Assignment-2

(1) $X_1 \ldots X_n | \mu, \sigma^2 \overset{iid}{\sim} N(\mu, \sigma^2)$

$\pi(\mu, \sigma^2) = \pi(\mu | \sigma^2) \cdot \pi(\sigma^2)$

$\sigma^2 \sim IG(a, b) \quad \mu | \sigma^2 \sim N(0, c\sigma^2)$

Marginal posterior distribution $\pi(\mu | X_1 \ldots X_n) = \int_0^\infty L(\mu, \sigma^2 | X_1 \ldots X_n)\, \pi(\mu|\sigma^2) \cdot \pi(\sigma^2)\, d\sigma^2$

$$L(\mu, \sigma^2 | X_1 \ldots X_n) = \frac{1}{(\sqrt{2\pi}\,\sigma)^n} \exp\left(\frac{-1}{2} \frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2}\right)$$

$$\pi(\mu | \sigma^2) = \frac{1}{\sqrt{2\pi c \sigma^2}} \cdot \exp\left(-\frac{1}{2} \frac{\mu^2}{c\sigma^2}\right)$$

$$\pi(\sigma^2) = \frac{b^a}{\Gamma(a)} \cdot (\sigma^2)^{-a-1} \exp\left(\frac{-b}{\sigma^2}\right)$$

$$\pi(\mu | X_1 \ldots X_n) \propto \int_0^\infty \frac{1}{(\sigma^2)^{n/2}} \exp\left(\frac{-1}{2} \frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2}\right) \frac{1}{(\sigma^2)^{1/2}} \exp\left(\frac{-1}{2} \frac{\mu^2}{c\sigma^2}\right) \cdot (\sigma^2)^{-a-1} \exp\left(\frac{-b}{\sigma^2}\right) d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-\frac{n}{2} - \frac{1}{2} - a - 1} \exp\left(-\frac{1}{2}\left[\frac{\sum (X_i - \mu)^2}{\sigma^2} + \frac{\mu^2}{c\sigma^2} + \frac{2b}{\sigma^2}\right]\right) d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-\left(\frac{n+1+2a}{2}\right) - 1} \exp\left[-\frac{1}{\sigma^2}\left(\frac{\sum (X_i - \mu)^2}{2} + \frac{\mu^2}{2c} + b\right)\right] d\sigma^2$$

$$= \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\sqrt{2\pi c}} \cdot \frac{b^a}{\Gamma(a)} \int_0^\infty (\sigma^2)^{-\left(\frac{n+1+2a}{2}\right) - 1} \exp\left[\frac{-1}{\sigma^2}\left(\frac{\sum (X_i - \mu)^2}{2} + \frac{\mu^2}{2c} + b\right)\right] d\sigma^2$$

we give cdf of

$$= \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\sqrt{2\pi c}} \cdot \frac{b^a}{\Gamma(a)} \cdot \frac{\Gamma\left(\frac{n+1+2a}{2}\right)}{B^{\frac{n+1+2a}{2}}} \cancel{\int_0^\infty \frac{B^A}{\Gamma(A)} (\sigma^2)^{-\left(\frac{n+1+2a}{2}\right)-1} \exp\left[-\frac{1}{\sigma^2}\left(\frac{\Sigma(x_i-\mu)^2}{2} + \frac{\mu^2}{2c} + b\right)\right] d\sigma^2} \quad 1$$

$$\pi(\mu \mid x_1 \dots x_n) = \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\sqrt{2\pi c}} \cdot \frac{b^a}{\Gamma(a)} \cdot \frac{\Gamma\left(\frac{n+1+2a}{2}\right)}{\left[\frac{\Sigma(x_i-\mu)^2}{2} + \frac{\mu^2}{2c} + b\right]^{\left(\frac{n+1+2a}{2}\right)}}$$

$$B = \frac{\Sigma(x_i-\mu)^2}{2} + \frac{\mu^2}{2c} + b$$

$$A = \frac{n+1+2a}{2}$$

posterior predictive distribution of X_{n+1}

$$\pi(X_{n+1}|X_1 \dots X_n) = \int \pi(X_{n+1}|\mu,\sigma^2) \cdot p(\mu,\sigma^2|X_1 \dots X_n)\, d(\mu,\sigma^2)$$

$\theta = (\mu,\sigma^2)$. $\quad X_{n+1} = x^* \quad \mathbf{X} = (X_1 \dots X_n)$

$$\pi(x^*|\mathbf{X}) = \int \pi(x^*|\theta)\, p(\theta|\mathbf{X})\, d\theta$$ (posterior predictive distribution)

$$\pi(x^*|\theta) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{1}{2}\frac{(x^*-\mu)^2}{\sigma^2}\right)$$

$$p(\theta|\mathbf{X}) \propto L(\theta|\mathbf{X}) \cdot \pi(\mu,\sigma^2)\,[\pi(\theta)]$$

$$\propto L(\theta|\mathbf{X}) \cdot \pi(\mu|\sigma^2)\cdot \pi(\sigma^2)$$

$$\propto \frac{1}{(\sigma)^n}\exp\left(-\frac{1}{2}\frac{\sum(x_i-\mu)^2}{\sigma^2}\right)\cdot\frac{1}{(c\sigma^2)^{1/2}}\exp\left(-\frac{1}{2}\frac{\mu^2}{c\sigma^2}\right)\cdot(\sigma^2)^{-a-1}\exp\left(-\frac{b}{\sigma^2}\right)$$

$$\propto (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}-a-1}\exp\left[-\frac{1}{\sigma^2}\left(\frac{\sum(x_i-\mu)^2}{2}+\frac{\mu^2}{2c}+b\right)\right]$$

$$p(\theta|\mathbf{X}) \sim IG\left(\frac{n+1+2a}{2},\ \frac{\sum(x_i-\mu)^2}{2}+\frac{\mu^2}{2c}+b\right)$$

$$\pi(x^*|\mathbf{X}) = \int_0^\infty \frac{1}{\sqrt{2\pi}\,\sigma}\cdot\exp\left(-\frac{1}{2}\frac{(x^*-\mu)^2}{\sigma^2}\right)\cdot\frac{1}{(\sqrt{2\pi})^{n+1}\sqrt{c}}\cdot\frac{b^a}{\Gamma(a)}$$
$$(\sigma^2)^{-\left(\frac{n+1+2a}{2}\right)-1}\exp\left\{-\frac{1}{\sigma^2}\left(\frac{\sum(x_i-\mu)^2}{2}+\frac{\mu^2}{2c}+b\right)\right\}d\theta$$

$$= \frac{1}{(\sqrt{2\pi})^{n+2}}\cdot\frac{b^a}{\sqrt{c}\cdot\Gamma(a)}\int_0^\infty(\sigma^2)^{-\left(\frac{n+1+2a}{2}\right)-1-1}$$
$$\exp\left\{-\frac{1}{\sigma^2}\left(\frac{(x^*-\mu)^2}{2}+\frac{\sum(x_i-\mu)^2}{2}+\frac{\mu^2}{2c}+b\right)\right\}d\theta$$

$$= \frac{1}{(\sqrt{2\pi})^{n+2}}\cdot\frac{b^a}{\sqrt{c}\,\Gamma(a)}\int_0^\infty(\sigma^2)^{-\left(\frac{n+3+2a}{2}\right)-1}$$
$$\exp\left\{-\frac{1}{\sigma^2}\left(\frac{(x^*-\mu)^2}{2}+\frac{\sum(x_i-\mu)^2}{2}+\frac{\mu^2}{2c}+b\right)\right\}d\theta$$

(2) Bayesian two sample T-test

$x_1 \ldots x_m \mid \mu_1, \sigma^2 \overset{iid}{\sim} N(\mu_1, \sigma^2)$ $\quad$ $y_1 \ldots y_n \mid \mu_2, \sigma^2 \overset{iid}{\sim} N(\mu_2, \sigma^2)$

$$\mathcal{L}(\mu_1, \sigma^2 \mid \mathbf{X}) = \frac{1}{(\sqrt{2\pi}\,\sigma)^m} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{m}(x_i-\mu_1)^2\right)$$

We Assume $m=n$

$$\mathcal{L}(\mu_2, \sigma^2 \mid \mathbf{Y}) = \frac{1}{(\sqrt{2\pi}\,\sigma)^n} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{\cancel{m}}(y_i-\mu_2)^2\right)$$

$\sigma^2 \sim IG(a,b)$

$\mu_1, \mu_2 \mid \sigma^2 \overset{iid}{\sim} N(0, c\sigma^2)$

Prior : $\pi(\mu_1, \mu_2, \sigma^2) = \pi(\mu_1 \mid \sigma^2) \cdot \pi(\mu_2 \mid \sigma^2) \cdot \pi(\sigma^2)$

$\mu_1 - \mu_2 \mid \sigma^2 \sim N(0, 2c\sigma^2)$

Let $\theta = \mu_1 - \mu_2$, then $\theta \mid \sigma^2 \sim N(0, 2c\sigma^2)$

Posterior : $\pi(\mu_1 - \mu_2 \mid x_1 \ldots x_m, y_1 \ldots y_n) = \int_0^\infty \mathcal{L}(\mu_1; \sigma^2 \mid \mathbf{X}) \cdot \mathcal{L}(\mu_2, \sigma^2 \mid \mathbf{Y})$
$\pi(\theta) \cdot \pi(\sigma^2)\, d\sigma^2$

$$\propto \int_0^\infty \frac{1}{(\sigma^2)^{m/2}} \cdot \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_1)^2\right) \cdot \frac{1}{(\sigma^2)^{n/2}} \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu_2)^2\right) \cdot \frac{1}{\cancel{2}(\sigma^2)^{-1}} \cdot \exp\left(\frac{-1\,\theta^2}{2\,4c\sigma^2}\right)$$
$$(\sigma^2)^{-a-1} \cdot \exp\left(-\frac{b}{\sigma^2}\right) d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-\frac{n}{2}+\frac{n}{2}-\frac{1}{2}-a-1} \exp\left\{\frac{-1}{\sigma^2}\left[\frac{\sum_{i=1}^{n}[(x_i-\mu_1)^2+(y_i-\mu_2)^2]}{2} + \frac{\theta^2}{4c} + b\right]\right\} d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-(n+a+\frac{1}{2})-1} \exp\left\{\frac{-1}{\sigma^2}\left[\frac{\sum_{i=1}^{n}(x_i-\mu_1)^2+(y_i-\mu_2)^2}{2} + \frac{(\mu_1-\mu_2)^2}{4c} + b\right]\right\} d\sigma^2$$

②

$\pi(\mu_1-\mu_2|x_1...x_n, y_1...y_n)$

Let B

$$\propto \int_0^\infty (\sigma^2)^{\underbrace{-(n+a+\frac{1}{2})}_{A}-1} \exp\left(\frac{-1}{\sigma^2}\left[\frac{\Sigma(x_i-\mu_1)^2+(y_i-\mu_2)^2}{2}+\frac{(\mu_1-\mu_2)^2}{4c}+b\right]\right) d\sigma^2$$

$A = n+a+1 \qquad B = \dfrac{\Sigma(x_i-\mu_1)^2+(y_i-\mu_2)^2}{2} + \dfrac{(\mu_1-\mu_2)^2}{4c} + b$

$$\propto \frac{\Gamma(A)}{B^A}\int_0^\infty \underbrace{\frac{B^A}{\Gamma(A)}(\sigma^2)^{-(n+a+\frac{1}{2})-1}\exp\left[-\frac{B}{\sigma^2}\right] d\sigma^2}_{1}$$

$\pi(\mu_1-\mu_2|x_1...x_n, y_1...y_n)$

$$= \frac{1}{(2\pi)^n}\cdot\frac{1}{\sqrt{2\pi c}}\cdot\frac{b^a}{\Gamma(a)}\cdot\frac{\Gamma(n+a+\frac{1}{2})}{\left[\frac{\sum_{i=1}^{n}(x_i-\mu_1)^2+(y_i-\mu_2)^2}{2}+\frac{(\mu_1-\mu_2)^2}{4c}+b\right]^{n+a+\frac{1}{2}}}$$

(3) $X_1 \ldots X_m | \lambda_1 \overset{iid}{\sim}$ Poisson (λ_1)

$Y_1 \ldots Y_n | \lambda_2 \overset{iid}{\sim}$ Poisson (λ_2)

$\lambda_1, \lambda_2 \overset{iid}{\sim}$ Gamma(a,b) [Prior]

Posterior of $\theta = \frac{\lambda_1}{\lambda_1+\lambda_2}$

$P(\theta | X_1 \ldots X_m, Y_1 \ldots Y_n)$

$\propto \mathcal{L}(\theta | X_1 \ldots X_m, Y_1 \ldots Y_n) \cdot \pi(\theta)$

$\pi(\theta) = \pi\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)$

~~As λ_1, λ_2 are gamma distribution of parameters a & b~~

~~then. $\frac{\lambda_1}{\lambda_1+\lambda_2} \sim$ Beta$(a, 2a)$~~

~~$\mathcal{L}(\theta | X_1 \ldots X_m, Y_1 \ldots Y_n)$~~

$P(\lambda_1 | X_1 \ldots X_m) \sim$ Gamma$(a + \Sigma x_i, b + m)$

$p(\lambda_1 | X_1 \ldots X_m) \propto \cancel{e^{-\Sigma x_i \lambda}}\ e^{-m\lambda} \cdot \lambda^{\Sigma x_i} \cdot \lambda^{a-1} \cdot e^{-\lambda b}$

$\propto e^{-\lambda(b+m)} \cdot \lambda^{(\Sigma x_i + a) - 1}$

Similarly, $p(\lambda_2 | Y_1 \ldots Y_n) \sim$ Gamma$(a + \Sigma Y_i, b+n)$

Then,

$$p(\theta | X_1 \ldots X_m, Y_1 \ldots Y_n) = \frac{p(\lambda_1 | X_1 \ldots X_m)}{p(\lambda_1 | X_1 \ldots X_m) + p(\lambda_2 | Y_1 \ldots Y_n)}$$

As λ_1 are independent of $Y_1 \ldots Y_n$
λ_2 are independent of $X_1 \ldots X_m$, we are not include it these posteriors.

⑤

$X, Y \sim MVN(\mu_1 = 0, \mu_2 = 0, \sigma_1 = 1, \sigma_2 = 2, \rho = 0.5)$

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$$

X, Y are not independent

$X = (x^{(0)}, x^{(1)} \ldots x^{(B)})$

$Y = (y^{(0)}, y^{(1)} \ldots y^{(B)})$

Let P, Q be linear combinations of X and Y such that P and Q are independent.

$$P = X + bY, \quad Q = Y$$

~~EfP~~

$E(PQ) = 0 \implies E((X + bY)Y) = 0$

$$E(XY) + bE(Y^2) = 0$$

$$\rho\sigma_1\sigma_2 + b\sigma_2^2 = 0$$

$$\boxed{b = -\frac{\rho\sigma_1}{\sigma_2}}$$

$$Z = \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\rho\sigma_1}{\sigma_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\begin{bmatrix} E(P) \\ E(Q) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\rho\sigma_1}{\sigma_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} \mu_1 - \rho\frac{\sigma_1}{\sigma_2}\mu_2 \\ \mu_2 \end{bmatrix}$$

$$\operatorname{cov}(Z) = \begin{bmatrix} EP^2 & EPQ \\ EPQ & EQ^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 - \rho^2\sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

$EP^2 = E(X^2 + b^2Y^2 + 2bXY)$

$= EX^2 + (EY^2)b^2 + 2bE(XY)$

$= \sigma_1^2 + \sigma_2^2 b^2 + 2b\rho\sigma_1\sigma_2$

$= \sigma_1^2 + \cancel{\sigma_2^2}\left(\frac{\rho^2\sigma_1^2}{\cancel{\sigma_2^2}}\right) + 2\rho\sigma_1\cancel{\sigma_2}\,b$

$= \sigma_1^2 + \rho^2\sigma_1^2 - 2\rho^2\sigma_1^2$

$= \sigma_1^2(1 - \rho^2)$

$\therefore P = X - \frac{\rho\sigma_1}{\sigma_2}Y \sim N\left(\mu_1 - \rho\frac{\sigma_1}{\sigma_2}\mu_2, (1-\rho^2)\sigma_1^2\right)$

$Q = Y \sim N(\mu_2, \sigma_2^2)$

$f_{P,Q}(p,q) = f_{X,Y}(x,y)\ |J| \qquad |J| = 1$

$f_{P,Q}(p,q) = f_P(p) \cdot f_Q(q)$ [As they are independent]

$$f_{P|Q}(p|q) = \frac{f_P(p) \cdot \cancel{f_Q(q)}}{\cancel{f_Q(q)}} = f_P(p)$$

$$P|Q = q \approx P \sim N\left(\mu_1 - \rho \frac{\sigma_1}{\sigma_2}\mu_2,\ (1-\rho^2)\sigma_1^2\right)$$

$$X + bY | Y = y \sim N\left(\mu_1 - \rho\frac{\sigma_1}{\sigma_2}\mu_2,\ (1-\rho^2)\sigma_1^2\right)$$

$$X | Y = y^{(b-1)} \sim N\left(\mu_1 - \rho\frac{\sigma_1}{\sigma_2}\left(y^{(b-1)} - \mu_2\right),\ (1-\rho^2)\sigma_1^2\right)$$

similarly,

$$Y | X = x^{(b)} \sim N\left(\mu_2 - \rho\frac{\sigma_2}{\sigma_1}\left(x^{(b)} - \mu_1\right),\ (1-\rho^2)\sigma_2^2\right)$$

(For this, $P = Y + bX$, $Q = X$ they can give it.)

(4) $X_1 \sim N(0,1)$ $\quad X_{t+1} | X_t \sim N(\rho X_t, 1-\rho^2)$ $\quad t = 1, 2 \dots T$

Likelihood $= f(X_T, X_{T-1}, \dots x_1 | \rho)$

$$= f(X_T | X_{T-1} \dots x_1, \rho) \cdot f(X_{T-1}, X_{T-2} \dots x_1 | \rho)$$

$$= \prod_{t=2}^{T} f(X_t | X_{t-1}) \cdot f(X_1)$$

$$= \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \cdot \exp\left(\frac{-1}{2} \frac{(X_t - \rho X_{t-1})^2}{1-\rho^2}\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2} X_1^2\right)$$

$$= \frac{1}{\left[\sqrt{2\pi(1-\rho^2)}\right]^{T-1}} \exp\left(\frac{-1}{2} \frac{\sum_{t=2}^{T} (X_t - \rho X_{t-1})^2}{1-\rho^2}\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2} X_1^2\right)$$

$$= \frac{1}{\sqrt{2\pi}\left[\sqrt{2\pi(1-\rho^2)}\right]^{T-1}} \exp\left\{\frac{-1}{2}\left[\frac{\sum_{t=2}^{T} (X_t - \rho X_{t-1})^2}{1-\rho^2} + X_1^2\right]\right\}$$

prior : $\rho \sim$ Uniform $(-1, 1)$ $\qquad \pi(\rho) = \frac{1}{1-(-1)} \cdot \mathbb{I}_{[\rho \in (-1,1)]}$

$$\pi(\rho) = \frac{1}{2} \mathbb{I}_{\rho \in (-1,1)}$$

Posterior distribution:-

$$P(\rho | x_1 \dots X_T) \propto f(X_T \dots x_1 | \rho) \cdot \pi(\rho)$$

$$\propto (1-\rho^2)^{-\frac{(T-1)}{2}} \exp\left\{\frac{-1}{2}\left[\frac{\sum_{t=2}^{T} (X_t - \rho X_{t-1})^2}{1-\rho^2} + X_1^2\right]\right\} \frac{1}{2} \mathbb{I}_{\rho \in (-1,1)}$$

$$P(\rho | X_1 \dots X_T) \propto (1-\rho^2)^{-\frac{(T-1)}{2}} \exp\left\{\frac{-1}{2}\left[\frac{\sum_{t=2}^{T} (X_t - \rho X_{t-1})^2}{1-\rho^2} + X_1^2\right]\right\} \mathbb{I}_{\rho \in (-1,1)}$$

So, we propose u = Uniform $(-1, 1)$ as proposal density.

when $u^* M <= P(\tilde{\rho} | X_1 \dots X_T)$ then we accept the ρ^*.

M = max of $P(\rho | X_1 \dots X_T)$

PPD $\pi(x_{T+1}, x_{T+2} | x_1 \dots x_T) = \int_{-\infty}^{\infty} \pi(x_{T+1}, x_{T+2} | \rho) \, P(\rho | x_1 \dots x_T) \, d\rho$

RW

$$\pi(x_{T+1}, x_{T+2} | \rho) = \pi(x_{T+2} | \rho, x_{T+1}) \cdot \pi(x_{T+1} | \rho)$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(\frac{-1}{2} \frac{(x_{T+2} - \rho x_{T+1})^2}{1-\rho^2}\right) \cdot \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left[\frac{-1}{2} \frac{(x_{T+1} - \rho x_T)^2}{1-\rho^2}\right]$$

$$\propto (1-\rho^2)^{-1} \cdot \exp\left(\frac{-1}{2(1-\rho^2)} \left[(x_{T+2} - \rho x_{T+1})^2 + (x_{T+1} - \rho x_T)^2\right]\right)$$

$$P(\rho | x_1 \dots x_T) \propto (1-\rho^2)^{\frac{T-2}{2}} \exp\left(\frac{-1}{2(1-\rho^2)} \left[\sum (x_t - \rho x_{t-1})^2 + (1-\rho^2) x_1^2\right]\right)$$

$$\pi(x_{T+1}, x_{T+2} | x_1 \dots x_T) \propto \int_{-\infty}^{\infty} (1-\rho^2)^{\left(\frac{T-2}{2}\right)-1} \exp\left[\frac{-1}{2(1-\rho^2)} \left[\sum_{t=2}^{T+2} (x_t - \rho x_{t-1})^2 + (1-\rho^2) x_1^2\right]\right] d\rho$$

$$\propto \int_{-1}^{1} (1-\rho^2)^{\frac{T-2}{2}-1} \exp\left(\frac{-1}{2(1-\rho^2)} \left(\sum_{t=2}^{T+2} (x_t - \rho x_{t-1})^2 + (1-\rho^2) x_1^2\right)\right] d\rho$$

$$\propto \exp\left(-\frac{x_1^2}{2}\right) \int_{-1}^{1} (1-\rho^2)^{\left(\frac{T-2}{2}\right)-1} \exp\left(\frac{-1}{(1-\rho^2)} \sum_{t=2}^{T+2} (x_t - \rho x_{t-1})^2\right) d\rho$$