Assignment-3

② $y_i|\mu \overset{ind}{\sim} N(\mu, \sigma_i^2) \quad \forall\ i=1, \ldots n$

$\pi(\mu) = 1$

a) $\hat{\mu} = \arg\max \log p(\mu | y_1 \ldots y_n, \sigma_1^2, \ldots \sigma_n^2)$

$= \arg\max [\log f(y_1 \ldots y_n | \mu, \sigma_1^2 \ldots \sigma_n^2) + \log \pi(\mu)]$

$= \arg\max [\sum_{i=1}^{n} \log f(y_i|\mu)]$

$= \arg\max \left[\sum_{i=1}^{n} \log \left(\frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(y_i-\mu)^2}{2\sigma_i^2}\right)\right) \right]$

$\hat{\mu} = \arg\max_{\mu} \left[\sum_{i=1}^{n} \log \left(\frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(y_i-\mu)^2}{2\sigma_i^2}\right)\right) \right]$

b) $\hat{\mu} = \arg\max_{\mu} \left[\sum_{i=1}^{n} \log \left(\frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(\frac{-(y_i-\mu)^2}{2\sigma_i^2}\right)\right) \right]$

$$\mathcal{L} = \sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2\sigma_i^2}$$

$$\frac{d\mathcal{L}}{d\mu} = \sum_{i=1}^{n} \frac{2(y_i-\mu)(-1)}{2\sigma_i^2} \overset{set}{=} 0$$

$y_1 = 12,\ y_2 = 10,\ y_3 = 22 \quad , \sigma_1 = 3, \sigma_2 = 3 \quad \sigma_3 = 10 \quad n=3$

$$\sum_{i=1}^{3} \frac{y_i}{\sigma_i^2} - \mu \sum_{i=1}^{3} \frac{1}{\sigma_i^2} = 0$$

$$\mu = \frac{\sum_{i=1}^{3} (y_i/\sigma_i^2)}{\sum_{i=1}^{3} (1/\sigma_i^2)}$$

$$\hat{\mu}_{MAP} = 11.473.$$

c) $E(\mu) = \int \mu \, p(\mu | y_1 \ldots y_n) \, d\mu$

$$p(\mu | y_1 \ldots y_n) \propto \prod_{i=1}^{n} f(y_i|\mu) \cdot \pi(\mu)$$

$$\propto \exp\left[-\sum_{i=1}^{n} \frac{1}{2\sigma_i^2}(y_i - \mu)^2\right]$$

$$\propto \exp\left[-\sum_{i=1}^{n}\left(\frac{n}{2\sigma_i^2}\right)(\mu - \bar{y})^2\right]$$

$$\mu | y_1 \ldots y_n \sim N\left(\bar{y}, \frac{\sum_{i=1}^{n}(1/\sigma_i^2)}{n}\right)$$

(3) $y_i | \sigma_i^2 \overset{ind}{\sim} N(0, \sigma_i^2) \quad i = 1, 2 \ldots n$

$\sigma_i^2 | b \sim \text{Inv Gamma}(a, b)$

$b \sim \text{Gamma}(1,1)$

~~Poster~~ FC of σ_i^2 :-

$$p(\sigma_1^2 | \sigma_2^2 \ldots \sigma_n^2, b, y_1 \ldots y_n) = p(\sigma_1^2 | y_1, b)$$

$$p(\sigma_1^2 | y_1, b) \propto \mathcal{L}(\sigma_1^2 | y_1) \cdot \pi(\sigma_1^2 | b)$$

$$\propto (\sigma_1^2)^{-1/2} \cdot \exp\left(-\frac{y_1^2}{2\sigma_1^2}\right) \cdot (\sigma_1^2)^{-a-1} \exp\left(\frac{-b}{\sigma_1^2}\right)$$

$$\propto (\sigma_1^2)^{-(a+\frac{1}{2})-1} \exp\left[-\frac{1}{\sigma_1^2}\left(\frac{y_1^2}{2} + b\right)\right]$$

$$\sigma_1^2 | y_1, b \sim \text{Inv Gamma}\left(a + \frac{1}{2}, \frac{y_1^2}{2} + b\right)$$

FC of b :-

$$p(b | \sigma_1^2 \ldots \sigma_n^2, y_1 \ldots y_n) = p(b | \sigma_1^2 \ldots \sigma_n^2)$$

$$p(b | \sigma_1^2 \ldots \sigma_n^2) \propto \left[\prod_{i=1}^{n} \pi(\sigma_i^2 | b)\right] \cdot \pi(b)$$

$$\propto \exp\left(-b\left[\sum_{i=1}^{n} \frac{1}{\sigma_i^2} + 1\right]\right)$$

b) $\sigma_1^2 \ldots \sigma_n^2 \sim \text{Exp}\left(\left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\right) + 1\right)$

(4)

c) $y_i | \theta_i \sim \text{Binomial}(n_i, \theta_i) \quad \forall\ i = 1 \ldots 10$

$\theta_i | m \sim \text{Beta}(e^m q_i, e^m(1-q_i))$

$m \sim N(0, 10)$

$$P(\theta_1 | \theta_2 \ldots \theta_n, y_1 \ldots y_{10}, m) = P(\theta_1 | y_1, m)$$

$$P(\theta_1 | y_1, m) \propto \mathcal{L}(\theta_1 | y_1) \cdot \pi(\theta_1 | m)$$

$$\propto \theta_1^{y_1} (1-\theta_1)^{n_1 - y_1} \theta_1^{e^m q_1 - 1} (1-\theta_1)^{e^m(1-q_1) - 1}$$

$$\propto \theta_1^{(y_1 + e^m q_1) - 1} (1-\theta_1)^{[n_1 - y_1 + e^m(1-q_1)] - 1}$$

$$\theta_1 | y_1, m) \sim \text{Beta}\left(y_1 + e^m q_1,\ n_1 - y_1 + e^m(1-q_1)\right)$$