Linear Algebra L4 - Matrices

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March 10, 2023

1 Learning Goals

- matrix properties

Task 1

Compute $A^\top A$ for

$$A = \begin{bmatrix} 2 & -3 \\ -6 & -9 \end{bmatrix}$$
$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

Task 2

Compute the inverse of

$$A_0 = \begin{bmatrix} -2 & -3 \\ -6 & -4 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$$

Use these inverses to solve

$$A_0 x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$A_1 x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

Note: it is not common to solve $Ax = b$ using matrix inversion.

Reasons:

- $Ax = b$ can be solvable when A is not invertible
- It is often slower / more costly see e.g. https://gregorygundersen.com/blog/2020/12/09/matrix-inversion/

Task 3

Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -4 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 3 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -5 & -10 & -15 \\ 6 & 12 & 18 \end{bmatrix}$$

- Are they invertible?
- Which of them has full rank ? Which of them has lower rank and which one ?

Task 4

For what value a the matrix is not invertible ?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & a & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

Task 5

Compute and apply the Householder matrix which makes transforms the first column of to a multiple of the first one-hot vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

and for (hint: here subtracting is nicer)

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 3 & 1 & 1 \\ \sqrt{6} & 3 & 1 \end{bmatrix}$$

Task 6

Verify that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

satisfies being an orthogonal matrix.

Task 7 (extra)

Because some of you had troubles with it
What is the cosine of the angle between

$$(6, -6, -4, \sqrt{12}),\ (6, 4, 2, \sqrt{25})\ ?$$

Task 8 (extra)

another 3x3 affine system

- show the intermediate result when the first column is the one hot vector $[1, 0, 0]$ for the first time
- show the intermediate result when the matrix has row echelon form for the first time
- get the solution

$$\begin{aligned} 2x - 3y + 2z &= -4 \\ 7x + 4.5y - 1z &= 16 \\ 4x + 3y + z &= 2 \end{aligned}$$

Task 1

Compute $A^\top A$ for

$$A = \begin{bmatrix} 2 & -3 \\ -6 & -9 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & -6 \\ -3 & -9 \end{bmatrix} \qquad A^T \cdot A = \begin{bmatrix} 2 & -6 \\ -3 & -9 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -6 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times2 + (-6)\times(-6) & 2\times(-3) + (-6)\times(-9) \\ -3\times2 + (-9)\times(-6) & -3\times(-3) + (-9)\times(-9) \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 48 \\ 48 & 90 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 3 \\ -1 & 2 & 1 \end{bmatrix} \qquad A^T \cdot A = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 3 \\ -1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0\times0 + 2\times2 + 4\times4 & 0\times1 + 2\times0 + 4\times3 & 0\times(-1) + 2\times2 + 4\times1 \\ 1\times0 + 0\times2 + 3\times4 & 1\times1 + 0\times0 + 3\times3 & 1\times(-1) + 0\times2 + 3\times1 \\ -1\times0 + 2\times2 + 1\times4 & -1\times1 + 2\times0 + 1\times3 & -1\times(-1) + 2\times2 + 1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 12 & 8 \\ 12 & 10 & 2 \\ 8 & 2 & 6 \end{bmatrix}$$

Task 2

Compute the inverse of

$$A_0 = \begin{bmatrix} -2 & -3 \\ -6 & -4 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$$

Use these inverses to solve

$$A_0 x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$A_1 x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

Note: it is not common to solve $Ax = b$ using matrix inversion.

Reasons:

- $Ax = b$ can be solvable when A is not invertible
- It is often slower / more costly see e.g. https://gregorygundersen.com/blog/2020/12/09/matrix-inversion/

$$|A_0| = \begin{vmatrix} -2 & -3 \\ -6 & -4 \end{vmatrix} = (-2)(-4) - (-3)(-6) = -10$$

$$A_0^{-1} = -\frac{1}{10}\begin{bmatrix} -4 & 3 \\ 6 & -2 \end{bmatrix}$$

$$A_0 x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$x = A_0^{-1}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$= -\frac{1}{10}\begin{bmatrix} -4 & 3 \\ 6 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$= -\frac{1}{10}\begin{bmatrix} -9 \\ 16 \end{bmatrix}$$
$$= \begin{bmatrix} 0.9 \\ -1.6 \end{bmatrix}$$

$$|A_1| = \begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} = (3 \times 2) - (1 \times (-2)) = 8$$

$$A_1^{-1} = \frac{1}{8}\begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}$$

$$A_1 x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$x = A_1^{-1} \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 9 \\ -11 \end{bmatrix}$$

$$= \begin{bmatrix} 1.125 \\ -1.375 \end{bmatrix} //$$

Task 3

Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -4 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 3 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -5 & -10 & -15 \\ 6 & 12 & 18 \end{bmatrix}$$

- Are they invertible?
- Which of them has full rank ? Which of them has lower rank and which one ?

(i) $$|A| = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{pmatrix} \begin{matrix} 1 & 2 \\ 2 & 0 \\ 4 & 3 \end{matrix}$$

$$= (1 \times 0 \times 1) + (2 \times 2 \times 4) + (-1 \times 2 \times 3) - (2 \times 2 \times 1) - (1 \times 2 \times 3) - (-1 \times 0 \times 4)$$

$$= 0 + 16 - 6 - 4 - 6$$

$= 0$ (Not invertible) //

REF:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$R_0 \times (-2) + R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 4 \\ 4 & 3 & 1 \end{bmatrix} \quad R_0 \times (-4) + R_2$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 4 \\ 0 & -5 & 5 \end{bmatrix} \quad R_1 \times \left(-\frac{5}{4}\right) + R_2$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Ans : There are two non-zero rows

$\therefore$ rank = 2 //

(ii) $|A| = \begin{pmatrix} 2 & -4 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix} \begin{matrix} 2 & -4 \\ 1 & 1 \\ 3 & 3 \end{matrix}$

$= (2 \times 1 \times 1) + (-4 \times 1 \times 3) + (3 \times 1 \times 3) - (-4 \times 1 \times 1) - (2 \times 1 \times 3) - (3 \times 1 \times 3)$

$= 2 - 12 + 9 + 4 - 6 - 9$

$= -12$ (invertible) //

REF :

$$\begin{bmatrix} 2 & -4 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix} \quad \text{Swap } R_1 \text{ and } R_0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -4 & 3 \\ 3 & 3 & 1 \end{bmatrix} \quad R_0 \times (-2) + R_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -6 & 1 \\ 3 & 3 & 1 \end{bmatrix} \quad R_0 \times (-3) + R_2$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad R_1 \div (-9)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/9 \\ 0 & 0 & -2 \end{bmatrix} \quad R_2 \div (-2)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/9 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \times (-1) + R_0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/9 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \times 1/9 + R_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \times (-1) + R_0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans : Rank = 3 //

(iii) $|A| = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 3 & 2 \end{pmatrix} \begin{matrix} 1 & 0 \\ 2 & 3 \\ -1 & 3 \end{matrix}$

$$(1\times3\times2) + (0\times2\times(-1)) + (1\times2\times3) - (0\times2\times2) - (1\times2\times3)$$
$$- (1\times3\times(-1))$$
$$= 6 + 6 - 6 + 3$$
$$= 9 \text{ (Invertible)}$$

Ans: Rank 3 //

(iv) $$|A| = \begin{pmatrix} 1 & 2 & 3 \\ -5 & -10 & -15 \\ 6 & 12 & 18 \end{pmatrix} \begin{matrix} 1 & 2 \\ -5 & -10 \\ 6 & 12 \end{matrix}$$

$$= (1\times(-10)\times18) + (2\times(-15)\times6) + (3\times(-5)\times12) - (2\times(-5)\times18)$$
$$- (1\times(-15)\times12) - (3\times(-10)\times6)$$
$$= -180 - 180 - 180 + 180 + 180 + 180$$
$$= 0 \text{ (Not invertible)}$$ //

REF:

$$\begin{bmatrix} 1 & 2 & 3 \\ -5 & -10 & -15 \\ 6 & 12 & 18 \end{bmatrix} \quad R_0\times5 + R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 6 & 12 & 18 \end{bmatrix} \quad R_0\times(-6) + R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Ans: Rank 1 //

Task 4

For what value a the matrix is not invertible ?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & a & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

$$|A| = \begin{pmatrix} 1 & 2 & 1 \\ 2 & a & 4 \\ -3 & 1 & 2 \end{pmatrix} \begin{matrix} 1 & 2 \\ 2 & a \\ -3 & 1 \end{matrix}$$

$$= (1 \times a \times 2) + (2 \times 4 \times (-3)) + (1 \times 2 \times 1) - (2 \times 2 \times 2) - (1 \times 4 \times 1) - (1 \times a \times (-3))$$

$$= 2a - 24 + 2 - 8 - 4 + 3a$$

$$= 5a - 34$$

$$34 = 5a$$

$$a = 34/5 \,//$$

Task 5

Compute and apply the Householder matrix which makes transforms the first column of to a multiple of the first one-hot vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

and for (hint: here subtracting is nicer)

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 3 & 1 & 1 \\ \sqrt{6} & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

$$u = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \pm \|(1,2,2)\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \pm \sqrt{1^2 + 2^2 + 2^2} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \pm 3 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

$$H_u = I - \frac{2}{u \cdot u} uu^T$$

$$u \cdot u = [-2, 2, 2] \cdot [-2, 2, 2]$$

$$= -2 \times -2 + 2 \times 2 + 2 \times 2$$

$$= 12$$

$$H = I - \frac{2}{12} uu^T = I - \frac{1}{6} \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} [-2\ 2\ 2]$$

$$= I - \frac{1}{6} \begin{bmatrix} 4 & -4 & -4 \\ -4 & 4 & 4 \\ -4 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2/3 & -2/3 & -2/3 \\ -2/3 & 2/3 & 2/3 \\ -2/3 & 2/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$H \cdot A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2.67 & 1.67 \\ 0 & -0.67 & -0.67 \\ 0 & 2.33 & -1.67 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 3 & 1 & 1 \\ \sqrt{6} & 3 & 1 \end{bmatrix}$$

$$u = \begin{bmatrix} 1 \\ 3 \\ \sqrt{6} \end{bmatrix} \pm \|(1, 3, \sqrt{6})\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u = \begin{bmatrix} 1 \\ 3 \\ \sqrt{6} \end{bmatrix} \pm \sqrt{1^2 + 3^2 + (\sqrt{6})^2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u = \begin{bmatrix} 1 \\ 3 \\ \sqrt{6} \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$u = \begin{bmatrix} -3 \\ 3 \\ \sqrt{6} \end{bmatrix}$$

$$u \cdot u = \begin{bmatrix} -3 & 3 & \sqrt{6} \end{bmatrix} \cdot \begin{bmatrix} -3 & 3 & \sqrt{6} \end{bmatrix}$$

$$= 24$$

$$H = I - \frac{2}{24} \begin{bmatrix} -3 \\ 3 \\ \sqrt{6} \end{bmatrix} \begin{bmatrix} -3 & 3 & \sqrt{6} \end{bmatrix} \qquad \underbrace{3 \times 1 \times 1 \times 3}_{3 \times 3 \text{ Matrix}}$$

$$H = I - \frac{1}{12} \begin{bmatrix} 9 & -9 & -3\sqrt{6} \\ -9 & 9 & 3\sqrt{6} \\ -3\sqrt{6} & 3\sqrt{6} & 6 \end{bmatrix}$$

$$H = I - \begin{bmatrix} 0.75 & -0.75 & -1/4\sqrt{6} \\ -0.75 & 0.75 & 1/4\sqrt{6} \\ -1/4\sqrt{6} & 1/4\sqrt{6} & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.75 & -0.75 & -1/4\sqrt{6} \\ -0.75 & 0.75 & 1/4\sqrt{6} \\ -1/4\sqrt{6} & 1/4\sqrt{6} & 0.5 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.25 & 0.75 & 1/4\sqrt{6} \\ 0.75 & 0.25 & -1/4\sqrt{6} \\ 1/4\sqrt{6} & -1/4\sqrt{6} & 0.5 \end{bmatrix}$$

$$H \cdot A = \begin{bmatrix} 0.25 & 0.75 & 1/4\sqrt{6} \\ 0.75 & 0.25 & -1/4\sqrt{6} \\ 1/4\sqrt{6} & -1/4\sqrt{6} & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 & 3 \\ 3 & 1 & 1 \\ \sqrt{6} & 3 & 1 \end{bmatrix}$$

= Answer use calculator :)

Task 6

Verify that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

Matrix A is orthogonal if $A^T A = AA^T = I$

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ 0 & -\sin a & \cos a \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & -\sin a \\ 0 & \sin a & \cos a \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2 a + \sin^2 a & -\sin a \cos a + \sin a \cos a \\ 0 & -\sin a \cos a + \sin a \cos a & \cos^2 a + \sin^2 a \end{bmatrix}$$

$\downarrow$ equals to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$