Linear Algebra L3 - Linear mappings

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Learning Goals

- Understanding Linear mappings

Task 1

Draw these affine spaces (you can contribute to tree murdering via pen and paper, thats ok.)

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot x + 1.5 = 0$$
$$\begin{bmatrix} -5 \\ -1 \end{bmatrix} \cdot x + 6 = 0$$
$$\begin{bmatrix} -5 \\ -1 \end{bmatrix} \cdot x - 6 = 0$$

Task 2

Find two non-parallel vectors x solving

$$w \cdot x = 3$$
$$w = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

Task 3

Show that the line given by

$$f(t) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

does not intersect the plane given by

$$2x + z = 9$$

Note:

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Task 4

Show that the line given by

$$f(t) = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$$

has an intersection with the plane given by

$$3x - 2y + 2z = 18$$

Note:

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Task 5

Check whether the plane given by

$$f(s,t) = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

has an intersection with the line given by

$$x + 2y - z = 3$$
$$2x - y + z = 6$$

Note:

$$f(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}$$

Task 6

Convert the plane equation into the form $Ax = b$ for

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Steps:

- what is the dimensionality of the whole vector space in which these equations are defined?
- what is the dimensionality of the affine space spanned by the plane equation?
- how does the matrix B look like for which we seek solutions x such that $Bx = 0$?
- Conclude based on the dimensionality of the whole vector space and the dimensionality of the plane, what is the dimensionality of solutions x which we are searching for ?
- find a basis for these solutions. Turn it into a matrix A
- get the correct bias vector b based the A which you found

Task 7

Convert the plane equation into the form $Ax = b$ for

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \\ -4 \end{bmatrix}$$

Task 8

plot 2d planes in a 3d space using e.g. matplotlib.

Task 1

Draw these affine spaces (you can contribute to tree murdering via pen and paper, thats ok.)

$$\begin{bmatrix}2\\-1\end{bmatrix} \cdot x + 1.5 = 0$$
$$\begin{bmatrix}-5\\-1\end{bmatrix} \cdot x + 6 = 0$$
$$\begin{bmatrix}-5\\-1\end{bmatrix} \cdot x - 6 = 0$$

(i) $w = \begin{bmatrix}2\\-1\end{bmatrix}$; $x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

$w \cdot x = b$ $(Ax = b)$, $\begin{bmatrix}2\\-1\end{bmatrix} \cdot x = -1.5$

$\begin{bmatrix}2\\-1\end{bmatrix} \begin{bmatrix}x_1\\x_2\end{bmatrix} + 1.5 = 0$

$2x_1 - x_2 + 1.5 = 0$

$x_2 = 2x_1 + 1.5$ $(y = mx + c)$

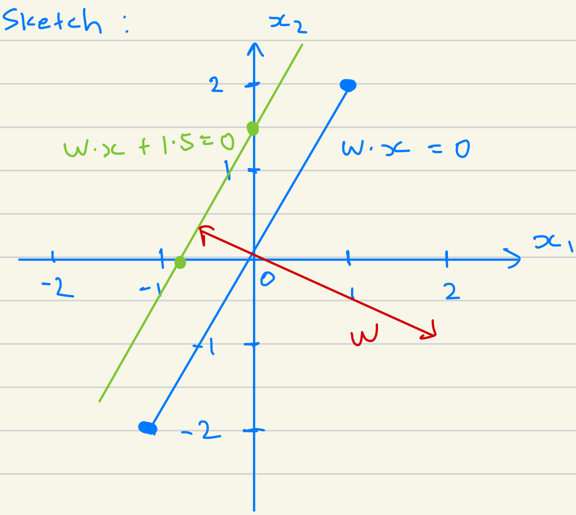
Affine space is a line $(y = mx + c)$:

Orthogonal to $(2, -1)$

offset in the direction of $(2,-1)$ by $b \| w = \frac{b}{w \cdot w} w = \frac{-1.5}{2^2 + 1^2} \cdot [2, -1]$

$= [-0.6, 0.3]$

Sketch:



(ii) $w = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$; $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$w \cdot x = b$ $(Ax = b)$, $\begin{bmatrix} -5 \\ -1 \end{bmatrix} \cdot x = -6$

$\begin{bmatrix} -5 \\ -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 6 = 0$

$-5x_1 - x_2 + 6 = 0$

$-5x_1 + 6 = x_2$

$x_2 = -5x_1 + 6$ $(y = mx + c)$

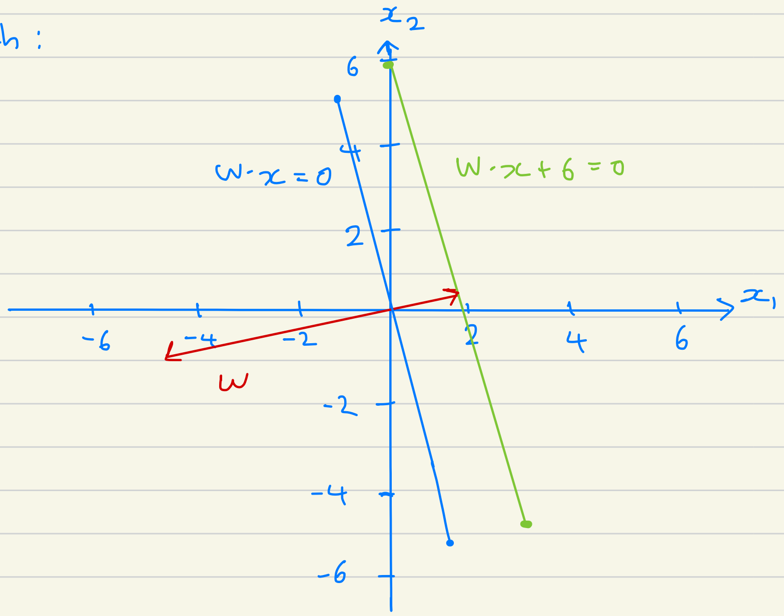
Affine space is a line $(y = mx + c)$

Orthogonal to $(-5, -1)$

offset in the direction of $(-5, -1)$ by $b \| w = \frac{b}{w \cdot w} w = \frac{-6}{5^2 + 1^2} \cdot [-5, -1]$

$= [1.15, 0.23]$

Sketch :



(iii) $w = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$; $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$w \cdot x = b$ $(Ax = b)$, $\begin{bmatrix} -5 \\ -1 \end{bmatrix} \cdot x = 6$

$$\begin{bmatrix} -5 \\ -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 6 = 0$$

$$-5x_1 - x_2 - 6 = 0$$

$$-5x_1 - 6 = x_2$$

$$x_2 = -5x_1 - 6 \quad (y = mx + c)$$

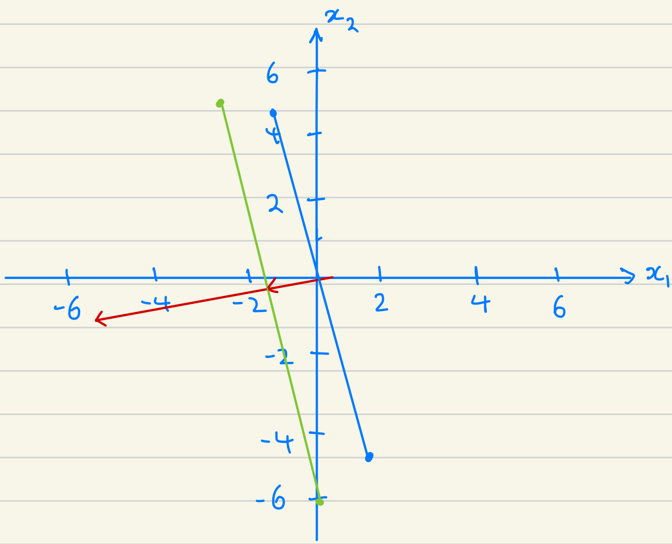
Affine space is a line $(y = mx + c)$

orthogonal to $(-5, -1)$

offset in the direction of $(-5,-1)$ by $b \| w = \frac{b}{w \cdot w} w = \frac{6}{5^2+1^2} \cdot [-5,-1]$

$= [-1.15, -0.23]$

Sketch :



Task 2

Find two non-parallel vectors x solving

$$w \cdot x = 3$$

$$w = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$W \cdot x = 3$

$\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = 3$

$x_0 - 2x_1 + 4x_2 = 3$

$x_0 = 3 + 2x_1 - 4x^2$

$x_0 = 3 + 2s - 4t$

Vector form : $\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

let $s=1, t=0$; $\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

let $s=0, t=1$; $\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ They are linearly independent , [5,1,0] and [-1,0,1] is a solution but not unique .

Task 3

Show that the line given by

$$f(t) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

does not intersect the plane given by

$$2x + z = 9$$

Note:

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Substitute f(t) into the equation system:

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

Affine System in t:

$x(t) = 2 + t(-1)$

$y(t) = 3 + 4t$

$z(t) = 1 + 2t$

Solve for t by substituting into $2x + z$:

$2x + z = 2(2-t) + 1 + 2t$

$2x + z = 4 - 2t + 1 + 2t$

$2x + z = 5$ //

Task 4

Show that the line given by

$$f(t) = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$$

has an intersection with the plane given by

$$3x - 2y + 2z = 18$$

Note:

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Substitute $f(t)$ into the equation system:

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$$

Affine system in t:

$x(t) = 1 + 2t$

$y(t) = -3 + 3t$

$z(t) = 2 - 5t$

Solve for t by substituting into $3x - 2y + 2z$:

$3(1+2t) - 2(-3+3t) + 2(2-5t) = 18$

$3 + 6t + 6 - 6t + 4 - 10t = 18$

$13 - 10t = 18$

$-10t = 18 - 13$

$t = 5/-10$

$t = -0.5$

$$f(-0.5) = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 0.5 \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -4.5 \\ 4.5 \end{bmatrix} //$$

Task 5

Check whether the plane given by

$$f(s,t) = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

has an intersection with the line given by

$$x + 2y - z = 3$$
$$2x - y + z = 6$$

Note:

$$f(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}$$

Substitute $f(s,t)$ into the equation system:

$$f(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Affine System in t and s:

$$x(s,t) = -2t + s$$
$$y(s,t) = 1 + 2s$$
$$z(s,t) = -3 + t - s$$

Solve for t and s by substituting into $x + 2y - z = 3$
$2x - y + z = 6$:

$$-2t + s + 2(1 + 2s) - (-3 + t - s) = 3$$
$$2(-2t + s) - (1 + 2s) + (-3 + t - s) = 6$$

$$\downarrow$$

$$-2t + s + 2 + 4s + 3 - t + s = 3$$
$$-4t + 2s - 1 - 2s - 3 + t - s = 6$$

$$\downarrow$$

$$-3t + 6s = -2$$
$$-3t - s = 10$$

$$\downarrow$$

$6s - 3t = -2$

$-s - 3t = 10$

using gauss - jordan elimination method solve for s and t:

$$\left[\begin{array}{cc|c} 6 & -3 & -2 \\ -1 & -3 & 10 \end{array}\right] = \left[\begin{array}{cc|c} -1 & -3 & 10 \\ 6 & -3 & -2 \end{array}\right]$$

Swap r_0 and r_1

$R_0 \div (-1)$

$$= \left[\begin{array}{cc|c} 1 & 3 & -10 \\ 6 & -3 & -2 \end{array}\right]$$

$R_0 \times (-6) + R_1$

$$= \left[\begin{array}{cc|c} 1 & 3 & -10 \\ 0 & -21 & 58 \end{array}\right]$$

$R_1 \div (-21)$

$$= \left[\begin{array}{cc|c} 1 & 3 & -10 \\ 0 & 1 & 58/-21 \end{array}\right]$$

$R_1 \times (-3) + R_0$

$$= \left[\begin{array}{cc|c} 1 & 0 & -10 + 58/7 \\ 0 & 1 & -58/21 \end{array}\right]$$

$s = -12/7, \quad t = -58/21$

Intersection point:

$$f(-12/7, -58/21) = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} - \frac{58}{21}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{12}{7}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 80/21 \\ -17/7 \\ -85/21 \end{bmatrix}$$

Task 6

Convert the plane equation into the form $Ax = b$ for

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Steps:

- what is the dimensionality of the whole vector space in which these equations are defined?
- what is the dimensionality of the affine space spanned by the plane equation?
- how does the matrix B look like for which we seek solutions x such that $Bx = 0$?
- Conclude based on the dimensionality of the whole vector space and the dimensionality of the plane, what is the dimensionality of solutions x which we are searching for ?
- find a basis for these solutions. Turn it into a matrix A
- get the correct bias vector b based the A which you found

Let $V_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $V_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$V_0 \cdot w = 0$

$V_1 \cdot w = 0$

$$\begin{bmatrix} V_0^T \\ V_1^T \end{bmatrix} w = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \end{bmatrix} w = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} w_0 + 2w_1 + 3w_2 \\ 2w_0 - w_1 + 2w_2 \end{bmatrix} = 0$$

$w_0 + 2w_1 + 3w_2 = 0$

$2w_0 - w_1 + 2w_2 = 0$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -1 & 2 & 0 \end{array}\right]$$ $R_0 \times (-2) + R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -4 & 0 \end{array}\right]$$ $R_0 \div (-5)$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4/5 & 0 \end{array}\right]$$ $R_1 \times (-2) + R_0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7/5 & 0 \\ 0 & 1 & 4/5 & 0 \end{array}\right]$$

$w_0 = -\frac{7}{5} w_2$

$w_1 = -\frac{4}{5} w_2$

Parametrize $w_2 = P$

$w_0 = -\frac{7}{5} P$

$w_1 = -\frac{4}{5} P$

$w_2 = P$

$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = P \begin{bmatrix} -\frac{7}{5} \\ -\frac{4}{5} \\ 1 \end{bmatrix}$$

Let $P = 1$, $\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -7/5 \\ -4/5 \\ 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} -7/5 \\ -4/5 \\ 1 \end{bmatrix}^T = \begin{bmatrix} -7/5 & -4/5 & 1 \end{bmatrix}$ //

Next $Ax = b$,

$b = \begin{bmatrix} -7/5 & -4/5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 & -3 \end{bmatrix}$

$= -8.4$ //

Task 7

Convert the plane equation into the form $Ax = b$ for

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \\ -4 \end{bmatrix}$$

$$V_0 = \begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix} \quad V_1 = \begin{bmatrix} 2 \\ -4 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} V_0^T \\ V_1^T \end{bmatrix} w = 0$$

$$\begin{bmatrix} -3 & 1 & 6 \\ 2 & -4 & -4 \end{bmatrix} w = 0$$

$$\begin{bmatrix} -3 & 1 & 6 \\ 2 & -4 & -4 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3w_0 + w_1 + 6w_2 \\ 2w_0 - 4w_1 - 4w_2 \end{bmatrix} = 0$$

$$-3w_0 + w_1 + 6w_2 = 0$$
$$2w_0 - 4w_1 - 4w_2 = 0$$

$$\left[\begin{array}{ccc|c} -3 & 1 & 6 & 0 \\ 2 & -4 & -4 & 0 \end{array}\right] \quad R_0 \times \frac{2}{3} + R_1$$

$$\left[\begin{array}{ccc|c} -3 & 1 & 6 & 0 \\ 0 & -\frac{10}{3} & 0 & 0 \end{array}\right] \quad R_0 \div (-3)$$

$$\left[\begin{array}{ccc|c} 1 & -1/3 & -2 & 0 \\ 0 & -10/3 & 0 & 0 \end{array}\right] \quad R_1 \div (-10/3)$$

$$\left[\begin{array}{ccc|c} 1 & -1/3 & -2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \quad R_1 \times (\tfrac{1}{3}) + R_0$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$w_0 = 2w_2$
$w_1 = 0$
Parametrize $w_2 = p$
$w_0 = 2p$
$w_1 = 0$
$w_2 = p$

$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = p\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Let $p=1$,

$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} //$$

Next $Ax=b$,
$b = [2\ 0\ 1] \cdot [-1\ 0\ 1]$
$= -1 //$