MTH686: Non-Linear Regression

Prof. Debasis Kundu

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Submitted by: **Dasari Charithambika(210302)**

1 Introduction

The main goal of this project is to fit the best model to a dataset that includes t and y(t). There are three models, and we must select the best model based on the data.

- Model-1 : $y(t) = \alpha_0 + \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t} + \epsilon(t)$
- Model-2 : $y(t) = \frac{\alpha_0 + \alpha_1 t}{\beta_0 + \beta_1 t} + \epsilon(t)$
- Model-3 : $y(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4 + \epsilon(t)$

Assume $\epsilon(t)$ is a sequence of i.i.d. normal random variable with mean zero and variance σ^2.

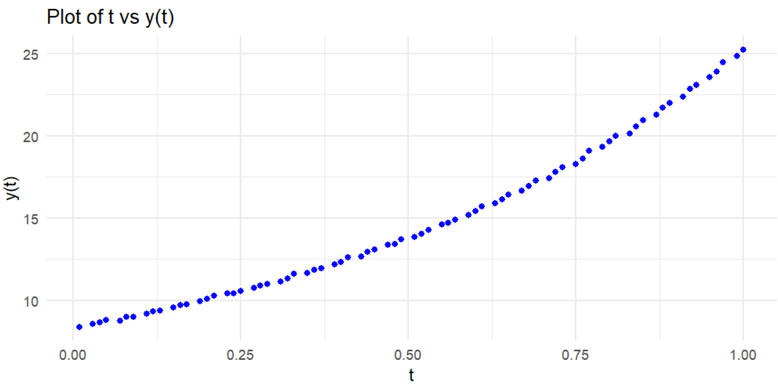

Figure 1: Scatter plot between t and y(t)

2 Least Square Estimators of unknown parameters

- Model - 1 : $y(t) = \alpha_0 + \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t} + \epsilon(t)$

  The least squares estimators of $\alpha_0, \alpha_1, \beta_1, \alpha_2, \beta_2$ in Model-1 were obtained using the `optim()` function with the **quasi-Newton method (BFGS)** from the `stats` R package, which iteratively refines the initial parameter estimates to achieve convergence. After fitting the model, the estimated values are

Parameter	Initial Guess	Estimated Value
α_0	1	3.446468
α_1	2	2.442989
β_1	-0.01	1.466948
α_2	2	2.453436
β_2	-0.01	1.516616

Table 1: Model-1 Parameter Estimates

- Model - 2 : $y(t) = \frac{\alpha_0 + \alpha_1 t}{\beta_0 + \beta_1 t} + \epsilon(t)$

  The least squares estimators of $\alpha_0, \alpha_1, \beta_0, \beta_1$ in Model-2 were obtained using the `optim()` function with the **quasi-Newton method (BFGS)** from the `stats` R package, which iteratively refines the initial parameter estimates to achieve convergence. After fitting the model, the estimated values are

Parameter	Initial Guess	Estimated Value
α_0	1	2.2302123
α_1	0.1	1.1155839
β_0	1	0.2717326
β_1	0.1	-0.1398105

Table 2: Model-2 Parameter Estimates

- Model - 3 : $y(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4 + \epsilon(t)$

  The least squares estimators of $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$ in Model-3 were obtained using the `optim()` function with the **quasi-Newton method (BFGS)** from the `stats` R package, which iteratively refines the initial parameter estimates to achieve convergence. After fitting the model, the estimated values are

Parameter	Initial Guess	Estimated Value
β_0	mean(y)	8.311840
β_1	1	7.639197
β_2	1	4.731818
β_3	1	2.763021
β_4	1	1.780748

Table 3: Model-3 Parameter Estimates

3 Best fitted model

The best-fitted model is selected based on

- Residual Sum of Squares (RSS)
- Akaike Information Criterion (AIC)

A lower RSS indicates a better fit of the model to the data, while a lower AIC suggests a more efficient model with fewer parameters. Thus, the model with the lowest values for both RSS and AIC is considered the best.

Model	RSS	AIC
Model-1	0.4712605	-370.2374
Model-2	0.6451833	-348.6782
Model-3	0.466042	-371.0726

Table 4: Comparison of RSS and AIC Values for Different Models

Model-3 provides the lowest RSS(0.466) and AIC(-371.07), making it the best-fitting model.

4 Estimate of σ^2

$$\sigma^2 = \frac{1}{n-p} \sum_{i}^{n} (y - \hat{y})^2$$

where y = observed values, $\hat{y}$ = fitted values, n = number of observations, p = number of parameters.

Estimated σ^2 is 0.006657743

5 Associated confidence intervals

Associated confidence intervals based on the Fisher information matrix:

To determine the 95% confidence interval based on the Fisher information matrix, the `optim()` function with `hessian = TRUE` was used, where the inverse of the Hessian matrix provides the Fisher information matrix. The confidence interval for each β_i is calculated as:

$$\hat{\beta}_i \pm z_{0.975}\, \mathrm{SE}(\hat{\beta}_i), \quad i = 0, 1, 2, 3, 4$$

Parameter	Lower_CI	Upper_CI
$\hat{\beta}_0$	8.2414361	8.382243
$\hat{\beta}_1$	6.6906262	8.587767
$\hat{\beta}_2$	0.9591805	8.504455
$\hat{\beta}_3$	-2.8147552	8.340796
$\hat{\beta}_4$	-0.9507982	4.512294

Table 5: 95% Confidence intervals

6 Residual Plot

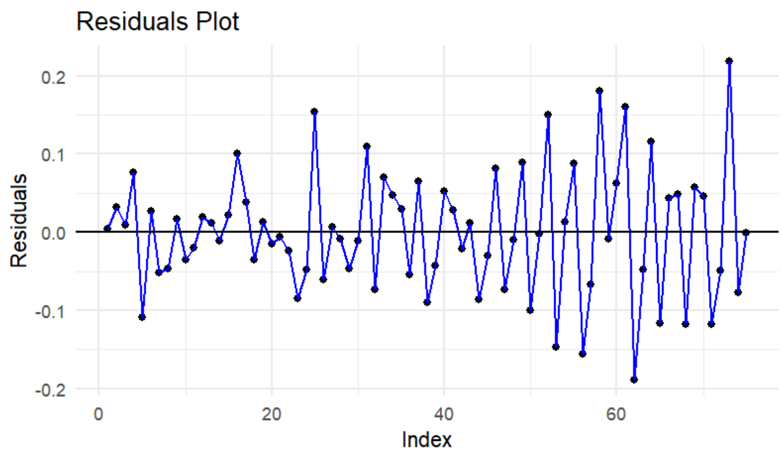

Figure 2: Residual plot

7 Normality assumption Verification

To test if the residuals of the best-fit model (Model-3) satisfy the normality assumption, we can conduct a few diagnostic tests and visualizations.

- Shapiro-Wilk Test : A p-value greater than 0.05 suggests residuals are likely normal, while a p-value less than 0.05 indicates they are not.
- Q-Q plot : In a Q-Q plot, normally distributed residuals align with the reference line, while deviations, especially in the tails, indicate non-normality.

Applied the `shapiro.test()` function to assess normality by checking if the p-value is greater than 0.05, and used `qqnorm()` to generate a Q-Q plot for visual inspection.

- Shapiro-Wilk Test p-value is 0.8971403(> 0.05), So the residuals are normally distributed.
- Q-Q plot : The residuals are align with reference line that implies normality

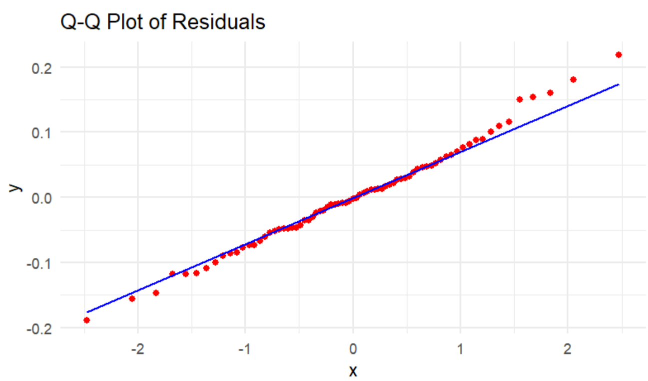

Figure 3: Q-Q plot

8 Plot the observed data points and fitted curve

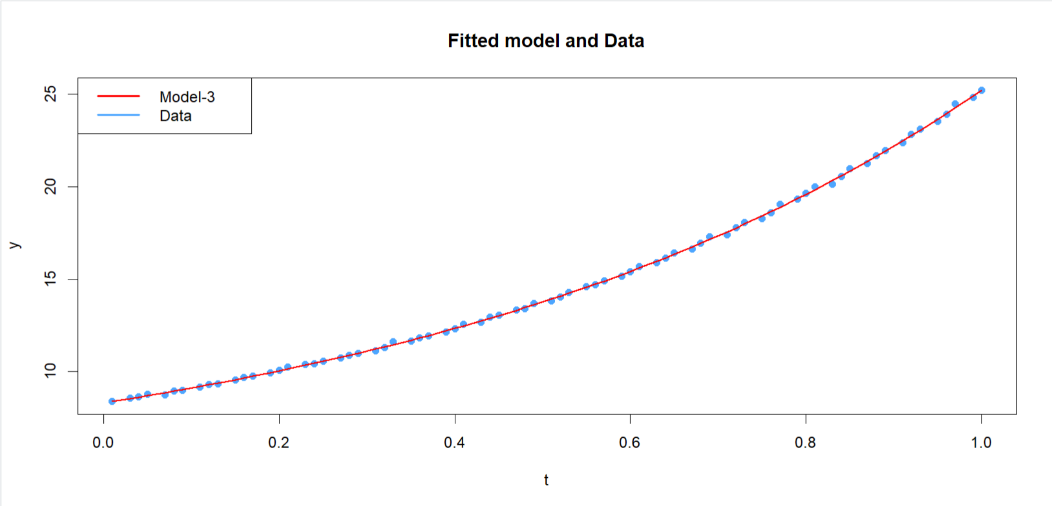

Figure 4: Plot between data and fitted curve