Linear Algebra L2 - Affine equation systems

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Learning Goals

- Solving affine equation systems

Task 1

Do this one together:

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - 3y + 2z &= 5 \\ -3x + y + 5z &= 13 \end{aligned}$$

Solve the following affine equation systems

$$\begin{aligned} 3x + 4y &= 1 \\ 2x + 3y &= 12 \end{aligned}$$

$$\begin{aligned} 3x - 2y &= 4 \\ -6x + 4y &= 7 \end{aligned}$$

$$\begin{aligned} 2x + y + z - 6 &= 0 \\ 4y + z + x &= 5 \\ 2x + z + 3y &= 7 \end{aligned}$$

Task 2

Modelling Problem: Solve using Gauss-Jordan elimination.

- are there linear relationships ?
- If so, understanding and deriving constraints

In 2010, the average salary for all accountants together in the two cities San Diego, California, and Salt Lake City, Utah, was $\$45091.50$.
The average salary in San Diego alone, however, was $\$5231$ greater than the average salary in Salt Lake City alone. What is the average salary of an accountant in each city, assuming that there are the same number of accountants in each city?

Task 3

Modelling Problem: Solve using Gauss-Jordan elimination. A chemist has prepared two acid solutions, one of which is 2% by volume, the other 7% by volume. How many cubic centimetres of each should the chemist mix together to obtain $40cm^3$ of a 3.2% acid solution?

Hint: If we multiply acidity per volume with a certain volume, we get a total amount of acid in this volume. If we sum 2 total amounts, we get another total amount of acid - which is the total amount for the union of the two volumes.
In order to get back to an acidity per volume, we have to divide by the volume.

Task 4

In a hack and slay game, you need bags of three items which you can use to increase your attack, defence and dexterity points. The counts of each item are x, y and z respectively.
The contributions of each item are shown below:

item	Attack	Defense	Dex
Aunties Old Table Cloth (x)	-20	40	10
Rusty old looking dagger (y)	50	10	-10
Geylang Gift Shop Crystal (z)	10	10	60

In order to clear a final boss, you need to have 320 attack and 280 defense stats. Note that the stats scale linearly on the item equipped. Also you have 16 slots, which allow you to equip 16 items in total.

PS: Isnt it weird how a small monster can drop a huge item on death ?
a) Derive an affine equation. - Let us check in to your progress after 5 minutes .
b) and solve it.

Task 5

Solve the following affine equation systems. Follow these steps:
a) Write down the augmented matrix [A|b] of the equation system above
b) Compute the reduced row echelon form.

- Show as an intermediate step the augmented matrix when for the first time the zero-th column A[:, 0] became a one-hot vector after performing transformations .
- Show as an intermediate step the augmented matrix when for the first time the augmented matrix is in row echelon form.
- Show as final answer the augmented matrix in reduced row echelon form.

c) Provide one solution which solves the equation system.
d) Write the set of all solutions as a single vector like this, if there is only one solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$

or an affine equation, if there is more than one solution, like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

or like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} + t \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

or state that there is no solution, if it has no solution.

$$\begin{aligned} x + y + z &= 1 \\ 2x - y + z &= -1 \\ x + 3y - z &= 7 \end{aligned}$$

$$\begin{aligned} 3x - 4y &= 8 \\ x + y + z &= 2 \\ 2x - 5y - z &= 6 \end{aligned}$$

Task 6

Solve these affine equation systems.

Again, write the set of all solutions as a single vector like this, if there is only one solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$

or an affine equation, if there is more than one solution, like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

or like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} + t \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

or state that there is no solution, if it has no solution.

$$\begin{aligned} 3x_0 - 9x_1 - 6x_2 + 2x_3 &= 5 \\ -2x_0 + 3x_1 + 4x_2 - 2x_3 &= -2 \end{aligned}$$

$$\begin{aligned} 2x_0 - 6x_1 - 6x_2 + 3x_3 &= 5 \\ -x_0 - 2x_1 + 3x_2 - 2x_3 &= -2 \\ 2x_0 + 4x_1 - 6x_2 + 4x_3 &= 7 \end{aligned}$$

Task 7

Define 2 or 3 equations in 3 variables with bias terms of your own choosing and solve it!

Verify your obtained solution x by checking that it satisfies $Ax = b$.

Do you need a Prof to write such things down?

Task 1

Do this one together:

$$x + 2y - z = 3$$
$$2x - 3y + 2z = 5$$
$$-3x + y + 5z = 13$$

Solve the following affine equation systems

$$3x + 4y = 1$$
$$2x + 3y = 12$$

$$3x - 2y = 4$$
$$-6x + 4y = 7$$

$$2x + y + z - 6 = 0$$
$$4y + z + x = 5$$
$$2x + z + 3y = 7$$

pivot

(i) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -3 & 2 & 5 \\ -3 & 1 & 5 & 13 \end{array}\right]$ $R_0 \times (-2) + R_1$

$(-2)+2=0$
$(-4)+(-3)=-7$
$2+2=4$
$(-6)+5=-1$

$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 4 & -1 \\ -3 & 1 & 5 & 13 \end{array}\right]$ $R_0 \times 3 + R_2$

$3+(-3)=0$
$6+1=7$
$(-3)+5=2$
$9+13=22$

$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 4 & -1 \\ 0 & 7 & 2 & 22 \end{array}\right]$ $R_1 \div (-7)$

$(-7)\div(-7)=1$
$4\div(-7)=-\frac{4}{7}$
$(-1)\div(-7)=\frac{1}{7}$

$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -\frac{4}{7} & \frac{1}{7} \\ 0 & 7 & 2 & 22 \end{array}\right]$ $R_1 \times (-7) + R_2$

$(-7)+(7)=0$
$4+2=6$
$(-1)+22=21$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -\frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 6 & 21 \end{array}\right] \quad R_2 \div 6$$

$6 \div 6 = 1$

$21 \div 6 = \frac{21}{6}$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -\frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 1 & \frac{21}{6} \end{array}\right] \quad R_2 \times \frac{4}{7} + R_1$$

$\frac{4}{7} + (-\frac{4}{7}) = 0$

$2 + \frac{1}{7} = \frac{15}{7}$

* Gaussian elimination ends here, apply backwards substitution from bottom of matrix to find values of x, y and z

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & \frac{15}{7} \\ 0 & 0 & 1 & \frac{21}{6} \end{array}\right] \quad R_2 \times 1 + R_0$$

$1 + (-1) = 0$

$\frac{21}{6} + 3 = \frac{13}{2}$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 0 & \frac{13}{2} \\ 0 & 1 & 0 & \frac{15}{7} \\ 0 & 0 & 1 & \frac{21}{6} \end{array}\right] \quad R_1 \times (-2) + R_0$$

$(-2) + 2 = 0$

$-\frac{30}{7} + \frac{13}{2} = 2.21$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2.21 \\ 0 & 1 & 0 & 2.14 \\ 0 & 0 & 1 & 3.5 \end{array}\right]$$

$\therefore x = 2.21, \; y = 2.14, \; z = 3.5$

* Gauss-Jordan elimination method

(ii) $\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 2 & 3 & 12 \end{array}\right]$ $R_0 \div 3$

$3 \div 3 = 1$
$4 \div 3 = \frac{4}{3}$
$1 \div 3 = \frac{1}{3}$

$= \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{1}{3} \\ 2 & 3 & 12 \end{array}\right]$ $R_0 \times (-2) + R_1$

$(-2) + 2 = 0$
$-\frac{8}{3} + 3 = 0.33$
$-\frac{2}{3} + 12 = 11.33$

$= \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{1}{3} \\ 0 & 0.33 & 11.33 \end{array}\right]$ $R_1 \div 0.33$

$0.33 \div 0.33 = 1$
$11.33 \div 0.33 = 34.33$

$= \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 34.33 \end{array}\right]$ $R_1 \times (-\frac{4}{3}) + R_0$

$-\frac{4}{3} + \frac{4}{3} = 0$
$-45.77 + \frac{1}{3} = -45.44$

$= \left[\begin{array}{cc|c} 1 & 0 & -45.44 \\ 0 & 1 & 34.33 \end{array}\right]$

$\therefore x = -45.44, \ y = 34.33$

(iii) $\left[\begin{array}{cc|c} 3 & -2 & 4 \\ -6 & 4 & 7 \end{array}\right]$ $R_0 \div 3$

$3 \div 3 = 1$
$-2 \div 3 = -\frac{2}{3}$
$4 \div 3 = \frac{4}{3}$

$= \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{4}{3} \\ -6 & 4 & 7 \end{array}\right]$ $R_0 \times 6 + R_1$

$6 + (-6) = 0$
$-4 + 4 = 0$
$8 + 7 = 15$

$= \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{4}{3} \\ 0 & 0 & 15 \end{array}\right]$

Ans : No solution!

if 0 then there is a solution via parametrization c(iv) $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 6 \\ 1 & 4 & 1 & 5 \\ 2 & 3 & 1 & 7 \end{array}\right]$ $R_0 \div 2$

$2 \div 2 = 1$
$1 \div 2 = \frac{1}{2}$
$1 \div 2 = \frac{1}{2}$
$6 \div 2 = 3$

$= \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 3 \\ 1 & 4 & 1 & 5 \\ 2 & 3 & 1 & 7 \end{array}\right]$ $R_0 \times (-1) + R_1$

$(-1)+1=0$
$(-\frac{1}{2})+4=3.5$
$(-\frac{1}{2})+1=0.5$
$(-3)+5=2$

$= \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 3 \\ 0 & 3.5 & 0.5 & 2 \\ 2 & 3 & 1 & 7 \end{array}\right]$ $R_0 \times (-2) + R_2$

$(-2)+2=0$
$(-1)+3=2$
$(-1)+1=0$
$(-6)+7=1$

$= \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 3 \\ 0 & 3.5 & 0.5 & 2 \\ 0 & 2 & 0 & 1 \end{array}\right]$ Swap R_1 and R_2

$= \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 3 \\ 0 & 2 & 0 & 1 \\ 0 & 3.5 & 0.5 & 2 \end{array}\right]$ $R_1 \div 2$

$2 \div 2 = 1$
$1 \div 2 = \frac{1}{2}$

$= \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 3 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 3.5 & 0.5 & 2 \end{array}\right]$ $R_1 \times (-3.5) + R_2$

$(-3.5)+(3.5)=0$
$-1.75+2=0.25$

$= \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 3 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0.5 & 0.25 \end{array}\right]$ $R_2 \times 0.5$

$0.5 \times 2 = 1$
$0.25 \times 2 = 0.5$

$= \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 3 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0.5 \end{array}\right]$ $R_2 \times (-\frac{1}{2}) + R_0$

$(-\frac{1}{2})+\frac{1}{2}=0$
$-0.25+3=2.75$

$$= \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 2.75 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0.5 \end{array}\right] \quad R_1 \times (-\tfrac{1}{2}) + R_0$$

$$(-\tfrac{1}{2}) + (\tfrac{1}{2}) = 0$$

$$-0.25 + 2.75 = 2.5$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2.5 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$\therefore\ x = 2.5,\ y = 0.5,\ z = 0.5$

Task 2

Modelling Problem: Solve using Gauss-Jordan elimination.

- are there linear relationships ?
- If so, understanding and deriving constraints

In 2010, the average salary for all accountants together in the two cities San Diego, California, and Salt Lake City, Utah, was $\$45091.50$.
The average salary in San Diego alone, however, was $\$5231$ greater than the average salary in Salt Lake City alone. What is the average salary of an accountant in each city, assuming that there are the same number of accountants in each city?

Unfinished business :)

Task 3

Modelling Problem: Solve using Gauss-Jordan elimination. A chemist has prepared two acid solutions, one of which is 2% by volume, the other 7% by volume. How many cubic centimetres of each should the chemist mix together to obtain $40cm^3$ of a 3.2% acid solution?

Hint: If we multiply acidity per volume with a certain volume, we get a total amount of acid in this volume. If we sum 2 total amounts, we get another total amount of acid - which is the total amount for the union of the two volumes.
In order to get back to an acidity per volume, we have to divide by the volume.

$$2\% \cdot V_0 + 7\% \cdot V_1 = 3.2\% (V_0 + V_1)$$
$$V_0 + V_1 = 40$$

$$\downarrow$$

$$2\% \cdot V_0 - 3.2\% \cdot V_0 + 7\% \cdot V_1 - 3.2\% \cdot V_1 = 0$$
$$V_0 + V_1 = 40$$

$$\downarrow$$

$$-1.2\% \cdot V_0 + 3.8\% \cdot V_1 = 0$$
$$V_0 + V_1 = 40$$

$$\left[\begin{array}{cc|c} -1.2 & 3.8 & 0 \\ 1 & 1 & 40 \end{array}\right] \quad \text{Swap } R_0 \text{ and } R_1$$

$$= \left[\begin{array}{cc|c} 1 & 1 & 40 \\ -1.2 & 3.8 & 0 \end{array}\right] \quad R_0 \times 1.2 + R_1$$

$$1.2 + (-1.2) = 0$$
$$1.2 + 3.8 = 5$$
$$1.2 \times 40 = 48$$

$$= \left[\begin{array}{cc|c} 1 & 1 & 40 \\ 0 & 5 & 48 \end{array}\right] \quad R_1 \div 5$$

$$5 \div 5 = 1$$
$$48 \div 5 = 9.6$$

$$= \left[\begin{array}{cc|c} 1 & 1 & 40 \\ 0 & 1 & 9.6 \end{array}\right] \quad R_1 \times (-1) + R_0$$

$$(-1) + 1 = 0$$
$$(-9.6) + 40 = 30.4$$

$$= \left[\begin{array}{cc|c} 1 & 0 & 30.4 \\ 0 & 1 & 9.6 \end{array}\right]$$

$\therefore$ Ans = Take $30.4\,cm^3$ of V_0 and $9.6\,cm^3$ of V_1

Task 4

In a hack and slay game, you need bags of three items which you can use to increase your attack, defence and dexterity points. The counts of each item are x, y and z respectively.

The contributions of each item are shown below:

item	Attack	Defense	Dex
Aunties Old Table Cloth (x)	-20	40	10
Rusty old looking dagger (y)	50	10	-10
Geylang Gift Shop Crystal (z)	10	10	60

In order to clear a final boss, you need to have 320 attack and 280 defense stats. Note that the stats scale linearly on the item equipped. Also you have 16 slots, which allow you to equip 16 items in total.

PS: Isnt it weird how a small monster can drop a huge item on death ?

a) Derive an affine equation. - Let us check in to your progress after 5 minutes .

b) and solve it.

$$-20x + 50y + 10z = 320$$
$$40x + 10y + 10z = 280$$
$$x + y + z = 16$$

$$\left[\begin{array}{ccc|c} -20 & 50 & 10 & 320 \\ 40 & 10 & 10 & 280 \\ 1 & 1 & 1 & 16 \end{array}\right]$$

Swap R_2 and R_0

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 40 & 10 & 10 & 280 \\ -20 & 50 & 10 & 320 \end{array}\right]$$

$R_0 \times (-40) + R_1$

$-40 + 40 = 0$

$-40 + 10 = -30$

$-40 + 10 = -30$

$-640 + 280 = -360$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & -30 & -30 & -360 \\ -20 & 50 & 10 & 320 \end{array}\right]$$

$R_0 \times (20) + R_2$

$20 + (-20) = 0$

$20 + 50 = 70$

$20 + 10 = 30$

$320 + 320 = 640$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & -30 & -30 & -360 \\ 0 & 70 & 30 & 640 \end{array}\right]$$

$R_1 \div (-30)$

$-30 \div (-30) = 1$

$-30 \div (-30) = 1$

$-360 \div (-30) = 12$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 1 & 12 \\ 0 & 70 & 30 & 640 \end{array}\right]$$

$R_1 \times (-70) + R_2$

$-70 + 70 = 0$

$-70 + 30 = -40$

$-840 + 640 = -200$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 1 & 12 \\ 0 & 0 & -40 & -200 \end{array}\right] \quad R_2 \div (-40)$$

$-40 \div (-40) = 1$
$-200 \div (-40) = 5$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 1 & 12 \\ 0 & 0 & 1 & 5 \end{array}\right] \quad R_2 \times (-1) + R_1$$

$(-1) + (1) = 0$
$-5 + 12 = 7$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 5 \end{array}\right] \quad R_2 \times (-1) + R_0$$

$(-1) + 1 = 0$
$(-5) + 16 = 11$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 0 & 11 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 5 \end{array}\right] \quad R_1 \times (-1) + R_0$$

$(-1) + 1 = 0$
$(-7) + 11 = 4$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$\therefore x = 4, \ y = 7, \ z = 5$

Task 5

Solve the following affine equation systems. Follow these steps:
a) Write down the augmented matrix [A|b] of the equation system above
b) Compute the reduced row echelon form.

- Show as an intermediate step the augmented matrix when for the first time the zero-th column A[:, 0] became a one-hot vector after performing transformations .
- Show as an intermediate step the augmented matrix when for the first time the augmented matrix is in row echelon form.
- Show as final answer the augmented matrix in reduced row echelon form.

c) Provide one solution which solves the equation system.
d) Write the set of all solutions as a single vector like this, if there is only one solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$

or an affine equation, if there is more than one solution, like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

or like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} + t \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

or state that there is no solution, if it has no solution.

$$\begin{aligned} x + y + z &= 1 \\ 2x - y + z &= -1 \\ x + 3y - z &= 7 \end{aligned}$$

$$\begin{aligned} 3x - 4y &= 8 \\ x + y + z &= 2 \\ 2x - 5y - z &= 6 \end{aligned}$$

(i) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & -1 \\ 1 & 3 & -1 & 7 \end{array}\right]$ $R_0 \times (-2) + R_1$

$(-2) + 2 = 0$
$(-2) + (-1) = -3$
$(-2) + 1 = -1$
$(-2) + (-1) = -3$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & -3 \\ 1 & 3 & -1 & 7 \end{array}\right]$ $R_0 \times (-1) + R_2$

$(-1) + 1 = 0$
$(-1) + 3 = 2$
$(-1) + (-1) = -2$
$(-1) + 7 = 6$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & -3 \\ 0 & 2 & -2 & 6 \end{array}\right] \quad R_1 \div (-3)$$

$(-3) \div (-3) = 1$
$(-1) \div (-3) = \frac{1}{3}$
$(-3) \div (-3) = 1$

Ans: First time one-hot vector

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} & 1 \\ 0 & 2 & -2 & 6 \end{array}\right] \quad R_1 \times (-2) + R_2$$

$(-2) + 2 = 0$
$(-\frac{2}{3}) + (-2) = -\frac{8}{3}$
$(-2) + 6 = 4$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} & 1 \\ 0 & 0 & -\frac{8}{3} & 4 \end{array}\right] \quad R_2 \div (-\frac{8}{3})$$

$(-\frac{8}{3}) \div (-\frac{8}{3}) = 1$
$4 \div (-\frac{8}{3}) = -1.5$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} & 1 \\ 0 & 0 & 1 & -1.5 \end{array}\right]$$

Ans: First time row echelon form

* Can continue to solve with Gauss-Jordan elimination

(ii)

$$\left[\begin{array}{ccc|c} 3 & -4 & 0 & 8 \\ 1 & 1 & 1 & 2 \\ 2 & -5 & -1 & 6 \end{array}\right] \quad \text{Swap } R_0 \text{ and } R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 3 & -4 & 0 & 8 \\ 2 & -5 & -1 & 6 \end{array}\right] \quad R_0 \times (-3) + R_1$$

$(-3) + 3 = 0$
$(-3) + (-4) = -7$
$(-3) + 0 = -3$
$(-6) + 8 = 2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -7 & -3 & 2 \\ 2 & -5 & -1 & 6 \end{array}\right] \quad R_0 \times (-2) + R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -7 & -3 & 2 \\ 0 & -7 & -3 & 2 \end{array}\right] \quad R_1 \div (-7)$$

$(-2)+2=0$
$(-2)+(-5)=-7$
$(-2)+(-1)=-3$
$(-4)+6=2$

Ans: First time one hot vector

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{3}{7} & -\frac{2}{7} \\ 0 & -7 & -3 & 2 \end{array}\right] \quad R_1 \times 7 + R_2$$

$(-7)\div(-7)=1$
$(-3)\div(-7)=\frac{3}{7}$
$2\div(-7)=-\frac{2}{7}$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{3}{7} & -\frac{2}{7} \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \times (-1) + R_0$$

$7+(-7)=0$
$3+(-3)=0$
$(-2)+2=0$

Ans: First time row echelon form

$$= \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{7} & \frac{16}{7} \\ 0 & 1 & \frac{3}{7} & -\frac{2}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$(-1)+1=0$
$(-\frac{3}{7})+1=\frac{4}{7}$
$(\frac{2}{7})+2=\frac{16}{7}$

$\rightarrow$ One-dim affine space $(z=t)$

$$x = \frac{16}{7} - \frac{4}{7}t$$
$$y = -\frac{2}{7} - \frac{3}{7}t$$
$$z = t$$

Sum of vectors $\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{16}{7} \\ -\frac{2}{7} \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{4}{7} \\ -\frac{3}{7} \\ 1 \end{bmatrix}$

Task 6

Solve these affine equation systems.

Again, write the set of all solutions as a single vector like this, if there is only one solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$

or an affine equation, if there is more than one solution, like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

or like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} + t \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

or state that there is no solution, if it has no solution.

$$\begin{aligned} 3x_0 - 9x_1 - 6x_2 + 2x_3 &= 5 \\ -2x_0 + 3x_1 + 4x_2 - 2x_3 &= -2 \end{aligned}$$

$$\begin{aligned} 2x_0 - 6x_1 - 6x_2 + 3x_3 &= 5 \\ -x_0 - 2x_1 + 3x_2 - 2x_3 &= -2 \\ 2x_0 + 4x_1 - 6x_2 + 4x_3 &= 7 \end{aligned}$$

(i) $\left[\begin{array}{cccc|c} 3 & -9 & -6 & 2 & 5 \\ -2 & 3 & 4 & -2 & -2 \end{array}\right]$ $R_0 \div 3$

$= \left[\begin{array}{cccc|c} 1 & -3 & -2 & 2/3 & 5/3 \\ -2 & 3 & 4 & -2 & -2 \end{array}\right]$ $R_0 \times 2 + R_1$

$= \left[\begin{array}{cccc|c} 1 & -3 & -2 & 2/3 & 5/3 \\ 0 & -3 & 0 & -2/3 & 4/3 \end{array}\right]$ $R_1 \div (-3)$

Ans: First time one-hot vector

$= \left[\begin{array}{cccc|c} 1 & -3 & -2 & 2/3 & 5/3 \\ 0 & 1 & 0 & 2/9 & -4/9 \end{array}\right]$ $R_1 \times (3) + R_0$

Ans: First time Row echelon form

$= \left[\begin{array}{cccc|c} 1 & 0 & -2 & 4/3 & 1/3 \\ 0 & 1 & 0 & 2/9 & -4/9 \end{array}\right]$

Ans: First time Reduced Row echelon form

$3 \div 3 = 1$
$-9 \div 3 = -3$
$-6 \div 3 = -2$
$2 \div 3 = 2/3$
$5 \div 3 = 5/3$

$2 + (-2) = 0$
$-6 + 3 = -3$
$-4 + 4 = 0$
$4/3 + (-2) = -2/3$
$10/3 + (-2) = 4/3$

$-3 \div (-3) = 1$
$-2/3 \div (-3) = 2/9$
$4/3 \div (-3) = -4/9$

$3 + (-3) = 0$
$2/3 + 2/3 = 4/3$
$-4/3 + 5/3 = 1/3$

→ The solution is a two-dim affine space $\left(\begin{matrix} x_2 = t \\ x_3 = u \end{matrix} \right)$

$x_0 = 1/3 + 2t - 4/3\,u$
$x_1 = -4/9 - 2/9\,u$

sum of vector $= \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -4/9 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -4/3 \\ -2/9 \\ 0 \\ 1 \end{bmatrix}$

(ii) $\left[\begin{array}{cccc|c} 2 & -6 & -6 & 3 & 5 \\ -1 & -2 & 3 & -2 & -2 \\ 2 & 4 & -6 & 4 & 7 \end{array}\right]$ $R_0 \times \frac{1}{2} + R_1$

$= \left[\begin{array}{cccc|c} 2 & -6 & -6 & 3 & 5 \\ 0 & -5 & 0 & -1/2 & 1/2 \\ 2 & 4 & -6 & 4 & 7 \end{array}\right]$ $R_0 \times (-1) + R_2$

$= \left[\begin{array}{cccc|c} 2 & -6 & -6 & 3 & 5 \\ 0 & -5 & 0 & -1/2 & 1/2 \\ 0 & 10 & 0 & 1 & 2 \end{array}\right]$ $R_0 \div 2$

$= \left[\begin{array}{cccc|c} 1 & -3 & -3 & 3/2 & 5/2 \\ 0 & -5 & 0 & -1/2 & 1/2 \\ 0 & 10 & 0 & 1 & 2 \end{array}\right]$ $R_1 \div (-5)$

Ans : First time one-hot vector

$1 + (-1) = 0$
$-3 + (-2) = -5$
$-3 + 3 = 0$
$3/2 + (-2) = -1/2$
$5/2 + (-2) = 1/2$

$-2 + 2 = 0$
$6 + 4 = 10$
$6 + (-6) = 0$
$-3 + 4 = 1$
$-5 + 7 = 2$

$2 \div 2 = 1$
$-6 \div 2 = -3$
$-6 \div 2 = -3$
$3 \div 2 = 3/2$
$5 \div 2 = 5/2$

$(-5) \div (-5) = 1$
$-1/2 \div (-5) = \frac{1}{10}$
$1/2 \div (-5) = -\frac{1}{10}$

$$= \left[\begin{array}{cccc|c} 1 & -3 & -2 & 3/2 & 5/2 \\ 0 & 1 & 0 & 1/10 & -1/10 \\ 0 & 10 & 0 & 1 & 2 \end{array}\right] \quad R_1 \times (-10) + R_2$$

$$= \left[\begin{array}{cccc|c} 1 & -3 & -3 & 3/2 & 5/2 \\ 0 & 1 & 0 & 1/10 & -1/10 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

NO SOLUTION! Not zero

$(-10) + 10 = 0$

$-1 + 1 = 0$

$1 + 2 = 3$