Linear Algebra L1 - Vectors

Alexander Binder

February 17, 2023

Learning Goals

- vectors of real numbers
- norms of vectors and their properties
- inner products, their interpretation and properties
- representing a vector as a linear combination
- vector spaces
- independent sets of vectors
- orthogonal sets of vectors
- projecting onto a vector, removing the direction of a vector
- creating an orthogonal set of vectors

Task 1

Compute the euclidean vector norm for vectors

$$[1, 0, 2],\ [3, 4],\ [-7, 2, -4, \sqrt{12}]$$

Task 2

Compute the corresponding unit length vector for these:

$$[3, 4],\ [-1, -2, 3],\ [-7, 2, -4, \sqrt{12}]$$

Task 3

Compute the inner product between these vectors and their angle in degrees:

$$[3, -2, 2], [1, 2, 2]$$

Compute the inner product between these vectors and their angle in degrees:

$$[1, 0, 1], [2, 1, -2], [\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}]$$

Task 4

- What is the projection of $[5, 2]$ onto the subspace spanned by vector $[1, 1]$?
- What is the projection of $[0, 2, 1]$ onto the subspace spanned by vector $[1, -1, -1]$?
- Project $[5, 2]$ onto the subspace spanned by vectors $[2, 3], \ [1, 1]$
- What is the projection of $[1, -1, 1]$ onto the subspace spanned by vectors $[0, 0, -1], \ [2, 0, 1]$? Hint: this one is more tricky. Reason: $[0, -1, -1] \cdot [2, 0, 1] \neq 0$

Task 5

compute these matrix multiplications:

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \end{bmatrix}$$

Question: Do you need more of them to practice? If so, you can do at home:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 2.5 \\ -3.5 & 1.5 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Task 6

- Project $[5, 2]$ onto the orthogonal space of vector $[2, -3]$
- Project $[1, -1, 3]$ onto the orthogonal space of vector $[-3, 1, 1]$
- Project $[1, -1, 3, 1]$ onto the orthogonal space of vectors $[-2, 2, 0, 0]$, $[0, 0, \sqrt{2}, \sqrt{2}]$

Task 7

run Gram-Schmid-orthogonalization on the vectors

$$[12, 12, 6], [2, -2, 4], [-2, -2, 1]$$

Task 8: understanding distances coming from ℓ_p-norms

Coding: plot in python or similar the set of points $x \in \mathbb{R}^2$ such that $\|x\|_p = 1$ for

- $p = 0.2$
- $p = 0.5$
- $p = 1$
- $p = 1.5$
- $p = 2$

- $p = 4$
- $p = 8$
- $p = 16$

Hint: in 2 dimensions for $p = 2$ the solution is given by

$$x(t) = (\cos(t), \sin(t))$$

due to $\cos^2(t) + \sin^2(t)$ =1.
You can use the same idea with different powers. You can start by considering $(\cos^r(t), \sin^r(t))$. One thing to note: $\cos(t)^r, \sin(t)^r$ is not always defined for negative values and certain r.

For $p \neq 2$ you can consider this, which deals with the signs:

$$x(t) = (sign(\cos(t))|\cos(t)|^r, sign(\sin(t))|\sin(t)|^r)$$

for the right choice of r. Find out which r is suitable for a general $p > 0$ such that $\|x\|_p = 1$. Then plot it in python.

Task 1

Compute the euclidean vector norm for vectors

$$[1, 0, 2],\ [3, 4],\ [-7, 2, -4, \sqrt{12}]$$

$x = [1,0,2] \rightarrow \|x\|_2 = (|1|^2 + |0|^2 + |2|^2)^{\frac{1}{2}} = \sqrt{1+0+4}$

$= \sqrt{5}$

$= 2.24$ (2 d.p.)

$x = [3,4] \rightarrow \|x\|_2 = (|3|^2 + |4|^2)^{\frac{1}{2}} = \sqrt{9+16}$

$= \sqrt{25}$

$= 5$

$x = [-7, 2, -4, \sqrt{12}] \rightarrow \|x\|_2 = (|-7|^2 + |2|^2 + |-4|^2 + |\sqrt{12}|^2)$

$= \sqrt{49+4+16+12}$

$= \sqrt{81}$

$= 9$

Task 2

Compute the corresponding unit length vector for these:

$$[3, 4],\ [-1, -2, 3],\ [-7, 2, -4, \sqrt{12}]$$

$\vec{a} = [3,4]$

$\|\vec{a}\| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

Ans: $[\frac{3}{5}, \frac{4}{5}]$

$\vec{a} = [-7, 2, -4, \sqrt{12}]$

$\|\vec{a}\| = \sqrt{7^2+2^2+4^2+(\sqrt{12})^2} = \sqrt{81} = 9$

Ans: $[-\frac{7}{9}, \frac{2}{9}, -\frac{4}{9}, \frac{\sqrt{12}}{9}]$

$\vec{a} = [-1, -2, 3]$

$\|\vec{a}\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

Ans: $[-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}]$

Task 3

Compute the inner product between these vectors and their angle in degrees:

$$[3,-2,2],[1,2,2]$$

Compute the inner product between these vectors and their angle in degrees:

$$[1,0,1],[2,1,-2],[\frac{1}{2\sqrt{2}},-\frac{\sqrt{3}}{2},\frac{1}{2\sqrt{2}}]$$

(i) $[3,-2,2]\cdot[1,2,2] = 3-4+4$
$= 3$

$$\cos\theta = \frac{u\cdot v}{\|u\|_2\|v\|_2} = \frac{3}{\sqrt{3^2+(-2)^2+2^2}\sqrt{1^2+2^2+2^2}}$$
$$= \frac{3}{\sqrt{17}\sqrt{9}}$$
$$= \frac{\cancel{3}}{\sqrt{17}\,\cancel{3}}$$
$$= \frac{1}{\sqrt{17}}$$

$\theta = \cos^{-1}\left(\frac{1}{\sqrt{17}}\right)$
$= 75.96^\circ$ (2 d.p.)

(ii) let $v_1 = [1,0,1]$ let $v_2 = [2,1,-2]$

Let $v_3 = [\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}]$

$v_1\cdot v_2 = [1,0,1]\cdot[2,1,-2]$
$= 2+0+(-2)$
$= 0$

$$\cos\theta = \frac{v_1\cdot v_2}{\|v_1\|_2\|v_2\|_2} = \frac{0}{\sqrt{1^2+0^2+1^2}\sqrt{2^2+1^2+(-2)^2}} = \frac{0}{\sqrt{2}\sqrt{9}}$$
$= 0$

$$\theta = \cos^{-1} 0$$
$$= 90^\circ \ (2 d.p.)$$

$$V_1 \cdot V_3 = [1, 0, 1] \cdot [\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}]$$
$$= \frac{1}{2\sqrt{2}} + 0 + \frac{1}{2\sqrt{2}}$$
$$= 0.71$$

$$\cos\theta = \frac{V_1 \cdot V_3}{\|V_1\|_2 \|V_3\|_2} = \frac{0.71}{\sqrt{1^2+0^2+1^2}\sqrt{(\frac{1}{2\sqrt{2}})^2 + (-\frac{\sqrt{3}}{2})^2 + (\frac{1}{2\sqrt{2}})^2}}$$
$$= \frac{0.71}{\sqrt{2}\sqrt{0.125 + 0.75 + 0.125}}$$
$$= \frac{0.71}{1.41}$$
$$\theta = \cos^{-1}(\frac{0.71}{1.41}) = 60^\circ \ (2 d.p.)$$

$$V_2 \cdot V_3 = [2, 1, -2] \cdot [\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}]$$
$$= 0.71 + (-0.87) + (-0.71)$$
$$= -0.87$$

$$\cos\theta = \frac{V_2 \cdot V_3}{\sqrt{2^2+1^2+(-2)^2}\sqrt{(\frac{1}{2\sqrt{2}})^2 + (-\frac{\sqrt{3}}{2})^2 + (\frac{1}{2\sqrt{2}})^2}}$$
$$= \frac{-0.87}{\sqrt{9}\sqrt{1}}$$
$$= -0.29$$
$$\theta = \cos^{-1}(-0.29) = 106.86^\circ \ (2 d.p.)$$

Task 4

- What is the projection of $[5, 2]$ onto the subspace spanned by vector $[1, 1]$?
- What is the projection of $[0, 2, 1]$ onto the subspace spanned by vector $[1, -1, -1]$?
- Project $[5, 2]$ onto the subspace spanned by vectors $[2, 3],\ [1, 1]$
- What is the projection of $[1, -1, 1]$ onto the subspace spanned by vectors $[0, 0, -1],\ [2, 0, 1]$? Hint: this one is more tricky. Reason: $[0, -1, -1] \cdot [2, 0, 1] \neq 0$

(i) Let $x = [5,2]$ and $v = [1,1]$

$$x\|_v = \left(x \cdot \frac{v}{\|v\|_2}\right) \cdot \frac{v}{\|v\|_2}$$

$$= \frac{[5,2][1,1]}{1^2+1^2} \cdot [1,1]$$

$$= \frac{5+2}{2} \cdot [1,1]$$

$$= \frac{7}{2} \cdot [1,1]$$

$$= \left[\frac{7}{2}, \frac{7}{2}\right]$$

$$= [3.5, 3.5]$$

(ii) Let $x = [0,2,1]$ and $v = [1,-1,-1]$

$$x\|v_2 = \left(x \cdot \frac{v}{\|v\|_2}\right) \cdot \frac{v}{\|v\|_2}$$

$$= \frac{[0,2,1] \cdot [1,-1,-1]}{1^2 + (-1)^2 + (-1)^2} \cdot [1,-1,-1]$$

$$= \frac{0 + (-2) + (-1)}{3} \cdot [1,-1,-1]$$

$$= -\frac{3}{3} \cdot [1,-1,-1]$$

$$= -1 \cdot [1,-1,-1]$$

$$= [-1, 1, 1]$$

(iii) Let $x = [5,2]$ and $v_1 = [2,3]$ $v_2 = [1,1]$

Orthogonal basis, $v_1 \perp v_2 = v_1 - v_1 \parallel v_2$

$$[2,3] - \frac{[2,3]\cdot[1,1]}{1^2+1^2} \cdot [1,1]$$

$$= [2,3] - \frac{2+3}{2} \cdot [1,1]$$

$$= [2,3] - \frac{5}{2} \cdot [1,1]$$

$$= [2,3] - [2.5, 2.5]$$

$$= [-0.5, 0.5]$$

Project x onto subspace S,

$$x \parallel S = x \parallel v_2 + x \parallel v_1 \perp v_2$$

$$= \frac{[5,2]\cdot[1,1]}{1^2+1^2} \cdot [1,1] + \frac{[5,2]\cdot[-0.5,0.5]}{(-0.5)^2+(0.5)^2} \cdot [-0.5,0.5]$$

$$= \frac{5+2}{2} \cdot [1,1] + \frac{-2.5+1}{0.5} \cdot [-0.5,0.5]$$

$$= [3.5, 3.5] + [1.5, -1.5]$$

$$= [5,2]$$

(iv) Let $x = [1,-1,1]$ and $v_1 = [2,0,1]$
$v_2 = [0,0,-1]$

Orthogonal basis, $v_1 \perp v_2 = v_1 - v_1 \parallel v_2$

$$= [2,0,1] - \frac{[2,0,1]\cdot[0,0,-1]}{0^2+0^2+(-1)^2} \cdot [0,0,-1]$$

$$= [2,0,1] - \frac{0+0+(-1)}{1} \cdot [0,0,-1]$$

$$= [2,0,1] - (-1)\cdot[0,0,-1]$$

$$= [2,0,1] - [0,0,1]$$

$$= [2,0,0]$$

Project x onto subspace S,

$$x \parallel S = x \parallel v_2 + x \parallel v_1 \perp v_2$$

$$= \frac{[1,-1,1]\cdot[0,0,-1]}{0^2+0^2+(-1)^2} \cdot [0,0,-1] + \frac{[1,-1,1]\cdot[2,0,0]}{2^2+0^2+0^2} \cdot [2,0,0]$$

$$= \frac{0+0+(-1)}{1} \cdot [0,0,-1] + \frac{2+0+0}{4} \cdot [2,0,0]$$

$$= [0,0,1] + [1,0,0]$$

$$= [1,0,1]$$

Task 5

compute these matrix multiplications:

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \end{bmatrix}$$

Question: Do you need more of them to practice? If so, you can do at home:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 2.5 \\ -3.5 & 1.5 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

(i) $\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix} = \underbrace{2 \times 2 \times 2 \times 2}_{2\times 2}$

$$= \begin{bmatrix} 2\times(-1) + 1\times(-4) & 2\times 0 + 1\times(-2) \\ 3\times(-1) + (-2)\times(-4) & 3\times 0 + (-2)\times(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -2 \\ 5 & 4 \end{bmatrix}$$

(ii) $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \end{bmatrix} = \underbrace{3 \times 1 \times 1 \times 3}_{3\times 3}$

$$= \begin{bmatrix} -3\times 2 & -3\times 4 & -3\times(-2) \\ 2\times 2 & 2\times 4 & 2\times(-2) \\ 1\times 2 & 1\times 4 & 1\times(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -12 & 6 \\ 4 & 8 & -4 \\ 2 & 4 & -2 \end{bmatrix}$$

Task 6

- Project $[5,2]$ onto the orthogonal space of vector $[2,-3]$
- Project $[1,-1,3]$ onto the orthogonal space of vector $[-3,1,1]$
- Project $[1,-1,3,1]$ onto the orthogonal space of vectors $[-2,2,0,0]$, $[0,0,\sqrt{2},\sqrt{2}]$

(i) $$x - x\|v = [5,2] - \frac{[5,2]\cdot[2,-3]}{2^2+(-3)^2} \cdot [2,-3]$$

$$= [5,2] - \frac{10+(-6)}{13} \cdot [2,-3]$$

$$= [5,2] - \frac{4}{13} \cdot [2,-3]$$

$$\overbrace{5-8/13 \quad 2+12/13}^{=0}$$
$$\times \qquad \times$$

$$= \left[5 - \frac{8}{13},\ 2 + \frac{12}{13}\right]$$ * multiply by $[2,-3]$ and sum up, make sure total $= 0$

(ii) $$x - x\|v = [1,-1,3] - \frac{[1,-1,3]\cdot[-3,1,1]}{-3^2+1^2+1^2} \cdot [-3,1,1]$$

$$= [1,-1,3] - \frac{-3+(-1)+3}{11} \cdot [-3,1,1]$$

$$= [1,-1,3] - \left(-\frac{1}{11}\right) \cdot [-3,1,1]$$

$$= [1,-1,3] - \left[\frac{3}{11},\ -\frac{1}{11},\ -\frac{1}{11}\right]$$

$$= \left[1-\frac{3}{11},\ -1+\frac{1}{11},\ 3+\frac{1}{11}\right]$$ * multiply by $[-3,1,1]$ and sum up, make sure total $= 0$

(iii) Let $x = [1,-1,3,1]$ and $v_1 = [-2,2,0,0]$
$v_2 = [0,0,\sqrt{2},\sqrt{2}]$

$v_1 \cdot v_2 = 0$ $\quad x - x_{\parallel V} = x - (x_{\parallel v_1} + x_{\parallel v_2})$
$\therefore v_1 \perp v_2$

$$x_{\parallel v_1} = \frac{[1,-1,3,1]\cdot[-2,2,0,0]}{-2^2+2^2+0^2+0^2} \cdot [-2,2,0,0]$$

$$= \frac{-2+(-2)+0+0}{8} \cdot [-2,2,0,0]$$

$$= -\frac{4}{8} \cdot [-2,2,0,0]$$

$$= [1,-1,0,0]$$

$$x_{\parallel v_2} = \frac{[1,-1,3,1]\cdot[0,0,\sqrt{2},\sqrt{2}]}{0^2+0^2+(\sqrt{2})^2+(\sqrt{2})^2} \cdot [0,0,\sqrt{2},\sqrt{2}]$$

$$= \frac{1\times 0 + (-1)\times 0 + 3\times\sqrt{2} + 1\times\sqrt{2}}{4} \cdot [0,0,\sqrt{2},\sqrt{2}]$$

$$= \frac{5.66}{4} \cdot [0,0,\sqrt{2},\sqrt{2}]$$

$$= [0,0,2,2]$$

$$x_{\parallel v_1} + x_{\parallel v_2} = [1,-1,0,0] + [0,0,2,2]$$

$$= [1,-1,2,2]$$

$$x - x_{\parallel v} = [1,-1,3,1] - [1,-1,2,2]$$

$$= [0,0,1,-1]$$

Task 7

run Gram-Schmid-orthogonalization on the vectors

$$[12, 12, 6], [2, -2, 4], [-2, -2, 1]$$

Let $v_1 = [2,-2,4]$ and $v_2 = [-2,-2,1]$

$$v^{\{1\}} = [2,-2,4] - \frac{[2,-2,4] \cdot [12,12,6]}{[12,12,6] \cdot [12,12,6]} \cdot [12,12,6]$$

$$= [2,-2,4] - \frac{2 \times 12 + (-2) \times 12 + 6 \times 4}{12 \times 12 + 12 \times 12 + 6 \times 6} \cdot [12,12,6]$$

$$= [2,-2,4] - \frac{24}{324} \cdot [12,12,6]$$

$$= [2,-2,4] - \left[\frac{288}{324}, \frac{288}{324}, \frac{144}{324}\right]$$

$$= \left[2 - \frac{8}{9}, -2 - \frac{8}{9}, 4 - \frac{4}{9}\right]$$

Unfinished !! :(<\3