MTH422_Assignment-2

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Assignment - 2

1

Given $X_1, \ldots, X_n|\mu, \sigma^2 \sim N(\mu, \sigma^2)$ are i.i.d

- *Likelihood*

  $\mathcal{L}(\mu, \sigma^2|X_1, \ldots, X_n) \propto (\sigma^2)^{-n}\exp\{-\frac{\sum_{i=1}^n (X_i-\mu)^2}{2\sigma^2}\}$

- *prior*

  $\pi(\mu, \sigma^2) = \pi(\mu|\sigma^2)\pi(\sigma^2)$ \

  $\sigma^2 \sim \text{Inverse Gamma}(a, b)$ and $\mu|\sigma^2 \sim N(0, c\sigma^2)$ for large c and small a,b.

  - $\pi(\mu|\sigma^2) = \frac{1}{\sqrt{2\pi c\sigma^2}}\exp(-\frac{\mu^2}{2c\sigma^2})$
  - $\pi(\sigma^2) = \frac{b^a}{\Gamma(a)}(\sigma^2)^{-a-1}\exp(-\frac{b}{\sigma^2})$

- *Posterior distribution* $\pi(\mu|X_1, \ldots, X_n)$

$$\pi(\mu|X_1, \ldots, X_n) \propto \int_0^\infty \mathcal{L}(\mu, \sigma^2|X_1, \ldots, X_n) \text{ x } \pi(\mu, \sigma^2)\mathrm{d}\sigma^2$$
$$\propto \int_0^\infty (\sigma^2)^{-\frac{n+2a+1}{2}-1}\exp\{-\frac{1}{\sigma^2}(\frac{\sum_{i=1}^n (X_i-\mu)^2)}{2} + \frac{\mu^2}{2c} + b\}\mathrm{d}\sigma^2$$

On integrating,we give the final answer of Posterior distribution $\pi(\mu|X_1, \ldots, X_n)$

$$\pi(\mu|X_1,\ldots,X_n) = \frac{b^a}{(\sqrt{2\pi})^n\sqrt(2\pi c)\Gamma(a)}\frac{\Gamma(A)}{B^A}$$

here

- B = $\frac{\sum(X_i-\mu)^2}{2} + \frac{\mu^2}{2c} + b$
- A = $\frac{n+2a+1}{2}$

- *Posterior predictive distribution of X_{n+1} i.e $\pi(X_{n+1}|X_1,\ldots,X_n)$*

  Let $X_{n+1} = X^*$ and $\theta = (\mu, \sigma^2)$

$$\pi(X^*|\mathbf{X}) = \int f(X^*|\theta)p(\theta|\mathbf{X})d\theta$$

$$f(X^*|\theta) = \frac{1}{\sqrt{2\pi}\sigma}\exp\frac{-(X^*-\mu)^2}{2\sigma^2}$$

$$p(\theta|\mathbf{X}) \propto (\sigma^2)^{-(\frac{n+2a+1}{2})-1}\exp\{-\frac{1}{\sigma^2}(\frac{\sum(X_i-\mu)^2}{2} + \frac{\mu^2}{2c} + b)\}$$

Posterior predictive distribution

$$\begin{aligned}\pi(X^*|\mathbf{X}) &= \int_0^\infty f(X^*|\theta)p(\theta|\mathbf{X})d\theta \\ &\propto \int_0^\infty \frac{1}{\sigma^2}\exp\frac{-(X^*-\mu)^2}{2\sigma^2}(\sigma^2)^{-(\frac{n+2a+1}{2})-1}\exp\{-\frac{1}{\sigma^2}(\frac{\sum(X_i-\mu)^2}{2} + \frac{\mu^2}{2c} + b)\}d\theta \\ &\propto \int_0^\infty (\sigma^2)^{-\frac{n+2a+3}{2}-1}\exp\{-\frac{1}{\sigma^2}(\frac{(X^*-\mu)^2}{2} + \frac{\sum(X_i-\mu)^2}{2} + \frac{\mu^2}{2c} + b)\}d\theta\end{aligned}$$

2

Bayesian two sample test

- *Likelihood*

  $X_1,\ldots X_m|\mu_1,\sigma^2 \sim N(\mu_1,\sigma^2)$ and $Y_1,\ldots Y_n|\mu_2,\sigma^2 \sim N(\mu_2,\sigma^2)$

- *prior*

$\pi(\mu_1, \mu_2, \sigma^2) = \pi(\mu_1|\sigma^2)\pi(\mu_2|\sigma^2)\pi(\sigma^2)$

- *Posterior distribution of* $\mu_1 - \mu_2$

  Let $\theta = \mu_1 - \mu_2$ then distribution of θ will be Normal with mean = 0 and variance = $2c\sigma^2$

$$\pi(\theta|X_1, \ldots. X_n, Y_1, \ldots Y_n) = \int_0^\infty \mathcal{L}(\mu_1, \sigma^2|X_1, \ldots X_n)\mathcal{L}(\mu_2, \sigma^2|Y_1, \ldots Y_n)\pi(\theta)\pi(\sigma^2)d\sigma^2$$

On calculating, we give

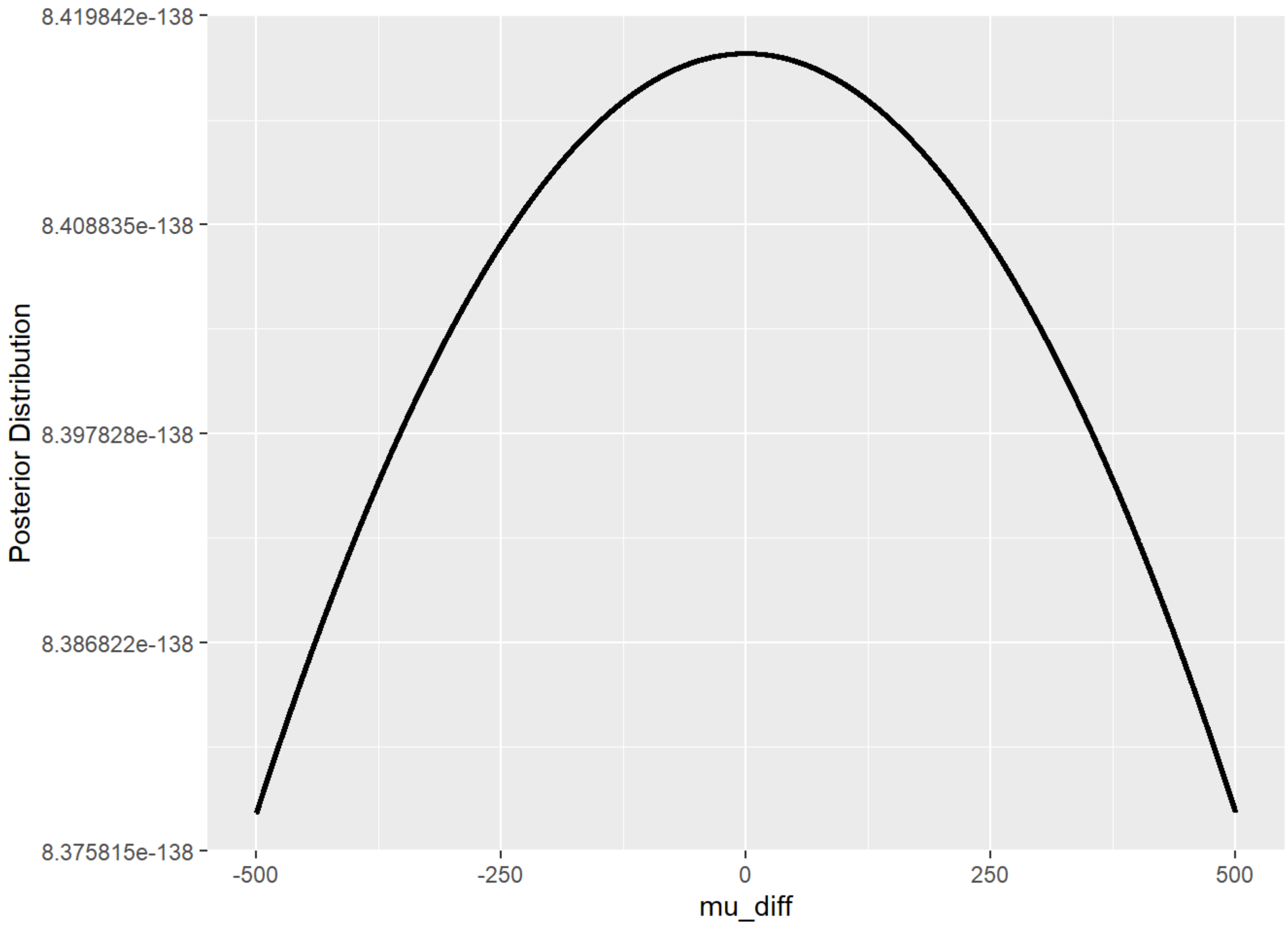
$$\pi(\theta|X_1, \ldots. X_n, Y_1, \ldots Y_n) = \frac{1}{(2\pi)^n}\frac{1}{\sqrt(2\pi c)}\frac{b^a}{\Gamma(a)}\frac{\Gamma(A)}{B^A}$$

Here

- B = $\frac{\sum[(X_i-\mu_1)^2+(Y_i-\mu_2)^2]}{2} + \frac{\theta^2}{4c} + b$
- A = $n + a + \frac{1}{2}$

```r
set.seed(123)
# given parameters
mu1_true <- 1
mu2_true <- 1.5
sigma_true <- 2
m <- 25
n <- 30

# Assumed Prior parameters
c <- 1e4
a <- 0.01
b <- 0.01

# Generate data
data_x <- rnorm(1e4,mu1_true,sigma_true)
data_y <- rnorm(1e4,mu2_true,sigma_true)

# gamma function
gamma <- function(alpha)
{
  ans <- integrate(function(x){x^(alpha-1)*exp(-x)},0,Inf)$value
  return(ans)
}

# calculated posterior for mu1-mu2
posterior <- function(mu_diff)
{
  B <- sum((data_x-mu1_true)^2+(data_y-mu2_true)^2)/2 + (mu_diff)^2/(4*c) + b
  A <- n+a+0.5
  den <- ((2*pi)^n)*(sqrt(2*pi*c))*(gamma(a))*(B^A)
  ans <- (b^a * gamma(A))/den
}

mu_diff_values <- seq(-500,500,length.out=1e4)
posterior_values <- sapply(mu_diff_values,posterior)

data <- data.frame(mu_diff_values,posterior_values)
library(ggplot2)
```

```
ggplot(data, aes(x = mu_diff_values,y=posterior_values)) +
  geom_line(size=1,color="black") +
  labs(x = "mu_diff",
       y = "Posterior Distribution")
```



3

- *Likelihood*

$X_1, \ldots, X_m|\lambda_1 \sim \text{Poisson}(\lambda_1)$ and $Y_1, \ldots, Y_n|\lambda_2 \sim \text{Poisson}(\lambda_2)$

- *Prior*

$\lambda_1, \lambda_2 \sim$ Gamma(a, b)

- *Posterior distribution of* $\theta = \frac{\lambda_1}{\lambda_1+\lambda_2}$
    - *Posterior of* λ_1

$$\pi(\lambda_1|X_1,\dots,X_m) \propto \mathcal{L}(\lambda_1|X_1,\dots,X_m)\pi(\lambda_1)$$

$$\pi(\lambda_1|X_1,\dots,X_m) \sim \text{Gamma}(a+\sum_i^m X_i, b+m)$$

- *Posterior of* λ_2

$$\pi(\lambda_2|Y_1,\dots,Y_n) \propto \mathcal{L}(\lambda_2|Y_1,\dots,Y_n)\pi(\lambda_1)$$

$$\pi(\lambda_2|Y_1,\dots,Y_n) \sim \text{Gamma}(a+\sum_i^n Y_i, b+n)$$

- *Posterior of* θ

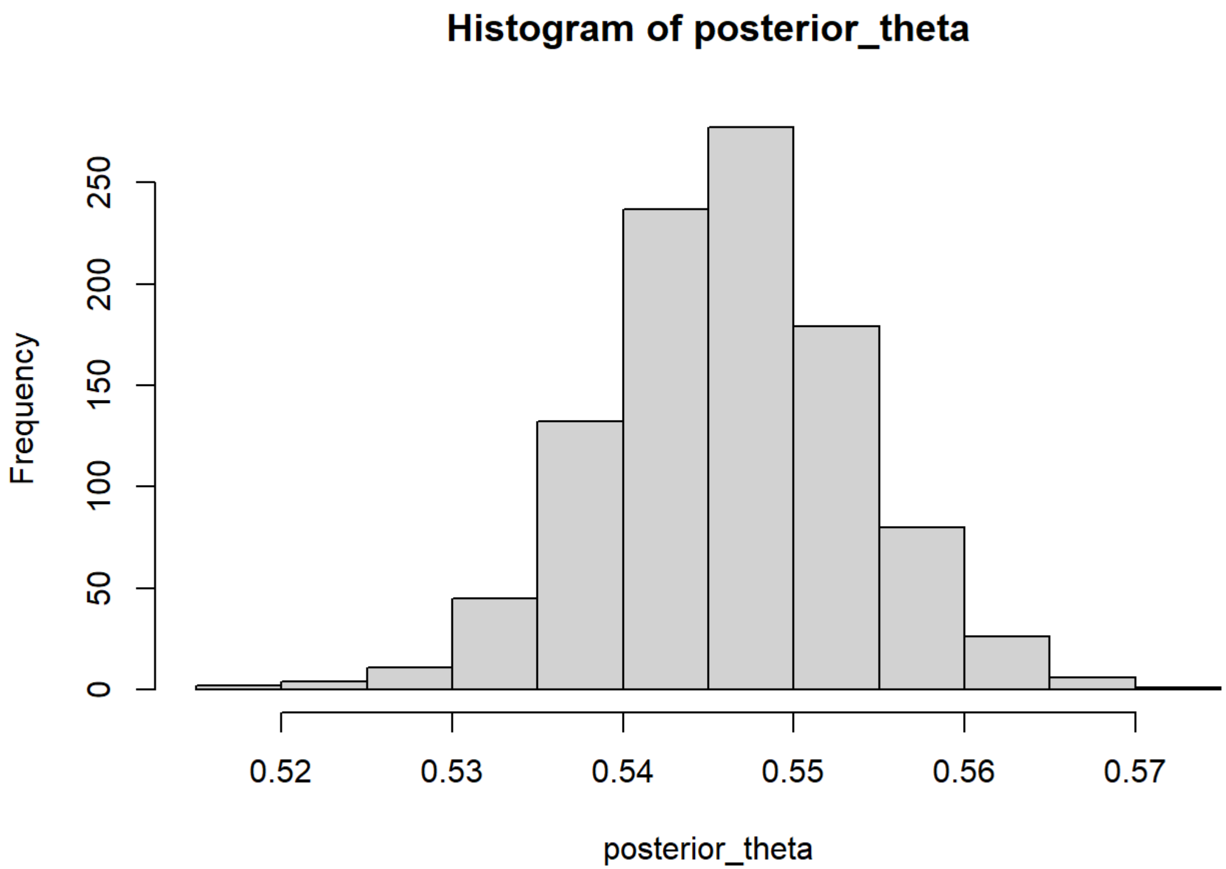
$$\pi(\theta|X_1,\dots,X_m,Y_1,\dots,Y_n) = \frac{\pi(\lambda_1|X_1,\dots,X_m)}{\pi(\lambda_1|X_1,\dots,X_m)+\pi(\lambda_2|Y_1,\dots,Y_n)}$$

$$\pi(\theta|X_1,\dots,X_m,Y_1,\dots,Y_n) \propto \frac{\text{Gamma}(a+\sum_i^m X_i, b+m)}{\text{Gamma}(a+\sum_i^m X_i, b+m)+\text{Gamma}(a+\sum_i^n Y_i, b+n)}$$

```r
set.seed(123)
# given parameters
lambda1 <- 2
lambda2 <- 2.5
m <- 10
n <- 15

# assumptions
a <- 0.1
b <- 0.1

x <- rpois(1e3,lambda1)
y <- rpois(1e3,lambda2)

posterior_lambda1 <- rgamma(1e3,shape=a+sum(x),rate=b+m)
posterior_lambda2 <- rgamma(1e3,shape=a+sum(y),rate=b+n)

posterior_theta <- posterior_lambda1/(posterior_lambda1+posterior_lambda2)

# summary of theta
summary(posterior_theta)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  0.5189  0.5411  0.5463  0.5461  0.5509  0.5726
```

```r
# histogram of posterior theta
hist(posterior_theta)
```




```
# 95% HPD credible interval of theta
library(MCMCpack)
```

```
## Loading required package: coda
```

```
## Loading required package: MASS
```

```
## ##
## ## Markov Chain Monte Carlo Package (MCMCpack)
```

```
## ## Copyright (C) 2003-2024 Andrew D. Martin, Kevin M. Quinn, and Jong Hee Park
```

```
## ##
## ## Support provided by the U.S. National Science Foundation
```

```
## ## (Grants SES-0350646 and SES-0350613)
## ##
```

```
hpd_interval <- HPDinterval(as.mcmc(posterior_theta), prob = 0.95)
hpd_interval
```

```
##          lower    upper
## var1 0.5312918 0.560722
## attr(,"Probability")
## [1] 0.95
```

```
cat("95% HPD credible interval of theta is ",hpd_interval)
```

```
## 95% HPD credible interval of theta is  0.5312918 0.560722
```

```
# hypothesis testing
mean(posterior_lambda1 == posterior_lambda2)
```

```
## [1] 0
```

- 95% HPD credible interval of θ is $(0.5312918, 0.560722)$
- In hypothesis testing,

  $H_0 : \lambda_1 = \lambda_2 \text{ v/s } H_A : \lambda_1 \neq \lambda_2$

  As the mean when posterior of λ_1 and λ_2 equal is 0, so H_0 is rejected. [probability $\propto$ acceptance of H_0]

4

Given $X_1 \sim N(0,1)$ and $X_{t+1}|X_t \sim N(\rho X_t, 1-\rho^2)$

- *Likelihood*

$$\begin{aligned}\mathcal{L}(\rho|X_1,.....,X_T) &= \prod_{t=2}^{T} f(X_t|X_{t-1})f(X_1) \\ &\propto (1-\rho^2)^{-\frac{T-1}{2}}\exp\{\frac{-1}{2}[\frac{\sum(X_t-\rho X_{t-1})^2}{1-\rho^2}+X_1^2]\}\end{aligned}$$

- *Prior*

  $\rho \sim Uniform(-1,1)$

  $\pi(\rho) = \frac{1}{2}\mathrm{I}_{\rho\in(-1,1)}$

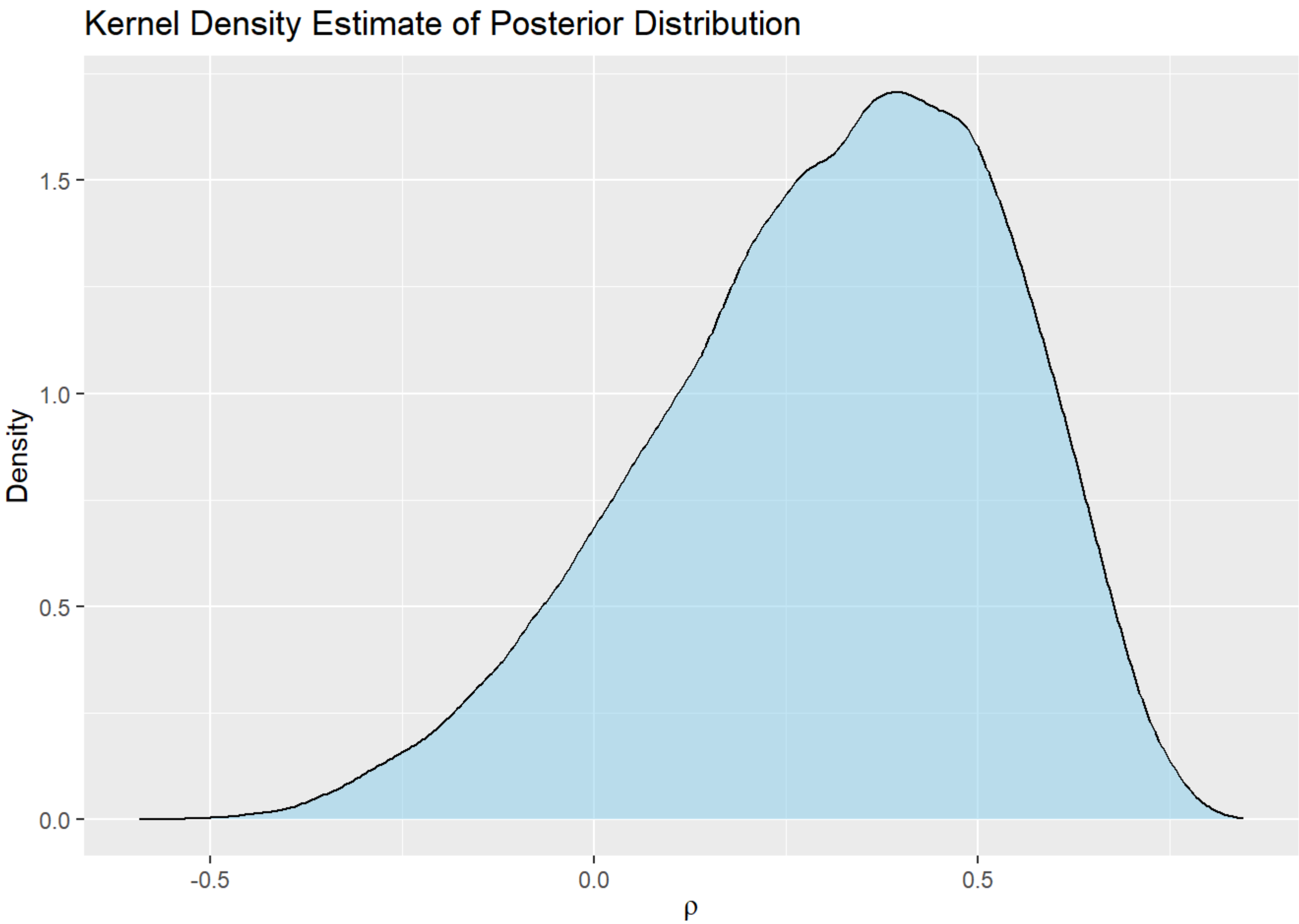
- *Posterior distribution*

$$\begin{aligned}P(\rho|X_1,....,X_T) &\propto \mathcal{L}(\rho|X_1,.....,X_T) \text{ x } \pi(\rho) \\ &\propto (1-\rho^2)^{-\frac{T-1}{2}}\exp\{\frac{-1}{2}[\frac{\sum(X_t-\rho X_{t-1})^2}{1-\rho^2}+X_1^2]\}\mathrm{I}_{\rho\in(-1,1)}\end{aligned}$$

So, we propose $u \sim uniform(-1,1)$ as proposal density, when $u * M \leq P(\rho^*|X_1,....,X_T)$ then we accept ρ^*. Here M = maximum value of $P(\rho|X_1,....,X_T)$

```r
library(mvtnorm)
# Data
T <- 10
rho <- 0.5
X <- numeric(T)
X[1] <- rnorm(1, 0, 1)
for (t in 2:T) {
  X[t] <- rnorm(1, rho * X[t - 1], sqrt(1 - rho^2))
}

# Function to calculate the likelihood
likelihood <- function(rho, X) {
  prod(dnorm(X[2:T], mean = rho * X[1:(T-1)], sd = sqrt(1 - rho^2)))
}

# Function to calculate the prior
prior <- function(rho) {
  dunif(rho, min = -1, max = 1)
}

# Function to calculate the unnormalized posterior
unnormalized_posterior <- function(rho, X) {
  likelihood(rho, X) * prior(rho)
}

# Parameters for acceptance-rejection sampling
M <- max(sapply(seq(-1,1,length.out=1000),function(rho){likelihood(rho,X)*prior(rho)}))
N <- 1e5  # Number of samples to draw

# Acceptance-rejection sampling
rho_samples <- numeric(N)
counter <- 1
while (counter <= N) {
  rho_proposal <- runif(1, min = -1, max = 1)
  u <- runif(1)
  if (u * M <= unnormalized_posterior(rho_proposal, X)) {
    rho_samples[counter] <- rho_proposal
    counter <- counter + 1
```

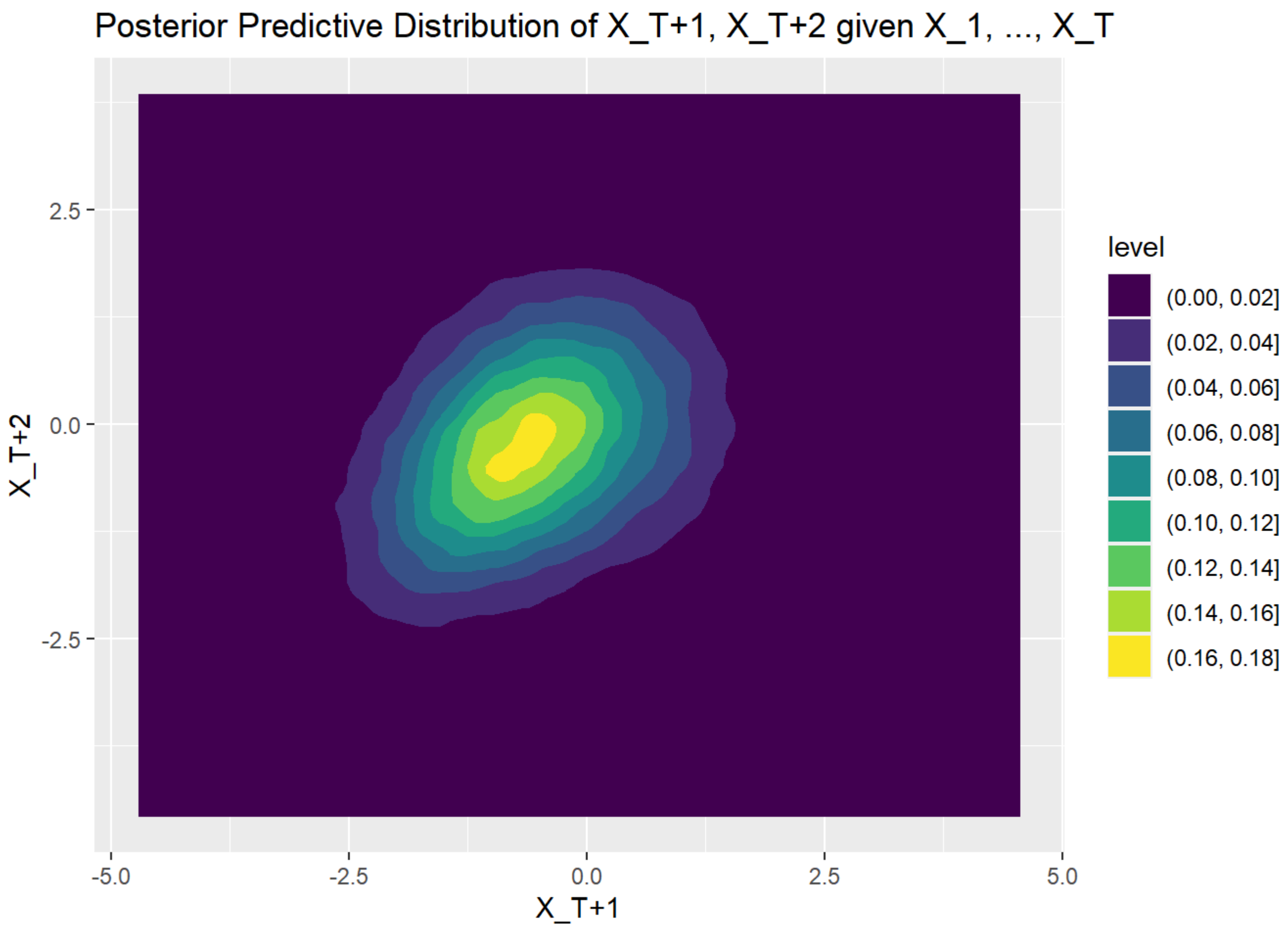
```
  }
}

# Plot kernel density estimate of the posterior distribution
library(ggplot2)
ggplot(data.frame(rho_samples), aes(x = rho_samples)) +
  geom_density(fill = "skyblue", alpha = 0.5) +
  labs(title = "Kernel Density Estimate of Posterior Distribution",
       x = expression(rho),
       y = "Density")
```



- *Joint posterior predictive density of* $\pi(X_{T+1}, X_{T+2}|X_1, ..., X_T)$

$$\pi(X_{T+1}, X_{T+2}|X_1, \ldots., X_T) = \int_{-\infty}^{\infty} \pi(X_{T+1}, X_{T+2}|\rho) P(\rho|X_1, \ldots., X_T) d\rho$$

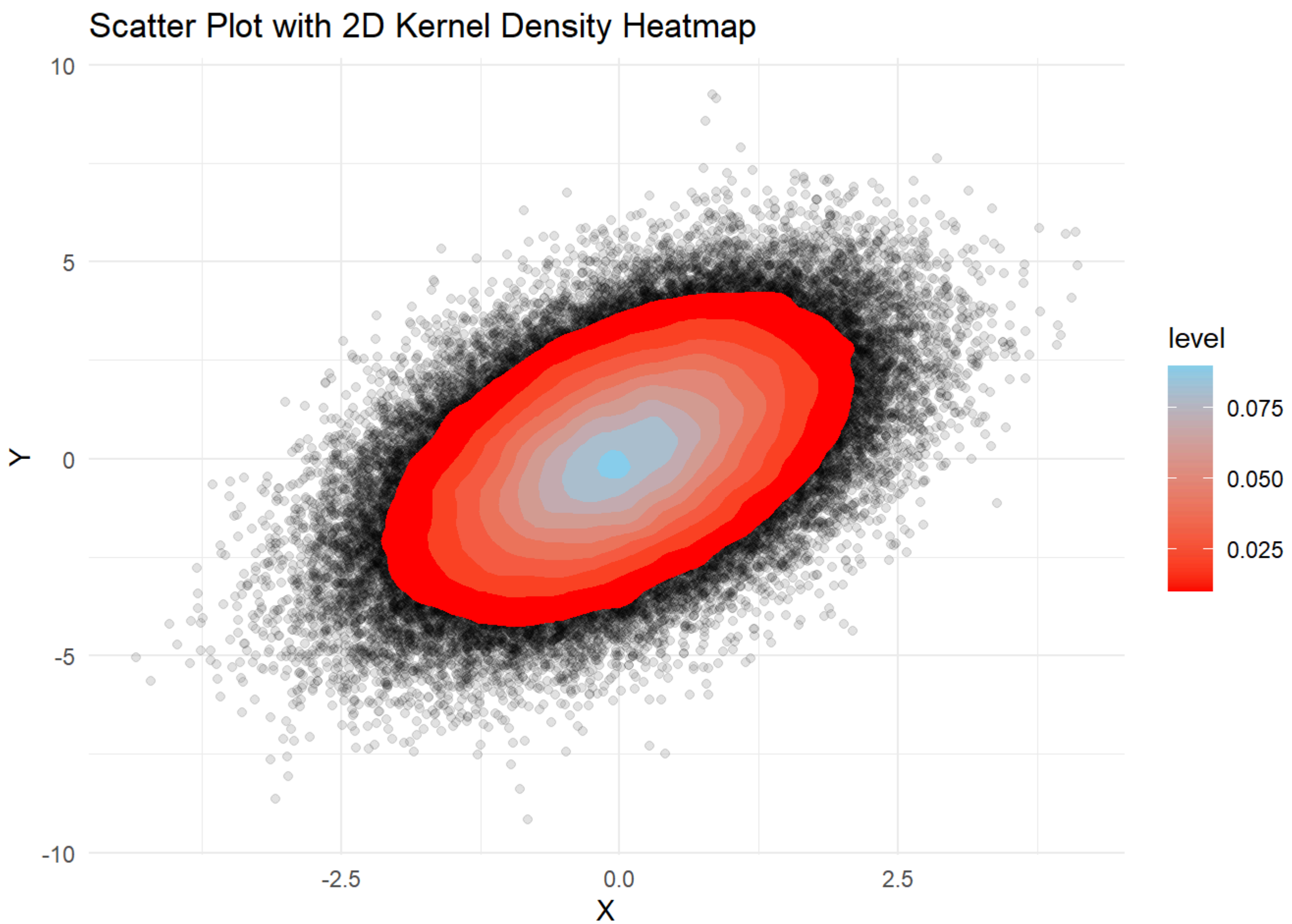
```
#drawing from posterior predictive distribution
ppd_X = matrix(NA, nrow = N, ncol = 2)

ppd_X[,1] = rnorm(N, rho_samples*X[T], sqrt(1 - rho_samples^2))
ppd_X[,2] = rnorm(N, rho_samples*ppd_X[,1], sqrt(1 - rho_samples^2))

#2D density plot of PPD
ggplot(as.data.frame(ppd_X), aes(V1, V2))+
  stat_density_2d_filled(aes(fill = after_stat(level)))+
  scale_color_viridis_c()+
  labs(x = 'X_T+1',
       y = 'X_T+2',
       title = 'Posterior Predictive Distribution of X_T+1, X_T+2 given X_1, ..., X_T')
```



5

- To draw samples from random vector (X, Y) following the bivariate normal distribution with means $\mu_1 = 0, \mu_2 = 0$ and standard deviations $\sigma_1 = 1, \sigma_2 = 2$ and correlation $\rho = 0.5$.

```r
library(mvtnorm)
library(ggplot2)

# given parameters
B <- 1e5
mu <- c(0,0)
sigma <- c(1,2)
rho <- 0.5

# generating samples
mat_sigma <- matrix(c(sigma[1]^2,rho*sigma[1]*sigma[2],rho*sigma[1]*sigma[2],sigma[2]^2),nrow=2,ncol=2)
samples <- rmvnorm(B,mean = mu,sigma = mat_sigma)

df <- data.frame(X = samples[,1],Y = samples[,2])

#  scatter plot with 2D-kernel density heatmap
p <- ggplot(df, aes(x = X, y = Y)) +
  geom_point(alpha = 0.1) + # Scatter plot
  stat_density_2d(aes(fill = ..level..), geom = "polygon") + # 2D kernel density heatmap
  scale_fill_gradient(low = "red", high = "skyblue") + # Color gradient
  theme_minimal() + # Minimal theme
  labs(title = "Scatter Plot with 2D Kernel Density Heatmap", x = "X", y = "Y")

p
```



- **Sampling by Markov fashion**

  $\{(X^{(b)}, Y^{(b)}), b = 1, 2...., B\}$

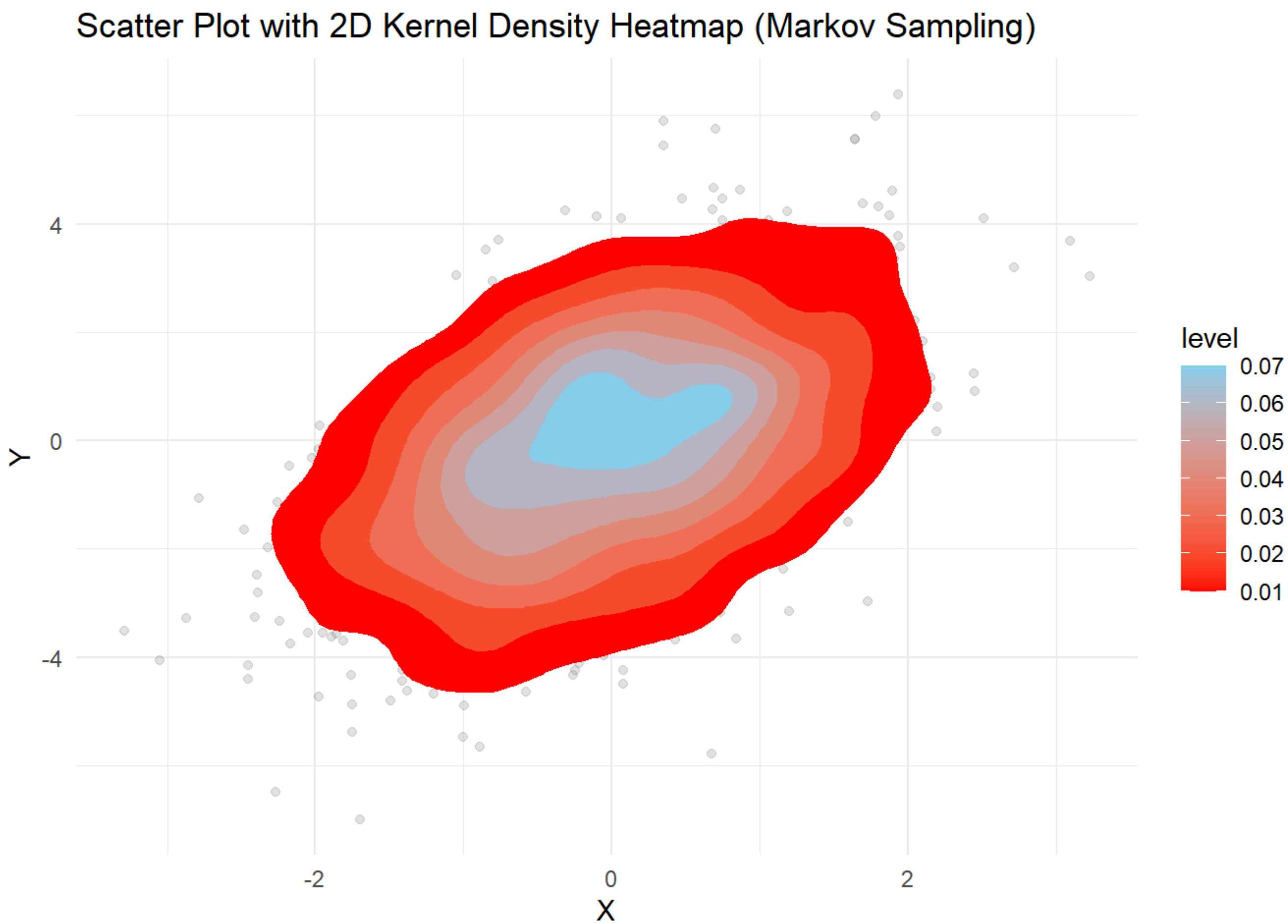
  Let initial samples be $X^{(0)} = 0, Y^{(0)} = 0$

  - $X^{(b)}$ from conditional distribution of X given $Y = Y^{(b-1)}$ then $X|Y = Y^{(b-1)}$ will follows Normal distribution with mean = $\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(Y^{(b-1)} - \mu_2)$ and variance = $(1-\rho^2)\sigma_1^2$.
  - $Y^{(b)}$ from conditional distribution of Y given $X = X^{(b)}$ then $Y|X = X^{(b)}$ will follows Normal distribution with mean = $\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(X^{(b)} - \mu_1)$ and variance = $(1-\rho^2)\sigma_2^2$.

```r
library(mvtnorm)
library(ggplot2)

# given parameters
B <- 1e5
B0 <- 1e3
mu <- c(0,0)
sigma <- c(1,2)
rho <- 0.5

# given initial samples
x0 <- 0
y0 <- 0
initial_sample <- c(x0,y0)

# making matrix to store samples
markov_samples <- matrix(NA,nrow=B+B0,ncol=2)

markov_samples[1,] <- initial_sample

# generate samples in Markov fashion
for(b in 2:(B+B0))
{
  # x(b) value
  markov_samples[b,1] <- rnorm(1,mean = mu[1] + (rho*sigma[1]/sigma[2])*(markov_samples[b-1,2] - mu[2]),sd = sqrt((1 - rho^
2) * sigma[1]^2))
  # y(b) value
  markov_samples[b,2] <- rnorm(1,mean = mu[2] + (rho*sigma[2]/sigma[1])*(markov_samples[b,1] - mu[1]),sd = sqrt((1 - rho^2)
* sigma[2]^2))
}

# first B0 samples
samples <- markov_samples[(B0+1):(B+B0), ]
head(samples)
```

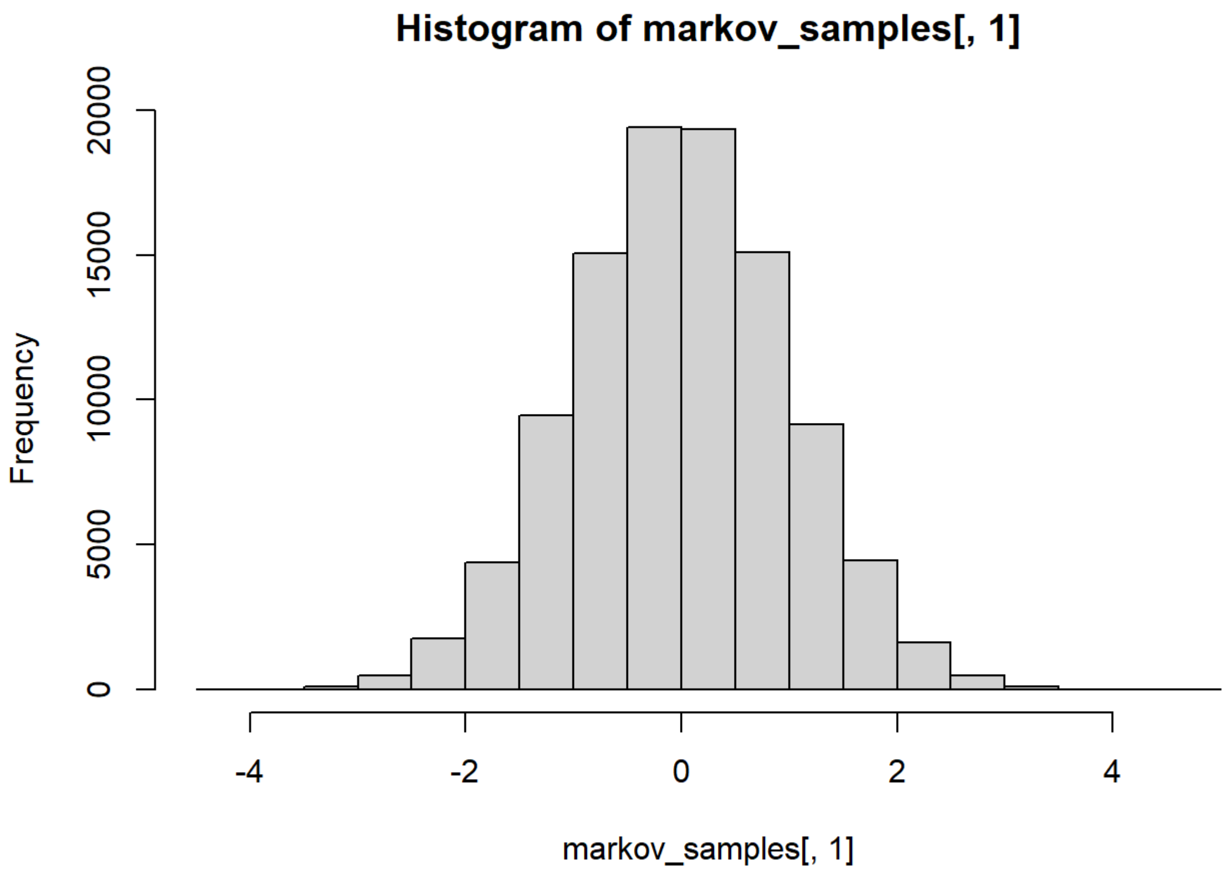
```
##             [,1]        [,2]
## [1,] -0.64061204 -2.16681738
## [2,] -0.06970878 -0.34266029
## [3,]  1.17757481 -0.03238815
## [4,]  2.96495855  0.67733749
## [5,] -0.52888067  1.67334922
## [6,]  0.78738299 -0.05346266
```

```
df_markov <- data.frame(X = markov_samples[-((B0+1):(B+B0)),1], Y = markov_samples[-((B0+1):(B+B0)),2])

# scatter plot with 2D kernel density heatmap
pm <- ggplot(df_markov, aes(x = X, y = Y)) +
  geom_point(alpha = 0.1) + # Scatter plot
  stat_density_2d(aes(fill = ..level..), geom = "polygon") + # 2D kernel density heatmap
  scale_fill_gradient(low = "red", high = "skyblue") + # Color gradient
  theme_minimal() + # Minimal theme
  labs(title = "Scatter Plot with 2D Kernel Density Heatmap (Markov Sampling)", x = "X", y = "Y")

pm
```

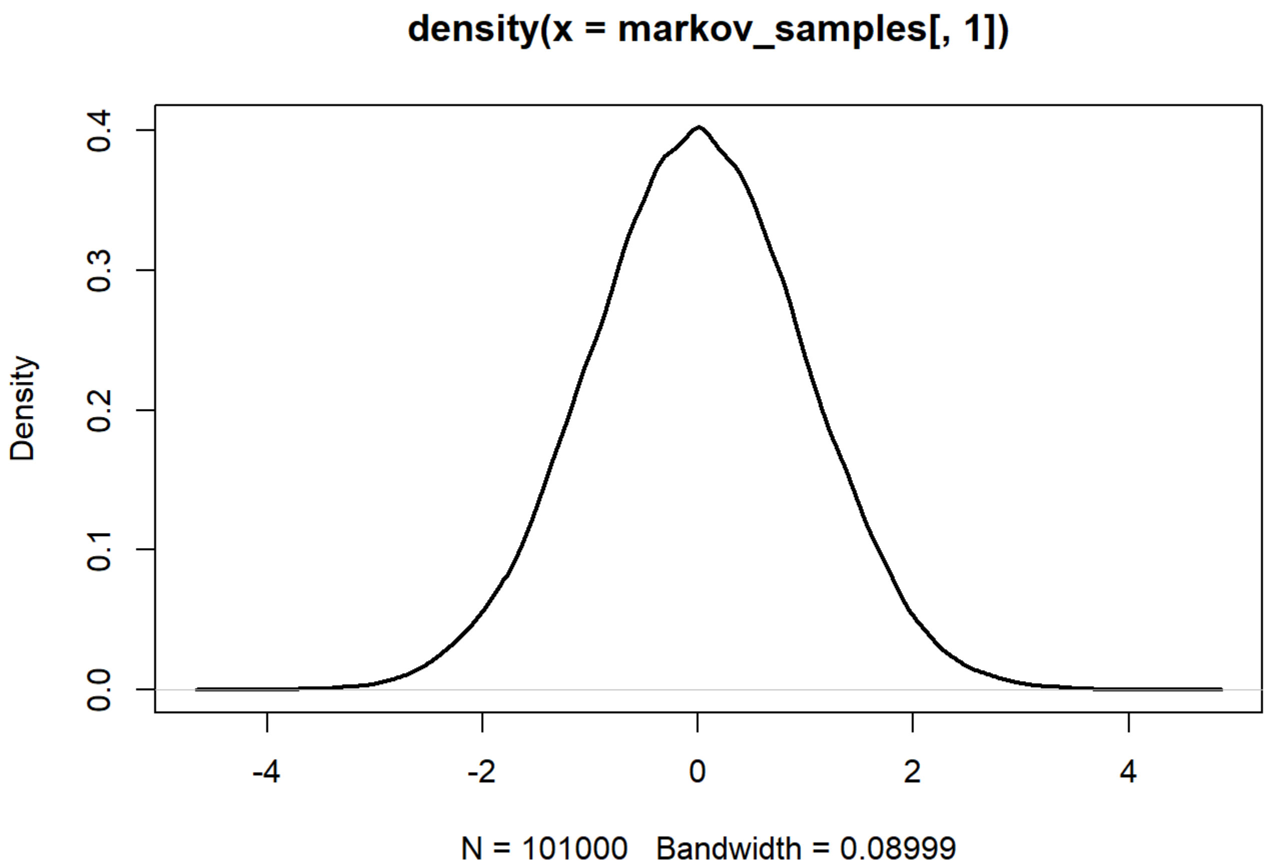


- **Histogram of X and Y** Individually for X and Y are following Normal distribution.
  - $X \sim \text{Normal}(\mu_1 = 0, \sigma_1^2 = 1)$
  - $Y \sim \text{Normal}(\mu_2 = 0, \sigma_2^2 = 4)$

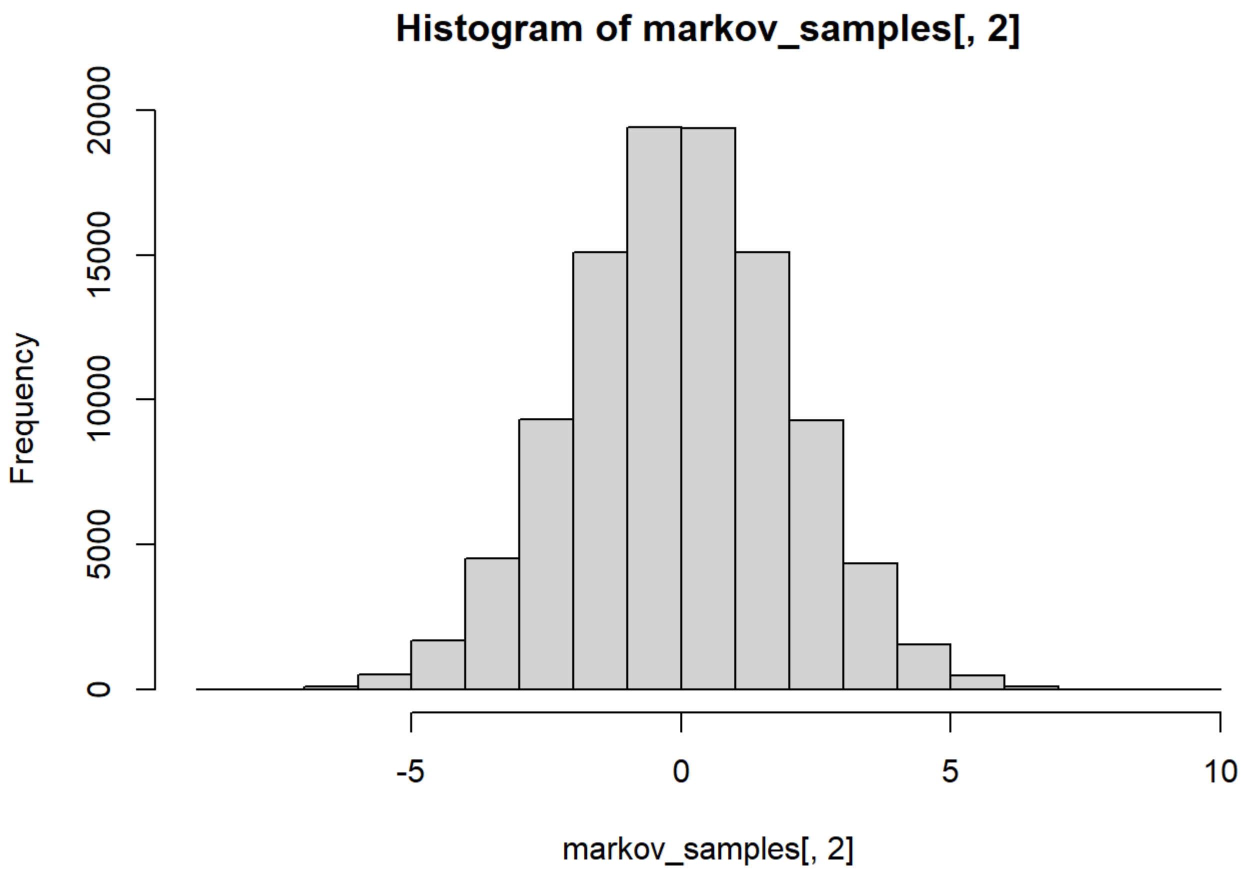
```
# histogram of X
hist(markov_samples[,1])
```



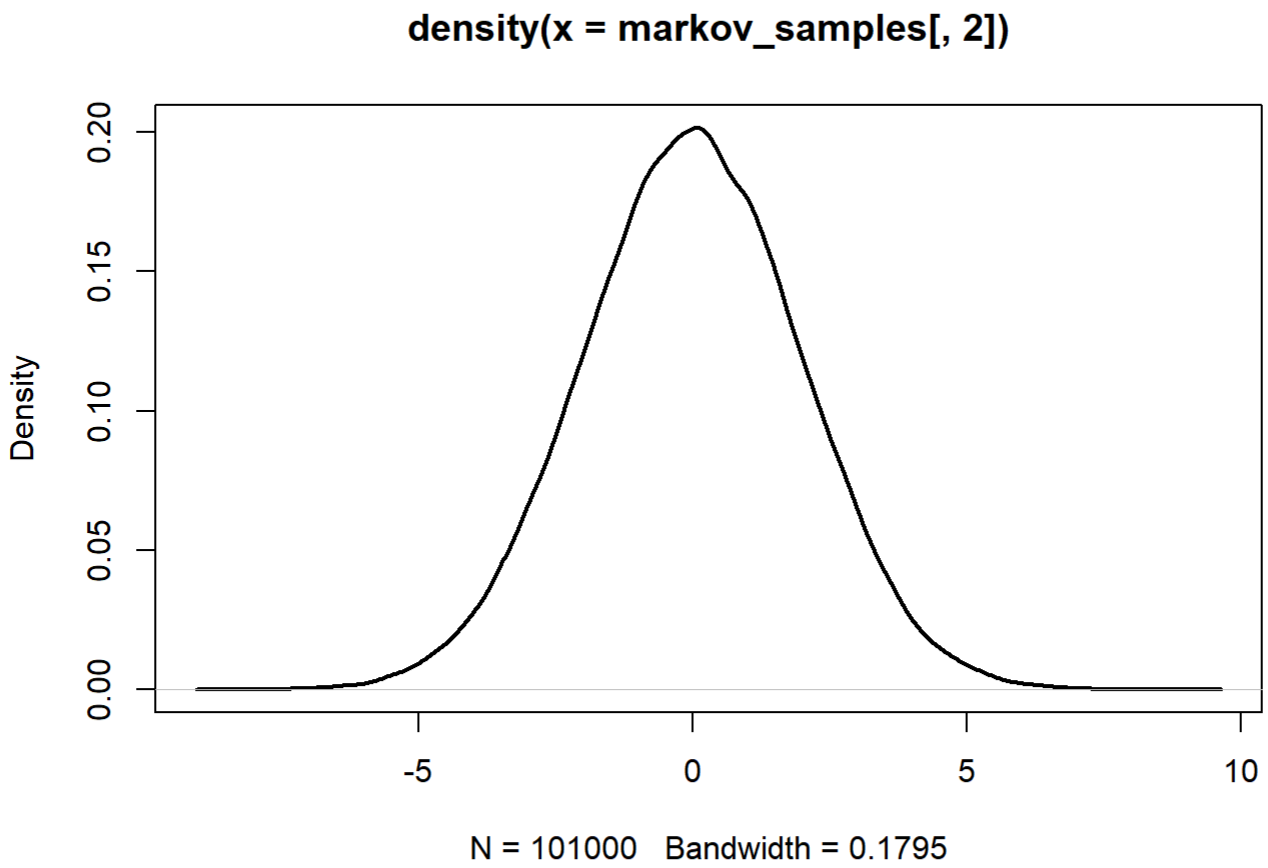

```
plot(density(markov_samples[,1]),lwd=2)
```




```
# histogram of Y
hist(markov_samples[,2])
```




```
plot(density(markov_samples[,2]),lwd=2)
```


density(x = markov_samples[, 2])
Density
0.00
0.05
0.10
0.15
0.20
-5
0
5
10
N = 101000   Bandwidth = 0.1795