# MTH422A

Charitha

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# Assignment - 1

## Question - 1

$X_1, ..X_N \sim bernolli(\theta)$ and $Y_N = \sum X_i \sim Binomal(N, \theta)$

$\bar{Y_N} = N^{-1}Y_N$ have a distribution whose moment generating function is $\{\theta e^{t/N} + (1-\theta)\}^N$.

Support = $\{0, \frac{1}{N}, \frac{2}{N}, ....1\}$

```
# Define the true distribution of Y
true_cdf <- function(y) {
  pnorm(y, mean = 0, sd = 1)
}

# Function to calculate CDF of sample mean based on CLT
clt_cdf <- function(y, N) {
  pnorm(y, mean = 0, sd = 1/sqrt(N))
}

# Function to compute Kolmogorov-Smirnov distance
ks_distance <- function(cdf1, cdf2, y_values) {
  max(abs(cdf1(y_values) - cdf2(y_values)))
}

# Define the values of N
Ns <- c(10, 50, 500)

# Calculate and print KS distances for each N
for (N in Ns) {
  # Generate N random samples from standard normal distribution
  Y_samples <- rnorm(N, mean = 0, sd = 1)

  # Compute sample mean
  Y_mean <- mean(Y_samples)

  # Calculate KS distance
  ks_dist <- ks_distance(true_cdf, function(y) clt_cdf(y, N), Y_mean)

  cat("KS distance for N =", N, ":", ks_dist, "\n")
}
```

```
## KS distance for N = 10 : 0.2320841
## KS distance for N = 50 : 0.2589532
## KS distance for N = 500 : 0.04522503
```

## Question - 2

Given $X_i \sim \text{Gamma}(\alpha, \lambda)$ where $\alpha$ is shape parameter and $\lambda$ is scale parameter.\ By using the characteristic function, we found that the distribution of $\bar{X}_N = N^{-1} \sum_{n=1}^{N} X_n \text{ follows Gamma}(\alpha N, \frac{\lambda}{N})$

## Question - 3

Given $X$ is random variable with density function $f_X(x) = 2K_0(2\sqrt{x})$. $K_0$ is Bessel function.

To check if $f_X(x)$ is a valid pdf or not.

- $f_X(x) \geq 0$
- $\int f_X(x) = 1$

```
fx<- function(x)
{
  a <- 2*besselK(2*sqrt(x),nu=0)
  return(a)
}

## integrate the function from 0 to inf as function is even, we took 2 times the bessel function
ans <- integrate(fx,lower=0,upper=Inf)
ans
```

```
## 1 with absolute error < 8.1e-06
```

$f_X(x)$ is a valid pdf as $\int f_X(x) = 1$.

A hierarchical representation of $X$ is $X|Y \sim \text{Exp}(1/y)$ and $Y \sim \text{Inverse Gamma}(2, 1)$.

## Question - 4

Given $X_1 \sim N(0, 1)$ and $X_t|X_{t-1} \sim N(\rho X_{t-1}, 1 - \rho^2)$ for all $t = 2, 3, \ldots, T$.

- The joint distribution of $\mathbf{X} = (X_1, \ldots\ldots, X_T)$

$$f_{\mathbf{X}}(x_1, \ldots, x_T) = f(x_1)\Pi_{t=2}^{T} f(x_t|x_{t-1}) \qquad = \frac{1}{2\pi(1-\rho^2)^{1/2}} exp\{\frac{1}{2(1-\rho^2)} \sum_{t=2}^{T} (x_t - \rho x_{t-1})^2 - \frac{1}{2} x_1^2\}$$

- Conditional distribution of $X_t$ given $X_1, X_2, \ldots, X_{t-1}, X_{t+1}, \ldots, X_T$

$$f(x_t|x_1, ..x_{t-1}, x_{t+1}, ..x_T) = f(x_t|x_{t-1}) \qquad = \frac{1}{2\pi(1-\rho^2)^{1/2}} exp\{\frac{1}{2(1-\rho^2)} (x_t - \rho x_{t-1})^2\}$$

## Question - 5

$\mathbf{X} \sim MVD(0, \Sigma)$, also individually $X_i$ follows uni-variate normal distribution.

Thus any linear combination of $X_i$ is also normal distribution.

[$X_i|Z \sim N(Z, 1 - \rho) \text{ and } Z \sim N(0, \rho)$]

## Question - 6

Given $Y_1, \ldots Y_T|X \sim Poisson(X)$ and $X \sim Gamma(a, b)$.\ By Bayes theorem, the conditional distribution of $X$ given $Y_1, \ldots\ldots, Y_T$ follows Gamma distribution of shape parameter $\sum y_i + a$ and scale parameter $b + T$.

$X|Y_1, \ldots Y_T \sim \text{Gamma}(\sum y_i + a, b + T)$

# Question - 7

We will be drawing $10^4$ IID samples from the uniform distribution within the curvature $(x^2 + y^2 - 1)^3 \leq x^2 y^3$ using acceptance-rejection sampling.

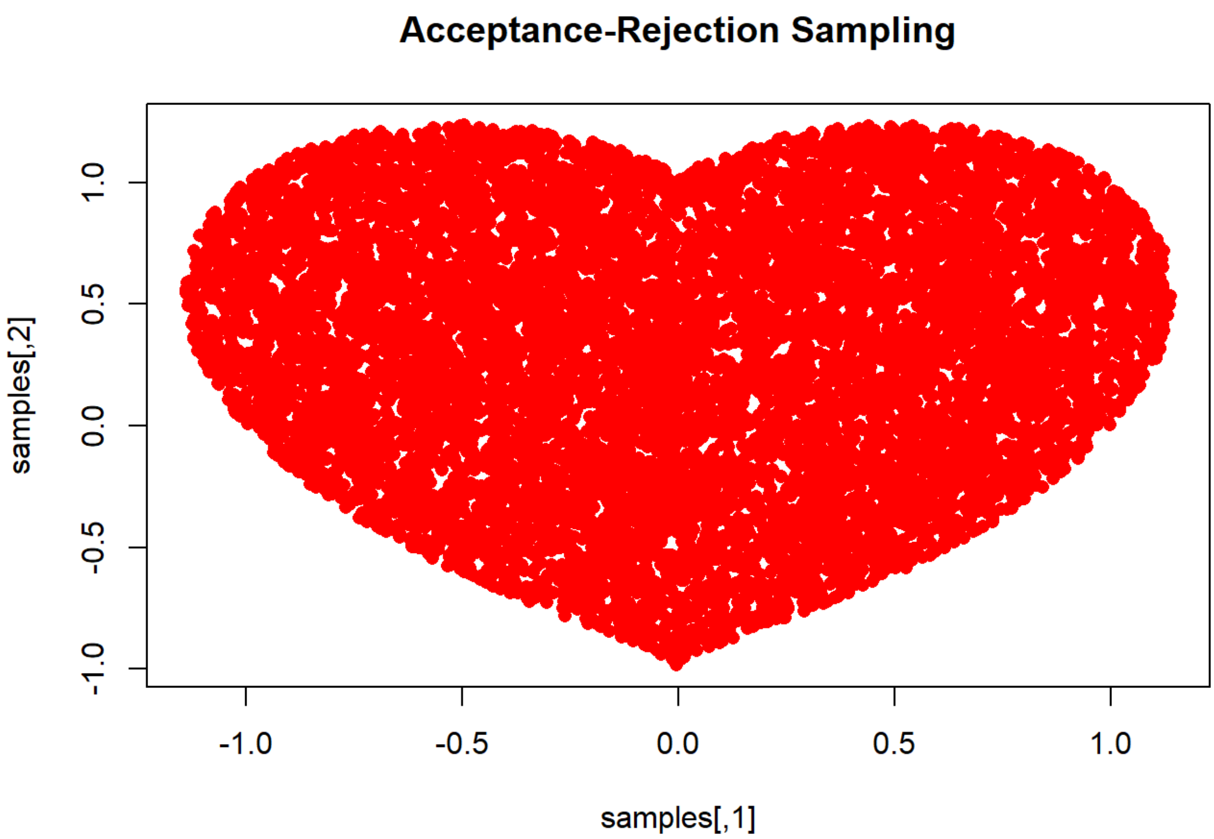
```
# function to check if the given point(x,y) is satisfy this condition
curvature <- function(x,y)
{
  return((x^2+y^2-1)^3 <= x^2 * y^3)
}

# generate samples
generate <- function(n)
{
  samples <- matrix(NA, ncol = 2, nrow = n)
  count <- 0
  # in place of (-1,1) we are taking (-2,2) box as it cover the overall graph.
  while (count < n) {
    x <- runif(n=1,min=-2,max=2)
    y <- runif(n=1,min=-2,max=2)

    if (curvature(x, y)) {
      count <- count + 1
      samples[count, ] <- c(x, y)
    }
  }
  return(samples)
}

samples <- generate(1e4)

plot(samples,pch=16,main="Acceptance-Rejection Sampling",col="red")
```




To check the time required to draw these samples

```
library(tictoc)
# Start the timer
tic()

# Your code goes here
Sys.sleep(1)

# Stop the timer and print elapsed time
toc()
```

```
## 1.01 sec elapsed
```