ST : GATE 2024

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I. 40-52

1) At a single-staff checkout counter of a supermarket store, the time taken in minutes to complete the service of a customer is a random variable having probability density function given by

$$f(x) = \begin{cases} \frac{1}{10} e^{\frac{-x}{10}} & \text{if} x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

When you arrive at the counter, you observe that there is already one person in service. You are also informed that the person has been in service for 5 minutes. Assuming that the service times of different customers are independent of each other, the probability that your waiting time (which is the sum of your waiting time in queue and service time) is more than 15 minutes equals

a) $\frac{5}{2}e^{-\frac{3}{2}}$ b) $\frac{3}{2}e^{-\frac{3}{2}}$ c) $\frac{3}{2}e^{-\frac{5}{2}}$ d) $\frac{5}{2}e^{-\frac{5}{2}}$

2) Let X be a random variable having discrete uniform distribution on $\{1, 3, 5, 7, \ldots, 99\}$. Then $E\left(X|X \text{is not a multip}\right.$ equals

a) $\frac{2365}{47}$ b) $\frac{2365}{50}$ c) 50 d) 47

3) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having $N\left(\mu, \sigma^2)\right)$ distribution, where $\mu \in \mathbb{R}$ and $\sigma > 0$. Consider the following statements:
   I If $\frac{\sqrt{5}}{\sigma(2n+1)} \sum_{i=1}^{n} i(X_i - \mu)$ has $N(0, 1)$ distribution, then $n = 2$.
   II $E\left(\left(-\ln\left(\Phi\left(\frac{X_1-\mu}{\sigma}\right)\right)\right)^3\right) = 6$, where $\Phi(\cdot)$ denotes the cumulative distribution function of an $N(0, 1)$ random variable.

   Which of the above statements is/are true?

a) Only (I) b) Only (II) c) Both (I) and (II) d) Neither (I) nor (II)

4) Let $\{X_n\}_{n\geq1}$ be a sequence of independent and identically distributed random variables with probability density function

$$f(x) = \begin{cases} e^{-(x-\theta)}, & x \geq 0, \theta \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where $\theta > 0$. Consider the following statements:
   I $\frac{1}{n} \sum_{i=1}^{n} X_i$ converges in probability to $\frac{\theta+1}{2}$ as $n \to \infty$.
   II $\lim_{n\to\infty} \mathbb{E}(\min\{X_1, X_2, \ldots, X_n\}) = \theta$.

   Which of the above statements is/are true?

(A) Only (I) (B) Only (II) (C) Both (I) and (II) (D) Neither (I) nor (II)

5) Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables such that X_n has Poisson distribution with mean λ_n, where $\lambda_n = \lambda + \frac{1}{2^n}, n \geq 1$, and $\lambda > 0$ is an unknown parameter. Which one of the following statements is true?
   a) $\frac{1}{n}\sum_{i=1}^{n} X_i$ is an unbiased estimator of λ
   b) $\frac{1}{n}\sum_{i=1}^{n} X_i$ is a consistent estimator of λ
   c) $\sum_{i=1}^{n} X_i$ is a consistent estimator of λ
   d) $\frac{1}{n^2}\sum_{i=1}^{n} X_i$ is an unbiased estimator of λ
6) Let $X_1, X_2, \ldots, X_{25}$ be a random sample of size 25 from a population having $N_3(\mu, \Sigma)$ distribution, where μ and non-singular Σ are unknown parameters. Let $S = \frac{1}{24}\sum_{i=1}^{25}(X_i - \bar{X})(X_i - \bar{X})^T$, where $\bar{X} = \frac{1}{25}\sum_{i=1}^{25} X_j$. and $B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$ Then which one of the following statements is true?
   a) $24BSB^T$ follows a Wishart distribution of order 3 with 24 degrees of freedom.
   b) $24BSB^T$ follows a Wishart distribution of order 2 with 25 degrees of freedom.
   c) $24BSB^T$ follows a Wishart distribution of order 2 with 24 degrees of freedom.
   d) $24BSB^T$ follows a Wishart distribution of order 3 with 25 degrees of freedom.
7) Consider the simple linear regression model
$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, 2, \ldots, n, \tag{3}$$
   where β_0, β_1 are unknown parameters, and the ϵ_i 's are uncorrelated random variables with mean 0 and finite variance $\sigma^2 > 0$. Let $\hat{\beta}_i$ be the least squares estimators of $\beta_i, i = 0, 1$. Consider the following statements:
   (I) A 95% joint confidence region for (β_0, β_1) is the region bounded by an ellipse.
   (II) The expression for the covariance between $\hat{\beta}_0$ and $\hat{\beta}_1$ does not involve σ^2.
   Which of the above statements is/are true?

   a) Only (I) b) Only (II) c) Both (I) and (II) d) Neither (I) nor (II)

8) If $f : [-2, 2] \to \mathbb{R}$ is a continuous function, then which of the following statements is/are true?
   a) $F : [0, 2] \to \mathbb{R}$ defined by $F(x) = \int_0^x f(t)\, dt$ is differentiable on $(0, 2)$.
   b) For any $x_1, x_2, \ldots, x_{10} \in [-2, 2]$, there exists a point $x_0 \in [-2, 2]$ such that
$$f(x_0) = \frac{1}{10}\left(f(x_1) + f(x_2) + \cdots + f(x_{10})\right).$$
   c) f is bounded on $[-2, 2]$.
   d) If f is differentiable at 0 and $f(0) = 0$, then
$$\lim_{x\to 0} \frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + \cdots + f\left(\frac{x}{10}\right)\right) = 10 f'(0),$$
   where f' is the derivative of f at 0.
9) Let A be an $n \times n$ real matrix. Which of the following statements is/are true?
   a) If A is a symmetric matrix such that $A+ \in I_n$ is positive semi-definite for every $\epsilon > 0$, where I_n is the $n \times n$ identity matrix, then A is positive semi-definite
   b) If n is odd, then $A - A^T$ is not inevitable
   c) If A is a symmetric matrix such that the singular values of A are all strictly positive, then A is positive definite
   d) If 1 is the only singular value of , then A is orthogonal
10) Which of the following statements is/are true?

a) If A is a 3×3 real matrix with 3 distinct eigenvalues, then A is diagonizable
b) If A is a 3×3 real matrix such that A^2 is diagonizable, then A is diagonizable.
c) For real numbers a, b, c, d, e, f, if $A = \begin{bmatrix} a & b & c \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$ is diagonizable, then $a = b = c = 0$
d) For real numbers a, b, c, d, e, f if $A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ is diagonalizable, then $AA^T = A^T A$

11) Let $\Omega = \{1, 2, 3, \dots\}$ and $\mathcal{H}$ be the collection of all subsets of Ω. Let P be a probability measure on $\mathcal{H}$ such that $P\{k\} = \frac{1}{2^k}, k \geq 1$. Further, let $X : \Omega \to \mathbb{R}$ be defined by $X(\omega) = \omega$ for all $\omega \in \Omega$. Then which of the following statements is/are true?
a) There exists $k \in \Omega$ such that $P(X = k) < 10^{-6}$
b) $\lim_{n\to\infty} P\left(X \geq 4 + \frac{1}{n}\right) = \frac{1}{16}$
c) $\lim_{n\to\infty} P\left(4 + \frac{1}{n^2} \leq X < 5 - \frac{1}{n}\right) = \frac{1}{16}$
d) If $x_n = 3 + \frac{(-1)^n}{n}, n \geq 1$, then $lim_{n\to\infty} P(X \leq x_n) = \frac{7}{8}$

12) Let (X, Y) be a random vector having joint probability density function given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{3}{4} & if x^2 \leq y \leq 1 \text{and} -1 \leq x \leq 1 \\ 0 & otherwise. \end{cases} \tag{4}$$

a) X has the same distribution as $-X$
b) $E(Y|X = 0) = \frac{1}{2}$
c) The correlation coefficient between X and Y is 0
d) X and Y are independent

13) Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables such that the probability density function of X_n is given by

$$f_n(x) = \begin{cases} \frac{1}{\lambda_n} e^{-\frac{x}{\lambda_n}} & if x \geq 0 \\ 0 & otherwise \end{cases} \tag{5}$$

where $\lambda_n = 10 - \sum_{i=1}^{n} \frac{5}{2^{i-1}}$. Which of the following statements is/are true?
a) $\{X_n\}_{n\geq 1}$ converges in distribution to the zero random variable
b) $\{X_n\}_{n\geq 1}$ converges in probability to the zero random variable
c) $\{X_n\}_{n\geq 1}$ converges in distribution to a random variable having Poisson distribution with mean 10
d) $\{X_n\}_{n\geq 1}$ converges in probability to a random variable having Poisson distribution with mean 10