Heat Conduction

ME210 Heat Transfer —Lect #9

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A General Heat Conduction Problem

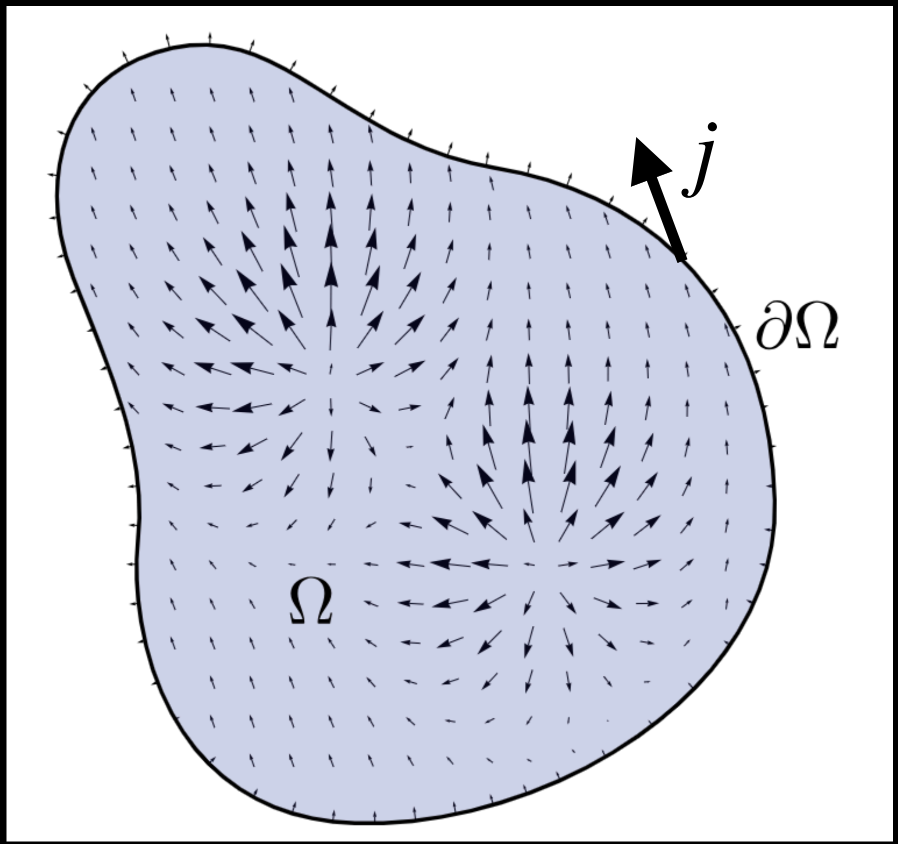
HE → Solve $\dfrac{1}{D}\dfrac{\partial T}{\partial t} = \nabla^2 T + \gamma(t,p)$ in the domain Ω

Diritchlet BCs → such that $T(t,p) = T_{BC}(p)$ for $p \in \partial\Omega$

ICs → and $T(0,p) = T_0(p)$

$$\mathscr{F} = \left\{ \begin{array}{l} \text{Space of square integrable} \\ \text{functions on } \Omega \\ \text{that satisfy the boundary conditions} \end{array} \right\}$$

For each $t \in \mathbb{R}$ $\quad T(t,\cdot\,) \in \mathscr{F}$

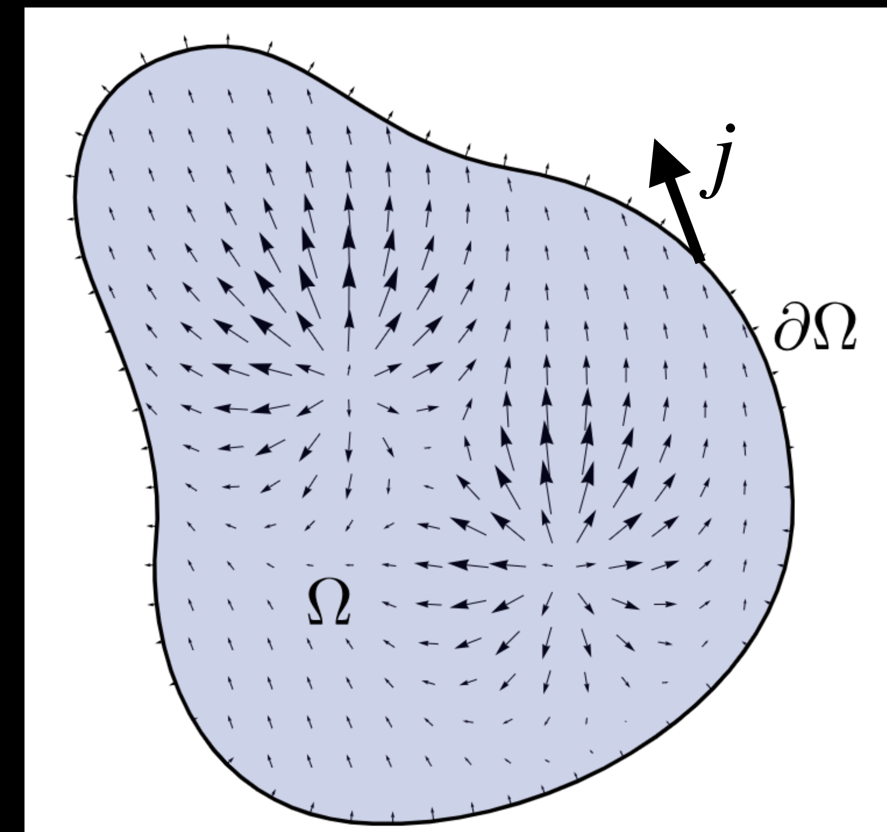


Special Case: Steady State Solution

Solve $\frac{1}{D}\frac{\partial T}{\partial t} = \nabla^2 T + \gamma_s(p)$ in the domain Ω

such that $T(t,p)$ where $p \in \Omega$ is specified on $\partial\Omega$

and satisfies the IC $T(0,p) = T_0(p)$



Define $T_s(p) \triangleq \lim_{t\to\infty} T(t,p)$

Then $T_s(p) \in \mathscr{F}$ must satisfy $\nabla^2 T + \gamma_s(p) = 0$

$\big($For example in the 1D case with $\gamma_s(x) \equiv 0 \;\; \rightarrow T_s(x) = C_1 x + C_0\big)$

General Solution

$$T(t,p) = \sum_{n=1}^{\infty} \alpha_n(t)\phi_n(p)$$

where

$$\boxed{\alpha_n(t) = e^{-D\lambda_n^2 t}\alpha_n(0) + D\int_0^t e^{-D\lambda_n^2 (t-\tau)}\gamma_n(\tau)\, d\tau}$$

and

$$\boxed{\alpha_n(0) = \langle \phi_n(p)\, , T_0(p)\rangle}$$

This is possible only if

If there exists $\phi_n \in \mathcal{F}$ **such that**

$$\nabla^2 \phi_n = -\lambda_n^2 \phi_n$$

and $\langle \phi_n, \phi_m \rangle = \delta_{nm}$

for some appropriate inner product

and $\{\phi_n\}_{n=1}^{\infty}$ spans $\mathcal{F}$

This process is in general very hard !!

General Solution Procedure

HE → Solve $\dfrac{1}{D}\dfrac{\partial T}{\partial t} = \nabla^2 T + \gamma(t,p)$ in the domain Ω

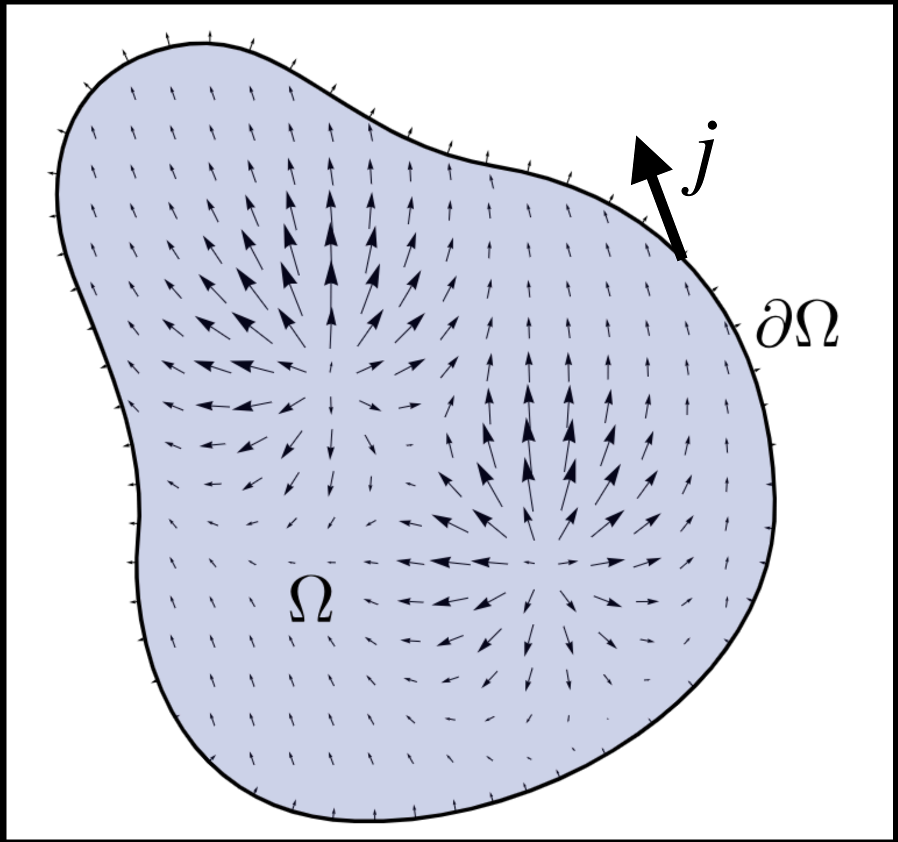
Diritchlet BCs → such that $T(t,p) = T_{BC}(p)$ for $p \in \partial\Omega$

ICs → and $T(0,p) = T_0(p)$

Let $T_{SH}(p)$ be such that $\nabla^2 T = 0$

and $T_{SH}(p)$ satisfies the BCs

($T_{SH}(p)$ is the steady state homogeneous solution)



Reduction to a Heat Conduction Problem with Homogeneous BCs

Let $\widetilde{T}(t, p) \triangleq T(t, p) - T_{SH}(p)$

Let $\mathscr{F} = \left\{ \begin{array}{l} \text{Space of square integrable} \\ \text{functions on } \Omega \text{ that satisfy} \\ \text{homogeneous boundary conditions} \end{array} \right\}$

Then we can verify that

$$\boxed{\begin{array}{c} \dfrac{1}{D}\dfrac{\partial \widetilde{T}}{\partial t} = \nabla^2 \widetilde{T} + \gamma(t, p) \\ \text{and } \widetilde{T}(t, \cdot\,) \in \mathscr{F} \text{ for each } t \in \mathbb{R} \end{array}}$$

Now we look for

$\phi_n \in \mathcal{F}$ **such that**

$$\nabla^2 \phi_n = -\lambda_n^2 \phi_n$$

and $\langle \phi_n , \phi_m \rangle = \delta_{nm}$

for some appropriate inner product

and $\{\phi_n\}_{n=1}^{\infty} \quad \text{spans} \quad \mathcal{F}$

This process is possible in many cases

General Solution to the reduced problem

$$\widetilde{T}(t,p) = \sum_{n=1}^{\infty} \alpha_n(t)\phi_n(p)$$

where

$$\alpha_n(t) = e^{-D\lambda_n^2 t}\alpha_n(0) + D\int_0^t e^{-D\lambda_n^2 (t-\tau)}\gamma_n(\tau)\, d\tau$$

and

$$\gamma_n(t) = \langle \phi_n(p)\,, \gamma(t,p)\rangle$$

and

$$\alpha_n(0) = \langle \phi_n(p)\,, T_0(p) - T_s(p)\rangle$$

Hence General Solution

$$T(t,p) = T_{SH}(p) + \sum_{n=1}^{\infty} \alpha_n(t)\phi_n(p)$$

where

$$\alpha_n(t) = e^{-D\lambda_n^2 t}\alpha_n(0) + D\int_0^t e^{-D\lambda_n^2 (t-\tau)}\gamma_n(\tau)\, d\tau$$

and

$$\gamma_n(t) = \langle \phi_n(p)\, , \gamma(t,p)\rangle$$

and

$$\alpha_n(0) = \langle \phi_n(p)\, , T_0(p) - T_s(p)\rangle$$

Back to 1-D Homogenous Problem

$$\boxed{\phi_n(x) \triangleq \sin\left(\frac{n\pi x}{L}\right) \in \mathscr{F}}$$

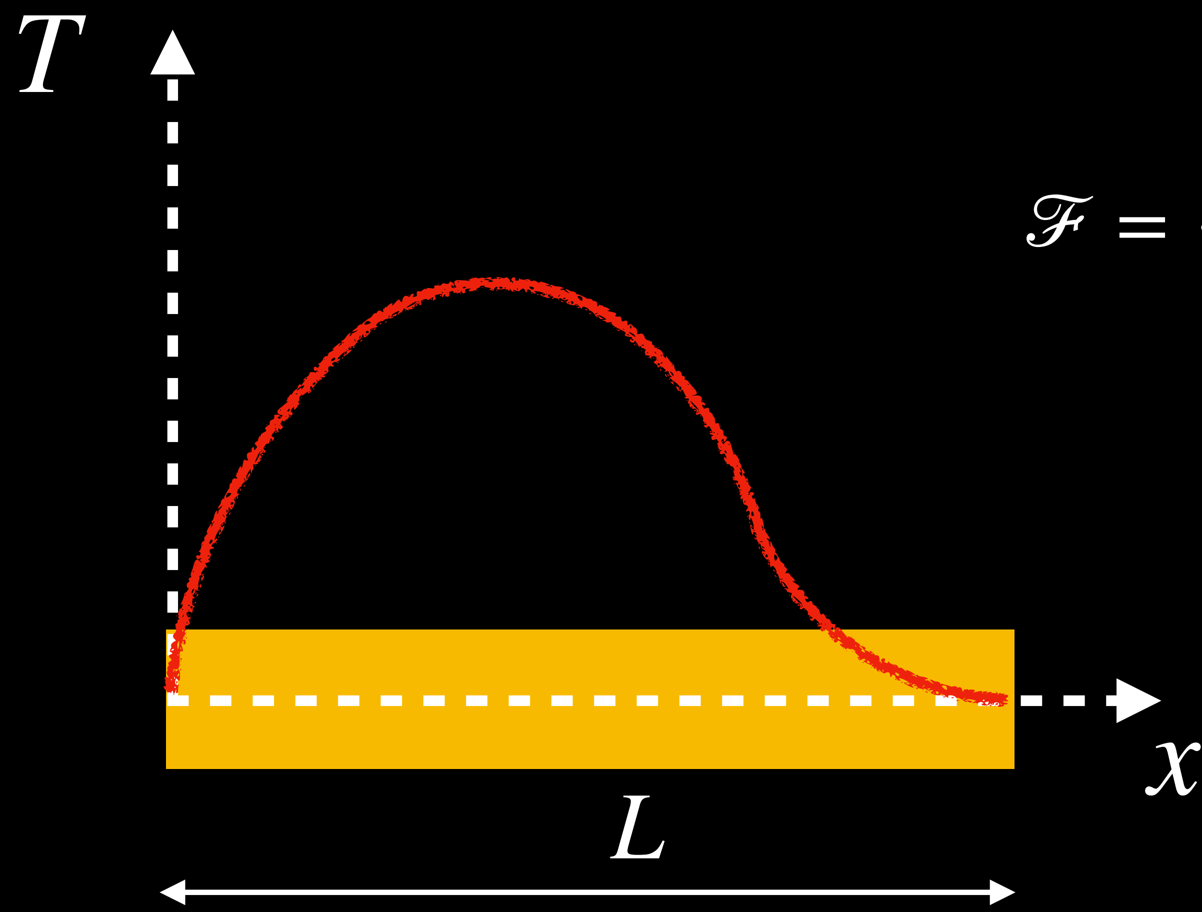
$$\nabla^2 \phi_n = -\lambda_n^2 \phi_n$$

$$\lambda_n^2 = \left(\frac{n\pi}{L}\right)^2$$

$$\boxed{\langle \phi_n, \phi_m \rangle = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \delta_{nm}}$$

and $\{\phi_n\}_{n=1}^{\infty}$ spans $\mathscr{F}$

Back to 1-D Heat Conduction



$$\mathscr{F} = \left\{ \begin{array}{l} \text{Space of infinitely differentiable functions} \\ \text{on } [0,L] \text{ that vanish at the endpoints} \end{array} \right\}$$

Non-homogeneous BCs

$T(t,0) = T_0$ and $T(t,L) = T_L$

$T(0,x) = T_0(x)$ → **ICs**

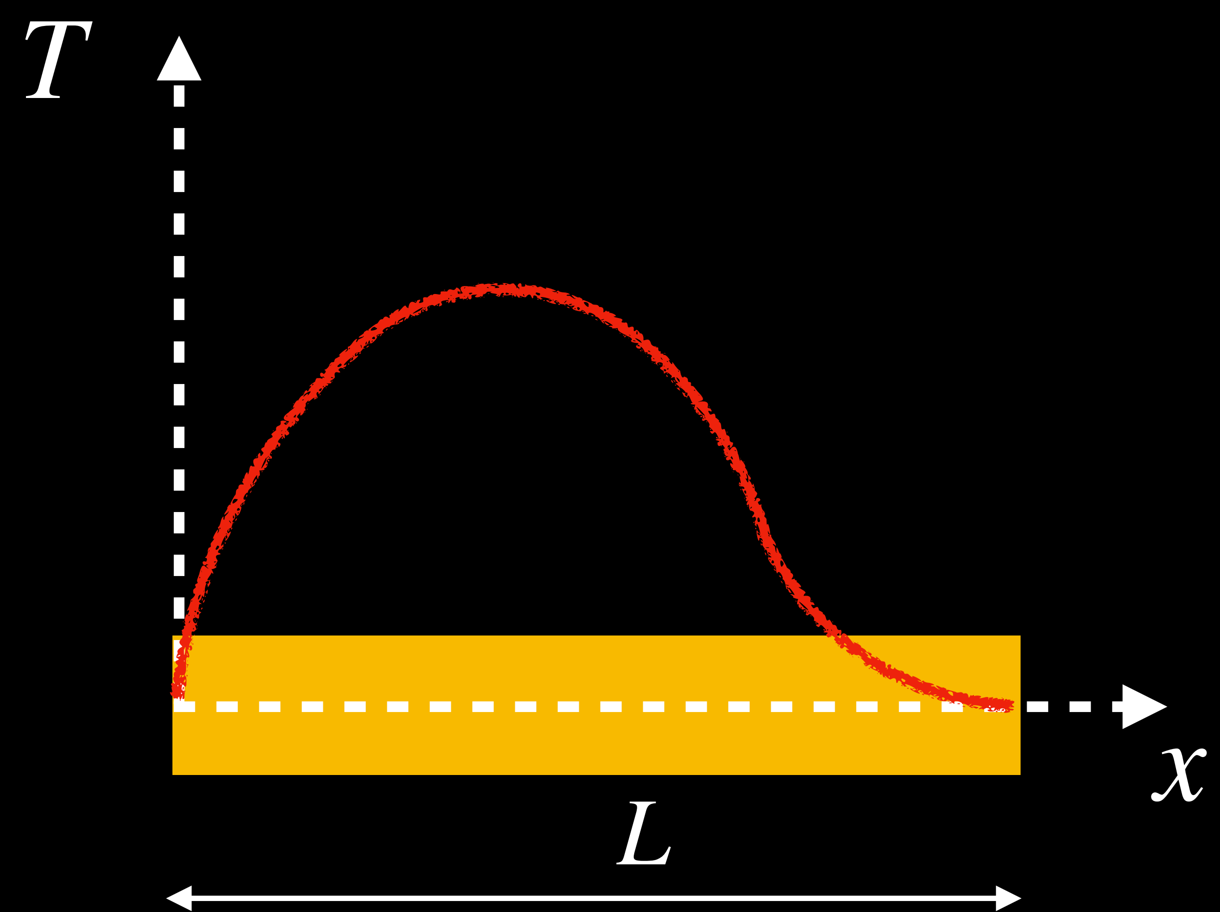
Solve HE → $\dfrac{1}{D}\dfrac{\partial T}{\partial t} = \nabla^2 T + \gamma(t,x)$

for $f, g \in \mathscr{F}$

$$\langle f, g \rangle \triangleq \frac{2}{L}\int_0^L f(x)g(x)\,dx$$

defines an inner product on $\mathscr{F}$

Steady State Homogeneous Solution



non-homogeneous BCs

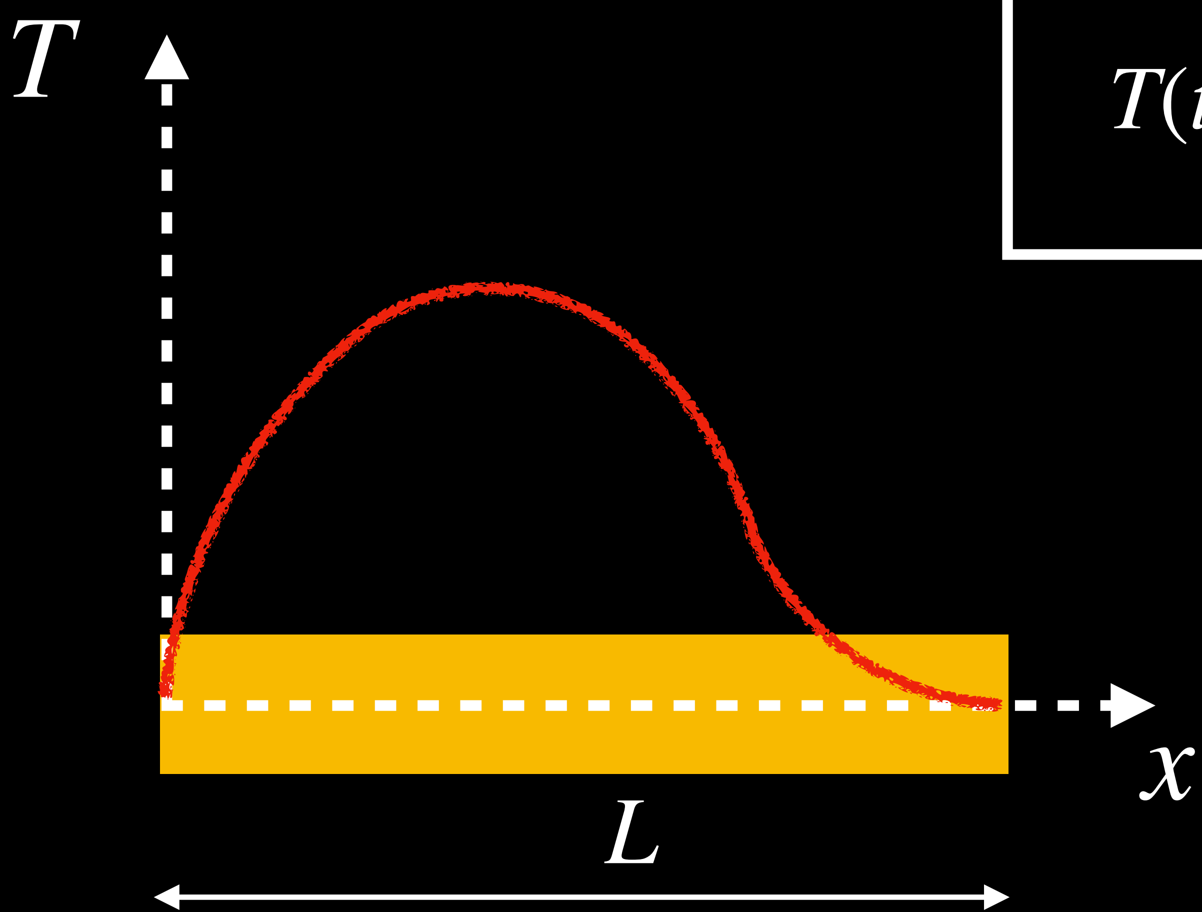
$$T_{SH}(0) = T_0 \quad \text{and} \quad T_{SH}(L) = T_L$$

Solve → $$\nabla^2 T_{SH} = 0$$

The solution is

$$T_{SH}(x) = T_0 - (T_0 - T_L)\frac{x}{L}$$

1-D Heat Conduction



$$T(t,x) = T_{SH}(x) + \sum_{n=1}^{\infty} \alpha_n(t)\ \sin\left(\frac{n\pi x}{L}\right)$$

$$\alpha_n(t) = e^{-D\lambda_n^2 t}\alpha_n(0) + D\int_0^t e^{-D\lambda_n^2 (t-\tau)}\gamma_n(\tau)\, d\tau$$

Where

$$\gamma_n(t) = \frac{2}{L}\int_0^L \gamma(t,x)\ \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\alpha_n(0) = \frac{2}{L}\int_0^L \left(T_0(x) - T_{SH}(x)\right)\ \sin\left(\frac{n\pi x}{L}\right) dx$$

Non-homogeneous BCs

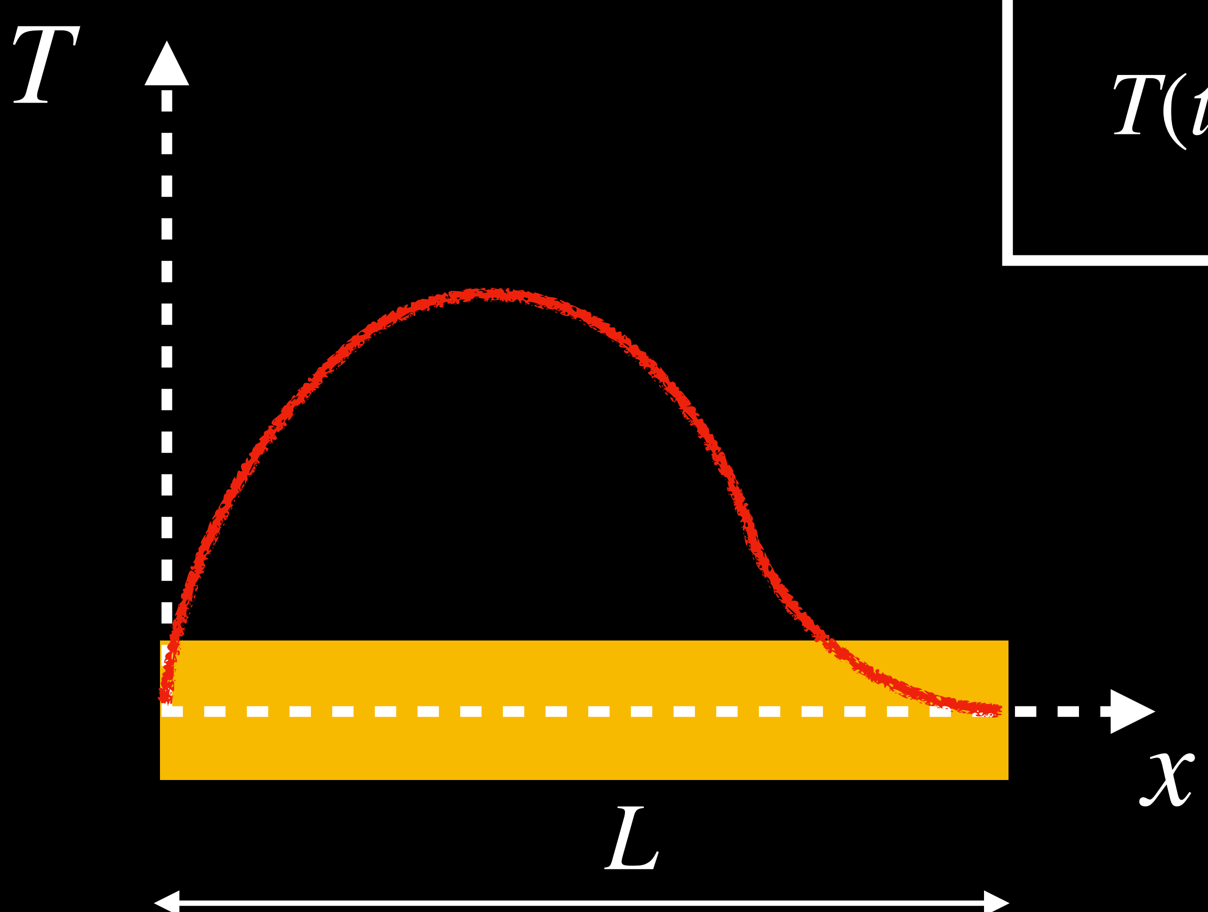
$T(t,0) = T_0$ and $T(t,L) = T_L$

$T(0,x) = T_0(x)$ → **ICs**

Solve HE → $\dfrac{1}{D}\dfrac{\partial T}{\partial t} = \nabla^2 T + \gamma(t,x)$

1-D Heat Conduction

$$\boxed{T(t,x) = T_{SH}(x) + \sum_{n=1}^{\infty} \alpha_n(t)\,\sin\left(\frac{n\pi x}{L}\right)}$$

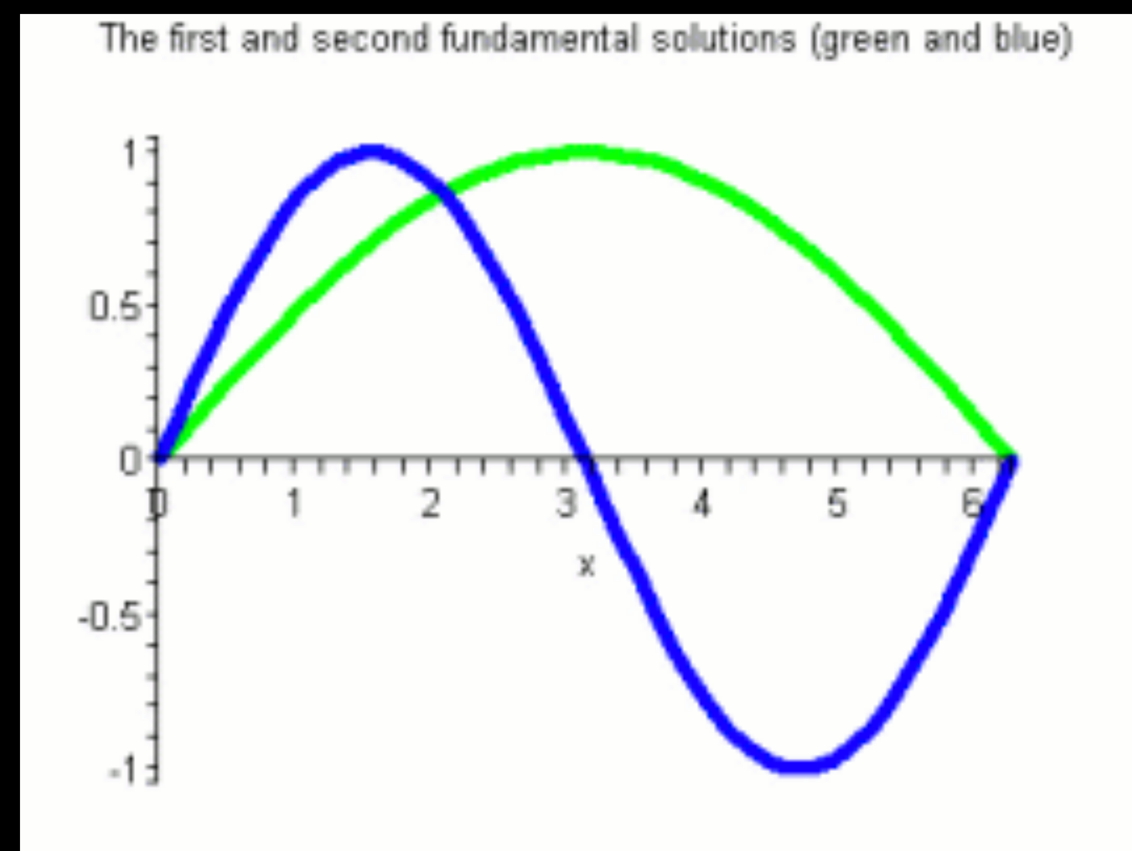


Non-homogeneous BCs

$T(t{,}0) = T_0$ and $T(t,L) = T_L$

$T(0{,}x) = T_0(x)$ → **ICs**

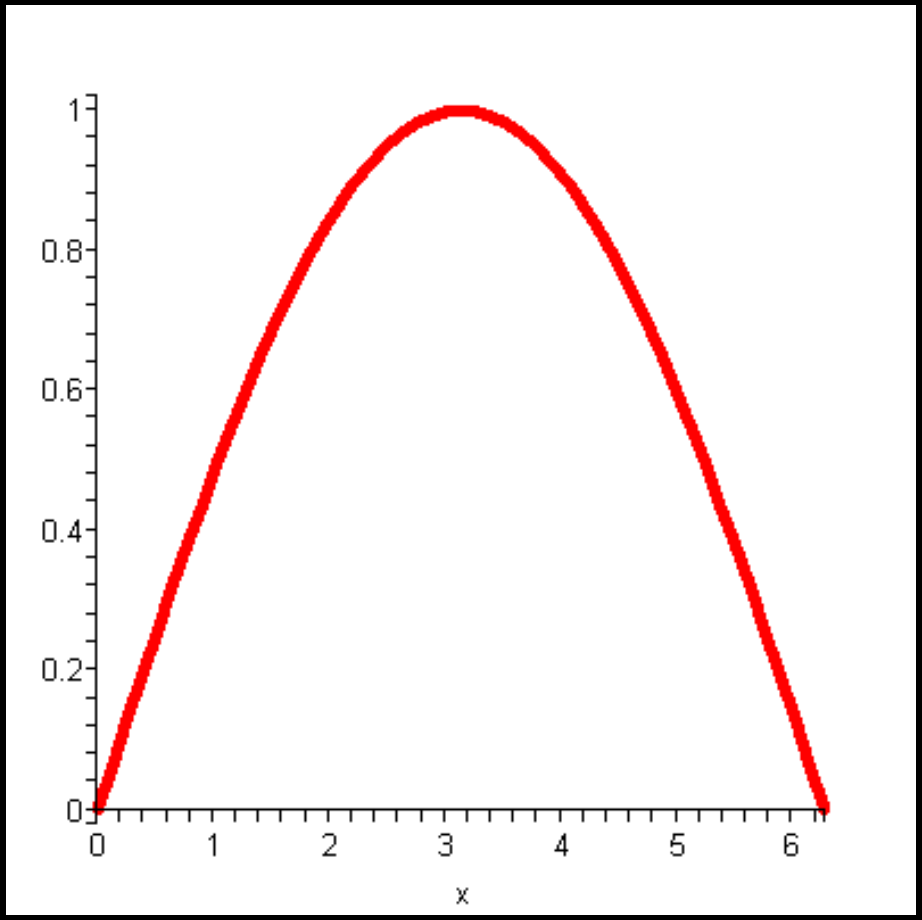
Solve HE → $$\frac{1}{D}\frac{\partial T}{\partial t} = \nabla^2 T + \gamma(t,x)$$



1-D Heat Diffusion: Example 1

$$\gamma(t,x) \equiv 0$$

What happens if $T_0(x) = \sin\left(\frac{\pi x}{L}\right)$?



Then

$$a_n = \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$
$$= \delta_{1n}$$

$$T(t,x) = e^{-\left(\frac{\pi}{L}\right)^2 t}\sin\left(\frac{\pi x}{L}\right)$$

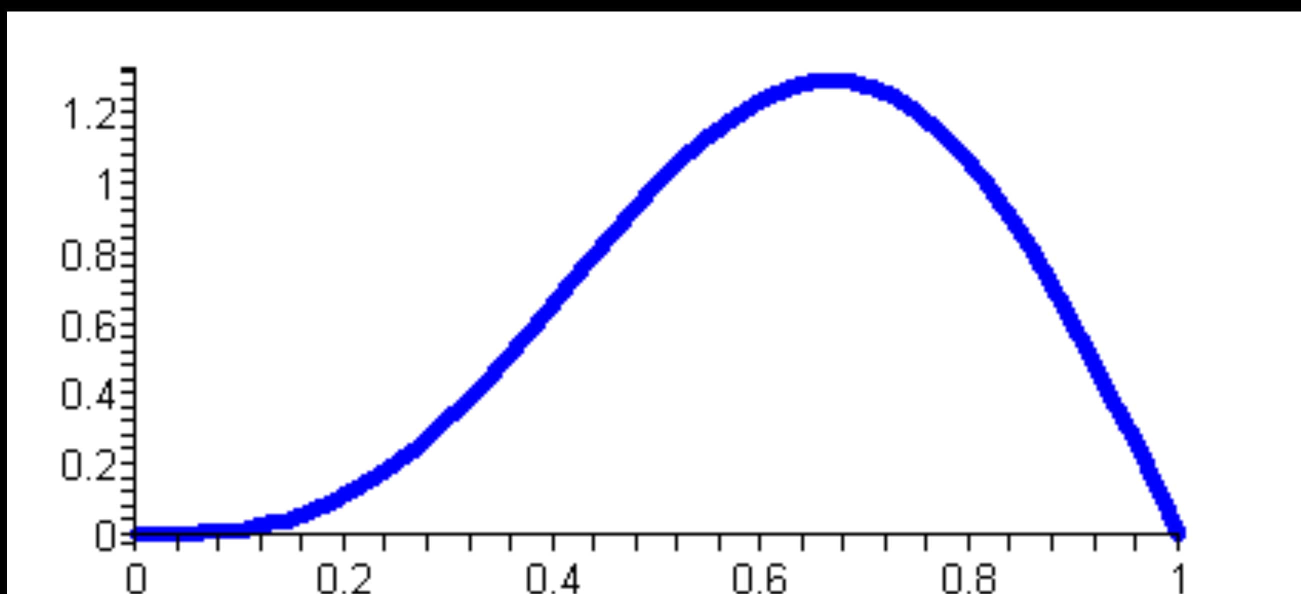
$T(t,0) = 0$ and $T(t,L) = 0$ → **BCs**

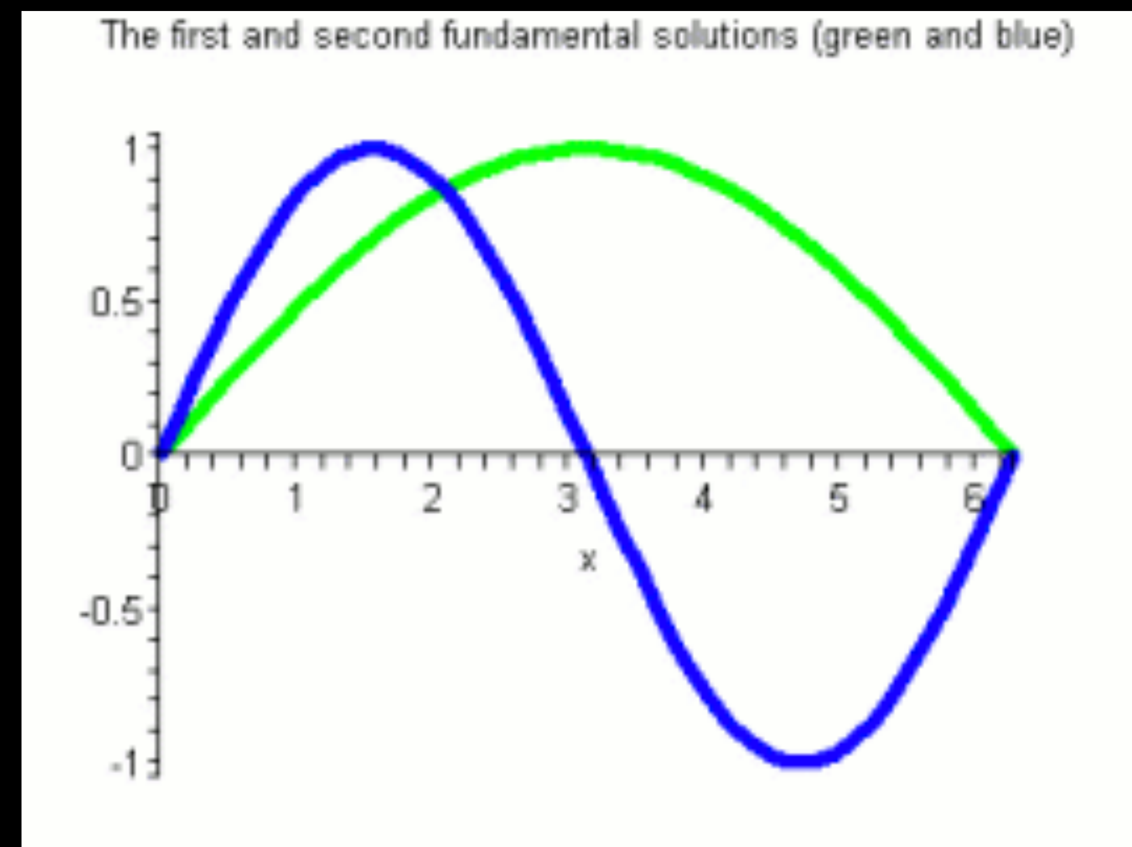
$T(0,x) = T_0(x) = \sin\left(\frac{\pi x}{L}\right)$ → **ICs**

Solve HE → $\frac{1}{D}\frac{\partial T}{\partial t} = \nabla^2 T$

1-D Heat Diffusion: Example 2

What happens if $T_0(x) = \sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right)$?





$$\boxed{T(t,x) = e^{-D\left(\frac{\pi}{L}\right)^2 t}\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}e^{-D\left(\frac{2\pi}{L}\right)^2 t}\sin\left(\frac{2\pi x}{L}\right)}$$

1-D Diffusion in an infinite domain

Solve HE $\Rightarrow$ $$\frac{1}{D}\frac{\partial T}{\partial t} = \nabla^2 T$$

In the domain $\Omega = \mathbb{R}$

With Initial Condition $T(0,x) = \delta(x)$

$$\frac{d}{dt}T(t,k) + D\,k^2\,T(t,k) = 0$$

and $T(0,k) = 1$

$$\Downarrow$$

$$T(t,k) = e^{-tDk^2}$$

$$\Downarrow$$

Spatial Fourier Transform

$$T(t,k) = \int_{-\infty}^{\infty} e^{-ikx}\,T(t,x)\,dx$$

$$T(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,T(t,k)\,dk$$

$$T(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-tDk^2}e^{ikx}\,dk$$

$$= \frac{1}{\sqrt{4\pi Dt}}\,e^{\frac{-x^2}{4Dt}}$$

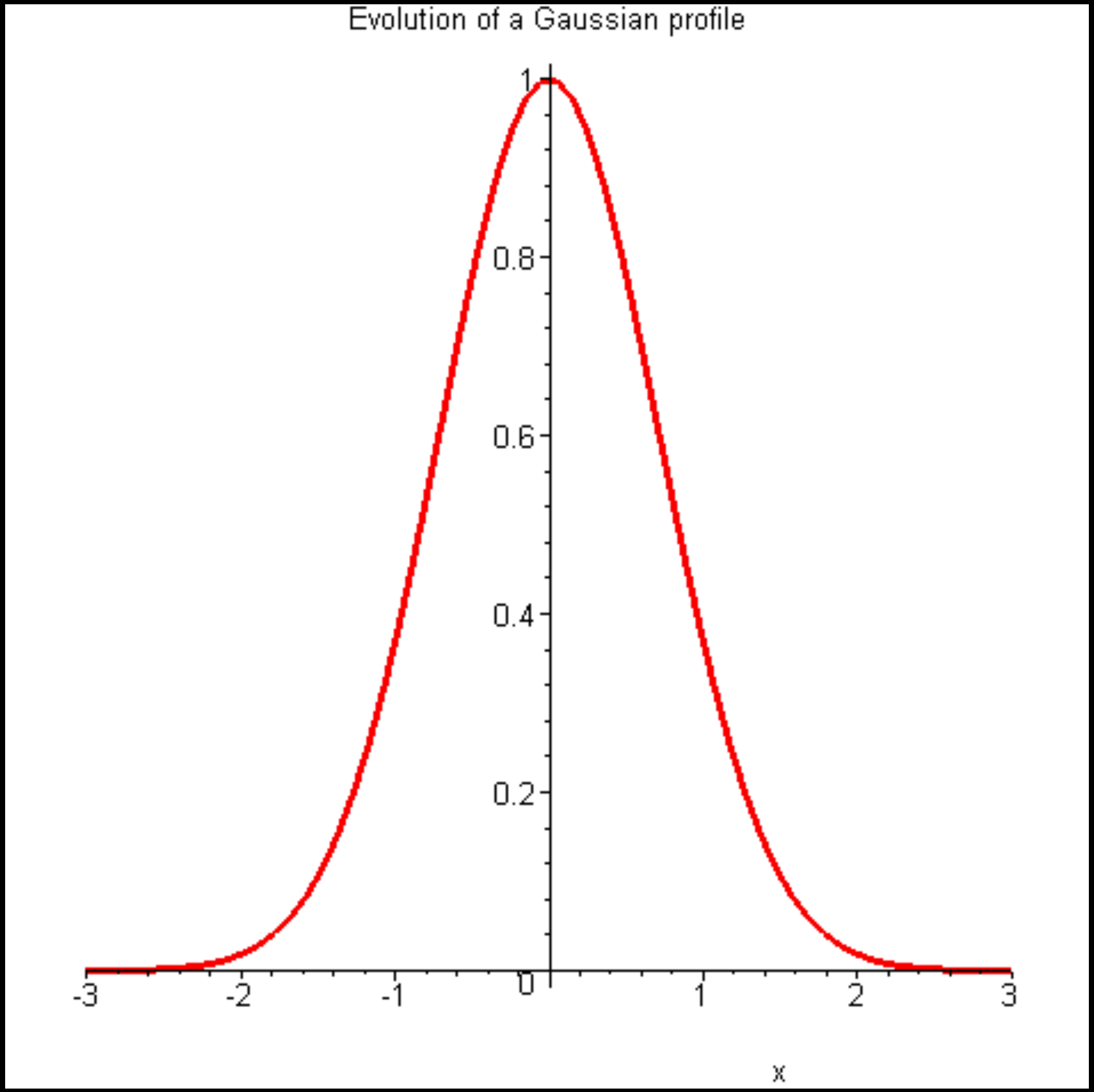
1-D Diffusion in an infinite domain

Solve HE $\Rightarrow$ $$\frac{1}{D}\frac{\partial T}{\partial t} = \nabla^2 T$$

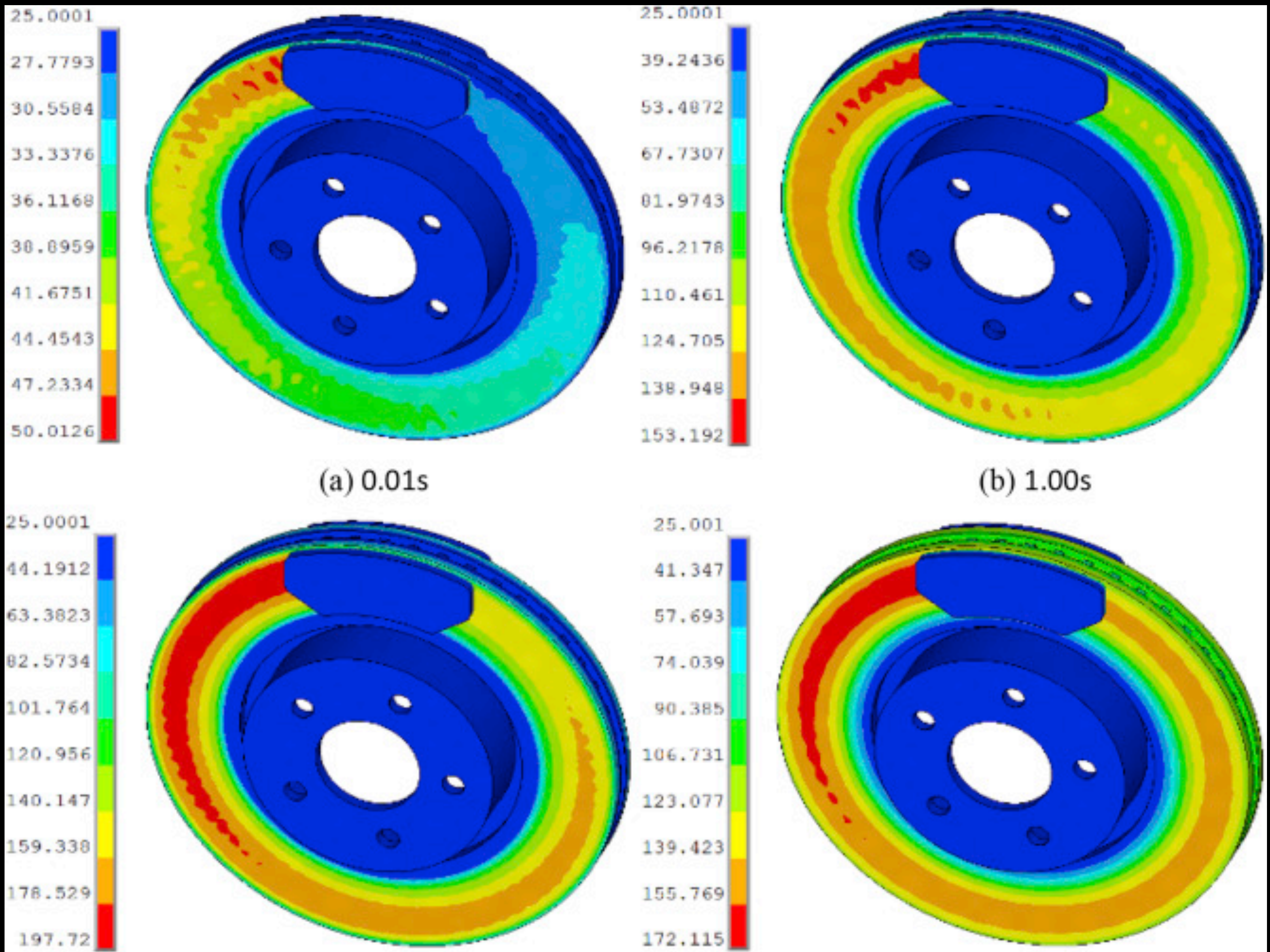
In the domain $\Omega = \mathbb{R}$

With Initial Condition $T(0,x) = \delta(x)$

$$T(t,x) = \frac{1}{\sqrt{4\pi D t}}\, e^{\frac{-x^2}{4Dt}}$$



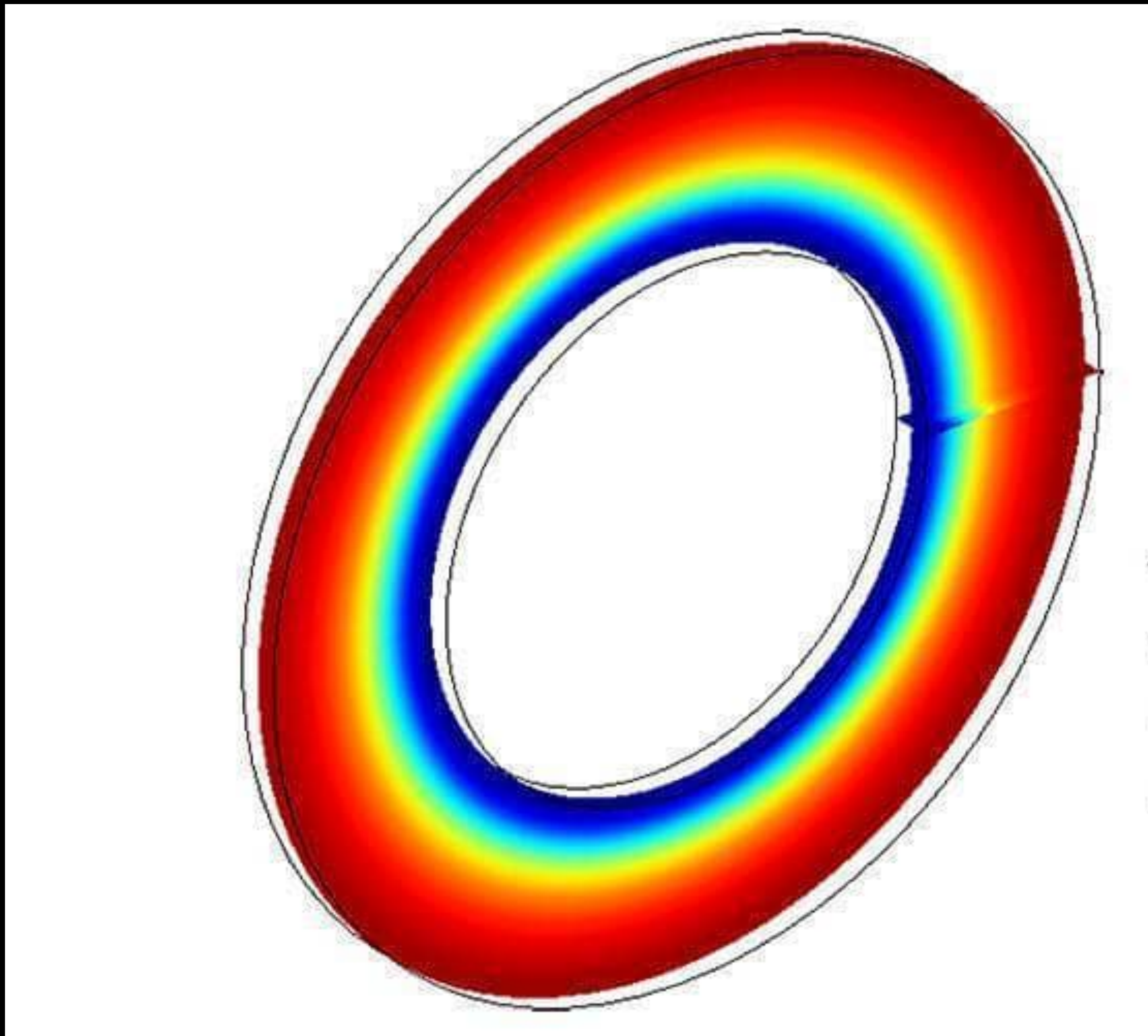
2D-Example: Heat Transfer in a Disk Brake



Example: 2D Axi-Symmetric Heat Conduction

$$T(t, r, \theta, z) \equiv T(t, r) \text{ for 2D axi} - \text{symmetric problems}$$

$$\nabla^2 = \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) \right) = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}$$



Steady State Homogenous Solution

$$\nabla^2 T = 0 \longrightarrow T_{SH}(r) = a_1 \ln r + a_2$$

2D Axi-Symmetric Heat Conduction

Solve $$\frac{1}{D}\frac{\partial T}{\partial t} = \nabla^2 T + \gamma(t,p)$$

Subject to BCs and ICs

$$\nabla^2 = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right)$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

Eigenvectors of ∇^2 $\Rightarrow$

$$\nabla^2 \varphi_n = -\lambda_n^2 \varphi_n$$

$$\Downarrow$$

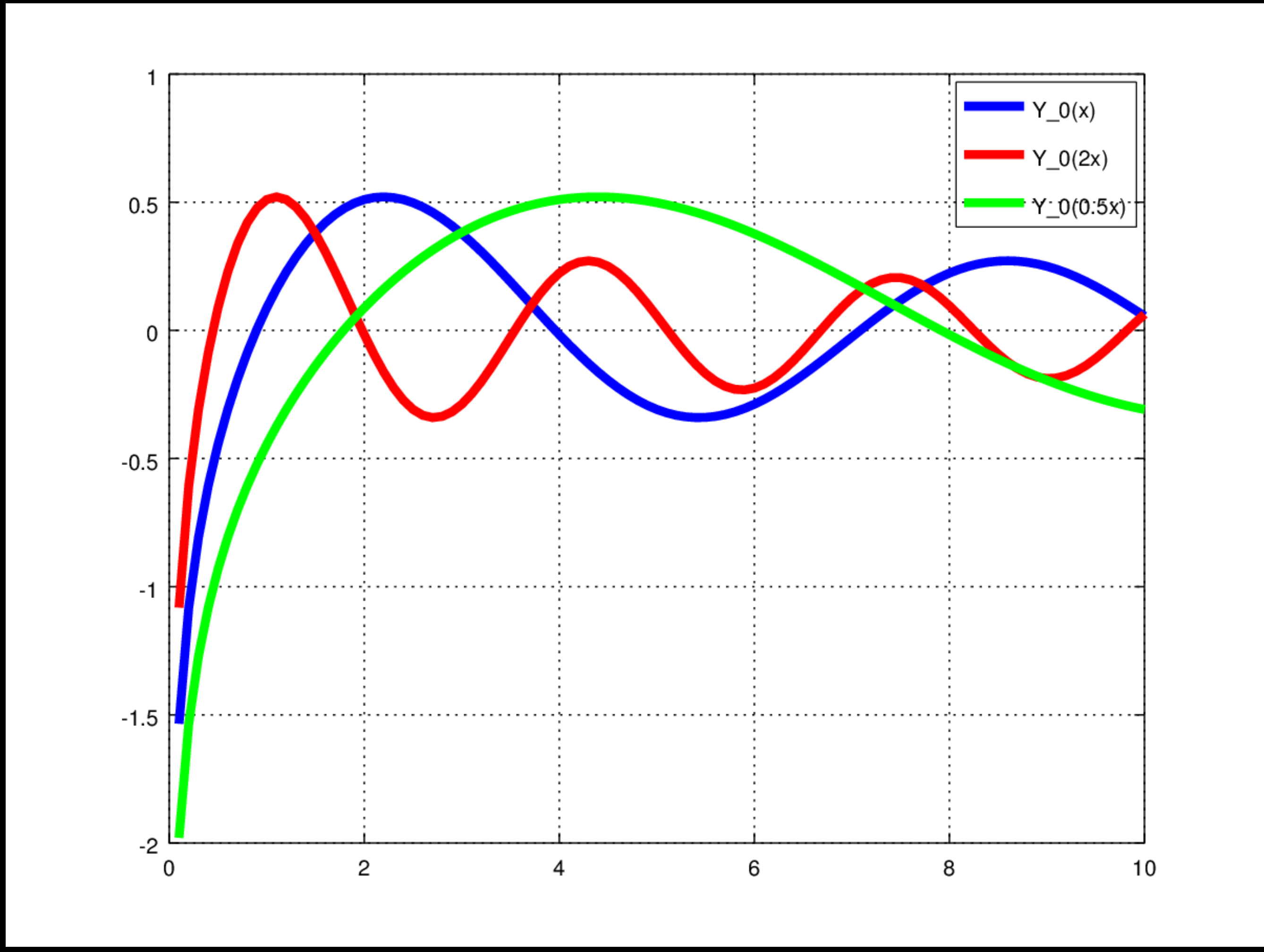
$$\frac{d^2\varphi_n}{dr^2} + \frac{1}{r}\frac{d\varphi_n}{dr} + \lambda_n^2 = 0$$

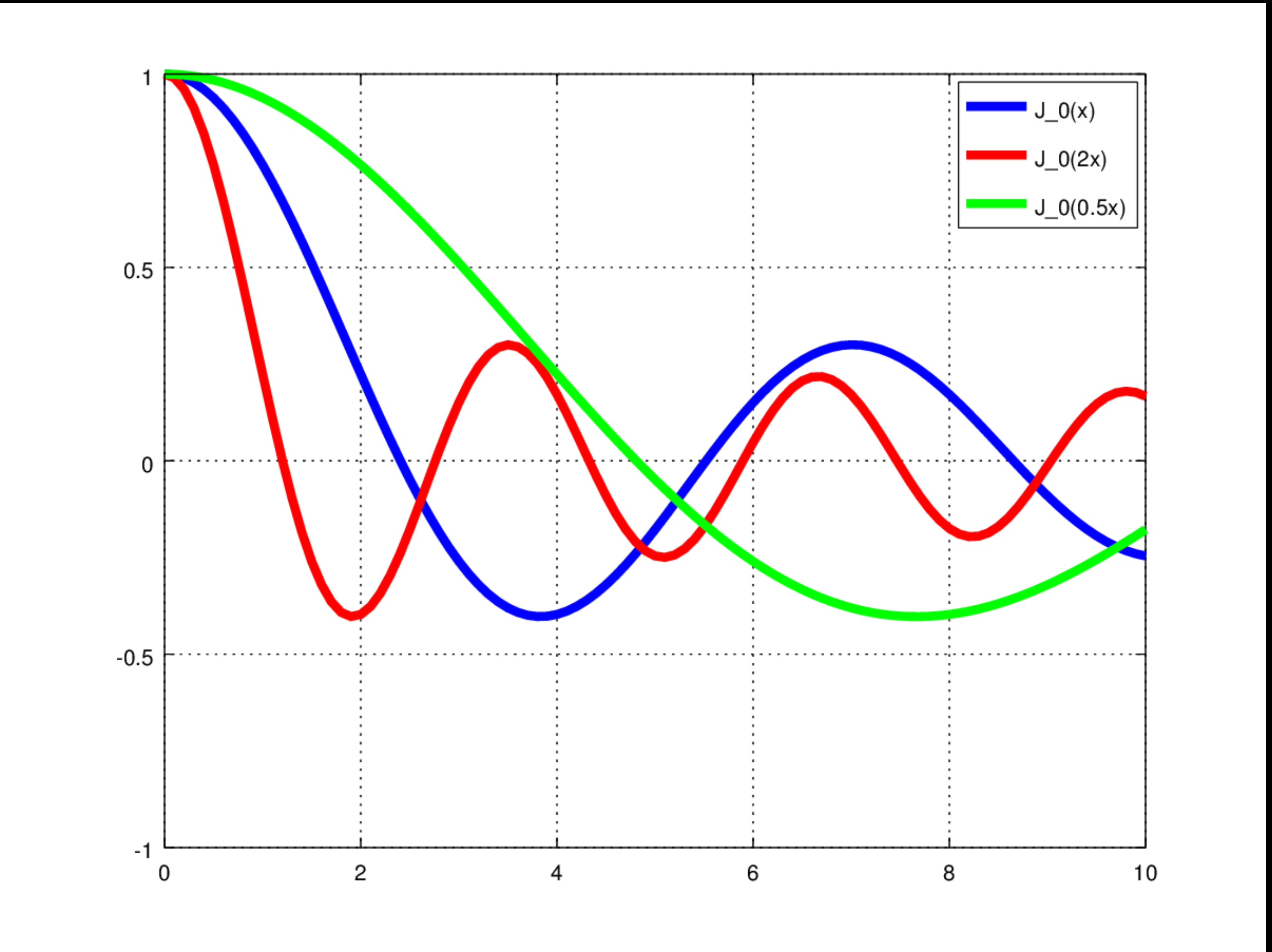
Bessel's DE

Solutions to BE

$$\frac{d^2\varphi_n}{dr^2} + \frac{1}{r}\frac{d\varphi_n}{dr} + \lambda_n^2\varphi_n = 0$$

$$\varphi_n(r) = b_{1n}J_0(\lambda_n r) + b_{2n}Y_0(\lambda_n r)$$





Bessel Functions of the 1st Kind

$$\left\langle J_0\left(\lambda_m r\right),\ J_0\left(\lambda_n r\right)\right\rangle \triangleq \frac{2}{a^2 J_1(\lambda_m) J_1(\lambda_n)} \int_0^a r J_0\left(\lambda_m r\right) J_0\left(\lambda_n r\right)\, dr$$
$$= \delta_{mn}$$

General Solution

$$T(t,p) = T_{SH}(p) + \sum_{n=1}^{\infty} \alpha_n(t)\varphi_n(p)$$

where

$$\dot{\alpha}_n(t) + D\lambda_n^2\alpha_n = \langle \psi_n\,, \gamma(t,x) \rangle$$

$$\nabla^2 T_{SH} = 0$$

and

$$\alpha_n(0) = \langle \varphi_n(p)\,, T_0(p) - T_s(p) \rangle$$

For the general cylindrical problem

- http://mathworld.wolfram.com/HeatConductionEquationDisk.html
- http://mathworld.wolfram.com/Fourier-BesselSeries.html