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MTH422-A

Assignment-1

(6) Given $z_t | Y_t \overset{ind}{\sim} N(Y_t, 1)$ $\quad t = 1 \dots T$

$Y_1 \dots Y_T | X \overset{iid}{\sim} \text{Poisson}(X)$

$X \sim \text{Gamma}(a, b)$

The conditional distribution

$$P(X | Y_1 \dots Y_T) \propto P(Y_1 \dots Y_T | X) \cdot P(X)$$

$$\propto \frac{e^{-Tx} \cdot x^{\Sigma y_i}}{(\Sigma y_i)!} \cdot x^{a-1} \cdot e^{-bx}$$

$$\propto x^{(\Sigma y_i + a) - 1} e^{-(b+T)x}$$

$\therefore X | Y_1 \dots Y_T \sim \text{Gamma}(\Sigma y_i + a, b + T)$

(4) Given $X_1 \sim N(0,1)$ & $X_t | X_{t-1} \sim N(\rho X_{t-1}, 1-\rho^2)$

$t = 2, 3 \dots T$

Joint distribution of $(X_1, X_2 \dots X_T) = \mathbf{X}$ is

$$f_{X_1, \dots X_T}(x_1, x_2 \dots x_T) = \underbrace{f(x_T | x_1 \dots x_{T-1})}_{1} \underbrace{f(x_1 \dots x_{T-1})}_{2}$$

$$= f(x_T | x_{T-1}) f(x_{T-1} | x_1, x_2 \dots x_{T-2}) f(x_1 \dots x_{T-2})$$

$$= \vdots$$

$$= \prod_{t=2}^{T} f(x_t | x_{t-1}) \cdot f(x_1)$$

$$f(x_t|x_{t-1}) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(\frac{-1}{2(1-\rho^2)}(x_t - \rho x_{t-1})^2\right)$$

$$f(x_1) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2} x_1^2\right)$$

$$f_X(x_1 \dots x_T) = \prod_{t=2}^{T} f(x_t|x_{t-1}) \cdot f(x_1)$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left[\frac{-1}{2(1-\rho^2)} \sum_{t=2}^{T} (x_t - \rho x_{t-1})^2 - \frac{1}{2} x_1^2\right]$$

Conditional distribution of X_t given $X_1 \dots X_{t-1}, X_{t+1} \dots X_T$

$t = 2, 3 \dots T-1$

$$f(x_t| x_1 \dots x_{t-1}, x_{t+1} \dots x_T) = f(x_t|x_{t-1})$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left[\frac{-1}{2(1-\rho^2)} (x_t - \rho x_{t-1})^2\right]$$

2) $X_i \overset{iid}{\sim} \text{Gamma}(\alpha, \lambda)$

$$f(x_i) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} \cdot e^{-\lambda x_i} \qquad \bar{X}_n = N^{-1} \sum_{n=1}^{N} X_n$$

By characteristic function, $\phi_{x_i}(t) = (1 - i\lambda t)^{-\alpha}$

$$Y = \sum_{n=1}^{N} X_n = X_1 + \dots X_n$$

$$\phi_X(t) = \prod_{i=1}^{N} \phi_{x_i}(t) = \prod_{i=1}^{N} (1 - i\lambda t)^{-\alpha} = (1 - i\lambda t)^{-\alpha N}$$

$$\phi_{\bar{X}_n}(t) = \phi_Y(t/N) = \left(1 - i\frac{\lambda t}{N}\right)^{-\alpha N} = \left[1 - i\left(\frac{\lambda}{N}\right)t\right]^{-\alpha N}$$

$\therefore$ The distribution of $\bar{X}_n$ is Gamma $(\alpha N, \frac{\lambda}{N})$

here λ = scale parameter , α = shape parameter.

(3) $f_X(x) = 2\, K_0(2\sqrt{x})$ $\qquad K_0(2x) = \frac{1}{2}\int_0^\infty y^{-1} \exp(-y - x^2/y)\, dy$

To check $f_X(x)$ is a valid pdf or not.

Check $\int_0^\infty f_X(x)\, dx = 1$

$$\int_0^\infty 2\, K_0(2\sqrt{x})\, dx \Rightarrow \int_0^\infty \cancel{2} \cdot \frac{1}{\cancel{2}} \int_0^\infty y^{-1} \exp\left(-y - \frac{(\sqrt{x})^2}{y}\right) dy\, dx$$

$$\int_0^\infty \int_0^\infty y^{-1} \exp(-y - x/y)\, dy\, dx$$

$$\int_0^\infty y^{-1} \exp(-y) \left[\int_0^\infty \exp(-x/y)\, dx\right] dy$$

$$\int_0^\infty y^{-1} \exp(-y) \left[\exp(-x/y)(-y)\right]_0^\infty dy$$

$$\int_0^\infty \cancel{y}^{-1} \exp(-y)(-\cancel{y})[0-1]\, dy$$

$$\int_0^\infty \exp(-y)\, dy = \left[-\exp(-y)\right]_0^\infty = -[0-1] = 1$$

Hierarchical representation of X

i.e $X|Y \sim F$, $Y \sim G$

$f(x|y) \propto y^{-1} e^{-y} \cdot e^{-x/y}$

$\propto \underbrace{\frac{1}{y} e^{-y}}_{\text{constant}} \cdot e^{-x/y}$

As y is fixed value

$f(x|y) \propto e^{-1/y \cdot x}$

$X | Y = y \sim \text{EXP}[\frac{1}{y}]$

$f(y) \propto y^{-1} \cdot \exp(-y)$

$y^{-2+1} \cdot \exp(\cancel{\ldots})$

~~Assume x is fixed value~~

$f(y) \propto y^{-2+1} \exp(-y)$

$Y \sim$ Inverse Gamma$(2, 1)$

(5) $\underset{\sim}{X} \sim N(0, \Sigma)$

$$\Sigma = \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \vdots \\ \vdots & & 1 & \rho \\ \rho & \cdots & \rho & 1 \end{bmatrix}$$

$E(x_i) = 0$

$Var(X_i) = 1$

$Cov(x_i, x_j) = \rho \quad (i \neq j)$

$X_i = Z + \epsilon_i$ (model each univariate component)

$Z \sim N(0, \rho)$

$\epsilon_i \sim N(0, 1-\rho)$

$E(X_i) = E(Z) + E(\epsilon_i)$

$= 0 + 0$

$E(X_i) = 0$

$Cov(X_i, X_j) = Cov(Z + \epsilon_i, Z + \epsilon_j)$

$= V(Z) + Cov(\epsilon_i, \epsilon_j)$

$= \rho + 0 = \rho$

$Var(X_i) = \rho + 1 - \rho = 1$

$\therefore$ X_i follows an univariate normal distribution.

For any vector $\underline{b}$,

$$\underline{b}^T X = \sum_{i=1}^{N} b_i X_i = \sum_i b_i (Z + \epsilon_i)$$

$$= \sum_i b_i Z + \sum_i b_i \epsilon_i$$

$$= Z \sum_i b_i + \sum_i b_i \epsilon_i$$

which is also a sum of Z independent normal distn

∴ Any linear combination of X follows univariate normal.

(1) $X_i \overset{iid}{\sim}$ Bernoulli (θ)

$$Y_N = \sum_{n=1}^{N} X_n$$

Sum of independent Bernoulli is Binomial

$$Y_N \sim \text{Binomial}(N, \theta)$$

$\bar{Y}_N = N^{-1} Y_N$ → Support of $\bar{Y}_N = \{0, \frac{1}{N}, \frac{2}{N} \cdots 1\}$.

$$f_{\bar{Y}_N}(y) = {}^N C_y \, \theta^y (1-\theta)^{N-y} \qquad y \in \{0, \tfrac{1}{N}, \tfrac{2}{N} \cdots 1\}$$

$$M_{\bar{Y}_n}(t) = E_{\bar{Y}_n}(e^{ty}) = \sum_{y=0}^{1} e^{ty} \cdot {}^N C_y \, \theta^y (1-\theta)^{N-y}$$

$$= \{\theta e^{t/N} + (1-\theta)\}^N$$

Central limit theorem

$X_1 \ldots X_2 \sim \text{Bernolli}(\theta)$

$E(x_i) = \theta$

$\text{Var}(x_i) = E(x_i^2) - (Ex_i)^2$

$= \theta - \theta^2 = \theta(1-\theta)$