Chapter 2
Dynamics

2.1 Introduction

In mechanics we have seen how the application of physical principles such as Newton's equations or Euler's rigid body equations are used to mathematically describe certain physical systems. This process is generally referred to as mathematical modelling. Typically the mathematical object so derived is a set of differential equations. The sense in which these differential equations describe the physical system is that the solutions of the differential equations correspond to certain behaviour of the physical system[1]. Thus the study of the behaviour of the solutions of a system of ODEs is of paramount importance in understanding a behaviour of the physical system. The study of the behaviour of differential equations is referred to as *Dynamics*.

We will first introduce some terminology involved in dealing with dynamics. Mathematically a dynamic system consists of the following ingredients: A space $\mathscr{X}$ called the *state space*, and a first order differential equation defined on $\mathscr{X}$ by

$$\frac{dx(t)}{dt} = \dot{x}(t) = f(x(t)). \tag{2.1}$$

Here $x \in \mathscr{X}$ is called the *state* of the system and $f(x)$ is called a *vector field*. The fundamental theorem of differential equations tells us that there always exists a solution[2] of (2.1) for any given initial condition $x(0)$ and if $f(x)$ satisfies certain continuity conditions, this solution is unique. Most of the physical systems give rise to dynamic systems that satisfies the said continuity condition and hence desirably the solutions are always unique. Furthermore the solutions depend continuously on the initial data $x(0)$. What that means is that the solutions corresponding to two "nearby" initial conditions say "close" to each other for a "while".

A solution $x(t)$ of (2.1) can be visualized as a curve on $\mathscr{X}$ that is parameterized by t. In this context we refer to $x(t)$ as a *trajectory* of the system that begins at $x(0)$ at $t = 0$. Observe that the tangent vector at any point $x(t)$ on the trajectory coincides with the value of the vector field at $x(t)$. Having this picture in mind is extremely useful in dynamics.

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[1] Remember that this correspondence is approximate.

[2] Note that this solution may not be defined for all time t

In most of the examples considered in this class the state space $\mathscr{X} = \mathbb{R}^n$ for some n or is an open subset of $\mathbb{R}^n$. In this case the state x is an ordered n-tuple of variables $(x_1, x_2, \cdots, x_n)$ and $f(x)$ is a set of ordered n-tuple of functions $(f_1(x), f_2(x), \cdots, f_n(x))$ where each f_i is a real valued function defined on $\mathbb{R}^n$ (that is $f_i : \mathbb{R}^n \mapsto \mathbb{R}$). Writing the ordered n-tuples in a column matrix we can then write the dynamic system (2.1) as

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} f_1(x_1(t), x_2(t), \cdots, x_n(t)) \\ f_2(x_1(t), x_2(t), \cdots, x_n(t)) \\ \vdots \\ f_n(x_1(t), x_2(t), \cdots, x_n(t)) \end{bmatrix}, \tag{2.2}$$

Thus a dynamic system on $\mathbb{R}^n$ is nothing but a collection of n number of coupled first order ODEs. To solve this system we know that n initial conditions $x(0) = (x_1(0), x_2(0), \cdots, x_n(0))$ need to be specified.

Let us now look at several types of behaviour of a dynamical system. These behaviour correspond to special types of solutions, or trajectories, of (2.1). The most common behaviour that one sees in Engineering systems is steady state (or equilibrium) behaviour, periodic behaviour and chaotic behaviour. They correspond respectively to steady state, periodic and chaotic solutions of the corresponding dynamic system (2.1).

Steady State Solutions

A solution $x(t)$ is said to be an steady state or equilibrium solution of the dynamic system (2.1) if it remains a constant for all time. That is $x(t) \equiv \bar{x}$. Substituting this equilibrium solution in (2.1) it can be seen that such equilibria are solutions of $f(\bar{x}) = 0$.

Periodic Solutions

A solution $x(t)$ is said to be a periodic solution of the dynamic system (2.1) if $x(t+T) = x(t)$ for all t for which the solution is defined and some $T > 0$ called the period of the solution[3].

Chaotic Solutions

Chaotic solutions are characterized by their extreme sensitivity to initial conditions. Although the solutions depend continuously on the initial conditions they move apart at an exponential rate if they are chaotic.

[3] The smallest such value is chosen.

2.2 Linear Systems

When the state space $\mathscr{X}$ is a vector space and the vector field $f(x)$ is linear, that is when $f(x) = Ax$, where A is a constant $n \times n$ matrix, we say that the dynamic system is a linear system:

$$\dot{x} = Ax. \tag{2.3}$$

Linearity Property of Solutions: It can be easily verified that if $x_1(t)$ is a solution with initial condition $x_1(0)$ and $x_2(t)$ is a solution with initial condition $x_2(0)$ then $x(t) = x_1(t) + x_2(t)$ is a solution with initial condition $x_1(0) + x_2(0)$ and $x(t) = \lambda x_1(t)$ is a solution with initial condition $\lambda x_1(0)$ for any $\lambda \in \mathbb{R}$. This in fact can be used as the characterizing property of linear systems. Systems that do not satisfy this property are called *nonlinear* systems.

Linear systems have the simplest possible behaviour. We shall see that the eigenvalues of A completely characterize the entire behaviour of the system. For linear systems (2.3) the origin $\bar{x} = 0$ is always an equilibrium solution and no other equilibria exist if A has no zero eigenvalue[4]. The equilibrium $\bar{x} = 0$ is said to be *Hyperbolic* if none of the the eigenvalues of A are on the imaginary axis while it is called *Non-Hyperbolic* otherwise.

By direct verification we can see that a general solution of (2.3) is given by

$$x(t) = e^{At}x(0). \tag{2.4}$$

Here e^X is the matrix exponential function given by the infinite series

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots,$$

where X is a $n \times n$ matrix.

We will see how one may calculate this solution for a second order system. For a second order system $x \in \mathbb{R}^2$ and A is a 2×2 constatnt matrix. Let the eigenvalues of A be λ_1 and λ_2. From linear algebra we know that A is similar to one of the following normal forms:

$$A_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad A_3 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad A_4 = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}.$$

That is there exists a 2×2 invertible matrix T_i such that $A = T_i A_i T_i^{-1}$ for some A_i given above. The similarity between A and A_i is denoted symbolically as $A \sim A_i$. The particular normal form depends on the eigenvalues of A and their multiplicities. When λ_1 and λ_2 are real and distinct $A \sim A_1$, when $\lambda_1 = \lambda_2 = \lambda$ and their geometric multiplicity is two then $A \sim A_2$, when $\lambda_1 = \lambda_2 = \lambda$ and their geometric multiplicity is one then $A \sim A_3$, and finally if $\lambda_1 = \sigma + i\omega$ and $\lambda_2 = \sigma - i\omega$ then $A \sim A_4$. It is easy to show that

[4] Reason out why this is true.

$$e^{A_1 t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}, \quad e^{A_2 t} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix}, \quad e^{A_3 t} = \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}, \quad e^{A_4 t} = \begin{bmatrix} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{bmatrix}.$$

You are asked to show this in exercise **??**. Let $A \sim A_i$ then since

$$e^{T_i A_i T_i^{-1}} = T_i e^{A_i} T_i^{-1},$$

we have that

$$x(t) = e^{At}x(0) = e^{tT_i A_i T_i^{-1}}x(0) = T_i e^{tA_i} T_i^{-1} x(0)$$

Thus we can show that if λ_1 and λ_2 are real and distinct the solution takes the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_{11}\,e^{\lambda_1 t} + c_{12}\,e^{\lambda_2 t} \\ c_{21}\,e^{\lambda_1 t} + c_{22}\,e^{\lambda_2 t} \end{bmatrix}$$

where all the c_{ij} are constants that depend on the initial condition $x(0)$. Note that not all four of these constants are independent and only two of them are. If $\lambda_1 = \lambda_2 = \lambda$ we can show that a solution either takes the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_{11}\,e^{\lambda t} \\ c_{12}\,e^{\lambda t} \end{bmatrix}$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (c_{11} + c_{12}\,t)e^{\lambda t} \\ (c_{21} + c_{22}\,t)e^{\lambda t} \end{bmatrix}$$

where all the c_{ij} are constants that depend on the initial condition $x(0)$. Note that not all four of these constants are independent and only two of them are. Finally if $\lambda_1 = \sigma + i\omega$ and $\lambda_2 = \sigma - i\omega$ we can show that a solution takes the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{\sigma t}(c_{11}\,\sin\omega t + c_{12}\,\cos\omega t) \\ e^{\sigma t}(c_{21}\,\sin\omega t + c_{22}\,\cos\omega t) \end{bmatrix}$$

where all the c_{ij} are constants that depend on the initial condition $x(0)$. Note that not all four of these constants are independent and only two of them are.

From this we see that if all the eigenvalues of A are in the strict left half of the complex plane then all solutions asymptotically converge to the equilibrium solution $\bar{x} = 0$. That is $\lim_{t\to\infty} x(t) \to 0$ for all initial conditions $x(0)$. In this case we say that the equilibrium solution $\bar{x} = 0$ is globally asymptotically stable. On the other hand if both eigenvalues are in the strict right half plane then all solutions, except the equilibrium solution, diverges to infinity. That is $\lim_{t\to\infty} x(t) \to \infty$ for all initial conditions $x(0) \neq 0$. If only one eigenvalue is in the strict right half plane then almost all solutions diverge to infinity. If the eigenvalues are on the imaginary axis and are distinct then all solutions are periodic.

A linear system with *forcing* is given by

$$\dot{x} = Ax + u(t). \tag{2.5}$$

The general solution of this can be written down explicitly using the *Variation of Parameters* formula[5]:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}u(\tau)\,d\tau. \tag{2.6}$$

2.2.1 Examples of Linear Systems

2.2.1.1 RC Circuit

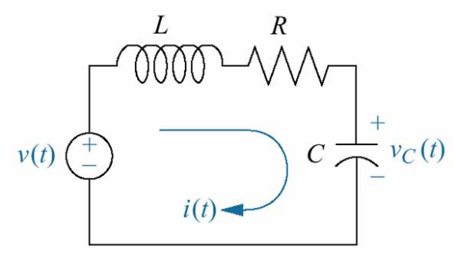


Fig. 2.1 RLC Circuit

If the inductance $L = 0$ then the circuit shown in figure 2.1 reduces to a RC circuit. Applying Kirchoff's voltage law we obtain

$$\dot{q} = -\frac{1}{RC}q + \frac{1}{R}v(t), \tag{2.7}$$

where q is the charge on the capacitor. Consider the case where the external voltage $v(t) \equiv 0$. Then the equation (2.7) becomes

$$\dot{q} = -\frac{1}{RC}q = -aq, \tag{2.8}$$

where we have set $a = \frac{1}{RC}$. This is a first order linear system where the state space is $\mathscr{X} = \mathbb{R}$. The equilibrium solution of the system is $\bar{q} = 0$. The equilibrium is a hyperbolic equilibrium as long as $a \neq 0$ and is non-hyperbolic when $a = 0$. In our elementary ODE class we have seen that the general solution of the system is

$$q(t) = e^{-at}q(0).$$

It can be seen that, when $a > 0$ $\lim_{t\to\infty} q(t) = 0$ and that when $a < 0$ $\lim_{t\to\infty} q(t) = 0$. This also shows that the equilibrium is globally asymptotically stable if $a > 0$ and is unstable when $a < 0$.

When the forcing $v(t)$ is non zero we have

[5] Show this as an exercise.

$$\dot{q} = -\frac{1}{RC}q = -a\,q + u(t), \tag{2.9}$$

where $u(t) = \frac{1}{R}v(t)$. The solution of this system can also be written down using the variation of parameters formula

$$q(t) = e^{-at}q(0) + \int_0^t e^{-a(t-\tau)}u(\tau)d\tau.$$

2.2.1.2 Spring Mass Damper System

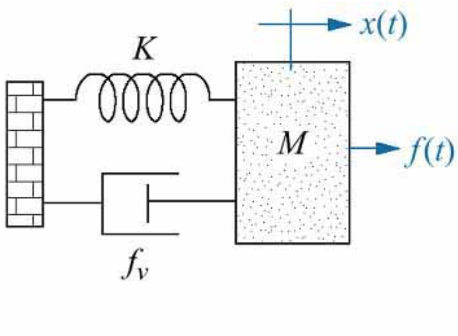


Fig. 2.2 Spring Mass Damper System.

For small deflections and small velocities the spring force and the viscous damping force can be approximated as $f_s = -Kx$ and $f_v = -C\dot{x}$ respectively where K and C are constants. Using Newton's equations for the mass M yields the following ODE model of the system.

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t). \tag{2.10}$$

Dividing by M and setting $\omega_n^2 = \frac{K}{M}$ and $2\zeta\omega_n = \frac{C}{M}$ in (2.10) we have

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = \frac{1}{M}f(t). \tag{2.11}$$

For small displacements of the mass, the solutions of this equation describes the motion quite accurately. Let us try to see what conclusions we can draw about the behavior of the system by studying the behavior of the solutions of the differential equation (2.11). For convenience we initially consider only the case $f(t) \equiv 0$.

By using the variable transformation $x_1 = x$, $x_2 = \dot{x}$ we can write (2.11) as a set of two first order ODE's. We write them in concise form using matrices as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.12}$$

We can also write this as $\dot{y} = Ay$ where

$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}.$$

This is a second order linear system. Recall that the eigenvalues of A determine the complete behavior of the system. The eigenvalues of A are the roots of the characteristic polynomial

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0.$$

The roots are

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm i\omega_d,$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Thus depending on ζ the eigenvalues will be real or complex.

The origin $\bar{y} = [0 \;\; 0]^T$ is an equilibrium solution. It is hyperbolic when none of the eigenvalues are on the imaginary axis, that is $\zeta \neq 0$, and is non-hyperbolic when at least one of them is on the imaginary axis, that is $\zeta = 0$.

Observe that the parameters ω_n, ω_d, and ζ, completely determine the eigenvalues. Thus we have special names for these parameters.

ζ	-	Damping ratio.
ω_n	-	Natural frequency.
ω_d	-	Damped natural frequency.

The meaning will be clear after we look at the solutions a little more closely. Depending on the damping ratio ζ the eigenvalues will be real and distinct, real and coinciding, complex and conjugate or purely imaginary. We will consider each of these cases separately.

Case I: Un-Damped: $\zeta = 0$

When $\zeta = 0$ the eigenvalues are

$$\lambda_1, \lambda_2 = \pm i\omega_n,$$

and are purely imaginary and conjugate. Then form the results of section 2.2 the solution takes the form

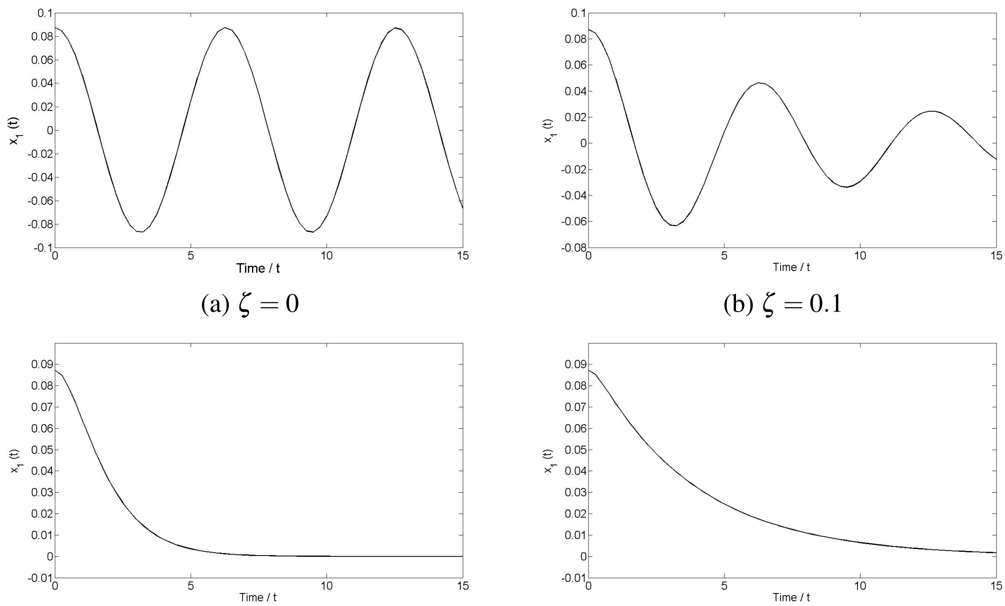
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_{11}\sin\omega_n t + c_{12}\cos\omega_n t \\ c_{21}\sin\omega_n t + c_{22}\cos\omega_n t \end{bmatrix}.$$

where all the c_{ij} are constants that depend on the initial condition $x(0)$. Note that $c_{22} = \omega_n c_{11}$ and $c_{21} = -\omega_n c_{12}$. A typical such solution is shown in figure 2.3 (a). We see that the solution is periodic with period $2\pi/\omega_n$ for all initial conditions. The frequency of oscillation is ω_n. ω_n is called the undamped natural frequency of the system.

Case II: Under Damped: $0 < \zeta < 1$

When $0 < \zeta < 1$ the eigenvalues are

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm i\omega_d,$$



(a) $\zeta = 0$ (b) $\zeta = 0.1$

(c) $\zeta = 1$ (d) $\zeta = 2$

Fig. 2.3 The displacement $x_1(t)$ Vs t for different values of ζ.

and are complex and conjugate. Figure 2.4(a) shows the location of two typical such eigenvalues. Then form the results of section 2.2 the solution takes the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-\zeta\omega_n t}(c_{11} \sin\omega_d t + c_{12} \cos\omega_d t) \\ e^{-\zeta\omega_n t}(c_{21} \sin\omega_d t + c_{22} \cos\omega_d t) \end{bmatrix}.$$

where all the c_{ij} are constants that depend on the initial condition $x(0)$. Note that $c_{21} = -\zeta\omega_n c_{11} - \omega_d c_{12}$ and $c_{22} = -\zeta\omega_n c_{12} + \omega_d c_{11}$. A typical solution for $x_1(t)$ is shown in figure 2.3(b). We see that the solution converges to $[0\ \ 0]^T$. However the convergence is oscillatory. The frequency of oscillation is ω_d and the rate of decay of the amplitude is $e^{-\zeta\omega_n t}$. The frequency ω_d is called the damped natural frequency. Figure 2.4(b) shows a typical underdamped response.

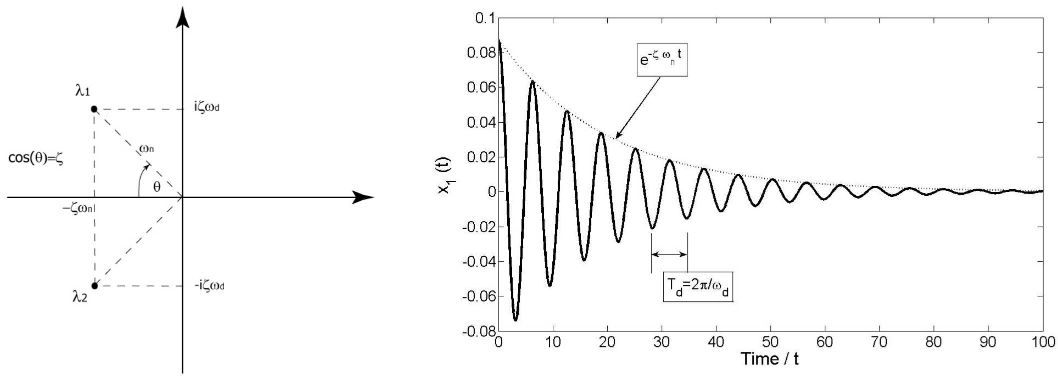
Case III: Over Damped: $\zeta > 1$

When $\zeta > 1$ the eigenvalues are

$$\lambda_1, \lambda_2 = \omega_n(-\zeta \pm \sqrt{1-\zeta^2}),$$

and are distinct and negative. Then form the results of section 2.2 the solution takes the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_{11}\, e^{\lambda_1 t} + c_{12}\, e^{\lambda_2 t} \\ \lambda_1 c_{11}\, e^{\lambda_1 t} + \lambda_2 c_{12}\, e^{\lambda_2 t} \end{bmatrix}$$



(a) Eigenvalue location. (b) A typical solution.

Fig. 2.4 Figure (a) shows the location of the eigenvalues λ_1 and λ_2 when $0 < \zeta < 1$. Figure (b) shows a typical underdamped response of $x_1(t)$.

where all the c_{ij} are constants that depend on the initial condition $x(0)$. Since both λ_1 and λ_2 are real and negative we see that for any initial condition $x(0)$ the solution converges to $[0\ \ 0]^T$ monotonically as well.

Case IV: Critically Damped: $\zeta = 1$

When $\zeta = 1$ the eigenvalues are

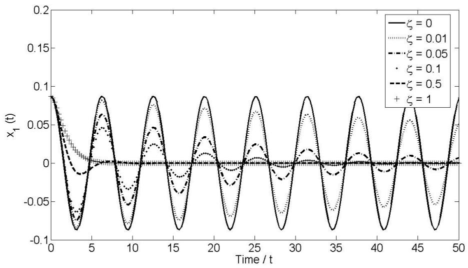
$$\lambda_1, \lambda_2 = -\omega_n, -\omega_n,$$

and are real coincident and negative. Then form the results of section 2.2 the solution takes the form

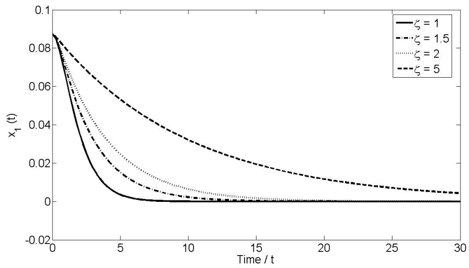
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-\omega_n t}(c_{11} + c_{12} t) \\ e^{-\omega_n t}(c_{21} + c_{22} t) \end{bmatrix}$$

where all the c_{ij} are constants that depend on the initial condition $x(0)$. Note that $c_{21} = -\omega_n c_{11} + c_{12}$ and $c_{22} = -\omega_n c_{12}$. The solution can be shown to converge to the origin $[0\ \ 0]^T$ monotonically. A typical solution for $x_1(t)$ is shown in figure 2.3(d). However the convergence is faster than for the over damped case. A typical solution for $x_1(t)$ is shown in figure 2.3(d).

For comparison purposes the response $x_1(t)$ for various different values of ζ are plotted in the same figure in 2.5. Observe that for $0 \leq \zeta \leq 1$ oscillations decay out faster when the damping ratio increases (see figure 2.5(a)). Also observe that when $\zeta \geq 1$ the solution is monotonic and that the convergence becomes slower as ζ increases (see figure 2.5(b)). Figure 2.6 shows the phase plot of the solution that begins at $[\pi/36\ \ 0]^T$ for several different values of ζ.



(a) $0 \leq \zeta \leq 1$



(b) $1 \leq \zeta$

Fig. 2.5 The displacement $x_1(t)$ Vs t for different values of ζ.

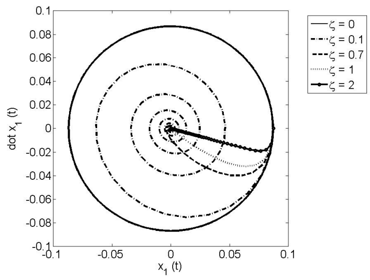


Fig. 2.6 Phase plot, $\dot{x}_1(t)$ Vs $x_1(t)$, for different values of ζ.

2.3 Nonlinear Systems

For general nonlinear systems the situation is more complicated. First of all linear super position does not work. That is, in general the linearity property described in the gray box at the beginning of section 2.2 does not hold. Second it is very rarely that one can explicitly write down the solution. Nonlinear systems for which explicit solutions can be written down are called *completely integrable* systems. The simple pendulum system and the rigid body system to be discussed subsequently are two examples of such systems. On the other extreme we have chaotic systems. The double pendulum and the Henon weather model are two examples that exhibit such behavior.

In Engineering the local behavior of a system near an equilibrium is of great importance. An equilibrium $\bar{x}$ is said to be stable if solutions that begin “sufficiently near” the equilibrium stay “sufficiently near” the equilibrium for all future time. If not the equilibrium is said to be unstable. If all solutions that begin “sufficiently near” the equilibrium “asymptotically” converge to the equilibrium (ie. as $t \to \infty$) then the equilibrium is said to be locally asymptotically stable.

In order to investigate the local behavior of solutions near an equilibrium $\bar{x}$ we can consider the perturbation $x = \bar{x} + \delta x$ where δx is a very small quantity. Substituting this in (2.1) and using the Taylor series expansion of $f(x)$ we have

$$\dot{\delta x} = f(\bar{x}) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} \delta x + H.O.T. \tag{2.13}$$

For small motions near the equilibrium the higher order terms $H.O.T$ are negligible compared to the δx term and recalling that $f(\bar{x}) = 0$ we have the perturbed dynamic system

$$\dot{\delta x} = A(\bar{x})\delta x, \tag{2.14}$$

where $A(\bar{x}) = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}}$. This is referred to as the linearization of (2.1) about the equilibrium $\bar{x}$.

The local behavior of the solutions of (2.17) about the equilibrium $\bar{x}$ is equivalent to the behavior of the solutions of the linearized system (2.14) provided that none of the eigenvalues of $A(\bar{x})$ lie on the imaginary axis (that is when the equilibrium $\bar{x}$ is hyperbolic). The behavior of solutions of a linear system (2.14) are completely characterized by the eigenvalues of $A(\bar{x})$. Thus the local behavior of the solutions of (2.1) near hyperbolic equilibria $\bar{x}$ are completely determined by the eigenvalues of $A(\bar{x})$. If all the eigenvalues of $A(\bar{x})$ are in the strict left half complex plane then the equilibrium $\bar{x}$ is locally asymptotically stable and if at least one eigenvalue of $A(\bar{x})$ is in the strict right half complex plane then the equilibrium $\bar{x}$ is unstable. When the equilibrium $\bar{x}$ is non-hyperbolic the local stability of $\bar{x}$ can not be determined by the stability properties of the linearized system.

The above procedure works when one needs to consider the stability of more general solutions than equilibrium solutions. However then the linearized system will in general be a time varying system and the eigenvalue analysis does not hold.

2.3.1 Simple Pendulum

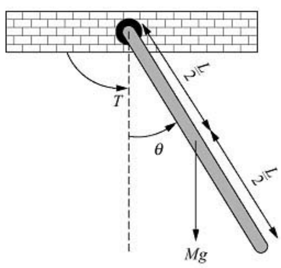


Fig. 2.7 A simple pendulum of length L and total mass M.

Consider the simple pendulum shown in figure 2.7. Assume that a viscous frictional torque, due to air resistance and bearing resistance, acts on the pendulum. Let the torque constant be CN/ms^{-1}. Using rigid body Euler's equations the governing equation of the system is given by,

$$\ddot{\theta} + \frac{3C}{ML^2}\dot{\theta} + \frac{3g}{2L}\sin\theta = 0. \tag{2.15}$$

Setting $\alpha_1 = \theta$ and $\alpha_2 = \dot{\alpha}_1 = \dot{\theta}$ the state space form of (2.15) is

$$\frac{d}{dt}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}\alpha_2\\ -\frac{3C}{ML^2}\alpha_2 - \frac{3g}{2L}\sin\alpha_1\end{bmatrix}. \tag{2.16}$$

This is also known as the dynamic system model of the system. Using the notation

$$\alpha = \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}, \quad f(\alpha) = \begin{bmatrix}\alpha_2\\ -\frac{3C}{ML^2}\alpha_2 - \frac{3g}{2L}\sin\alpha_1\end{bmatrix}$$

(2.16) can be written succinctly as

$$\dot{\alpha} = f(\alpha). \tag{2.17}$$

The right hand side of (2.17) is referred to as the vectorfield of the dynamic system. A solution $\alpha(t)$ is said to be an equilibrium solution of the dynamic system (2.17) if it remains a constant for all time. That is $\alpha(t) \equiv \bar{\alpha}$. Substituting this equilibrium solution in (2.17) it can be seen that such equilibria are solutions of $f(\bar{\alpha}) = 0$. For the simple pendulum the equilibria are given by $\bar{\alpha} = [0 \;\; 0]^T$ and $\bar{\alpha} = [\pi \;\; 0]^T$.

In order to investigate the local behavior of solutions near an equilibrium $\bar{\alpha}$ we can consider the linearized system

$$\dot{\delta\alpha} = A(\bar{\alpha})\delta\alpha, \tag{2.18}$$

where

$$A(\bar{\alpha}) = \left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=\bar{\alpha}} = \begin{bmatrix}0 & 1\\ -\frac{3g}{2L}\cos\bar{\alpha} & -\frac{3C}{ML^2}\end{bmatrix}.$$

For the simple pendulum it can be shown that the eigenvalues of $A(\bar{\alpha})$ are in the strict left half complex plane for the equilibrium $\bar{\alpha} = [0 \;\; 0]^T$ and that at least one eigenvalue is in the strict right half complex plane when the equilibrium $\bar{\alpha} = [\pi \;\; 0]^T$. Thus the vertically downward equilibrium $\bar{\alpha} = [0 \;\; 0]^T$ is 'locally' stable while the vertically upward equilibrium $\bar{\alpha} = [\pi \;\; 0]^T$ is unstable.

2.3.1.1 Analytical Solution of the Undamped Simple Pendulum

If the frictional torque is negligible then $C \simeq 0$ and the governing equation reduces to

$$\ddot{\theta} + \omega^2\sin\theta = 0, \tag{2.19}$$

where $\omega^2 = \frac{3g}{2L}$. The vectorfield of the dynamic system model is

$$f(\alpha) = \begin{bmatrix}\alpha_2\\ -\omega^2\sin\alpha_1\end{bmatrix}.$$

The total energy of the system is

$$T(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + 2\omega^2\sin^2\frac{\theta}{2}. \tag{2.20}$$

Differentiating T along the trajectories of the system (2.19) we have that

$$\frac{d}{dt}T = \dot{\theta}\,(\ddot{\theta} + \omega^2\sin\theta) = 0,$$

and hence that $T(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + 2\omega^2\sin^2\frac{\theta}{2} = E$ is a constant along the solutions of (2.19). This fact allows us to find the solutions of (2.19) analytically since we can write

$$\dot{\theta}^2 = 2E - 4\omega^2\sin^2\frac{\theta}{2} \tag{2.21}$$

$$\frac{d}{dt}\theta = \sqrt{2E - 4\omega^2\sin^2\frac{\theta}{2}} \tag{2.22}$$

The last of the above two expressions can be integrated using quadrature as follows:

$$\int \frac{1}{\sqrt{2E - 4\omega^2\sin^2\frac{\theta}{2}}}d\theta = \int dt.$$

Large Total Energy: $2\omega^2 \leq E$

A change of variables $u = \sin\frac{\theta}{2}$ reduces the above integral to

$$\int \frac{1}{\sqrt{(1-u^2)(1-ku^2)}}du = \int \Omega\, dt, \tag{2.23}$$

where $k^2 = 2\omega^2/E$ and $\Omega^2 = E/2$. For total Energy $2\omega^2 \leq E$ it can be seen that $0 \leq k \leq 1$. Then the left hand side is known as a Jacobi Elliptic integral of a first kind. Specifically a sn function of modulus k.

$$\mathrm{sn}^{-1}(u) = \int \frac{1}{\sqrt{(1-u^2)(1-ku^2)}}du. \tag{2.24}$$

Thus (2.23) becomes

$$\mathrm{sn}^{-1}(u) = \Omega t + c,$$

and hence

$$u(t) = \mathrm{sn}(\Omega t + c).$$

Thus the complete analytic solution of the simple pendulum equation is

$$\theta(t) = 2\arcsin u(t) = 2\arcsin\left(\mathrm{sn}(\Omega t + c)\right). \tag{2.25}$$

From (2.22) we also have that

$$\dot{\theta}(t) = 2\Omega\,\mathrm{dn}(\Omega t + c). \tag{2.26}$$

Two interesting limiting behaviors of the pendulum result when E is very large, (that is when $k \to 0$) and when $E = 2\omega^2$ (that is when $k \to 1$).

In the limit $k \to 0$ the integral (2.24) reduces to

$$\lim_{k\to 0} \mathrm{sn}^{-1}(u) = \int \frac{1}{\sqrt{1-u^2}} du = \arcsin u$$

and sn to the trigonometric sin function. It can be shown form properties of Elliptic functions that cn reduces to cos and dn $= 1$.
On the other hand in the limit $k \to 1$ the integral (2.24) reduces to

$$\lim_{k\to 1} \mathrm{sn}^{-1}(u) = \int \frac{1}{1-u^2} du = \frac{1}{2}\ln\frac{1+u}{1-u} = \mathrm{arctanh}(u).$$

It can be shown that in this case cn reduces to sech and dn $=$ sech.
Thus for large total energy E ($>> 2\omega^2$) the motion is given by

$$\theta(t) = 2(\Omega t + c), \tag{2.27}$$
$$\dot{\theta}(t) = 2\Omega. \tag{2.28}$$

Similarly when $E = 2\omega^2$ (ie. $k = 1$) the motion is given by

$$\theta(t) = 2\arcsin\left(\tanh(\omega t + c)\right) = \pi - 4\arctan\left(e^{-(\omega t + c)}\right), \tag{2.29}$$
$$\dot{\theta}(t) = 2\omega\,\mathrm{sech}(\omega t + c). \tag{2.30}$$

Small Total Energy: $E \leq 2\omega^2$

For total energy such that $E \leq 2\omega^2$ from equation (2.22) we have

$$\frac{d}{dt}\theta = 2\omega\sqrt{\frac{E}{2\omega^2} - \sin^2\frac{\theta}{2}}.$$

Now setting $k^2 = \frac{E}{2\omega^2}$ and $\sin\theta/2 = k\sin\psi$ we have

$$\frac{d}{dt}\psi = \omega\sqrt{1 - k^2\sin^2\psi},$$

where $0 \leq k \leq 1$. Now setting $u = \sin\psi$ we have that for small energy $E \leq 2\omega^2$ the solutions of the pendulum is given by the Elliptic integral of the first kind,

$$\int \frac{1}{\sqrt{(1-u^2)(1-ku^2)}} du = \int \omega\, dt. \tag{2.31}$$

Thus as before we have

$$\theta(t) = 2\arcsin ku(t) = 2\arcsin\left(k\,\mathrm{sn}(\omega t + c)\right), \tag{2.32}$$
$$\dot{\theta}(t) = \sqrt{2E}\,\mathrm{cn}(\omega t + c). \tag{2.33}$$

In summary we have shown that all the solutions of the simple pendulum can be completely written down using Jacobi Elliptic functions. For Energy $E \geq 2\omega^2$ the solutions are given by (2.25) and (2.26), while for energy $E \leq 2\omega^2$ the solutions are given by (2.32) and (2.33). Jacobi Elliptic sn and cn functions are periodic functions of period $4K$ where

$$K(k) = \mathrm{sn}^{-1}(1) = \int_0^1 \frac{1}{\sqrt{(1-u^2)(1-k^2u^2)}}\,du = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\psi}}\,d\psi.$$

Thus the motion of the pendulum is periodic of periodicity $T(k) = 4K/\Omega$ for $E \geq 2\omega^2$ and $T(k) = 4K/\omega$ for $E \leq 2\omega^2$. Notice that it is a function of k and that k is a function of the energy. Thus the period of periodicity is a function of energy and hence the initial conditions.

For $k <<< 1$ using the binomial expansion the right hand side of the last integral can be approximated to give,

$$K = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\psi}}\,d\psi = \int_0^{\pi/2}\left(1+\frac{k^2}{4}(1-\cos 2\psi) + \mathrm{H.O.T}\right)d\psi \approx \left(1+\frac{k^2}{4}\right)\frac{\pi}{2}.$$

Thus the period of the solution is

$$T \approx \left(\frac{2\pi}{\omega} + \frac{k^2\pi}{2\omega}\right) = \frac{2\pi}{\omega}\left(1+\frac{E}{8\omega^2}\right).$$

For small Energy $k \approx 0$ and θ is very small and $\sin\theta \approx \theta$. For $k \approx 0$ the Jacobi Elliptic functions approximate the trigonometric functions, $\mathrm{sn} \approx \sin$, $\mathrm{cn} \approx \cos$ and $\mathrm{dn} \approx 1$. Thus for very small Energy (2.32) and (2.33) reduce to

$$\theta(t) \approx \frac{\sqrt{2E}}{\omega}\,\sin(\omega t + c), \tag{2.34}$$
$$\dot{\theta}(t) \approx \sqrt{2E}\,\cos(\omega t + c). \tag{2.35}$$

These correspond to the solutions of the linearized system

$$\ddot{\theta} + \omega^2\theta = 0.$$

The period of the solution is in this case

$$T \approx \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2L}{3g}},$$

since $k \approx 0$.

2.3.2 van der Pol Oscillator

$$\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2 - 1), \tag{2.36}$$

$$\dot{x}_2 = x_1 - x_2(x_1^2 + x_2^2 - 1). \tag{2.37}$$

The state space of the system is $\mathscr{X} = \mathbb{R}^2$. Show that the origin $[0 \ 0]^T$ is a hyperbolic equilibrium and that it is unstable. Then show that the unit circle in the state space $\mathbb{R}^2$ is a periodic orbit. Figure 2.8 shows the phase portrait of the van da Pol oscillator where we have plotted a selected few trajectories.

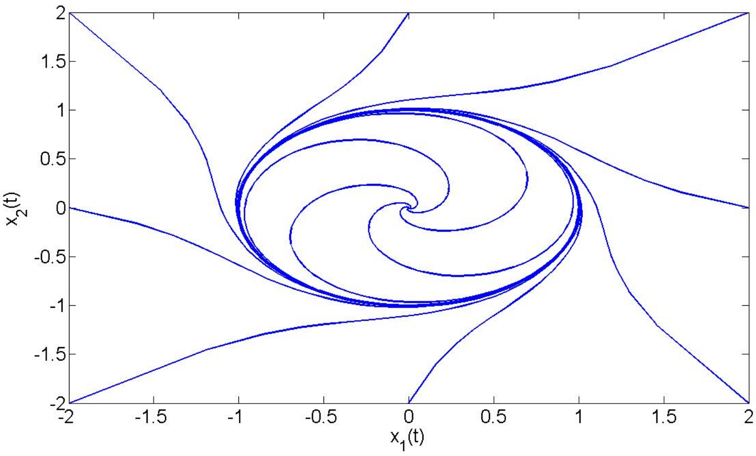


Fig. 2.8 The phase portrait of the van da Pol oscillator.

2.3.3 Double Pendulum

2.3.4 Lorenz Weather Model

$$\dot{x}_1 = \sigma(x_2 - x_1), \tag{2.38}$$

$$\dot{x}_2 = rx_1 - x_2 - x_1x_3, \tag{2.39}$$

$$\dot{x}_3 = x_1x_2 - bx_3. \tag{2.40}$$

The state space of the system is $\mathbb{R}^3$.

Figure 2.9 shows a trajectory of the attractor that begins at $[1 \ 0 \ 0]$. The parameters are chosen such that $\sigma = 10, r = 28, b = 8/3$. c

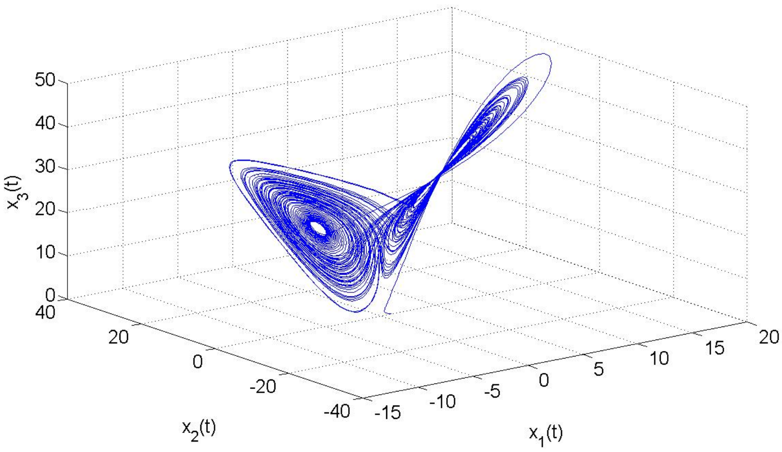


Fig. 2.9 The Lorenz Attractor.

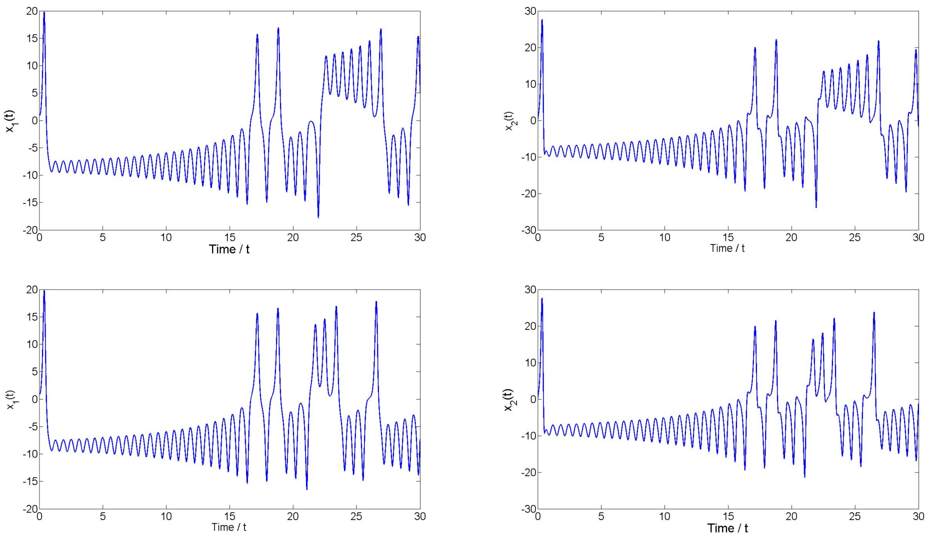


Fig. 2.10 $x_1(t)$ Vs t and $x_2(t)$ Vs t for very close initial conditions. The figures in the top row correspond to an initial condition $[1\ \ 0\ \ 0]$ while the figures in the top row correspond to an initial condition $[1.001\ \ 0\ \ 0]$. The solutions diverge rapidly around twenty seconds.