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Assignment for Basic Mathematical Tools

Exercise 1 Row Rank and Column Rank

Let $A \in \mathbb{R}^{m \times n}$. Show that the column rank of A is equal to the row rank of A.

Hint Consider a basis $b_1, \ldots, b_r \in \mathbb{R}^n$ for the row space of A, i.e. the subspace spanned by the row vectors of A, and show that the vectors $Ab_1, \ldots, Ab_r$ are linearly independent.

Exercise 2 (P) QR decomposition and Least Squares Problems

In this assignment we consider the reduced QR decomposition of a matrix $A \in \mathbb{R}^{m \times n}$, i.e.,

$$A = Q \cdot R = \left(q_1 \middle| q_2 \middle| q_3 \middle| \ldots \middle| q_n \right) \cdot \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & \ddots & & \vdots \\ & 0 & \ddots & \vdots \\ & & & r_{nn} \end{pmatrix},$$

where $Q \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{n \times n}$. The columns of Q, $q_1, \ldots, q_n \in \mathbb{R}^m$, comprise an orthonormal basis and Q is thus a *column-orthogonal matrix*[1], i.e.

$$Q^T Q = I \in \mathbb{R}^{n \times n}.$$

A QR-decomposition of A can be obtained by the so-called Gram-Schmidt method which works as follows.

Denoting the columns of A by $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ and the projection of a vector x onto a vector u by

$$P_u(x) = \frac{\langle x, u \rangle}{\langle u, u \rangle} u$$

we iterative compute an orthonormal basis from the columns of A:

$$\begin{aligned} u_1 &= a_1, \quad q_1 = \frac{u_1}{\|u_1\|}, \\ u_2 &= a_2 - P_{q_1}(a_2), \quad q_2 = \frac{u_2}{\|u_2\|}, \\ u_3 &= a_3 - P_{q_1}(a_3) - P_{q_2}(a_3), \quad q_3 = \frac{u_3}{\|u_3\|}, \\ &\vdots \\ u_n &= a_n - \sum_{i=1}^{n-1} P_{q_i}(a_n), \quad q_n = \frac{u_n}{\|u_n\|}. \end{aligned}$$

[1] Important: $QQ^T = I_m$ does only hold true for $m = n$.

Thus, we find that

$$a_k = u_k + \sum_{i=1}^{k-1} \langle a_k, q_i \rangle q_i = \|u_k\| q_k + \sum_{i=1}^{k-1} \langle a_k, q_i \rangle q_i,$$

for $k = 1, \ldots, n$, which can finally be written as a matrix-matrix multiplication:

$$A = \left(a_1 \middle| a_2 \middle| a_3 \middle| \ldots \middle| a_n \right) = \left(q_1 \middle| q_2 \middle| q_3 \middle| \ldots \middle| q_n \right) \cdot \begin{pmatrix} \|u_1\| & \langle a_2, q_1 \rangle & \ldots & \langle a_n, q_1 \rangle \\ & \|u_2\| & \ddots & \vdots \\ & & \ddots & \langle a_n, q_{n-1} \rangle \\ 0 & & & \|u_n\| \end{pmatrix}.$$

a) Implement a Python function `[Q, R] = my_qr(A)` that computes a QR-decomposition for a given matrix A. Does it work for any matrix?

b) Compare the result of your implementation with the one of Numpy, i.e., `[Q, R] = np.linalg.qr(A)`.

c) Consider the normal equations $A^T A x = A^T b$ for

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

What happens for $\varepsilon =$ `1e-9`? Try to solve the normal equations directly, i.e., `x=np.linalg.solve(np.matmul(A.T,A),np.matmul(A.T,b))`, and with the QR decomposition as shown in the lecture.

Exercise 3 LU-Decomposition

a) Find LU decomposition of the matrices A by hand.

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix}$$

b) The following LUP decomposition of the matrix A is given. Find the inverse of A.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \ L = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}, \ U = \begin{pmatrix} 2 & 3 \\ 0 & \frac{5}{2} \end{pmatrix}, \ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

c) Linear systems of equations can be solved using Gaussian elimination. Under which circumstances are decomposition methods better suited than Gaussian elimination to find the solution.