathbf{x}.
\end{aligned}$$

$'' \Leftarrow ''$ Consider an interval $I = [\mathbf{a}, \mathbf{b}) \subseteq [0,1]^s$ and its characteristic function $\mathbb{1}_I$.

As characteristic functions of such intervals can be approximated from below and from above by continuous, one-periodic functions, we have that for all $\varepsilon > 0$, there exists two continuous one-periodic functions $f_1, f_2 : [0,1]^s \to \mathbb{R}$ such that $f_1 \leq \mathbb{1}_I \leq f_2$ and $\int_{[0,1]^s} (f_2(\mathbf{x}) - f_1(\mathbf{x}))\, d\mathbf{x} < \varepsilon$.
By the exact same computations as in the proof of Theorem 2.2, we deduce that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{1}_I(\mathbf{x}_n) = \int_{[0,1]^s} \mathbb{1}_I(\mathbf{x}) d\mathbf{x} = \lambda_s(I).$$

Again since $\sum_{n=0}^{N-1} \mathbb{1}_I(\mathbf{x}_n) = A(I, (\mathbf{x}_n)_{n\in\mathbb{N}_0}, N)$ and because the interval $I \subseteq [0,1]^s$ is arbitrary, we deduce that $(\mathbf{x}_n)_{n\in\mathbb{N}_0}$ is uniformly distributed. $\square$

Theorem 2.4 (Weyl's criterion) *A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}_0} \subseteq [0,1)^s$ is uniformly distributed modulo one if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \mathbf{x}_n) = 0 \tag{2.2}$$

for all vectors $\mathbf{h} \in \mathbb{Z}^s \setminus \{0\}$.

Proof $'' \Rightarrow ''$ Let $(\mathbf{x}_n)_{n\in\mathbb{N}_0}$ be a sequence uniformly distributed modulo one. We consider the continuous function $f : [0,1]^s \to \mathbb{C}$ given by $f(\mathbf{x}) = \exp(2\pi i \mathbf{h} \cdot \mathbf{x})$. Then f is also one-periodic since for $\mathbf{x} \in \partial[0,1]^s$ and $h \in \mathbb{Z}^s$ we have $\mathbf{h} \cdot \mathbf{x} \in \mathbb{Z}$ and therefore $f(\mathbf{x}) = \exp(2\pi i \mathbf{h} \cdot \mathbf{x}) = 1$ for all $\mathbf{x} \in \partial[0,1]^s$. We can therefore apply Theorem 2.3 to f and deduce that

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \mathbf{x}_n) &= \int_{[0,1]^s} \exp\left(2\pi i \mathbf{h} \cdot \mathbf{x}\right) d\mathbf{x} \\
&= \int_{[0,1]^s} \exp\left(2\pi i \sum_{k=1}^{s} h_k x_k\right) dx_1 ... dx_s \\
&= \prod_{k=1}^{s} \int_{[0,1]} \exp\left(2\pi i h_k x_k\right) dx_k \\
&= \prod_{\substack{k=1 \\ h_k \neq 0}}^{s} \frac{1}{2\pi i h_k} \exp\left(2\pi i h_k x_k\right) \Big|_{x_k=0}^{1} = 0
\end{aligned}$$

where we used that $\mathbf{h} \in \mathbb{Z}^s \setminus \{0\}$ so that $\{k \in \{1, ..., s\} \mid h_k \neq 0\} \neq \emptyset$ and so the last product is not empty.

$'' \Leftarrow ''$ Let $f : [0,1]^s \to \mathbb{C}$ be a continuous one-periodic function and $\varepsilon > 0$. By Theorem A.7, there exists a function $P : [0,1]^s \to \mathbb{C}$ of the form

$$P(\mathbf{x}) = \sum_{j=1}^{m} a_j \exp(2\pi i \mathbf{h_j} \cdot \mathbf{x})$$

for some $\mathbf{h_j} \in \mathbb{Z}^s$ and $a_j \in \mathbb{C}, j = 1, ..., m$, such that $\|f - P|\|_{L^\infty} < \varepsilon/3$.
By the triangle inequality, we then have

$$\begin{aligned}\left|\int_{[0,1]^s} f(\mathbf{x})d\mathbf{x} - \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n)\right| \leq &\left|\int_{[0,1]^s} f(\mathbf{x})d\mathbf{x} - \int_{[0,1]^s} P(\mathbf{x})d\mathbf{x}\right| \\ &+ \left|\int_{[0,1]^s} P(\mathbf{x})d\mathbf{x} - \frac{1}{N}\sum_{n=0}^{N-1} P(\mathbf{x}_n)\right| \\ &+ \left|\frac{1}{N}\sum_{n=0}^{N-1} P(\mathbf{x}_n) - \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n)\right|.\end{aligned}$$

We can bound the first and last terms as follows:

$$\left|\int_{[0,1]^s} f(\mathbf{x})d\mathbf{x} - \int_{[0,1]^s} P(\mathbf{x})d\mathbf{x}\right| \leq \|f - P\|_{L^\infty} \leq \varepsilon/3$$

and

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} P(\mathbf{x}_n) - \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n)\right| \leq \|f - P\|_{L^\infty} \leq \varepsilon/3.$$

For the second term, it follows from (2.2) that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} P(\mathbf{x}_n) = \sum_{j=1}^{m} a_j \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h_j} \cdot \mathbf{x}_n) = \sum_{\substack{j=1 \\ h_j=0}}^{s} a_j.$$

Finally, since $\int_{[0,1]^s} \exp(2\pi i \mathbf{h_j} \cdot \mathbf{x})d\mathbf{x} = \begin{cases} 0 & \text{if } h \neq 0 \\ 1 & \text{if } h = 0 \end{cases}$, we compute that

$$\int_{[0,1]^s} P(\mathbf{x})d\mathbf{x} = \sum_{j=1}^{m} a_j \int_{[0,1]^s} \exp(2\pi i \mathbf{h_j} \cdot \mathbf{x})d\mathbf{x} = \sum_{\substack{j=1 \\ h_j=0}}^{s} a_j.$$

Therefore, for N large enough, we have

$$\left|\int_{[0,1]^s} P(\mathbf{x})d\mathbf{x} - \frac{1}{N}\sum_{n=0}^{N-1} P(\mathbf{x}_n)\right| < \varepsilon/3.$$

Overall, for N large enough, we deduce that

$$\left|\int_{[0,1]^s} f(\mathbf{x})d\mathbf{x} - \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n)\right| < \varepsilon.$$

Therefore the condition from Theorem 2.3 is satisfied and the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}_0}$ is uniformly distributed modulo 1. $\square$

Using Weyl's criterion, we give our first example of a uniformly distributed sequence.
For $x \in \mathbb{R}$ we will need its fractional part, denoted as $\{x\} \in [0,1)$ and defined as $\{x\} = x - \lfloor x \rfloor$.

Proposition 2.5 ($n\boldsymbol{\alpha}$-sequences) *Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_s) \in \mathbb{R}^s$. The sequence $(\{n\boldsymbol{\alpha}\})_{n\in\mathbb{N}_0}$ where the fractional part $\{\cdot\}$ is applied component-wise, is uniformly distributed modulo one if and only if the numbers $1, \alpha_1, ..., \alpha_s$ are linearly independent over the rationals.*
In particular, the one-dimensional sequence $(\{n\alpha\})_{n\in\mathbb{N}_0}$ is uniformly distributed modulo one if and only if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Proof " $\Leftarrow$ " Let $h \in \mathbb{Z}^s \setminus \{0\}$. We have that

$$\begin{aligned} \exp(2\pi i \mathbf{h} \cdot \{n\boldsymbol{\alpha}\}) &= \exp(2\pi i \mathbf{h} \cdot (n\boldsymbol{\alpha} - \lfloor n\boldsymbol{\alpha} \rfloor)) \\ &= \exp(2\pi i \mathbf{h} \cdot n\boldsymbol{\alpha}) \exp(-2\pi i \mathbf{h} \cdot \lfloor n\boldsymbol{\alpha} \rfloor) \\ &= \exp(n 2\pi i \mathbf{h} \cdot \boldsymbol{\alpha}) \\ &= \exp(2\pi i \mathbf{h} \cdot \boldsymbol{\alpha})^n \end{aligned} \tag{2.3}$$

where we used that $\mathbf{h} \cdot \lfloor n\boldsymbol{\alpha} \rfloor \in \mathbb{Z}$ so that $\exp(-2\pi i \mathbf{h} \cdot \lfloor n\boldsymbol{\alpha} \rfloor) = 1$.
Now since $1, \alpha_1, ..., \alpha_s$ are linearly independent over $\mathbb{Q}$, we know that $\mathbf{h} \cdot \boldsymbol{\alpha} \notin \mathbb{Q}$ and therefore $\exp(2\pi i \mathbf{h} \cdot \boldsymbol{\alpha}) \neq 1$.
Using the formula for geometric sums, we compute that

$$\begin{aligned} \left| \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \boldsymbol{\alpha})^n \right| &= \frac{1}{N} \left| \frac{\exp(2\pi i N \mathbf{h} \cdot \boldsymbol{\alpha}) - 1}{\exp(2\pi i \mathbf{h} \cdot \boldsymbol{\alpha}) - 1} \right| \\ &\leq \frac{1}{N} \frac{|\exp(2\pi i N \mathbf{h} \cdot \boldsymbol{\alpha})| + 1}{|\exp(2\pi i \mathbf{h} \cdot \boldsymbol{\alpha}) - 1|} \\ &\leq \frac{1}{N} \frac{2}{|\exp(2\pi i \mathbf{h} \cdot \boldsymbol{\alpha}) - 1|}. \end{aligned}$$

Overall, we have that

$$\begin{aligned} 0 \leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \{n\boldsymbol{\alpha}\}) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \boldsymbol{\alpha})^n \\ &\leq \lim_{N\to\infty} \frac{1}{N} \frac{2}{|\exp(2\pi i \mathbf{h} \cdot \boldsymbol{\alpha}) - 1|} = 0. \end{aligned}$$

Using Theorem 2.4 (Weyl's criterion), we conclude that $(\{n\boldsymbol{\alpha}\})_{n\in\mathbb{N}_0}$ is uniformly distributed modulo one.

" $\Rightarrow$ " Suppose now that $(\{n\boldsymbol{\alpha}\})_{n\in\mathbb{N}_0}$ is uniformly distributed modulo one and assume, by contradiction, that $1, \alpha_1, ..., \alpha_s$ are not linearly independent over $\mathbb{Q}$. Then there exists $\mathbf{h} \in \mathbb{Z}^s \setminus \{0\}$ such that $\mathbf{h} \cdot \boldsymbol{\alpha} \in \mathbb{Q}$.

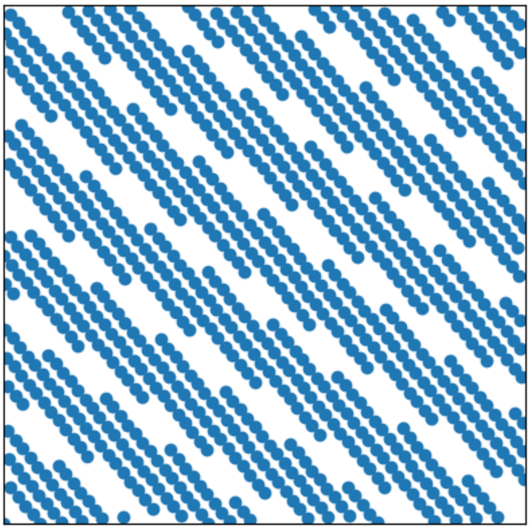


Figure 2.1: First 1000 points of the $n\alpha$-sequence, with $\alpha = (\sqrt{2}, \sqrt{3})$

We write $\mathbf{h} \cdot \boldsymbol{\alpha} = \frac{a}{b}$ for some $a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}$.
We have

$$\begin{aligned}\sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \boldsymbol{\alpha}) = \sum_{n=0}^{N-1} \exp(2\pi i n \frac{a}{b}) \geq \sum_{\substack{n=0 \\ n \in b\mathbb{Z}}}^{N-1} \exp(2\pi i n \frac{a}{b}) \\ &= |\{0, ..., N-1\} \cap b\mathbb{Z}| \\ &= \left| \left\{ k \in \mathbb{Z} \mid 0 \leq k \leq \left\lfloor \frac{N-1}{b} \right\rfloor \right\} \right| \\ &= \left\lceil \frac{N-1}{b} \right\rceil \geq \frac{N-1}{b}.\end{aligned}$$

We deduce that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \boldsymbol{\alpha}) \geq \frac{N-1}{Nb} = \frac{1}{b} > 0.$$

Finally, using (2.3), we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \{n\boldsymbol{\alpha}\}) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \boldsymbol{\alpha}) > 0.$$

This is in contradiction with Theorem2.4 (Weyl's criterion). $\square$

Before moving to the next section, we provide another example of uniformly distributed sequence which will be useful later.

Definition 2.6 *Let $b \in \mathbb{N}_{\geq 2}$. We define the b-adic radical inverse function φ_b : $\mathbb{N}_0 \to [0,1)$ as*

$$\varphi_b(n) = \frac{n_0}{b} + \frac{n_1}{b^2} + \frac{n_3}{b^3} + ...$$

where $n \in \mathbb{N}_0$ has a b-adic digit expansion $n = n_0 + n_1 b + n_2 b^2 + ...$ for some $n_i \in \{0, 1, ..., b-1\}$.

Example 2.7 *Symbolically, the b-adic radical inverse $\varphi_b(n)$ of $n \in \mathbb{N}_0$ is the reflection of the b-adic digit expansion of n at the comma.*
So if n has a digit expansion $(n)_b = n_k ... n_0$ in base base b then we have $(\varphi_b(n))_b = 0.n_0 ... n_k$.

Proposition 2.8 (Van der Corput sequence) *We define the Van der Corput sequence $(x_n)_{n \in \mathbb{N}_0}$ in base b as $x_n = \varphi_b(n)$. Then this sequence is uniformly distributed modulo one.*

Proof To prove the result, we will directly check Definition 2.1.
Fix $m \in \mathbb{N}$. For every $a \in \{0, 1, ..., b^m - 1\}$ we can write its b-adic expansion with m terms as $a = a_0 b^{m-1} + a_1 b^{m-2} + ... + a_{m-2} b + a_{m-1}$ with $a_i \in \{0, 1, ..., b-1\}, i = 0, ..., m-1$.
We first consider intervals of the form $J_a = [\frac{a}{b^m}, \frac{a+1}{b^m}) \subseteq [0,1)$ and we then proceed by approximation. For $n \in \mathbb{N}_0$ with b-adic digit expansion $n = n_0 + n_1 b + n_2 b^2 + ...$, the element $\varphi_b(n)$ belongs to J_a if and only if

$$\frac{a}{b^m} \leq \frac{n_0}{b} + \frac{n_1}{b^2} + \frac{n_3}{b^3} + ... < \frac{a+1}{b^m}.$$

Multiplying both sides by b^m gives

$$a \leq n_0 b^{m-1} + n_1 b^{m-2} + ... + n_{m-1} + \frac{n_m}{b} + ... \leq a + 1.$$

Since $n_0 b^{m-1} + n_1 b^{m-2} + ... + n_{m-1} \in \mathbb{N}_0$ and $\frac{n_m}{b} + ... \in [0,1)$, we deduce that $a = n_0 b^{m-1} + n_1 b^{m-2} + ... + n_{m-1}$. In terms of the b-adic digit expansion

$$n_0 = a_0, n_1 = a_1, ..., n_{m-1} = a_{m-1}.$$

Since we can write $n = n_0 + n_1 b + ... + n_{m-1} b^{m-1} + b^m (n_m + n_{m+1} b + ...)$, the above is also equivalent to

$$n \equiv a' \pmod{b^m}, \quad a' = a_0 + a_1 b + ... + a_{m-1} b^{m-1}.$$

So far we proved that an element $x_n = \varphi_b(n)$ is in J_a if and only if $n \equiv a'$ (mod b^m). Since this congruence has a unique solution modulo b^m, it follows that exactly one of b^m consecutive elements of the Van der Corput sequence belongs to J_a. Hence, for $N \in \mathbb{N}$ it holds that

$$A(J_a, N) = \left\lfloor \frac{N}{b^m} \right\rfloor + \theta$$

for some $\theta \in \{0,1\}$.
Since $\lim_{N\to\infty} \frac{1}{N} \left\lfloor \frac{N}{b^m} \right\rfloor = \frac{1}{b^m}$, we obtain

$$\lim_{N\to\infty} \frac{A(J_a, N)}{N} = \frac{1}{b^m} = \lambda_1(J_a).$$

We now generalize this result for any $J = [\alpha, \beta) \subseteq [0,1)$ via approximation. Let

$$\underline{J} = \bigcup_{a:J_a \subseteq J} J_a, \quad \text{and} \quad \overline{J} = \bigcup_{a:J_a \cap J \neq \emptyset} J_a.$$

Then, by construction, we have $\underline{J} \subseteq J \subseteq \overline{J}$ and $\lambda_1(\overline{J}) - \lambda(\underline{J}) = \lambda_1(\overline{J} \setminus \underline{J}) < \frac{2}{2^m}$.
From the inclusions, we deduce

$$\frac{A(\underline{J}, N)}{N} - \lambda_1(\overline{J}) \leq \frac{A(J,N)}{N} - \lambda_1(J) \leq \frac{A(\overline{J}, N)}{N} - \lambda_1(\underline{J}). \tag{2.4}$$

Lemma A.1 applied on (2.4) yields

$$\begin{aligned}
&\left| \frac{A(J,N)}{N} - \lambda_1(J) \right| \\
&\leq \lambda_1(\overline{J}) - \lambda_1(\underline{J}) + \max\left\{ \left| \frac{A(\underline{J},N)}{N} - \lambda_1(\underline{J}) \right|, \left| \frac{A(\overline{J},N)}{N} - \lambda_1(\overline{J}) \right| \right\} \\
&\leq \frac{2}{b^m} + \max\left\{ \left| \frac{A(\underline{J},N)}{N} - \lambda_1(\underline{J}) \right|, \left| \frac{A(\overline{J},N)}{N} - \lambda_1(\overline{J}) \right| \right\}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\lim_{N\to\infty} \frac{A(\underline{J},N)}{N} = \lim_{N\to\infty} \sum_{a:J_a\subseteq J} \frac{A(J_a,N)}{N} &= \sum_{a:J_a\subseteq J} \lim_{N\to\infty} \frac{A(J_a,N)}{N} \\
&= \sum_{a:J_a\subseteq J} \lambda_1(J_a) = \lambda_1(\underline{J})
\end{aligned}$$

where in the last inequality we used that the $\{J_a : a \in \{0, ..., b^m - 1\}\}$ are disjoint.
In the same manner, we deduce that $\lim_{N\to\infty} \left| \frac{A(\overline{J},N)}{N} - \lambda_1(\overline{J}) \right| = 0$. It follows that

$$\lim_{N\to\infty} \left| \frac{A(J,N)}{N} - \lambda_1(J) \right| \leq \frac{2}{b^m}.$$

Since $m \in \mathbb{N}$ was arbitrary, we obtain

$$\lim_{N\to\infty} \left| \frac{A(J,N)}{N} - \lambda_1(J) \right| = 0$$

and we deduce that the Van der Corput sequence in base b is uniformly distributed modulo one. ☐

Chapter 3

Discrepancy

Now that we have several examples of uniformly distributed sequences at hand, we want to define a measure for the quality of such sequences.

3.1 Definitions and properties

Definition 3.1 *Let $\mathcal{P} \subseteq [0,1)^s$ be an N-element point set. The extreme discrepancy D_N of this point set is defined as*

$$D_N(\mathcal{P}) = \sup_{\substack{\mathbf{a},\mathbf{b}\in[0,1]^s \\ \mathbf{a}\leq\mathbf{b}}} \left| \frac{A([\mathbf{a},\mathbf{b}),\mathcal{P},N)}{N} - \lambda_s([\mathbf{a},\mathbf{b})) \right|.$$

For an infinite sequence $\mathcal{S}$, $D_N(\mathcal{S})$ is the discrepancy of the first N elements of S.
We also define a weaker notion of discrepancy.

Definition 3.2 *Let $\mathcal{P} \subseteq [0,1)^s$ be an N-element point set. The star discrepancy D_N^* of this point set is defined as*

$$D_N^*(\mathcal{P}) = \sup_{\mathbf{a}\in[0,1]^s} \left| \frac{A([\mathbf{0},\mathbf{a}),\mathcal{P},N)}{N} - \lambda_s([\mathbf{0},\mathbf{a})) \right|.$$

We have the following relation between extreme and star discrepancy:

Proposition 3.3 *For every N-element point set $\mathcal{P} \subseteq [0,1)^s$,*

$$D_N^*(\mathcal{P}) \leq D_N(\mathcal{P}) \leq 2^s D_N^*(\mathcal{P})$$

Proof The left inequality is clear from the definitions of extreme and star discrepancy.

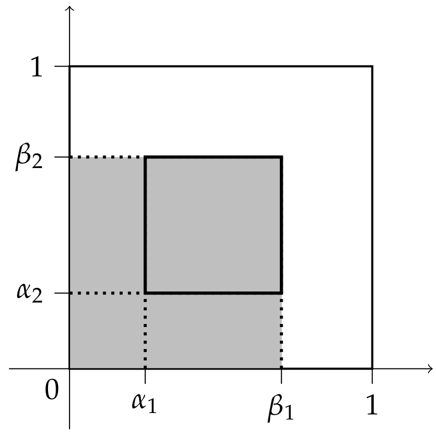


Figure 3.1: Illustration of the decomposition of $J = [\alpha_1, \beta_1) \times [\alpha_2, \beta_2)$

To fix ideas, we first prove the right inequality for $s = 2$.
Let $J = [\alpha_1, \beta_1) \times [\alpha_2, \beta_2) \subseteq [0, 1)^2$. We have the following decomposition (see Figure 3.1):

$$J = ([0, \beta_1) \times [0, \beta_2)) \setminus \{[0, \alpha_1) \times [0, \beta_2) \cup [0, \beta_1) \times [0, \alpha_2)\}.$$

Hence, by the inclusion-exclusion principle:

$$\begin{aligned} A(J, N) = A(&[0, \beta_1) \times [0, \beta_2), N) \\ &- A([0, \alpha_1) \times [0, \beta_2), N) - A([0, \beta_1) \times [0, \alpha_2), N) \\ &+ A([0, \alpha_1) \times [0, \alpha_2), N) \end{aligned}$$

and similarly

$$\begin{aligned} \lambda_2(J) = \lambda_2(&[0, \beta_1) \times [0, \beta_2)) \\ &- \lambda_2([0, \beta_1) \times [0, \beta_2)) - \lambda_2([0, \beta_1) \times [0, \alpha_2)) \\ &+ \lambda_2([0, \alpha_1) \times [0, \alpha_2)). \end{aligned}$$

We deduce that

$$\begin{aligned} \frac{A(J, N)}{N} - \lambda_2([\alpha, \beta)) = &\frac{A([0, \beta_1) \times [0, \beta_2), N)}{N} - \lambda_2([0, \beta_1) \times [0, \beta_2)) \\ &- \left(\frac{A([0, \beta_1) \times [0, \beta_2), N)}{N} - \lambda_2([0, \beta_1) \times [0, \beta_2)) \right) \\ &- \left(\frac{A([0, \beta_1) \times [0, \alpha_2), N)}{N} - \lambda_2([0, \beta_1) \times [0, \alpha_2)) \right) \\ &+ \left(\frac{A([0, \alpha_1) \times [0, \alpha_2), N)}{N} - \lambda_2([0, \alpha_1) \times [0, \alpha_2)) \right). \end{aligned}$$

Taking the absolute value, the triangle inequality yields

$$\left| \frac{A(J, N)}{N} - \lambda_2(J) \right| = 2^2 D_N^*(\mathcal{P}).$$

Since $\alpha \leq \beta$ were arbitrary, we deduce that

$$D_N(\mathcal{P}) \leq 2^2 D_N^*(\mathcal{P}).$$

This proves the case $s = 2$.
For the general case, we consider $J = [\boldsymbol{\alpha}, \boldsymbol{\beta}) \subseteq [0,1)^s$. We introduce the following notation

$$\Gamma := \{(\gamma_1, .., \gamma_s) \in \{\alpha_1, \beta_1\} \times ... \times \{\alpha_s, \beta_s\} \mid \#\{i \in \{1, .., s\} \mid \gamma_i = \alpha_i\} = 1\}$$

and

$$B_{(\gamma_1,..,\gamma_s)} := [0, \gamma_1) \times ... \times [0, \gamma_s) \quad \text{for } (\gamma_1, .., \gamma_s) \in \Gamma.$$

Then we can write

$$J = [\mathbf{0}, \boldsymbol{\beta}) \setminus \left(\bigcup_{(\gamma_1,..,\gamma_s) \in \Gamma} B_{(\gamma_1,..,\gamma_s)} \right).$$

As $|\Gamma| = s$, let $\varphi : \{1, 2, ..., s\} \to \Gamma$ be a bijection, then we have by the inclusion-exclusion principle,

$$A(J, N) = A([\mathbf{0}, \boldsymbol{\beta}), N) - \sum_{\emptyset \neq S \subseteq \{1,...,s\}} (-1)^{|S|-1} A\left(\bigcap_{k \in S} B_{\varphi(k)}, N \right).$$

and similarly for λ_s

$$\lambda_s(J) = \lambda_s([\mathbf{0}, \boldsymbol{\beta})) - \sum_{\emptyset \neq S \subseteq \{1,...,s\}} (-1)^{|S|-1} \lambda_s\left(\bigcap_{k \in S} B_{\varphi(k)} \right).$$

We deduce that,

$$\begin{aligned} \left| \frac{A(J,N)}{N} - \lambda_s(J) \right| &\leq \left| \frac{A([\mathbf{0}, \boldsymbol{\beta}), N)}{N} - \lambda_s([\mathbf{0}, \boldsymbol{\beta})) \right| \\ &\quad + \sum_{\emptyset \neq S \subseteq \{1,...,s\}} \left| \frac{A\left(\bigcap_{k \in S} B_{\varphi(k)}, N \right)}{N} - \lambda_s\left(\bigcap_{k \in S} B_{\varphi(k)} \right) \right| \end{aligned}$$

Finally, since for all $(\gamma_1, .., \gamma_s) \in \Gamma$, the set $B_{(\gamma_1,..,\gamma_s)}$ is a rectangle anchored at 0, we deduce that also $\bigcap_{k \in S} B_{\varphi(k)}$ is a rectangle anchored at 0 for all $\emptyset \neq S \subseteq \{1, ..., s\}$.
We deduce that

$$\left| \frac{A\left(\bigcap_{k \in S} B_{\varphi(k)}, N \right)}{N} - \lambda_s\left(\bigcap_{k \in S} B_{\varphi(k)} \right) \right| \leq D_N^*(\mathcal{P})$$

and therefore

$$\left|\frac{A(J,N)}{N} - \lambda_s(J)\right| \leq D_N^*(\mathcal{P}) + (2^s - 1)D_N^*(\mathcal{P}) = 2^s D_N^*(\mathcal{P}).$$

Since $J = [\boldsymbol{\alpha}, \boldsymbol{\beta}) \subseteq [0,1)^s$ was arbitrary, we conclude that

$$D_N(\mathcal{P}) \leq 2^s D_N^*(\mathcal{P}). \qquad \square$$

The next theorem shows a criterion for a sequence to be uniformly distributed modulo one, based on the discrepancy.

Theorem 3.4 *Let $\mathcal{S} \subseteq [0,1)^s$ be an infinite sequence. The following assertions are equivalent:*

(a) *$\mathcal{S}$ is uniformly distributed modulo one;*

(b) $\lim_{N\to\infty} D_N^*(\mathcal{S})$*;*

(c) $\lim_{N\to\infty} D_N(\mathcal{S})$.

Proof The equivalence of (b) and (c) follows from Proposition 3.3.
$''(c) \implies (a)''$: Assume that $\lim_{N\to\infty} D_N(\mathcal{S})$, i.e.

$$\lim_{N\to\infty} \sup_{\substack{\mathbf{a},\mathbf{b}\in[0,1]^s \\ \mathbf{a}\leq\mathbf{b}}} \left|\frac{A([\mathbf{a},\mathbf{b}),\mathcal{S},N)}{N} - \lambda_s([\mathbf{a},\mathbf{b}))\right| = 0.$$

Then, in particular, for all intervals $[\mathbf{a},\mathbf{b}) \subseteq [0,1)^s$ we have

$$\lim_{N\to\infty} \left|\frac{A([\mathbf{a},\mathbf{b}),\mathcal{S},N)}{N} - \lambda_s([\mathbf{a},\mathbf{b}))\right| = 0.$$

We conclude that $\mathcal{S}$ is uniformly distributed modulo one.
$''(a) \implies (b)''$: Assume now that the sequence $\mathcal{S}$ is uniformly distributed modulo one. Let $J = [\mathbf{0}, \mathbf{a}) \subseteq [0,1)^s$ and choose $m \in \mathbb{N}_{\geq 2}$.
We consider the intervals $Q_{\mathbf{k}} = \prod_{j=1}^s [\frac{k_j}{m}, \frac{k_j+1}{m})$ for $\mathbf{k} = (k_1, ..., k_s) \in \{0, 1, ..., m-1\}$ and we define

$$\underline{J} = \bigcup_{\mathbf{k}:Q_{\mathbf{k}}\subseteq J} Q_{\mathbf{k}} \quad \text{and} \quad \overline{J} = \bigcup_{\mathbf{k}:Q_{\mathbf{k}}\cap J\neq\emptyset} Q_{\mathbf{k}}.$$

By construction, we have $\underline{J} \subseteq J \subseteq \overline{J}$ and $\underline{J}, \overline{J}$ are also intervals anchored at 0 ($\underline{J}$ may be empty) and their edge-lengths differ by at most $\frac{1}{m}$ (see Figure 3.2).

We can therefore apply Lemma A.2 and we obtain

$$\lambda_s(\overline{J}) - \lambda_s(\underline{J}) \leq s\frac{1}{m}.$$

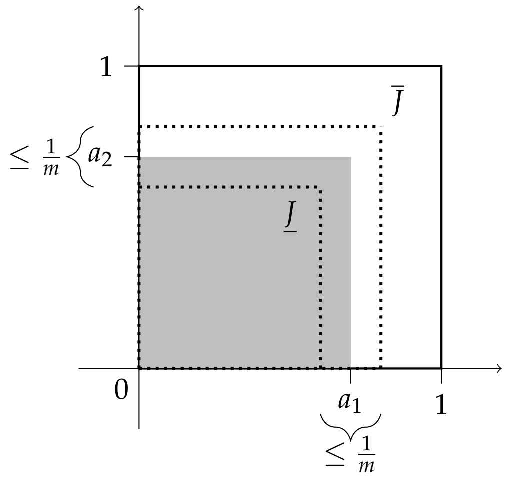


Figure 3.2: Approximation of the interval $J = [\mathbf{0}, \mathbf{a})$ by $\underline{J}$ and $\overline{J}$

We deduce that

$$\frac{A(\underline{J}, \mathcal{S}, N)}{N} - \lambda_s(\overline{J}) \leq \frac{A(J, \mathcal{S}, N)}{N} - \lambda_s(J) \leq \frac{A(\overline{J}, \mathcal{S}, N)}{N} - \lambda_s(\underline{J}).$$

To use Lemma A.1, we first write

$$\frac{A(J, \mathcal{S}, N)}{N} - \lambda_s(J) \begin{cases} \geq & \frac{A(\underline{J}, \mathcal{S}, N)}{N} - \lambda_s(\underline{J}) - (\lambda_s(\overline{J}) - \lambda_s(\underline{J})) \\ \leq & \frac{A(\overline{J}, \mathcal{S}, N)}{N} - \lambda_s(\overline{J}) + (\lambda_s(\overline{J}) - \lambda_s(\underline{J})) \end{cases}$$

and we now apply Lemma A.1 to have

$$\begin{aligned} \left| \frac{A(J, \mathcal{S}, N)}{N} - \lambda_s(J) \right| &\leq \lambda_s(\overline{J}) - \lambda_s(\underline{J}) \\ &\quad + \max \left\{ \left| \frac{A(\overline{J}, \mathcal{S}, N)}{N} - \lambda_s(\overline{J}) \right|, \left| \frac{A(\underline{J}, \mathcal{S}, N)}{N} - \lambda_s(\underline{J}) \right| \right\} \\ &\leq \frac{s}{m} + \max_{\mathbf{k} \in \{0,1,\dots,m\}^s} \left\{ \left| \frac{A([\mathbf{0}, \frac{1}{m}\mathbf{k}), \mathcal{S}, N)}{N} - \lambda_s([\mathbf{0}, \frac{1}{m}\mathbf{k})) \right| \right\}. \end{aligned} \tag{3.1}$$

Note that the final term does not depend on J.
Since $\mathcal{S}$ is uniformly distributed modulo one, we know that for all $\mathbf{k} \in \{0, 1, \dots, m\}^s$,

$$\lim_{N \to \infty} \left| \frac{A([\mathbf{0}, \frac{1}{m}\mathbf{k}), \mathcal{S}, N)}{N} - \lambda_s([\mathbf{0}, \frac{1}{m}\mathbf{k})) \right| = 0$$

and so also,

$$\lim_{N \to \infty} \max_{\mathbf{k} \in \{0,1,\dots,m\}^s} \left\{ \left| \frac{A([\mathbf{0}, \frac{1}{m}\mathbf{k}), \mathcal{S}, N)}{N} - \lambda_s([\mathbf{0}, \frac{1}{m}\mathbf{k})) \right| \right\} = 0.$$

Taking the supremum over all $J = [\mathbf{0}, \mathbf{a}) \subseteq [0,1)^s$ in (3.1), we deduce that

$$0 \leq \limsup_{N \to \infty} D_N^*(\mathcal{S}) \leq \frac{s}{m}.$$

Finally, since $m \in \mathbb{N}$ was arbitrary, we have

$$\lim_{N \to \infty} D_N^*(\mathcal{S}) = 0.$$

This concludes the proof. □

Theorem 3.4 shows that a sequence is uniformly distributed modulo one if and only if its discrepancy (extreme or star) converges to 0. The next theorem shows that the order of convergence cannot be faster than $\frac{1}{N}$.

Proposition 3.5 *For every N-element point set $\mathcal{P} \subseteq [0,1)^s$ we have*

$$D_N(\mathcal{P}) \geq \frac{1}{N} \quad \textit{and} \quad D_N^*(\mathcal{P}) \geq \frac{1}{2^s N}.$$

Proof Let $0 < \varepsilon < \frac{1}{N}$ and let $\mathbf{x} = (x_1, ..., x_s) \in \mathcal{P}$. We consider the interval

$$J = ([x_1, x_1 + \varepsilon^{\frac{1}{s}}) \times ... \times [x_s, x_s + \varepsilon^{\frac{1}{s}})) \cap [0,1]^s.$$

By construction, we have $\mathbf{x} \in J$ so that $A(J, N) \geq 1$. It follows that

$$D_N(\mathcal{P}) \geq \frac{A(J,N)}{N} - \lambda_s(J) \geq \frac{1}{N} - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $D_N(\mathcal{P}) \geq \frac{1}{N}$.
The result for the star discrepancy follows from Proposition 3.3. □

To see how tight the lower bound from Proposition 3.5 is, we compute the discrepancy of a very simple sequence on the unit cube, namely the regular lattice.

Definition 3.6 (Regular lattice) *Let $m \in \mathbb{N}_{\geq 2}$. The regular lattice in $[0,1)^s$ is defined as*

$$\Gamma_{m,s} = \left\{ \left(\frac{2n_1 + 1}{2m}, ..., \frac{2n_s + 1}{2m} \right) : n_1, ..., n_s \in \{0, ..., m-1\} \right\}.$$

Theorem 3.7 *Let $s, m \in \mathbb{N}, m \geq 2$. For the start discrepancy of the regular lattice $\Gamma_{m,s}$ with $N = m^s$ elements in $[0,1)^s$ it holds that*

$$D_N^*(\Gamma_{m,s}) = 1 - \left(1 - \frac{1}{2m}\right)^s.$$

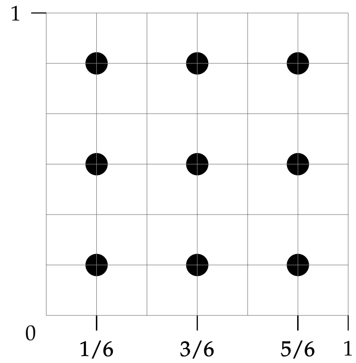


Figure 3.3: Regular lattice $\Gamma_{3,2}$

Proof $'' \geq ''$: The most extreme value a coordinate of a point $x \in \Gamma_{m,s}$ can be is $\frac{2(m-1)+1}{2m} = 1 - \frac{1}{2m}$. It follows that $\Gamma_{m,s} \subseteq [0, 1 - \frac{1}{2m}]^s =: B_{m,s}$ and therefore

$$D_N^*(\Gamma_{m,s}) \geq \left| \frac{A(B_{m,s}, \Gamma_{m,s}, N)}{N} - \lambda_s(B_{m,s}) \right| = 1 - \left(1 - \frac{1}{2m}\right)^s.$$

$'' \leq ''$: Let $J = [0, \alpha_1) \times ... \times [0, \alpha_s) \subseteq [0,1)^s$. For $x \in \Gamma_{m,s}$, we have $x_i \geq \frac{1}{2m}$ for all $i = 1, ..., s$. Therefore if $\min_{j=1,...,s} \alpha_j < \frac{1}{2m}$, then

$$A(J, \Gamma_{m,s}, N) = 0 \quad \text{and} \quad \lambda_s(J) \leq \frac{1}{2m} \leq 1 - \left(1 - \frac{1}{2m}\right)^s.$$

Otherwise let $a_j \in \{0, ..., m-1\}$ for $j = 1, ..., s$ such that $\frac{2a_j+1}{2m} < \alpha_j \leq \frac{2a_j+3}{2m}$. Then $A(J, \Gamma_{m,s}, N) = \prod_{i=1}^s (a_j + 1)$ and

$$\prod_{i=1}^s \frac{2a_j+1}{2m} \leq \lambda_s(J) = \prod_{i=1}^s \alpha_j \leq \prod_{i=1}^s \frac{2(a_j+1)+1}{2m} \leq \prod_{i=1}^s \frac{2(a_j+1)}{2m} = \frac{A(J, \Gamma_{m,s}, N)}{N}.$$

We deduce that

$$0 \leq \frac{A(J, \Gamma_{m,s}, N)}{N} - \lambda_s(J) \leq \prod_{i=1}^s \frac{a_j+1}{m} - \prod_{i=1}^s \frac{2a_j+1}{2m}.$$

Therefore, Lemma A.2 yields

$$\left| \frac{A(J, \Gamma_{m,s}, N)}{N} - \lambda_s(J) \right| \leq \left| \prod_{i=1}^s \frac{a_j+1}{m} - \prod_{i=1}^s \frac{2a_j+1}{2m} \right| \leq 1 - \left(1 - \frac{1}{2m}\right)^s.$$

Since the upper bound is independent of J, we deduce that

$$D_N^*(\Gamma_{m,s}) \leq 1 - \left(1 - \frac{1}{2m}\right)^s.$$

This concludes the proof. $\square$

For $s = 1$, we obtain $D_N^*(\Gamma_{m,1}) = \frac{1}{2m}$ which is optimal, by Proposition 3.5. Therefore the one-dimensional regular lattice, is a uniformly distributed sequence modulo one with the lowest star discrepancy among all N-element point sets.
For dimension $s \geq 2$, we have

$$\frac{1}{2N^{1/s}} = \frac{1}{2m} \leq 1 - \left(1 - \frac{1}{2m}\right)^s \leq \frac{s}{2m} = \frac{s}{2N^{1/s}}$$

where the right inequality comes from the Lemma A.2. We deduce that the convergence rate $D_N^*(\Gamma_{m,s})$ is at least $1/s$ which is rather weak compared to the rate 1 of Proposition 3.5. We have therefore hope to find sequences with much lower discrepancy, and we will indeed construct such examples later on.

We finish this section by giving another definition of discrepancy, which will be useful in the next section.
Given a point set $\mathcal{P}$, we can write the star discrepancy $D_N^*(\mathcal{P})$ as the supremum norm of the function $\Delta_{\mathcal{P},N} : [0,1]^s \to \mathbb{R}$,

$$\Delta_{\mathcal{P},N}(\mathbf{y}) = \frac{A([\mathbf{0},\mathbf{y}),\mathcal{P},N)}{N} - \lambda_s([\mathbf{0},\mathbf{y})).$$

It is then natural to extend this definition for other p-norms.

Definition 3.8 (L_p discrepancy) *Let $\mathcal{P} \subseteq [0,1)^s$ be an N-element point set and $p \in [1,\infty)$. The L_p discrepancy of $\mathcal{P}$ is defined as*

$$L_{p,N}(\mathcal{P}) = \|\Delta_{\mathcal{P},N}\|_{L_p} = \left(\int_{[0,1]^s} \left|\frac{A([\mathbf{0},\mathbf{y}),\mathcal{P},N)}{N} - \lambda_s([\mathbf{0},\mathbf{y}))\right|^p d\mathbf{y}\right)^{1/p}.$$

For an infinite sequence $\mathcal{S}$, the L_p discrepancy $L_{p,N}(\mathcal{S})$ is defined as the discrepancy of the first N elements of $\mathcal{S}$.

From the monotonicity of p-norms, it follows that for $p_1 \leq p_2$ we have $L_{p_1,N}(\mathcal{P}) \leq L_{p_2,N}(\mathcal{P})$.
The following theorem gives the relation between L_p and star discrepancy. We state it without proof.

Proposition 3.9 *For every N-element point set $\mathcal{P} \subseteq [0,1]^s$, we have*

$$L_{p,N}(\mathcal{P}) \leq D_N^*(\mathcal{P}) \leq c(s,p) L_{p,N}(\mathcal{P})^{\frac{p}{p+s}}$$

where $c(s,p) > 0$ only depends on s and p.

The following corollary follows directly from Proposition 3.9.

Corollary 3.10 *Let $p \in [0,\infty]$. A sequence $\mathcal{S} \subseteq [0,1)^s$ is uniformly distributed modulo one if and only if $\lim_{N\to\infty} L_{p,N}(\mathcal{S}) = 0$.*

3.2 Bounds on the discrepancy

In this section, we want to find bounds for the discrepancy. Lower bounds are useful to have an idea of how low can the discrepancy be and therefore to make sense of an ”optimal” discrepancy. Upper bounds are needed to quantify the order of convergence of the discrepancy of a particular sequence. We will prove one upper bound that can be applied to many sequences but often it is computed on a case by case basis.

3.2.1 Roth's lower bounds

Proposition 3.5 already gives a lower bound, but the next theorem gives us a much tighter lower bound, especially for high dimensions.

Theorem 3.11 (K.F. Roth, 1954) *For every dimension $s \in \mathbb{N}$ there exists a constant $c_s > 0$ with the following property: for every N-element point set $\mathcal{P} \in [0,1)^s$,*

$$D_N(\mathcal{P}) \geq D_N^*(\mathcal{P}) \geq L_{2,N}(\mathcal{P}) \geq c_s \frac{(\log N)^{\frac{s-1}{2}}}{N}.$$

Proof We only prove the theorem for the case of dimension $s = 2$.
For $\mathbf{x} \in [0,1]^2$, let

$$D(\mathbf{x}) = N\lambda_2([\mathbf{0},\mathbf{x})) - A([\mathbf{0},\mathbf{x}),\mathcal{P},N).$$

Using the Cauchy-Schwartz inequality, we have for every $F : [0,1]^2 \to \mathbb{R}$

$$\int_{[0,1]^2} F(\mathbf{x})D(\mathbf{x})d\mathbf{x} \leq \left(\int_{[0,1]^2} F^2(\mathbf{x})d\mathbf{x}\right)^{1/2} \left(\int_{[0,1]^2} D^2(\mathbf{x})d\mathbf{x}\right)^{1/2}$$

and hence, if $F \neq 0$,

$$NL_{2,N}(\mathcal{P}) = \left(\int_{[0,1]^2} D^2(\mathbf{x})d\mathbf{x}\right)^{1/2} \geq \frac{\int_{[0,1]^2} F(\mathbf{x})D(\mathbf{x})d\mathbf{x}}{\left(\int_{[0,1]^2} F^2(\mathbf{x})d\mathbf{x}\right)^{1/2}}. \tag{3.2}$$

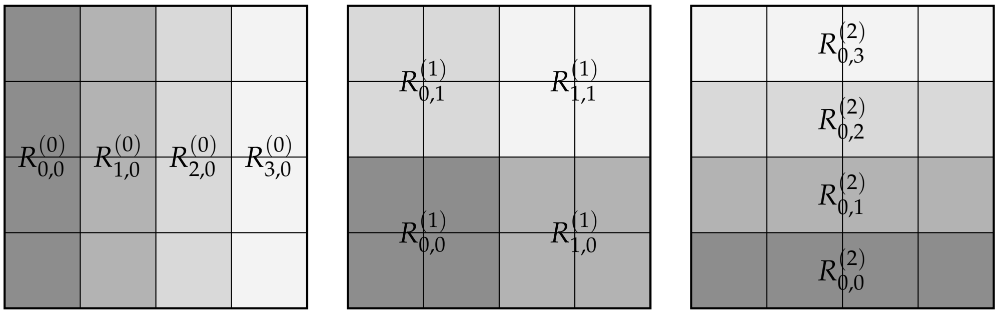
We therefore want to find a function F such that $\int_{[0,1]^2} F^2(\mathbf{x})d\mathbf{x} = \mathcal{O}(logN)$ and $\int_{[0,1]^2} F(\mathbf{x})D(\mathbf{x})d\mathbf{x}$ is at least of the order of magnitude $\log N$.

We first define the auxiliary function $\psi : \mathbb{R} \to \mathbb{R}$ via

$$\psi(x) = \begin{cases} 1 & \text{if } x \in [k, k+1/2) \text{ for some } k \in \mathbb{Z} \\ -1 & \text{if } x \in [k+1/2, k+1] \text{ for some } k \in \mathbb{Z} \end{cases}.$$

We note that, for $i \in \mathbb{N}_0$ and $a \in \{0,1,...,2^i - 1\}$, we have

$$\int_{a/2^i}^{(a+1)/2^i} \psi(2^i y)dy = \frac{1}{2^i}\int_a^{a+1} \psi(y)dy = \frac{1}{2^i}(1/2 - 1/2) = 0.$$



(a) Partition $j = 0$ **(b)** Partition $j = 1$ **(c)** Partition $j = 2$

Figure 3.4: Partitions of the unit square $[0,1)^2$ in the case $m = 2$.

We now choose $m \in \mathbb{N}$ such that $1 + \log_2 N \leq m < 2 + \log_2 N$, i.e. such that $2N \leq 2^m < 4N$. We consider the sets $R_{a,b}^{(j)}$ define as

$$R_{a,b}^{(j)} = \left[\frac{a}{2^{m-j}}, \frac{a+1}{2^{m-j}}\right) \times \left[\frac{b}{2^j}, \frac{b+1}{2^j}\right)$$

where $a \in \{0,1,...,2^{m-j}-1\}$, $b \in \{0,1,...,2^j-1\}$, $j = 0,1,...,m$. Note that for a fixed $j \in \{0,1,...,m\}$ the sets $\{R_{a,b}^{(j)}\}_{a,b}$ form a partition of $[0,1)^2$ (see Figure 3.4).

For $j \in \{0,1,...,m\}$ define the functions $f_j = [0,1]^2 \to \{-1,0,1\}$ as follows: for $\mathbf{y} = (y_1, y_2) \in R_{a,b}^{(j)}$, we set

$$f_j(\mathbf{y}) = \begin{cases} 0 & \text{if } R_{a,b}^{(j)} \cap \mathcal{P} \neq \emptyset \\ \psi(2^{m-j}y_1)\psi(2^j y_2) & \text{if } R_{a,b}^{(j)} \cap \mathcal{P} = \emptyset \end{cases}. \tag{3.3}$$

Now we show that the functions $\{f_j\}_{j=0,...,m}$ are mutually orthogonal. Let $i < j$ and consider the partitions of $[0,1)$ given by

$$\left\{\left[\frac{a}{2^j}, \frac{a+1}{2^j}\right)\right\}_{a=0,...,2^j-1} \quad \text{and} \quad \left\{\left[\frac{2b}{2^{i+1}}, \frac{2b+1}{2^{i+1}}\right), \left[\frac{2b+1}{2^{i+1}}, \frac{2(b+1)}{2^{i+1}}\right)\right\}_{b=0,...,2^i-1}.$$

Since $i < j$ the first partition is finer than the first one, and therefore for all $a = 0,...,2^j-1$ there exists some $b \in \{0,...,2^i-1\}$ such that either

$$\left[\frac{a}{2^j}, \frac{a+1}{2^j}\right) \subseteq \left[\frac{2b}{2^{i+1}}, \frac{2b+1}{2^{i+1}}\right) \quad \text{or} \quad \left[\frac{a}{2^j}, \frac{a+1}{2^j}\right) \subseteq \left[\frac{2b+1}{2^{i+1}}, \frac{2(b+1)}{2^{i+1}}\right).$$

We deduce that $\psi(2^i y) = c \in \{-1,1\}$ is constant for $y \in [a/2^j, (a+1)/2^j)$ and therefore

$$\int_{a/2^j}^{(a+1)/2^j} \psi(2^j y)\psi(2^i y)dy = c\int_{a/2^j}^{(a+1)/2^j} \psi(2^j y) = 0. \tag{3.4}$$

We now set

$$R_{a,b}^{(i,j)} = \left[\frac{a}{2^{m-i}}, \frac{a+1}{2^{m-i}}\right) \times \left[\frac{b}{2^j}, \frac{b+1}{2^j}\right)$$

for $i,j \in \{0,1,...,m\}, a \in \{0,1,...,2^{m-i}-1\}$ and $b \in \{0,1,...,2^j-1\}$.
Since $\{R_{a,b}^{(i,j)}\}_{a,b}$ is a partition of $[0,1)^2$ we have

$$\left|\int_{[0,1]^2} f_i(\mathbf{y}) f_j(\mathbf{y}) d\mathbf{y}\right| = \left|\sum_{a=0}^{2^{m-i}-1} \sum_{b=0}^{2^j} \int_{R_{a,b}^{(i,j)}} f_i(\mathbf{y}) f_j(\mathbf{y}) d\mathbf{y}\right|.$$

By Lemma A.4, we have that $R_{a,b}^{(i,j)} \subseteq R_{a,\lfloor b2^{i-j}\rfloor}^{(i)}$ and also $R_{a,b}^{(i,j)} \subseteq R_{\lfloor a2^{i-j}\rfloor,b}^{(j)}$. We therefore have

$$\begin{aligned}
&\left|\int_{[0,1]^2} f_i(\mathbf{y}) f_j(\mathbf{y}) d\mathbf{y}\right| \\
&= \left|\sum_{a=0}^{2^{m-i}-1} \sum_{b=0}^{2^j} \int_{R_{a,b}^{(i,j)}} f_i(\mathbf{y}) f_j(\mathbf{y}) d\mathbf{y}\right| \\
&= \left|\sum_{a=0}^{2^{m-i}-1} \sum_{b=0}^{2^j} \int_{R_{a,b}^{(i,j)}} f_i(\mathbf{y}) f_j(\mathbf{y}) \mathbb{1}_{y \in R_{a,\lfloor b2^{i-j}\rfloor}^{(i)}} \mathbb{1}_{y \in R_{\lfloor a2^{i-j}\rfloor,b}^{(j)}} d\mathbf{y}\right| \\
&= \left|\sum_{a=0}^{2^{m-i}-1} \sum_{b=0}^{2^j} \int_{R_{a,b}^{(i,j)}} \psi(2^{m-i}y_1)\psi(2^i y_2)\psi(2^{m-j}y_1)\psi(2^j y_2) \mathbb{1}_{R_{a,\lfloor b2^{i-j}\rfloor}^{(i)} \cap \mathcal{P} = \emptyset} \mathbb{1}_{R_{\lfloor a2^{i-j}\rfloor,b}^{(j)} \cap \mathcal{P} = \emptyset} dy_1 dy_2\right| \\
&\leq \sum_{a=0}^{2^{m-i}-1} \sum_{b=0}^{2^j} \mathbb{1}_{R_{a,b}^{(i,j)} \cap \mathcal{P} = \emptyset} \left|\int_{R_{a,b}^{(i,j)}} \psi(2^{m-i}y_1)\psi(2^i y_2)\psi(2^{m-j}y_1)\psi(2^j y_2) dy_1 dy_2\right| \\
&= \sum_{a=0}^{2^{m-i}-1} \sum_{b=0}^{2^j} \mathbb{1}_{R_{a,b}^{(i,j)} \cap \mathcal{P} = \emptyset} \left|\int_{a/2^{m-i}}^{(a+1)/2^{m-i}} \psi(2^{m-i}y_1)\psi(2^{m-j}y_1) dy_1 \underbrace{\int_{b/2^j}^{(b+1)/2^j} \psi(2^i y_2)\psi(2^j y_2) dy_2}_{=0}\right| \\
&= 0.
\end{aligned}$$

We deduce that, for $i \neq j$, $\int_{[0,1]^2} f_i(\mathbf{y}) f_j(\mathbf{y}) d\mathbf{y} = 0$ i.e. the $\{f_j\}_{j=0,...,m}$ are mutually orthogonal.
We now construct the function F as follows

$$F(\mathbf{x}) = f_0(\mathbf{x}) + f_1(\mathbf{x}) + ... + f_m(\mathbf{x}). \tag{3.5}$$

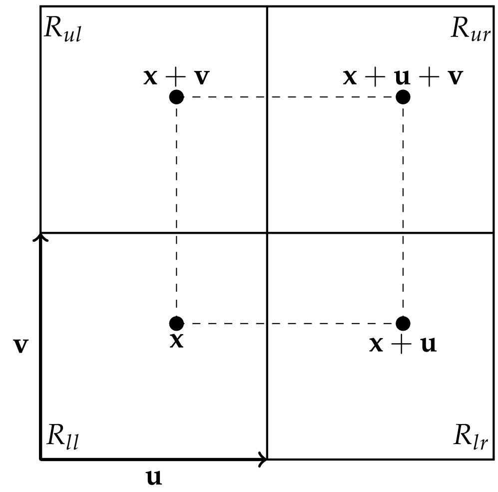


Figure 3.5: Empty rectangle $R = R^{(j)}_{a,b}$

Using the orthogonality and the fact that $2^m < 4N$, we deduce that

$$\begin{aligned}\int_{[0,1]^2} F^2(\mathbf{x})d\mathbf{x} &= \sum_{i,j=0}^{m} \int_{[0,1]^2} f_i(\mathbf{x})f_j(\mathbf{x})d\mathbf{x}\\ &= \sum_{i=0}^{m} \int_{[0,1]^2} \underbrace{f_i^2(\mathbf{x})}_{\leq 1} d\mathbf{x}\\ &\leq m+1\\ &< \log_2(4N)+1 = \log_2(N)+3.\end{aligned}$$

We therefore proved the first part of the proof, i.e. $\int_{[0,1]^2} F^2(\mathbf{x})d\mathbf{x} = \mathcal{O}(\log_2 N)$.

We now move on to the second part. Consider the function f_jD on the rectangles $R^{(j)}_{a,b}$. From now on, we will call such a rectangle empty if $R^{(j)}_{a,b} \cap \mathcal{P} = \emptyset$. We only have to consider empty $R^{(j)}_{a,b}$, since otherwise $f_j(\mathbf{x}) = 0$ for all $x \in R^{(j)}_{a,b}$.

We want to bound $\int_{[0,1]^2} f_j(\mathbf{x})D(\mathbf{x})d\mathbf{x}$ by $\frac{1}{2^8}$ from below. For a fixed $j \in \{0, 1, ..., m\}$, there $2^m \geq 2N$ rectangle of the form $R^{(j)}_{a,b}$ of which at most N contain a point from $\mathcal{P}$. Hence, there are at least N empty rectangles $R^{(j)}_{a,b}$ and it suffices to show that for every empty rectangle $R^{(j)}_{a,b}$ we have

$$\int_{R^{(j)}_{a,b}} f_j(\mathbf{x})D(\mathbf{x})d\mathbf{x} \geq \frac{1}{2^8 N}.$$

Let $R^{(j)}_{a,b}$ be an empty rectangle and let R_{ll} be the lower left quadrant, and accordingly R_{lr}, R_{ul}, R_{ur} (see Figure 3.5). Let $\mathbf{u}, \mathbf{v}$ be the vector as given in

Figure 3.5.
By definition of f_j, we have $f_j(\mathbf{x}) = 1$ for all $x \in R_{ll} \cup R_{ur}$ and $f_j(\mathbf{x}) = -1$ for all $x \in R_{lr} \cup R_{ul}$. We deduce that

$$\begin{aligned}\int_{R_{a,b}^{(j)}} f_j(\mathbf{x})D(\mathbf{x})d\mathbf{x} &= \int_{R_{ll}} D(\mathbf{x})d\mathbf{x} - \int_{R_{lr}} D(\mathbf{x})d\mathbf{x} - \int_{R_{ul}} D(\mathbf{x})d\mathbf{x} + \int_{R_{ur}} D(\mathbf{x})d\mathbf{x} \\ &= \int_{R_{ll}} [D(\mathbf{x}) - D(\mathbf{x}+\mathbf{u}) - D(\mathbf{x}+\mathbf{v}) + D(\mathbf{x}+\mathbf{u}+\mathbf{v})]\, d\mathbf{x}.\end{aligned} \tag{3.6}$$

Since the Lebesgue-measure is invariant under translation, we have

$$\begin{aligned}\lambda_2(R_{ll}) &= \lambda_2([\mathbf{x}, \mathbf{x}+\mathbf{u}+\mathbf{v})) \\ &= \lambda_2([\mathbf{0}, \mathbf{x})) - \lambda_2([\mathbf{0}, \mathbf{x}+\mathbf{u})) - \lambda_2([\mathbf{0}, \mathbf{x}+\mathbf{v})) + \lambda_2([\mathbf{0}, \mathbf{x}+\mathbf{u}+\mathbf{v}))\end{aligned}$$

and using that $R_{a,b}^{(j)}$ is an empty rectangle we also have

$$\begin{aligned}0 &= A([\mathbf{x}, \mathbf{x}+\mathbf{u}+\mathbf{v}), \mathcal{P}, N) \\ &= A([\mathbf{0}, \mathbf{x}), \mathcal{P}, N) - A([\mathbf{0}, \mathbf{x}+\mathbf{u}), \mathcal{P}, N) - A([\mathbf{0}, \mathbf{x}+\mathbf{v}), \mathcal{P}, N) + A([\mathbf{0}, \mathbf{x}+\mathbf{u}+\mathbf{v}), \mathcal{P}, N).\end{aligned}$$

Altogether, it follows that

$$D(\mathbf{x}) - D(\mathbf{x}+\mathbf{u}) - D(\mathbf{x}+\mathbf{v}) + D(\mathbf{x}+\mathbf{u}+\mathbf{v}) = N\lambda_2(R_{ll}).$$

We therefore have

$$\begin{aligned}\int_{R_{a,b}^{(j)}} f_j(\mathbf{x})D(\mathbf{x})d\mathbf{x} = \int_{R_{ll}} N\lambda_2(R_{ll}) &= N\lambda_2(R_{ll})^2 \\ &= \frac{N}{(2^{m+2})^2} > \frac{2^{m-2}}{2^{2m+4}} = \frac{1}{2^{m+6}} > \frac{1}{2^8 N}.\end{aligned} \tag{3.7}$$

where in the first and last inequality, we used that $4N > 2^m$ i.e. $N > 2^{m-2}$ and $\frac{1}{2^m} > \frac{1}{4N}$.
Overall, we have

$$\int_{[0,1]^2} f_j(\mathbf{x})D(\mathbf{x})d\mathbf{x} = \sum_{a=0}^{2^{m-j}-1} \sum_{b=0}^{2^j-1} \int_{R_{a,b}^{(j)}} f_j(\mathbf{x})D(\mathbf{x})d\mathbf{x} \geq N\frac{1}{2^8 N} = \frac{1}{2^8}$$

and we obtain

$$\int_{[0,1]^2} F(\mathbf{x})D(\mathbf{x})d\mathbf{x} \geq \frac{m+1}{2^8} \geq \frac{\log_2 N}{2^8}. \tag{3.8}$$

Altogether, we deduce from (3.2) that

$$NL_{2,N}(\mathcal{P}) \geq \frac{\log_2 N}{2^8\sqrt{\log_2 N + 3}} \geq \frac{\sqrt{\log_2 N}}{2^9}.$$

where the last inequality comes from Lemma A.3. □

It is known, though not proved here, that Roth's lower bound is the best, in the order of magnitude, for the L_2-discrepancy but not for the star discrepancy. We will study the best lower bound in the special case where $s = 2$ in the next section. For higher dimensions $s \geq 3$, the exact asymptotic order of the star discrepancy is still unknown. In Section 4.2, we will construct a N-element point set in $[0,1)^s$ such that its star discrepancy is in $\mathcal{O}((\log N)^{s-1}/N)$. It is conjectured that this rate gives a sharp lower bound. Whether this conjecture is true is still an open question.

To adapt Theorem 3.11 to infinite sequences, we will need the following lemma.

Lemma 3.12 *Let $s \in \mathbb{N}$, $s \geq 2$ and let $\mathcal{S} = (\mathbf{x}_n)_{n \in \mathbb{N}_0} \in [0,1)^{s-1}$ be an infinite sequence. Let $N \in \mathbb{N}$ and $\mathcal{P} = \{\mathbf{y}_0, ..., \mathbf{y}_{N-1}\}$, where $\mathbf{y}_n = (\frac{n}{N}, \mathbf{x}_n)$ for $n = 0, 1, ..., N-1$. Then*

$$D_N^*(\mathcal{P}) \leq \frac{1}{N} \left(\max_{M=1,...,N} M D_M^*(\mathcal{S}) + 1 \right).$$

Proof Consider a general interval of the form $J = \prod_{i=1}^{s} [0, \alpha_i) \subseteq [0,1)^s$. Then an element $\mathbf{y}_n \in \mathcal{P}$ belongs to J if and only if

$$0 \leq \frac{n}{N} < \alpha_1 \quad \text{and} \quad \mathbf{x}_n \in \prod_{i=2}^{s} [0, \alpha_i) =: \tilde{J}.$$

We choose $M \in \{1, ..., N\}$ such that for $\frac{M-1}{N} < \alpha_1 \leq \frac{M}{N}$. Then we have

$$0 \leq \frac{n}{N} < \alpha_1 \iff 0 \leq n < M.$$

We deduce that $A(J, \mathcal{P}, N) = A(\tilde{J}, \mathcal{S}, M)$. Consequently,

$$\begin{aligned} &|A(J, \mathcal{P}, N) - N\lambda_s(J)| \\ &\leq \left| A(\tilde{J}, \mathcal{S}, M) - M\lambda_{s-1}(\tilde{J}) \right| + \left| M\lambda_{s-1}(\tilde{J}) - N\lambda_s(J) \right|. \end{aligned}$$

Since $\left| M\lambda_{s-1}(\tilde{J}) - N\lambda_s(J) \right| = |(M - N\alpha_1) \prod_{i=2}^{s} a_i| \leq |1 \cdot \prod_{i=2}^{s} a_i| \leq 1$, it follows that

$$\begin{aligned} |A(J, \mathcal{P}, N) - N\lambda_s(J)| &\leq \left| A(\tilde{J}, \mathcal{S}, M) - M\lambda_{s-1}(\tilde{J}) \right| + 1 \\ &\leq M D_M^*(\mathcal{S}) + 1 \\ &\leq \max_{M=1,...,N} M D_M^*(\mathcal{S}) + 1. \end{aligned}$$

Taking the supremum over all intervals $J = [0, \boldsymbol{\alpha}) \in [0,1)^s$, we deduce that

$$N D_N^*(\mathcal{P}) \leq \max_{M=1,...,N} M D_M^*(\mathcal{S}) + 1.$$

□

We can now extend Theorem 3.11 to infinite sequence. Note that the result is slightly different as we only have the lower bound for infinitely many $N \in \mathbb{N}$.

Theorem 3.13 *For every $s \in \mathbb{N}$ there exists a constant $c'_s > 0$ such that for every infinite sequence $\mathcal{S} \in [0,1)^s$ we have*

$$D_N^*(\mathcal{S}) \geq c'_s \frac{(\log N)^{s/2}}{N} \quad \textit{for infinitely many } N \in \mathbb{N}.$$

Proof Let $\mathcal{S} = (\mathbf{x}_n)_{n \in \mathbb{N}_0}$. Let $N \in \mathbb{N}$, we consider the N-element point set $\mathcal{P} = \{\mathbf{y}_0, ..., \mathbf{y}_{N-1}\} \in [0,1)^{s+1}$ given by

$$\mathbf{y}_n := (n/N, \mathbf{x}_n), \quad \text{for } n = 0, ..., N-1.$$

Lemma 3.12 and Theorem 3.11 give

$$\max_{M=1,...,N} MD_M^*(\mathcal{S}) + 1 \geq ND_N^*(\mathcal{P}) \geq c_{s+1}(\log N)^{s/2}. \tag{3.9}$$

Let $c'_s \in \mathbb{R}$ such that $0 < c'_s < c_{s+1}$. For $N \in \mathbb{N}$ large enough we have

$$c'_s < c_{s+1} - \log(N)^{-s/2}$$

or equivalently

$$c'_s \log(N)^{s/2} < c_{s+1} \log(N)^{s/2} - 1.$$

We deduce from (3.9) that for $N \in \mathbb{N}$ large enough, there exists $M \in \{1, ..., N\}$ such that

$$MD_M^*(\mathcal{S}) \geq c_{s+1} \log(N)^{s/2} - 1 > c'_s \log(N)^{s/2} \geq c'_s \log(M)^{s/2}. \tag{3.10}$$

We now show that the inequality

$$MD_M^*(\mathcal{S}) \geq c'_s \log(M)^{s/2} \tag{3.11}$$

holds true for infinitely many $M \in \mathbb{N}$. By contradiction, assume that (3.11) holds true for finitely many $M \in \mathbb{N}$. Then let $M^* \in \mathbb{N}$ be the maximal integer with this property.
Then choose $N \in \mathbb{N}$ large enough such that

$$c'_s \log(N)^{s/2} > \max_{k=1,...,M^*} kD_k^*(\mathcal{S}).$$

Up to taking N even larger, as shown in (3.10), there exists $M \in \{1, ..., N\}$ such that

$$MD_M^*(\mathcal{S}) > c'_s \log(N)^{s/2} \geq c'_s \log(M)^{s/2}.$$

If $M \leq M^*$, then we have a contradiction since then

$$MD_M^*(\mathcal{S}) > c'_s \log(N)^{s/2} > \max_{k=1,...,M^*} kD_k^*(\mathcal{S}) \geq MD_M^*(\mathcal{S}).$$

Therefore $M > M^*$. But since $MD_M^*(\mathcal{S}) > c'_s \log(M)^{s/2}$ this contradicts the maximality of M^*.
We deduce that (3.11) holds for infinitely many $M \in \mathbb{N}$ and this concludes the proof. $\square$

3.2.2 Schmidt's lower bounds

As mentioned before, Roth's lower bound is not sharp. In 1972, W.M Schmidt improved this theorem for the particular case where $s = 2$ by giving a tighter lower bound. In Section 4.2, we construct the Hammersley point set and it will show that Schmidt's lower bound is actually the best possible in order of magnitude.

Theorem 3.14 (W.M. Schmidt, 1972) *There exists a constant $c > 0$ such that for any N-element point set $\mathcal{P} \subseteq [0,1)^2$ we have*

$$D_N^*(\mathcal{P}) \geq c\frac{\log N}{N}.$$

Proof We will decompose the proof in several lemmas. The proof is inspired by the proof of Theorem 3.11.
Let $n \in \mathbb{N}$ such that $2N < 2^n \leq 4N$. For $i = 1,..,n$ define the functions $f_i : [0,1)^2 \to \{-1,0,1\}$ as in (3.3).
We extend the orthogonality property by the following lemma:

Lemma 3.15 *For any set of integers $\{i_1,...,i_k\}$ satisfying $0 \leq i_1 < ... < i_k \leq n$ we have $\int_{[0,1]^2} f_{i_1}(\mathbf{y})...f_{i_k}(\mathbf{y})d\mathbf{y} = 0$.*

Proof Using Lemma A.4, we have $R_{a,b}^{(i_1,i_k)} \subseteq R^{(i_l)}_{\lfloor a2^{i_1-i_l}\rfloor,\lfloor b2^{i_l-i_k}\rfloor}$ for $l = 1,...,k$ and therefore we deduce

$$\begin{aligned}
&\left|\int_{[0,1]^2} f_{i_1}(\mathbf{y})...f_{i_k}(\mathbf{y})d\mathbf{y}\right| \\
&= \left|\sum_{a=0}^{2^{n-i_1-1}}\sum_{b=0}^{2^{i_k}}\int_{R_{a,b}^{(i_1,i_k)}}\prod_{l=1}^{k} f_{i_l}(\mathbf{y})\mathbb{1}_{y\in R^{(i_l)}_{\lfloor a2^{i_1-i_l}\rfloor,\lfloor b2^{i_l-i_k}\rfloor}}(\mathbf{y})d\mathbf{y}\right| \\
&= \left|\sum_{a=0}^{2^{n-i_1-1}}\sum_{b=0}^{2^{i_k}}\int_{R_{a,b}^{(i_1,i_k)}}\prod_{l=1}^{k} \psi(2^{n-i_l}y_1)\psi(2^{i_l}y_2)\mathbb{1}_{R^{(i_l)}_{\lfloor a2^{i_1-i_l}\rfloor,\lfloor b2^{i_l-i_k}\rfloor}\cap\mathcal{P}=\emptyset}d\mathbf{y}\right| \\
&\leq \sum_{a=0}^{2^{n-i_1-1}}\sum_{b=0}^{2^{i_k}}\mathbb{1}_{R_{a,b}^{(i_1,i_k)}\cap\mathcal{P}=\emptyset}\left|\int_{R_{a,b}^{(i_1,i_k)}}\prod_{l=1}^{k}\psi(2^{n-i_l}y_1)\psi(2^{i_l}y_2)d\mathbf{y}\right| \\
&= \sum_{a=0}^{2^{n-i_1-1}}\sum_{b=0}^{2^{i_k}}\mathbb{1}_{R_{a,b}^{(i_1,i_k)}\cap\mathcal{P}=\emptyset}\left|\int_{a/2^{n-i_1}}^{(a+1)/2^{n-i_1}}\prod_{l=1}^{k}\psi(2^{n-i_l}y_1)dy_1\int_{b/2^{i_k}}^{(b+1)/2^{i_k}}\prod_{l=1}^{k}\psi(2^{i_l}y_2)dy_2\right|.
\end{aligned}$$

For $l < k$ we have $i_l < i_k$ and therefore, as in the proof of Theorem 3.11, the functions $y \mapsto \psi(2^{i_l}y)$ are constant on the interval $[b/2^{i_k},(b+1)/2^{i_k}]$. We deduce, as in (3.4), that $\int_{b/2^{i_k}}^{(b+1)/2^{i_k}}\prod_{l=1}^{k}\psi(2^{i_l}y_2)dy_2 = 0$.
Therefore, we have

$$\left|\int_{[0,1]^2} f_{i_1}(\mathbf{y})...f_{i_k}(\mathbf{y})d\mathbf{y}\right| = 0$$

and this concludes the proof. □

For $\alpha \in (0, 1/2)$, we consider the auxiliary function $H : [0,1)^2 \to \mathbb{R}$ defined via

$$H(\mathbf{x}) = \prod_{i=0}^{n} (1 + \alpha f_i(\mathbf{x})) - 1.$$

Developing the product, we have

$$\prod_{i=0}^{n} (1 + \alpha f_i(\mathbf{x})) = 1 + \sum_{k=1}^{n+1} \alpha^k \sum_{0 \le i_1 < \ldots < i_k \le n} f_{i_1}(x) \ldots f_{i_k(x)}. \tag{3.12}$$

Let $F : [0,1)^2 \to \mathbb{R}$ be the function defined via $F(x) = \sum_{i=0}^{n} f_i(x)$, as in (3.5). Similarly, let $F_k : [0,1)^2 \to \mathbb{R}$ be defined as $F_k(x) = \sum_{0 \le i_1 < \ldots < i_k \le n} f_{i_1}(x) \ldots f_{i_k(x)}$ for $k = 2, \ldots, n+1$.
We can therefore rewrite (3.12) as

$$\prod_{i=0}^{n} (1 + \alpha f_i(\mathbf{x})) = 1 + \alpha F(x) + \sum_{k=2}^{n+1} \alpha^k F_k(x).$$

We can now rewrite $H(x)$ and find an upper bound

$$\begin{aligned}
\int_{[0,1]^2} |H(\mathbf{x})| \, d\mathbf{x} &\le 1 + \int_{[0,1]^2} \prod_{i=1}^{n} (1 + \alpha f_i(\mathbf{x})) \, d\mathbf{x} \\
&= 1 + \int_{[0,1]^2} \left(1 + \alpha F(\mathbf{x}) + \sum_{k=2}^{n+1} \alpha^k F_k(\mathbf{x}) \right) d\mathbf{x} \\
&= 2 + \alpha \int_{[0,1]^2} F(\mathbf{x}) d\mathbf{x} + \sum_{k=2}^{n+1} \alpha^k \int_{[0,1]^2} F_k(\mathbf{x}) d\mathbf{x}.
\end{aligned}$$

Using Lemma 3.15, we deduce that and $\int_{[0,1]^2} F = 0$ $\int_{[0,1]^2} F_k = 0$ and therefore We have

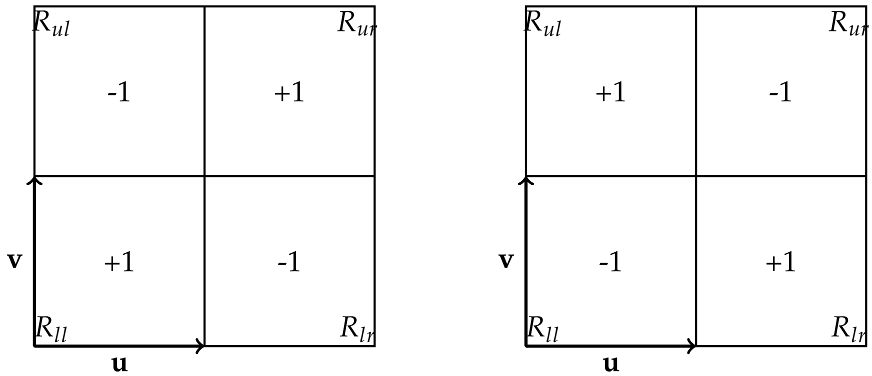
$$\int_{[0,1]^2} |H(\mathbf{x})| \, d\mathbf{x} \le 2. \tag{3.13}$$

We can now prove the following lemma:

Lemma 3.16 *For any set of integers $\{i_1, \ldots, i_k\}$ satisfying $0 \le i_1 < \cdots < i_k \le n$, we have*

$$\left| \int_{[0,1]^2} f_{i_1}(x) \ldots f_{i_k}(x) D(x) \, dx \right| \le N 2^{-n+i_1-i_k-4}.$$

Proof Let $R = R_{a,b}^{(i_1,i_k)}$ for some $a \in \{0, \ldots, 2^{n-i_1}\}$ and $b \in \{0, 1, \ldots, 2^{i_k}\}$.
For $l = 2, \ldots, k-1$ we have $i_1 < i_l < i_k$ and therefore $R_{a,b}^{(i_1,i_k)}$ is included in one quarter of the rectangle $R^{(i_l)}_{\lfloor a 2^{i_1 - i_l} \rfloor, \lfloor b 2^{i_l - i_k} \rfloor}$. We deduce that $\operatorname{sign}(f_2 \ldots f_{i_{k-1}})$ is constant on $R_{a,b}^{(i_1,i_k)}$.



(a) Case 1 **(b)** Case 2

Figure 3.6: Possible values for $\mathrm{sign}(f_1 ... f_{i_k})$ on $R = R_{a,b}^{(i_1,i_k)}$

Since the possible values for $\mathrm{sign}(f_1 f_{i_k})$ is either Case 1 or Case 2 (see Figure 3.6) on $R = R_{a,b}^{(i_1,i_k)}$, we therefore deduce that also $\mathrm{sign}(f_1 f_2 ... f_{i_k})$ is either Case 1 or Case 2.
If $\mathrm{sign}(f_1 f_2 ... f_{i_k})$ on R corresponds to Case 1, then we are in the exact same setting as for the proof of Theorem 3.11 and we deduce from (3.7) that

$$\int_{R_{a,b}^{(i_1,i_k)}} f_1(\mathbf{x}) ... f_k(\mathbf{x}) d\mathbf{x} = N \lambda_2(R_{ll})^2 = N \left(\frac{\lambda_2(R)}{4} \right)^2 .$$

If $\mathrm{sign}(f_1 f_2 ... f_{i_k})$ on R corresponds to Case 2, then by the exact same argument as in (3.6), we deduce that

$$\int_{R_{a,b}^{(i_1,i_k)}} f_1(\mathbf{x}) ... f_k(\mathbf{x}) d\mathbf{x} = -N \lambda_2(R_{ll})^2 = -N \left(\frac{\lambda_2(R)}{4} \right)^2 .$$

Overall, we have proved that

$$\left| \int_{R_{a,b}^{(i_1,i_k)}} f_1(\mathbf{x}) ... f_k(\mathbf{x}) d\mathbf{x} \right| = N \left(\frac{\lambda_2(R_{a,b}^{(i_1,i_k)})}{4} \right)^2 = N 2^{-2n+2i_1-2i_k-4}.$$

To conclude, we have by the triangle inequality

$$\begin{aligned} \left| \int_{[0,1]^2} f_1(\mathbf{x}) ... f_k(\mathbf{x}) D(\mathbf{x}) d\mathbf{x} \right| &\leq \sum_{a=0}^{2^{n-i_1}} \sum_{b=0}^{2^{i_k}} \left| \int_{R_{a,b}^{(i_1,i_k)}} f_1(\mathbf{x}) ... f_k(\mathbf{x}) D(\mathbf{x}) d\mathbf{x} \right| \\ &\leq 2^{n-i_1+i_k} N 2^{-2n+2i_1-2i_k-4} = N 2^{-n+i_1-i_k-4}. \quad \square \end{aligned}$$

We continue the proof with the following lemma:

Lemma 3.17 *For every $k = 2, \dots, n+1$, we have*

$$\left| \int_{[0,1]^2} F_k(x) D(x)\, dx \right| \le \sum_{i=0}^{n-k+1} \sum_{h=1}^{n-i} \frac{N}{2^{n+h+4}} \binom{h-1}{k-2}.$$

Proof Let $k \in \{2, ..., n+1\}$, then using Lemma 3.16 we have

$$\begin{aligned} \left| \int_{[0,1]^2} F_k(x) D(x)\, dx \right| &\le \sum_{0 \le i_1 < ... < i_k \le n} \left| \int_{[0,1]^2} f_{i_1}(x) ... f_{i_k(x)} D(x) dx \right| \\ &\le \sum_{0 \le i_1 < ... < i_k \le n} N 2^{-n+i_1-i_k-4} \\ &= \sum_{i_1=0}^{n-k+1} \sum_{i_2=i_1+1}^{n-k+2} ... \sum_{i_k=i_{k-1}+1}^{n} N 2^{-n+i_1-i_k-4}. \end{aligned}$$

Let $h := i_k - i_1$, because $0 \le i_1 < i_2 < ... < i_k \le n$ we have that h ranges over $\{k-1, ..., n-i\}$. Now if $i_1 \in \{0, ..., n-k+1\}$ and $h\{k_1, ..., n-i\}$ are fixed there are $\binom{h-1}{k-2}$ ways to choose the remaining $k-2$ indices $i_2 < ... < i_{k-1}$ among the $h-1$ integer between i_1 and $i_k = i_1 + h$.
We deduce that

$$\begin{aligned} \left| \int_{[0,1]^2} F_k(x) D(x)\, dx \right| &\le \sum_{i=0}^{n-k+1} \sum_{h=k-1}^{n-i} \binom{h-1}{k-2} N 2^{-n-h-4} \\ &\le \sum_{i=0}^{n-k+1} \sum_{h=1}^{n-i} \binom{h-1}{k-2} N 2^{-n-h-4}. \end{aligned}$$ □

Using Lemma 3.17 we can find an upper bound for $\sum_{k=2}^{n+1} \alpha^k \int_{[0,1]^2} F_k(x)D(x)dx$.

$$\begin{aligned}
\left|\sum_{k=2}^{n+1} \alpha^k \int_{[0,1]^2} F_k(x)D(x)dx\right| &\leq \sum_{k=2}^{n+1} \alpha^k \sum_{i=0}^{n-k+1} \sum_{h=1}^{n-i} \binom{h-1}{k-2} N2^{-n-h-4} \\
&= \sum_{i=0}^{n-1} \sum_{h=1}^{n-i} N2^{-n-h-4} \sum_{k=2}^{n-i+1} \alpha^k \binom{h-1}{k-2} \\
&= \sum_{i=0}^{n-1} \sum_{h=1}^{n-i} N2^{-n-h-4}\alpha^2 \sum_{k=0}^{n-i-1} \alpha^k \binom{h-1}{k} \\
&\leq \sum_{i=0}^{n-1} \sum_{h=1}^{n-i} N2^{-n-h-4}\alpha^2 \sum_{k=0}^{h-1} \alpha^k \binom{h-1}{k} \\
&= \sum_{i=0}^{n-1} \sum_{h=1}^{n-i} N2^{-n-h-4}\alpha^2 (\alpha+1)^{h-1} \\
&= \sum_{i=0}^{n-1} \frac{N}{2^{n+4}} \frac{\alpha^2}{\alpha+1} \sum_{h=1}^{n-i} \left(\frac{\alpha+1}{2}\right)^h \\
&= \sum_{i=0}^{n-1} \frac{N}{2^{n+4}} \frac{\alpha^2}{\alpha+1} \frac{\alpha+1}{1-\alpha} \left(1 - \left(\frac{\alpha+1}{2}\right)^{n-i}\right) \\
&= \frac{N}{2^{n+4}} \frac{\alpha^2}{1-\alpha} \left(n - \sum_{i=0}^{n-1} \left(\frac{\alpha+1}{2}\right)^{n-i}\right) \\
&= \frac{N}{2^{n+4}} \frac{\alpha^2}{1-\alpha} \left(n - \frac{\alpha+1}{\alpha-1} \left(\left(\frac{\alpha+1}{2}\right)^n - 1\right)\right).
\end{aligned}$$

Since $\alpha \in (0, 1/2)$ we have $\alpha - 1 < 0$, $\left(\frac{\alpha+1}{2}\right)^n < 1$ and therefore $\frac{\alpha+1}{\alpha-1}\left(\left(\frac{\alpha+1}{2}\right)^n - 1\right) > 0$. We can therefore find an upper bound as follows

$$\begin{aligned}
\left|\sum_{k=2}^{n+1} \alpha^k \int_{[0,1]^2} F_k(x)D(x)dx\right| &\leq \frac{N}{2^{n+4}} \frac{\alpha^2}{1-\alpha} \left(n - \frac{\alpha+1}{\alpha-1} \left(\left(\frac{\alpha+1}{2}\right)^n - 1\right)\right) \\
&\leq \frac{Nn\alpha^2}{2^{n+4}(1-\alpha)} \\
&= \alpha^2 N \frac{n}{2^{n+2}} \frac{1}{2^2(1-\alpha)} \leq \alpha^2 N \frac{n}{2^{n+2}}.
\end{aligned}$$

Using this upper bound, we now want to estimate $\left|\int_{[0,1]^2} H(x)D(x)dx\right|$ from below.

Recall that $H(x) = \prod_{i=0}^{n}(1 + \alpha f_i(x)) - 1 = \alpha F(x) + \sum_{k=2}^{n+1} \alpha^k F_k(x)$. Using the previous upper bound, we have

$$\begin{aligned}\left|\int_{[0,1]^2} H(x)D(x)dx\right| &= \left|\int_{[0,1]^2} \alpha F(x)D(x)dx + \sum_{k=2}^{n+1}\alpha^k \int_{[0,1]^2} F_k(x)D(x)dx\right| \\ &\geq \left|\alpha \int_{[0,1]^2} F(x)D(x)dx\right| - \left|\sum_{k=2}^{n+1}\alpha^k \int_{[0,1]^2} F_k(x)D(x)dx\right| \\ &\geq \alpha \left|\int_{[0,1]^2} F(x)D(x)dx\right| - \alpha^2 N\frac{n}{2^{n+2}}.\end{aligned}$$

Equation (3.8) gives us $\left|\int_{[0,1]^2} F(x)D(x)dx \geq \log_2(N)2^{-8}\right|$. Using that $2N \leq 2^n < 4N$ we deduce that $n < \log_2(4N)$ and therefore

$$\begin{aligned}\left|\int_{[0,1]^2} H(x)D(x)dx\right| &\geq \alpha \left|\int_{[0,1]^2} F(x)D(x)dx\right| - \alpha^2 N\frac{n}{2^{n+2}} \\ &\geq \alpha \log_2(N)2^{-8} - \alpha^2 \log_2(4N)2^{n-1}2^{-n-2} \\ &\geq \log_2(N)\left(\alpha 2^{-8} - \alpha^2\frac{\log_2(4N)}{\log_2(N)}2^{-3}\right).\end{aligned}$$

Finally, since $x \mapsto \frac{\log_2(4x)}{\log_2(x)}$ is decreasing on $(1, +\infty)$, we deduce that for $N \geq 2$ $\frac{\log_2(4N)}{\log_2(N)} \leq \frac{\log_2(8)}{\log_2(2)}$.

$$\begin{aligned}\left|\int_{[0,1]^2} H(x)D(x)dx\right| &\geq \log_2(N)\left(\alpha 2^{-8} - \alpha^2\frac{\log_2(4N)}{\log_2(N)}2^{-3}\right) \\ &\geq \log_2(N)\left(\alpha 2^{-8} - \alpha^2\frac{\log_2(8)}{\log_2(2)}2^{-3}\right).\end{aligned}$$

We conclude that for $0 < \alpha < 2^{-5}\frac{\log_2(2)}{\log_2(8)}$ we have that

$$\left|\int_{[0,1]^2} H(x)D(x)dx\right| \geq C\log(N)$$

for a certain $C > 0$.
Finally, using (3.13),

$$\begin{aligned}C\log(N) \leq \left|\int_{[0,1]^2} H(x)D(x)dx\right| &\leq \sup_{x\in[0,1]^2} |D(x)| \int_{[0,1]^2} |H(x)|\, dx \\ &= ND_N^*(\mathcal{P}) \int_{[0,1]^2} |H(x)|\, dx \leq 2ND_N^*(\mathcal{P}).\end{aligned}$$

We conclude that

$$ND_N^*(\mathcal{P}) > c\log N$$

for some constant $c > 0$. $\square$

As a remark, we note that Schmidt's lower bound is coherent with the conjecture that $\mathcal{O}((\log N)^{s-1}/N)$ gives a sharp lower bound for the asymptotic star discrepancy of a N-element point set.

To close this section, we adapt Theorem 3.14 to infinite sequences in dimension $s = 1$.

Theorem 3.18 (W.M. Schmidt, 1972) *There exists a constant $c > 0$ such that for every infinite sequence $\mathcal{S} \subseteq [0,1)$ we have*

$$D_N^*(\mathcal{S}) \geq c\frac{\log N}{N} \quad \textit{for infinitely many } N \in \mathbb{N}.$$

Proof This theorem follows from Theorem 3.14.
Indeed we can repeat the exact same proof as Theorem 3.13, but using the lower bound of Theorem 3.14 in (3.9). □

3.2.3 Niederreiter's upper bounds

We now turn to upper bounds on the discrepancy. Often this is done on a case by case basis, for example in Section 4 for the Halton sequence. In 1977, H. Niederreiter proved an upper bound for the discrepancy of point sets whose components are rational numbers. We prove it in this section.

We introduce the following notations:
For $M \in \mathbb{N}, M \geq 2$, put $C(M) = (-M/2, M/2] \cap \mathbb{Z}$ and $C_s(M) = C(M)^s$.
We also write $C_s^*(M) = C_s(M) \setminus \{0\}$.
For $h \in C(M)$, we write

$$r(h, M) = \begin{cases} M \sin(\pi\,|h|\,/M) & \text{if } h \neq 0 \\ 1 & \text{if } h = 0 \end{cases}$$

and for $\mathbf{h} = (h_1, ..., h_s) \in C_s(M)$, we write $r(\mathbf{h}, M) = \prod_{i=1}^{s} r(h_j, M)$.

Theorem 3.19 (Niederreiter) *For an integer $M \geq 2$ and $\mathbf{y}_0, \ldots, \mathbf{y}_{N-1} \in \mathbb{Z}^s$, let $P = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be the point set consisting of the fractional parts $\mathbf{x}_n = \{\mathbf{y}_n/M\}$ for $n = 0, \ldots, N-1$. Then*

$$D_N(\mathcal{P}) \leq 1 - \left(1 - \frac{1}{M}\right)^s + \sum_{\mathbf{h} \in C_s^*(M)} \frac{1}{r(\mathbf{h}, M)} \left| \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \mathbf{y}_n / M) \right|.$$

Proof For $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s$, let

$$A(\mathbf{k}) := \#\{n \in \mathbb{N}_0 \mid 0 \leq n < N \text{ and } \mathbf{y}_n \equiv \mathbf{k} \pmod{M}\},$$

where the congruence between vectors is understood component-wise. By Lemma A.5, we have

$$\frac{1}{M} \sum_{h \in C(M)} \exp(2\pi i h a/M) = \begin{cases} 1 & \text{if } a \equiv 0 \pmod{M} \\ 0 & \text{if } a \not\equiv 0 \pmod{M} \end{cases}.$$

Therefore,

$$\sum_{n=0}^{N-1} \frac{1}{M^s} \sum_{\mathbf{h} \in C_s(M)} \exp(2\pi i \mathbf{h} \cdot (\mathbf{y}_n - \mathbf{k})/M) = \sum_{n=0}^{N-1} \prod_{j=1}^{s} \underbrace{\left(\frac{1}{M} \sum_{h_j \in C(M)} \exp(2\pi i h_j (y_{n,j} - k_j)/M) \right)}_{= \begin{cases} 1 & \text{if } y_{n,j} \equiv k_j \pmod M \\ 0 & \text{otherwise} \end{cases}}$$

$$= A(\mathbf{k}).$$

Consequently,

$$\begin{aligned} A(\mathbf{k}) - \frac{N}{M^s} &= \sum_{n=0}^{N-1} \left(\frac{1}{M^s} \left(\sum_{\mathbf{h} \in C_s(M)} \exp(2\pi i \mathbf{h} \cdot (\mathbf{y}_n - \mathbf{k})/M) - 1 \right) \right) \\ &= \sum_{n=0}^{N-1} \frac{1}{M^s} \sum_{\mathbf{h} \in C_s^*(M)} \exp(2\pi i \mathbf{h} \cdot (\mathbf{y}_n - \mathbf{k})/M) \\ &= \frac{1}{M^s} \sum_{\mathbf{h} \in C_s^*(M)} \exp(-2\pi i \mathbf{h} \cdot \mathbf{k}/M) \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \mathbf{y}_n/M). \end{aligned}$$

Now, let $J = \prod_{j=1}^{s} [u_j, v_j) \subseteq [0,1]^s$.
For $j = 1, \ldots, s$, let $a_j \in \mathbb{Z}$ be minimal such that $u_j \leq a_j/M$ and $b_j \in \mathbb{Z}$ maximal such that $b_j/M < v_j$. In particular, we have $[a_j/M, b_j/M] \subseteq [u_j, v_j)$.

If $[a_j/M, b_j/M] = \emptyset$ for some $j \in \{1, \ldots, s\}$, then $A(J, \mathcal{P}, N) = 0$ and $v_j - u_j < 1/M$. Therefore,

$$\left| \frac{A(J, \mathcal{P}, N)}{N} - \lambda_s(J) \right| = \lambda_s(J) < \frac{1}{M} \leq 1 - \left(1 - \frac{1}{M}\right)^s.$$

Now assume that $[a_j/M, b_j/M] \neq \emptyset$ for all $j = 1, \ldots, s$.

We claim that

$$A(J, \mathcal{P}, N) = \sum_{\mathbf{a} \leq \mathbf{k} \leq \mathbf{b}} A(\mathbf{k}).$$

We prove the claim: By definition of the sequence $(\mathbf{x}_n)_{n \in \mathbb{N}_0}$, we have

$$\mathbf{x}_n \in J \iff \{\mathbf{y}_n/M\} \in J \iff u_j \leq \{y_{n,j}/M\} < v_j \quad \forall j = 1, ...s.$$

We prove that the latter condition is equivalent to $y_{n,j} \equiv k_j \pmod M$ for some $k_j \in \{a_j, ..., b_j\}$.
Suppose first that $u_j \leq \{y_{n,j}/M\} < v_j$ for $j = 1, ...s$. We write $y_{n,j} = lM + k_j$ for some $l \in \mathbb{Z}$ and $k_j \in \{0, ..., M-1\}$. Then $\{y_{n,j}/M\} = k_j/M$ and therefore $u_j \leq k_j/M < v_j$. By minimality of a_j and maximality of b_j we conclude that $a_j \leq k \leq b_j$.

Suppose now that $y_{n,j} \equiv k_j \pmod M$ for some $k_j \in \{a_j, ..., b_j\}$, then we can write $y_{n,j} = lM + k_j$ for some $l \in \mathbb{Z}$. Therefore we have $u_j \leq a_j/M \leq k_j/M = \{y_{n,j}/M\} \leq b_j/M < v_j$.
This concludes the proof of the claim.

Going back to the proof of the main theorem, we also have

$$\sum_{\mathbf{a} \leq \mathbf{k} \leq \mathbf{b}} \frac{1}{M^s} = \frac{1}{M^s} \prod_{j=1}^{s} (b_j - a_j + 1).$$

Hence,

$$\begin{aligned}
&\frac{A(J, \mathcal{P}, N)}{N} - \lambda_s(J) = \sum_{\mathbf{a} \leq \mathbf{k} \leq \mathbf{b}} \left(\frac{A(\mathbf{k})}{N} - \frac{1}{M^s} \right) + \frac{1}{M^s} \prod_{j=1}^{s} (b_j - a_j + 1) - \lambda_s(J) \\
&= \frac{1}{M^s} \sum_{\mathbf{h} \in C_s^*(M)} \left(\sum_{\mathbf{a} \leq \mathbf{k} \leq \mathbf{b}} \exp(-2\pi i \mathbf{h} \cdot \mathbf{k}/M) \right) \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \mathbf{y}_n/M) \\
&+ \prod_{j=1}^{s} \frac{b_j - a_j + 1}{M} - \prod_{j=1}^{s} (v_j - u_j).
\end{aligned}$$

For all $j \in \{1, \ldots, s\}$, we have by definition of a_j and b_j

$$\begin{aligned}
\left| \frac{b_j - a_j + 1}{M} - (v_j - u_j) \right| &= \left| (v_j - \frac{b_j}{M}) - (u_j - \frac{a_j}{M}) - \frac{1}{M} \right| \\
&\leq \left| \frac{1}{M} - (u_j - \frac{a_j}{M}) - \frac{1}{M} \right| < \frac{1}{M}.
\end{aligned}$$

Thus, it follows from Lemma A.2 that

$$\left| \prod_{j=1}^{s} \frac{b_j - a_j + 1}{M} - \prod_{j=1}^{s} (v_j - u_j) \right| \leq 1 - \left(1 - \frac{1}{M} \right)^s.$$

Consequently,

$$\begin{aligned}
&\left| \frac{A(J, \mathcal{P}, N)}{N} - \lambda_s(J) \right| \leq 1 - \left(1 - \frac{1}{M} \right)^s \\
&+ \frac{1}{M^s} \sum_{\mathbf{h} \in C_s^*(M)} \underbrace{\left| \sum_{\mathbf{a} \leq \mathbf{k} \leq \mathbf{b}} \exp(-2\pi i \mathbf{h} \cdot \mathbf{k}/M) \right|}_{=: r^*(\mathbf{h}, M)} \left| \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \mathbf{y}_n/M) \right|.
\end{aligned}$$

We compute further,

$$
\begin{aligned}
r^*(\mathbf{h}, M) &= \left| \prod_{j=1}^{s} \sum_{k=a_j}^{b_j} \exp\left(2\pi i h_j k_j / M\right) \right| \\
&= \prod_{j=1}^{s} \left| \sum_{k=0}^{b_j - a_j} \exp\left(2\pi i h_j k_j / M\right) \exp(2\pi i h_j a_j / M) \right| \\
&= \prod_{j=1}^{s} \left| \sum_{k=0}^{b_j - a_j} \exp\left(2\pi i h_j k_j / M\right) \right| .
\end{aligned}
$$

If $h_j = 0$ then, using that $a_j, b_j \in \{1, ..., M-1\}$,

$$
\left| \sum_{k=0}^{b_j - a_j} \exp\left(2\pi i h_j k_j / M\right) \right| = |b_j - a_j + 1| \leq M = \frac{M}{r(h_j, M)}.
$$

If now $h_j \neq 0$ then, using the trigonometric identity $|\exp(i\theta) - 1| = 2\,|\sin(\theta/2)|$ for $\theta \in \mathbb{R}$,

$$
\begin{aligned}
\left| \sum_{k=0}^{b_j - a_j} \exp\left(2\pi i h_j k_j / M\right) \right| &= \left| \frac{\exp(2\pi i h_j (b_j - a_j + 1)/M) - 1}{\exp(2\pi i h_j / M) - 1} \right| \\
&= \left| \frac{\sin\left(\pi h_j (b_j - a_j + 1)/M\right)}{\sin\left(\pi h_j / M\right)} \right| \\
&\leq \frac{1}{\sin(\pi \left|h_j\right| / M)} = \frac{M}{r(h_j, M)}.
\end{aligned}
$$

In any case,

$$
r^*(\mathbf{h}, M) \leq \prod_{j=1}^{s} \frac{M}{r(h_j, M)} = \frac{M^s}{r(\mathbf{h}, M)}.
$$

And we conclude that

$$
\begin{aligned}
\left| \frac{A(J, \mathcal{P}, N)}{N} - \lambda_s(J) \right| &\leq 1 - \left(1 - \frac{1}{M}\right)^s \\
&\quad + \sum_{\mathbf{h} \in C_s^*(M)} \frac{1}{r(\mathbf{h}, M)} \left| \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \mathbf{y}_n / M) \right| .
\end{aligned}
$$

Taking the supremum over all intervals $J = [\mathbf{a}, \mathbf{b}) \subseteq [0,1)^s$, the result follows. □

Chapter 4

A classical construction: The Halton sequence

In this section we will construct the Halton sequence, a generalization of the Van der Corput sequence introduced in Proposition 2.8, and compute an upper bound for its discrepancy. From the Halton sequence we will deduce its finite version, the Hammersley point set.

4.1 The Halton sequence

Recall that, for $b, n \in \mathbb{N}, b \geq 2$, $\varphi_b(n)$ denotes the b-adic radical inverse, as defined in Definition 2.6.

Definition 4.1 *Let $s \in \mathbb{N}$ and let $b_1, \dots, b_s \geq 2$ be integers. The Halton sequence in bases $b_1, \dots, b_s$ is the sequence $\mathcal{S}_{b_1,\dots,b_s} = (\mathbf{x}_n)_{n \in \mathbb{N}_0}$ whose n-th element is given by*

$$\mathbf{x}_n = (\varphi_{b_1}(n), \varphi_{b_2}(n), \dots, \varphi_{b_s}(n)) \quad \textit{for} \quad n \in \mathbb{N}_0.$$

For $s = 1$ we recover the van der Corput sequence $(\varphi_b(n))_{n \in \mathbb{N}_0}$ in base $b = b_1$.

Theorem 4.2 *Let $s \in \mathbb{N}$, and let $b_1, \dots, b_s \geq 2$ be pairwise coprime integers. For the star discrepancy of the Halton sequence $\mathcal{S}_{b_1,\dots,b_s}$ in bases $b_1, \dots, b_s$, we have*

$$D_N^*(\mathcal{S}_{b_1,\dots,b_s}) \leq \left(\prod_{j=1}^{s} \frac{b_j \log(b_j N)}{\log b_j} \right) \frac{1}{N} \quad \textit{for} \quad N \in \mathbb{N}.$$

Hence, asymptotically for $N \to \infty$ we have $D_N^*(\mathcal{S}_{b_1,\dots,b_s}) = \mathcal{O}((\log N)^s / N)$.

Proof We only prove the theorem in dimensions 1 and 2.
For $N < \max_{j=1,\ldots,s} b_j$, put $j^* = \arg\max_{j=1,\ldots,s} b_j$ then $N < b_{j^*}$ and we have

$$\left(\prod_{j=1}^{s} \frac{b_j \log(b_j N)}{\log b_j}\right) \frac{1}{N} \geq \frac{b_{j^*}}{b_{j^*}} \frac{\log(N b_{j^*})}{\log(b_{j^*})} \prod_{\substack{j=1 \\ j \neq j^*}}^{s} \frac{b_j \log(b_j N)}{\log b_j}$$

$$\geq 1 \cdot \prod_{\substack{j=1 \\ j \neq j^*}}^{s} b_j \cdot 1 \geq 1.$$

Since the discrepancy of sequence $\mathcal{S} \subseteq [0,1)^s$ is always less than 1, the identity holds trivially.
Hence we may assume in the following that $N \geq \max_{j=1,\ldots,s} b_j$.

The case $s = 1$**:** Let $b = b_1$. Let $\alpha \in (0,1]$ with infinite b-adic expansion

$$\alpha = \frac{\alpha_0}{b} + \frac{\alpha_1}{b^2} + \frac{\alpha_2}{b^3} + \ldots \quad \text{for } \alpha_i \in \{0, \ldots, b-1\}.$$

If the b-adic expansion of α is finite, say $\alpha = (0, \alpha_0 \ldots \alpha_M)_b$ with $\alpha_M \neq 0$, then we switch to the infinite expansion by writing

$$\alpha = (0, \alpha_0 \ldots \alpha_{M-1} \alpha'_M \alpha'_{M+1} \ldots)_b$$

with $\alpha'_M = \alpha_M - 1$ and $\alpha'_i = b - 1$ for $i \geq M+1$.
Let $k \in \{0, \ldots, N-1\}$ with b-adic expansion

$$k = k_0 + k_1 b + k_2 b^2 + \ldots + k_m b^m$$

where $k_i \in \{0, \ldots, b-1\}, k_m \neq 0$. Then $b^m \leq k < b^{m+1}$ and hence $m = \lfloor \frac{\log(k)}{\log b} \rfloor$.
This means that k has exactly $m + 1 = \left\lfloor \frac{\log_b k}{\log_b} \right\rfloor + 1 \leq M + 1$ digits in its b-adic expansion, where

$$M = \left\lfloor \frac{\log_b N}{\log_b} \right\rfloor. \tag{4.1}$$

Hence for $k \in \{0, \ldots, N-1\}$ we have

$$k = k_0 + k_1 b + \cdots + k_M b^M$$

with b-adic digits $k_0, k_1, \ldots, k_M \in \{0, \ldots, b-1\}$, and the corresponding value of the b-adic radical inverse function equals

$$\varphi_b(k) = \frac{k_0}{b} + \frac{k_1}{b^2} + \cdots + \frac{k_M}{b^{M+1}}.$$

Let us determine

$$A([0,\alpha), \mathcal{S}_b, N) = \#\{k \in \{0, \ldots, N-1\} : \varphi_b(k) \in [0,\alpha)\}.$$

According to the above definition of the radical inverse function $\varphi_b(k)$ and because we only work with infinite b-adic expansion, we have $\varphi_b(k) < \alpha$ if and only if one of the following conditions is satisfied:

$$\begin{array}{rl}
(1) & k_0 < \alpha_0 \\
(2) & k_0 = \alpha_0,\ k_1 < \alpha_1 \\
(3) & k_0 = \alpha_0,\ k_1 = \alpha_1,\ k_2 < \alpha_2 \\
 & \vdots \\
(M+1) & k_0 = \alpha_0, \dots, k_{M-1} = \alpha_{M-1},\ k_M < \alpha_M \\
(M+2) & k_0 = \alpha_0, \dots, k_M = \alpha_M.
\end{array}$$

Note that conditions $(1), \dots, (M+2)$ are mutually exclusive. Now rewrite these $M+2$ conditions as congruences in the following way:

$$\begin{array}{rl}
(1) & k \equiv k_0 \pmod{b},\ 0 \leq k_0 < \alpha_0 \\
(2) & k \equiv \alpha_0 + k_1 b \pmod{b^2},\ 0 \leq k_1 < \alpha_1 \\
(3) & k \equiv \alpha_0 + \alpha_1 b + k_2 b^2 \pmod{b^3},\ 0 \leq k_2 < \alpha_2 \\
 & \vdots \\
(M+1) & k \equiv \alpha_0 + \cdots + \alpha_{M-1} b^{M-1} + k_M b^M \pmod{b^{M+1}},\ 0 \leq k_M < \alpha_M \\
(M+2) & k \equiv \alpha_0 + \cdots + \alpha_M b^M \pmod{b^{M+2}}.
\end{array}$$

By Lemma A.6, the number of solutions of the respective congruences for $k \in \{0, \dots, N-1\}$ equals

$$\begin{array}{rl}
(1) & \alpha_0 \left(\lfloor N/b \rfloor + \theta_0 \right) \\
(2) & \alpha_1 \left(\lfloor N/b^2 \rfloor + \theta_1 \right) \\
 & \vdots \\
(M+1) & \alpha_M \left(\left\lfloor N/b^{M+1} \right\rfloor + \theta_M \right) \\
(M+2) & \left\lfloor N/b^{M+2} \right\rfloor + \theta_{M+1}
\end{array}$$

with appropriate $\theta_j \in \{0, 1\}$ for $j = 0, \dots, M+1$. For the last line, we have by (4.1) that $b^M \leq N < b^{M+1}$ and therefore $\lfloor N/b^{M+2} \rfloor = 0$.
Therefore, we have

$$A([0,\alpha), \mathcal{S}_b, N) = \sum_{i=0}^{M} \alpha_i \left(\left\lfloor \frac{N}{b^{i+1}} \right\rfloor + \theta_i \right) + \theta_{M+1}$$

and

$$\begin{aligned}\left|A([0,\alpha),\mathcal{S}_b,N) - N\sum_{i=0}^{M}\frac{\alpha_i}{b^{i+1}}\right| &= \left|\sum_{i=0}^{M}\alpha_i\left(\left\lfloor\frac{N}{b^{i+1}}\right\rfloor + \theta_i - \frac{N}{b^{i+1}}\right) + \theta_{M+1}\right| \\ &\leq \sum_{i=0}^{M}\alpha_i\left|\frac{N}{b^{i+1}} - \left\lfloor\frac{N}{b^{i+1}}\right\rfloor - \theta_i\right| + 1 \\ &\leq \sum_{i=0}^{M}\alpha_i + 1 \\ &\leq (M+1)(b-1)+1\end{aligned}$$

where we used that $\alpha_i \leq b-1$ in the last line.
We deduce that,

$$\begin{aligned}|A([0,\alpha),\mathcal{S}_b,N) - \alpha N| &\leq \left|A([0,\alpha),\mathcal{S}_b,N) - N\sum_{i=0}^{M}\frac{\alpha_i}{b^{i+1}}\right| + \left|N\sum_{i=M+1}^{\infty}\frac{\alpha_i}{b^{i+1}}\right| \\ &\leq (M+1)(b-1)+1+N\sum_{i=M+1}^{\infty}\frac{\alpha_i}{b^{i+1}}.\end{aligned}$$

Using again that $\alpha_i \leq b-1$, we obtain

$$N\sum_{i=M+1}^{\infty}\frac{\alpha_i}{b^{i+1}} \leq N\frac{b-1}{b^{M+2}}\sum_{i=0}^{\infty}\frac{1}{b^i} = \frac{N}{b^{M+1}} < 1 \tag{4.2}$$

and hence

$$|A([0,\alpha),\mathcal{S}_b,N) - \alpha N| \leq (M+1)(b-1)+2.$$

To conclude, we have by (4.1)

$$\begin{aligned}(M+1)(b-1)+2 &\leq \left(\frac{\log N}{\log b}+1\right)(b-1)+2 \\ &= b\frac{\log(Nb)}{\log b} + 2 - \frac{\log(Nb)}{\log b} \\ &\leq b\frac{\log(Nb)}{\log b} \end{aligned} \tag{4.3}$$

where, in the last line, we used $N \geq b$. Therefore

$$2 - \frac{\log(Nb)}{\log b} = \frac{\log b^2 - \log(Nb)}{\log b} = \frac{\log(b/N)}{\log b} < 0.$$

Thus we have shown that for all $\alpha \in (0,1]$ and all $N \geq b$:

$$\left|\frac{A([0,\alpha),S_b,N)}{N} - \alpha\right| \leq \frac{b}{N}\frac{\log(Nb)}{\log b}.$$

Since this bound is independent of the specific choice of α the result follows for $s = 1$ by taking the supremum over all $\alpha \in [0, 1)$.

The case $s = 2$: We write $b = b_1$ and $c = b_2$. Let $\alpha, \beta \in (0, 1]$ with infinite digit expansions

$$\alpha = (0, \alpha_0 \alpha_1 \ldots)_b \quad \text{and} \quad \beta = (0, \beta_0 \beta_1 \ldots)_c$$

in bases b and c, respectively. Put

$$M := \left\lfloor \frac{\log N}{\log b} \right\rfloor \quad \text{and} \quad L := \left\lfloor \frac{\log N}{\log c} \right\rfloor .$$

The number $A([0, \alpha) \times [0, \beta), \mathcal{S}_{b,c}, N)$ is the number of indices $k \in \{0, \ldots, N - 1\}$ with

$$\varphi_b(k) < \alpha \quad \text{and} \quad \varphi_c(k) < \beta.$$

Let

$$\begin{aligned} k &= k_0 + k_1 b + \cdots + k_M b^M \\ &= l_0 + l_1 c + \cdots + l_L c^L \end{aligned}$$

with $k_i \in \{0, ..., b - 1\}, l_i \in \{0, ..., c - 1\}$, respectively be the b-adic and c-adic expansion of k.
Then we have $\varphi_b(k) < \alpha$ and $\varphi_c(k) < \beta$ if and only if, one of the condition $(1), (2), ..., (M + 2)$ from the case $s = 1$ is fulfilled and one of the following conditions for the second component is satisfied:

$$\begin{aligned}
(\bar{1}) &\quad k \equiv l_0 \pmod{c},\ 0 \le l_0 < \beta_0 \\
(\bar{2}) &\quad k \equiv \beta_0 + l_1 c \pmod{c^2},\ 0 \le l_1 < \beta_1 \\
(\bar{3}) &\quad k \equiv \beta_0 + \beta_1 c + l_2 c^2 \pmod{c^3},\ 0 \le l_2 < \beta_2 \\
&\quad \vdots \\
(\overline{L+1}) &\quad l \equiv \beta_0 + \cdots + \beta_{L-1} c^{L-1} + l_L c^L \pmod{c^{L+1}},\ 0 \le l_L < \beta_L \\
(\overline{L+2}) &\quad k \equiv \beta_0 + \cdots + \beta_L c^L \pmod{c^{L+2}}.
\end{aligned}$$

Recall that by our global assumption we have $\gcd(b, c) = 1$ and thus also $\gcd(b^m, c^l) = 1$ for $m, l \in \mathbb{N}$.
According to Lemma A.6 and to the Chinese remainder theorem, the number of $k \in \{0, \ldots, N - 1\}$ which satisfy the congruences (m) and $(\bar{l})$ for

- $m \in \{1, \ldots, M + 1\}$ and $l \in \{1, \ldots, L + 1\}$ equals

  $$\alpha_{m-1} \beta_{l-1} \left(\left\lfloor \frac{N}{b^m c^l} \right\rfloor + \theta_{m-1,l-1} \right)$$

  where $\theta_{m-1,l-1} \in \{0, 1\}$;

- $m \in \{1, ..., M+1\}$ and $l = L+2$ equals

$$\alpha_{m-1}\left(\underbrace{\left\lfloor \frac{N}{b^m c^{L+2}} \right\rfloor}_{=0} + \theta_{m-1}\right) = \alpha_{m-1}\theta_{m-1}$$

  where $\theta_{m-1} \in \{0,1\}$;
- $m = M+2$ and $l \in \{1, ..., L+1\}$ equals $\beta_{l-1}\theta_{l-1}$, where $\theta_{l-1} \in \{0,1\}$;
- $m = M+2$ and $l = L+2$ equals $\theta_{M+1,L+1} \in \{0,1\}$.

Therefore,

$$\begin{aligned} A([0,\alpha)\times[0,\beta),\mathcal{S}_{b,c},N) = &\sum_{m=1}^{M+1}\sum_{l=1}^{L+1}\alpha_{m-1}\beta_{l-1}\left(\left\lfloor \frac{N}{b^m c^l}\right\rfloor + \theta_{m-1,l-1}\right) \\ &+ \sum_{m=1}^{M+1}\alpha_{m-1}\theta_{m-1} + \sum_{l=1}^{L+1}\beta_{l-1}\theta_{l-1} + \theta_{M+1,L+1}. \end{aligned}$$

We obtain

$$\begin{aligned} &\left| A([0,\alpha)\times[0,\beta),\mathcal{S}_{b,c},N) - \sum_{m=1}^{M+1}\sum_{l=1}^{L+1}\alpha_{m-1}\beta_{l-1}\frac{N}{b^m c^l}\right| \\ &\leq \sum_{m=1}^{M+1}\sum_{l=1}^{L+1}\alpha_{m-1}\beta_{l-1} + \sum_{m=1}^{M+1}\alpha_{m-1} + \sum_{l=1}^{L+1}\beta_{l-1} + 1 \\ &= \left(\sum_{m=1}^{M+1}\alpha_{m-1} + 1\right)\left(\sum_{l=1}^{L+1}\beta_{l-1} + 1\right). \end{aligned}$$

Finally, since $N \geq \max\{b, c\}$, we have using (4.2) and (4.3)

$$\begin{aligned} &|A([0,\alpha)\times[0,\beta),\mathcal{S}_{b,c},N) - \alpha\beta N| \\ &= \left| A([0,\alpha)\times[0,\beta),\mathcal{S}_{b,c},N) - N\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{\alpha_{m-1}}{b^m}\frac{\beta_{l-1}}{c^l}\right| \\ &\leq \left| A([0,\alpha)\times[0,\beta),\mathcal{S}_{b,c},N) - \sum_{m=1}^{M+1}\sum_{l=1}^{L+1}\alpha_{m-1}\beta_{l-1}\frac{N}{b^m c^l}\right| + N\sum_{m=M+2}^{\infty}\frac{\alpha_{m-1}}{b^m} \\ &+ N\sum_{l=L+2}^{\infty}\frac{\beta_{l-1}}{c^l} + N\sum_{m=M+2}^{\infty}\sum_{l=L+2}^{\infty}\frac{\alpha_{m-1}}{b^m}\frac{\beta_{l-1}}{c^l} \\ &\leq \left(\sum_{m=1}^{M+1}\alpha_{m-1} + 1\right)\left(\sum_{l=1}^{L+1}\beta_{l-1} + 1\right) + 3 \\ &\leq ((b-1)(M+1)+1)\,((c-1)(L+1)+1) + 3 \\ &\leq ((b-1)(M+1)+2)\,((c-1)(L+1)+2) \\ &\leq \frac{b\log(bN)}{\log b}\frac{c\log(cN)}{\log c}. \end{aligned}$$

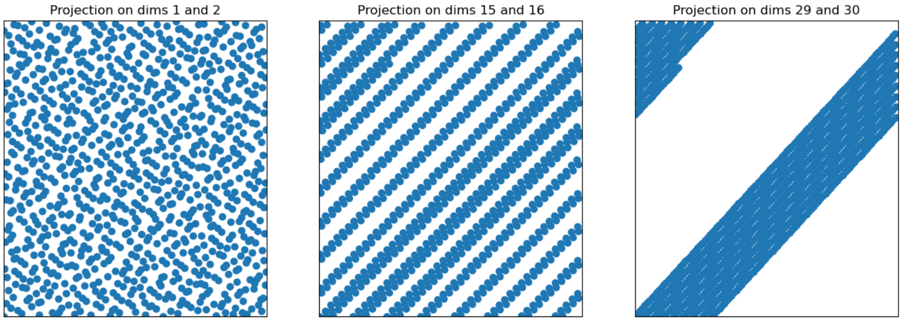


Figure 4.1: Projections of first 1000 points of the 30-dimensional Halton sequence on different pairs of coordinates. We observe that the sequence has very poor distribution properties when projected on "high" dimensional coordinates as large portions of the unit cube do not contain any points.

This finishes the proof for $s = 2$. $\square$

From now on, we say that an infinite sequence $\mathcal{S} \in [0,1)^s$ is of low-discrepancy if its star discrepancy is of order $\mathcal{O}((\log N)^s/N)$. In order to minimize the upper bound given by Theorem 4.2, we should choose $b_1, ..., b_s$ as small as possible. Since we need them to be pairwise coprime, we choose $b_1, ..., b_s$ to be the first s prime integers.
For high dimensions $s \geq 3$, we look at the distribution of the Halton sequence on pairs of coordinates. Unfortunately, when projecting the Halton sequence on "high" dimensional coordinates, we see that the sequence has very poor distribution properties, see Figure 4.1.
There are different ways to overcome this issue, one is a generalization of the Halton sequence to so called digital (t,s)-sequences. This is outside the scope of this project but we refer to [1, Chapter 5] for a study of such constructions.

4.2 The Hammersley Point Set

To finish this section, we use the infinite Halton sequence to construct a N-elements point set called the Hammersley point set.

Definition 4.3 *Let $s \in \mathbb{N}$ and let $b_1, ..., b_{s-1} \geq 2$ be pairwise coprime integers. The N-element Hammersley point set $\mathcal{H}_{N,b_1,...,b_{s-1}}$ in bases $b_1, ..., b_{s-1}$ is the finite point set $\{\mathbf{x}_0, ..., \mathbf{x}_{n-1}\} \subseteq [0,1)^s$ where*

$$\mathbf{x}_n := \left(\frac{n}{N}, \varphi_{b_1}(n), ..., \varphi_{b_{s-1}}(n)\right) \quad \text{for } n = 0, ..., N-1.$$

Theorem 4.4 *For the star discrepancy of the N-element Hammersley point set $\mathcal{H}_{N,b_1,...,b_{s-1}}$ with pairwise coprime bases $b_1, ..., b_{s-1}$ we have*

$$D_N^*(\mathcal{H}_{N,b_1,...,b_{s-1}}) \leq \frac{1}{N}\left(\prod_{j=1}^{s-1} \frac{b_j \log(b_j N)}{\log b_j} + 1\right).$$

Hence, asymptotically for $N \to \infty$ we have $D_N^*(\mathcal{H}_{N,b_1,\dots,b_{s-1}}) = \mathcal{O}((\log N)^{s-1}/N)$.

Proof Using Lemma 3.12 and Theorem 4.2 we have

$$\begin{aligned} D_N^*(\mathcal{H}_{N,b_1,\dots,b_{s-1}}) &\leq \frac{1}{N}\left(\max_{M=1,\dots,N} M D_M^*(\mathcal{S}_{b_1,\dots,b_{s-1}}) + 1\right) \\ &= \frac{1}{N}\left(\max_{M=1,\dots,N} \prod_{j=1}^{s-1} \frac{b_j \log(b_j M)}{\log b_j} + 1\right) \\ &= \frac{1}{N}\left(\prod_{j=1}^{s-1} \frac{b_j \log(b_j N)}{\log b_j} + 1\right). \end{aligned}$$

This finishes the proof. $\square$

Similarly to the case of infinite sequences, we say that a N-elements point set $\mathcal{P} \in [0,1)^s$ is of low-discrepancy if its star discrepancy is of order $\mathcal{O}((\log N)^{s-1}/N)$.

Chapter 5

Conclusion

In this project, we saw that sequences $(\mathbf{x}_n)_{n\in\mathbb{N}_0}$ satisfying the desired property

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) = \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x}$$

must have a particular distribution, namely they are uniformly distributed modulo 1.
We studied different characterizations of this property and we introduced the notion of discrepancy, which enables us to compare such sequences.
We then studied several bounds on the discrepancy, which show that there are natural limits on how "well" distributed such sequences can be.
Finally, we finished with the construction of a concrete example of a low-discrepancy sequence, the Halton sequence.

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Appendix A

Appendix

Lemma A.1 *Let $x, y, z \in \mathbb{R}$ and $\delta > 0$. Then we have*

$$y - \delta \leq x \leq z + \delta \implies |x| \leq \delta + \max\{|y|, |z|\}.$$

Proof If $x \geq 0$, then $|x| = x \leq z + \delta \leq \delta + |z| \leq \delta + \max\{|y|, |z|\}$.
If $x \geq 0$, then $|x| = -x \leq \delta - y \leq \delta + |y| \leq \delta + \max\{|y|, |z|\}$. $\square$

Lemma A.2 *For $j = 1, ..., s$ let $u_j, v_j \in [0, 1]$ with $\left|u_j - v_j\right| < \delta$ for some $\delta \leq 1$. Then*

$$\left|\prod_{j=1}^{s} u_j - \prod_{j=1}^{s} v_j\right| \leq 1 - (1 - \delta)^s \leq s\delta.$$

Proof We prove it by induction on s. Trivially, the assertion holds true for $s = 1$.
Assume that it holds for $s \in \mathbb{N}$. Then without loss of generality, we may assume that $u_{s+1} \geq v_{s+1}$, otherwise we permute the roles of u_{s+1} and v_{s+1} in the rest of the proof.
Using the triangle inequality and the assumption $u_j \in [0, 1]$ for $j = 1, ..., s$ we have

$$\begin{aligned} \left|\prod_{j=1}^{s+1} u_j - \prod_{j=1}^{s+1} v_j\right| &= \left|(u_{s+1} - v_{s+1}) \prod_{j=1}^{s} u_j + v_{s+1} \left(\prod_{j=1}^{s} u_j - \prod_{j=1}^{s} v_j\right)\right| \\ &\leq |u_{s+1} - v_{s+1}| \cdot 1 + v_{s+1}(1 - (1 - \delta)^s). \end{aligned}$$

Since $u_{s+1} \geq v_{s+1}$, we have $|u_{s+1} - v_{s+1}| = u_{s+1} - v_{s+1}$ and we therefore have

$$\begin{aligned} \left|\prod_{j=1}^{s+1} u_j - \prod_{j=1}^{s+1} v_j\right| &\leq |u_{s+1} - v_{s+1}| \cdot 1 + v_{s+1}(1 - (1 - \delta)^s) \\ &= u_{s+1} - v_{s+1}(1 - \delta)^s \\ &= u_{s+1}(1 - (1 - \delta)^s) + (u_{s+1} - v_{s+1})(1 - \delta)^s \\ &\leq 1 - (1 - \delta)^s + \delta(1 - \delta)^s = 1 - (1 - \delta)^s. \end{aligned}$$

This shows the left inequality.
To prove the right inequality, we start by applying the Mean Value Theorem to the function $f : \mathbb{R} \to \mathbb{R}$ defined via $f(y) = y^s$. We have for all $y, z \in \mathbb{R}$ with $z \leq y$ that

$$y^s - z^s \leq f'(\xi)(y - z) = s\xi^{s-1}(y - z)$$

for some $\xi \in (z, y)$.
Choosing $y = 1$ and $z = 1 - \delta$ yields

$$1 - (1 - \delta)^s = s\xi^{s-1}\delta \leq s\delta$$

for some $\xi \in (1 - \delta, 1)$, proving the right inequality. □

Lemma A.3 *Let $N \geq 1$, then we have*

$$\frac{\log_2 N}{2^8\sqrt{\log_2 N + 3}} \geq \frac{\sqrt{\log_2 N}}{2^9}.$$

Proof If $N = 1$, then both sides are equal to 0.
If $N \geq 2$, then $N^3 \geq 8$. The result then follows as

$$\begin{aligned}
&8 \leq N^3 \\
&\iff \log_2(8N) \leq \log_2(N^4) = 4\log_2(N) \\
&\iff \log_2(N)(\log_2(N) + 3) = \log_2(N)\log_2(8N) \leq 4(\log_2 N)^2 \\
&\iff \sqrt{\log_2(N)(\log_2(N) + 3)} \leq 2\log_2 N \\
&\iff \frac{\sqrt{\log_2 N}}{2^9} \leq \frac{\log_2 N}{2^8\sqrt{\log_2 N + 3}}.
\end{aligned}$$

□

Lemma A.4 *Let $i, j, k \in \{0, 1, ..., m\}$ with $i < j$ and $i \leq k \leq j$. We consider the sets*

$$R^{(j)}_{a,b} = \left[\frac{a}{2^{m-j}}, \frac{a+1}{2^{m-j}}\right) \times \left[\frac{b}{2^j}, \frac{b+1}{2^j}\right)$$

for $a \in \{0, 1, ..., 2^{m-j} - 1\}, b \in \{0, 1, ..., 2^j - 1\}$, and

$$R^{(i,j)}_{a,b} = \left[\frac{a}{2^{m-i}}, \frac{a+1}{2^{m-i}}\right) \times \left[\frac{b}{2^j}, \frac{b+1}{2^j}\right)$$

for $a \in \{0, 1, ..., 2^{m-i} - 1\}, b \in \{0, 1, ..., 2^j - 1\}$. Then we have

$$R^{(i,j)}_{a,b} \subseteq R^{(k)}_{\lfloor a2^{i-k} \rfloor, \lfloor b2^{k-j} \rfloor}.$$

Proof Let $\mathbf{x} = (x_1, x_2) \in R_{a,b}^{(i,j)}$. Then $\frac{a}{2^{m-i}} \leq x_1 < \frac{a+1}{2^{m-i}}$ and $\frac{b}{2^j} \leq x_2 < \frac{b+1}{2^j}$. Therefore we also have

$$\frac{\lfloor a2^{i-k} \rfloor}{2^{m-k}} \leq \frac{a2^{i-k}}{2^{m-k}} = \frac{a}{2^{m-i}} \leq x_1 < \frac{a+1}{2^{m-i}} = \frac{(a+1)2^{i-k}}{2^{m-k}}.$$

We are left to show that $(a+1)2^{i-k} \leq \lfloor a2^{i-k} \rfloor + 1$, i.e. that $a2^{i-k} - \lfloor a2^{i-k} \rfloor \leq 1 - 2^{i-k}$.
Let $r \in \mathbb{N}_0$ such that $b \in \{r2^{k-i}, r2^{k-i}+1, ..., (r+1)2^{k-i} - 1\}$. Then we have

$$\begin{aligned} a2^{i-k} - \lfloor a2^{i-k} \rfloor &= a2^{i-k} - r \\ &\leq \left((r+1)2^{k-i} - 1\right) 2^{i-k} - r = 1 - 2^{i-k}. \end{aligned}$$

We deduce that

$$\frac{\lfloor a2^{i-k} \rfloor}{2^{m-k}} \leq \frac{a2^{i-k}}{2^{m-k}} \leq x_1 < \frac{(a+1)2^{i-k}}{2^{m-k}} \leq \frac{\lfloor a2^{i-k} \rfloor + 1}{2^{m-k}}.$$

We can proceed similarly to show

$$\frac{\lfloor b2^{k-j} \rfloor}{2^k} \leq \frac{b2^{k-j}}{2^k} \leq x_2 < \frac{(b+1)2^{k-j}}{2^k} \leq \frac{\lfloor b2^{k-j} \rfloor + 1}{2^k}$$

and therefore $\mathbf{x} \in R^{(k)}_{\lfloor a2^{i-k} \rfloor, \lfloor b2^{k-j} \rfloor}$. $\square$

Lemma A.5 *Let $M \in \mathbb{N}$, $M \geq 2$ and $C(M) := (-M/2, M/2] \cap \mathbb{Z}$. We then have*

$$\frac{1}{M} \sum_{h \in C(M)} \exp(2\pi i h a / M) = \begin{cases} 1 & \text{if } a \equiv 0 \pmod{M} \\ 0 & \text{if } a \not\equiv 0 \pmod{M} \end{cases}.$$

Proof Since $|C(M)| = M$, the case where $a \equiv 0 \pmod{M}$ is clear.
Assume $a \not\equiv 0 \pmod{M}$. If M is odd, we can write $M = 2l + 1$ for some $l \in \mathbb{N}_0$ and therefore $C(M) = \{-l, -l+1, ..., 0, 1, ..., l\}$. We deduce that

$$\begin{aligned} \sum_{h \in C(M)} \exp(2\pi i h a / M) &= \sum_{h=-l}^{l} \exp(2\pi i h a / M) \\ &= \frac{\sin((l+1/2)2\pi \frac{a}{M})}{\sin(\pi \frac{a}{M})} \\ &= \frac{\sin(M\pi \frac{a}{M})}{\sin(\pi \frac{a}{M})} = 0. \end{aligned}$$

If M is even, we can write $M = 2l$ for some $l \in \mathbb{N}_0$ and therefore $C(M) = \{-l+1, ..., 0, 1, ..., l-1, l\}$. We deduce that

$$\begin{aligned}
\sum_{h \in C(M)} \exp(2\pi i h a/M) &= \sum_{h=-(l-1)}^{l-1} \exp(2\pi i h a/M) + \exp(2\pi i l a/M) \\
&= \frac{\sin\left((l-1/2)2\pi\frac{a}{M}\right)}{\sin(\pi\frac{a}{M})} + \exp(2\pi i l a/M) \\
&= \frac{\sin\left((M-1)\pi\frac{a}{M}\right)}{\sin(\pi\frac{a}{M})} + \exp(2\pi i l a/M) \\
&= \frac{\sin\left((M-1)\pi\frac{a}{M}\right)}{\sin(\pi\frac{a}{M})} + \exp(\pi i a).
\end{aligned}$$

Finally, we have

$$\sin\left((M-1)\pi\frac{a}{M}\right) = \sin\left(\pi a - \pi\frac{a}{M}\right) = \begin{cases} \sin(\pi\frac{a}{M}) & \text{if } a \text{ is odd} \\ -\sin(\pi\frac{a}{M}) & \text{if } a \text{ is even} \end{cases}$$

and

$$\exp(\pi i a) = \begin{cases} -1 & \text{if } a \text{ is odd} \\ 1 & \text{if } a \text{ is even} \end{cases}$$

so we conclude that $\sum_{h \in C(M)} \exp(2\pi i h a/M) = 0$. □

Lemma A.6 *Let $a, n, m \in \mathbb{Z}$. The congruence $x \equiv a \pmod{m}$ has exactly $\lfloor \frac{n}{m} \rfloor + \theta$ solutions $x \in \{0, \ldots, n-1\}$ where θ is either 0 or 1.*

Proof The result follows from the fact that for any m consecutive integers, the congruence problem has exactly one solution. □

Theorem A.7 (Weierstrass approximation for trigonometric polynomials) *Let $f : [0,1]^s \to \mathbb{C}$ be a continuous, one-periodic function. Then, for all $\varepsilon > 0$, there exist a function $P : [0,1]^s \to \mathbb{C}$ of the form*

$$P(\mathbf{x}) = \sum_{j=1}^{m} a_j \exp(2\pi i \mathbf{h_j} \cdot \mathbf{x})$$

for some $\mathbf{h_j} \in \mathbb{Z}^s$ and $a_j \in \mathbb{C}, j = 1, ..., m$, such that $\|f - P\|_{L^\infty} < \varepsilon$.

Proof A proof for the case $s = 1$ can be found in [2, Theorem 5.4.1]. □

Bibliography

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