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Assignment for Basic Mathematical Tools

Exercise 1 Convex Calculus

a) Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ convex functions. Show the following statements:

(i) The function $f + g$ is convex

(ii) The function $g \circ f$ is convex if g is monotonously increasing.

(iii) The function βf is strictly convex, if $\beta > 0$ and f strictly convex.

b) Find a new argument for why the well-known equation

$$\min_{\mathbf{x}} E(\mathbf{x}) = \min_{\mathbf{x}} \left(\frac{1}{2} \mathbf{x}^\top A \mathbf{x} + \mathbf{x}^\top \mathbf{b} \right)$$

with A symmetric positive definite has a unique solution.

Exercise 2 Convex Functions

Check if the following functions are convex:

a) $f(\mathbf{x}) = \|\mathbf{x}\|_p$ for $x \in \mathbb{R}^n$, $p \geq 1$

b) $g(x) = \sqrt{x}$ for $x \in \mathbb{R}^+$

c) $h(\mathbf{x}) = \mathbf{x}^\top \mathbf{x}$ for $x \in \mathbb{R}^2 \backslash B_1(0)$

Exercise 3 (T) Image Tracking

For many applications such as video editing and computer vision, it is highly interesting to track a certain image patch over time. In this assignment, we consider a simple first approach to solving this problem.[1]

Assume a sequence of images

$$I_i : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$$

and an initial location $\mathbf{x}^{(0)}$ selected in the first image I_0. We enlarge that location by a certain radius t, and define the square patch

$$P \subset \Omega = \{\mathbf{x} \in \Omega : |\mathbf{x} - \mathbf{x}^{(0)}| \leq t\}$$

We assume that this image patch moves – more or less – rigidly from frame I_0 to the subsequent frame I_1, that is, we search for a single update vector $\mathbf{v}$ which relates the image patch on the current frame $I_1(P)$ to it's origin $I_0(P + \mathbf{v})$.

[1] This method has already been discussed in the lecture.

Using the difference of intensities, this can be cast as minimization problem

$$\min_{\mathbf{v}\in\mathbb{R}^2} \int_P |I_0(\mathbf{x}+\mathbf{v}) - I_1(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x}$$

and, by means of *Taylor Expansion*, we get

$$\min_{\mathbf{v}\in\mathbb{R}^2} \int_P |\nabla I_0(\mathbf{x})^\top \mathbf{v} + I_0(\mathbf{x}) - I_1(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x}.$$

Working in a discrete domain, we can explicitly list the $n = (2\lfloor t \rfloor + 1)^2$ elements of

$$P = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$$

and replace the integral by a sum, yielding

$$\min_{\mathbf{v}\in\mathbb{R}^2} \sum_{\mathbf{x}\in P} |\nabla I_0(\mathbf{x})^\top \mathbf{v} + I_0(\mathbf{x}) - I_1(\mathbf{x})|^2 = \min_{\mathbf{v}\in\mathbb{R}^2} \|A\mathbf{v} - \mathbf{b}\|^2$$

where

$$A = \begin{pmatrix} \nabla I_0(\mathbf{x}_1)^\top \\ \vdots \\ \nabla I_0(\mathbf{x}_n)^\top \end{pmatrix} \in \mathbb{R}^{n\times 2} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} I_1(\mathbf{x}_1) - I_0(\mathbf{x}_1) \\ \vdots \\ I_1(\mathbf{x}_n) - I_0(\mathbf{x}_n) \end{pmatrix} \in \mathbb{R}^n.$$

This linear regression problem can be solved with the approaches covered earlier, the normal equation or – preferably – Golub's method. The solution $\mathbf{v}$ is exactly the update relating the image patches, and we can compute $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \mathbf{v}$.

In order to continue tracking over the following frames, we can simply re-initialize the patch P to be centered around $\mathbf{x}^{(1)}$, compute a new update yielding $\mathbf{x}^{(2)}$, and so on.

On our website, we provide you with two short movie sequences, `tracking-cars.mat` and `tracking-leukocyte.mat`. Implement the method described above, let the user select an initial point to track, and track the point over all the frames (the respective dummy functions and scripts are already available, cf. `tracking1.zip`). Note that you will have to use the method `scipy.ndimage.map_coordinates` for interpolating the image or the gradient at the position $\mathbf{x}^{(0)} + \mathbf{v}$. What behavior do you observe, and how would you explain it?

This method is inspired by Bruce D. Lucas and Takeo Kanade: *An Iterative Image Registration Technique with an Application to Stereo Vision*. In: *Proc. Image Understanding Workshop*, 1981, pp. 121–130. Today, this work is among the 'classical' *Optical Flow* publications and worth reading.