2 Analysis

2.1 Metric spaces

Arbitrary sets M can be equipped with a notion of "distance" via a metric.

Definition (Metric)**.** *Let M be a set, then a mapping $d : M \times M \to \mathbb{R}_0^+$ is called **metric on** M, if $\forall x, y, z \in M$ we have*

(1) $d(x, y) = 0 \quad \iff \quad x = y$

(2) $d(x, y) = d(y, x)$

(3) $d(x, z) \leq d(x, y) + d(y, z)$.

*The pair (M, d) is then called a **metric space**.*

Examples:

- Let $G = (V, E)$ be an undirected, connected graph with weights $w : E \to \mathbb{R}^+$, then for two vertices $x, y \in V$

$$d(x, y) := \text{"length of shortest path connecting } x \text{ and } y\text{"}$$

  is a metric.

- Let $(V, \|\cdot\|)$ be a normed linear space. Then

$$d(x, y) := \|y - x\|$$

  for $x, y \in V$ defines a metric.

  For the special case $V = \mathbb{R}^n$, there are two notable special cases:

  - ***p*–norm:** For $1 \leq p < \infty$ the p–norm of $x \in \mathbb{R}^n$ is defined as

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}},$$

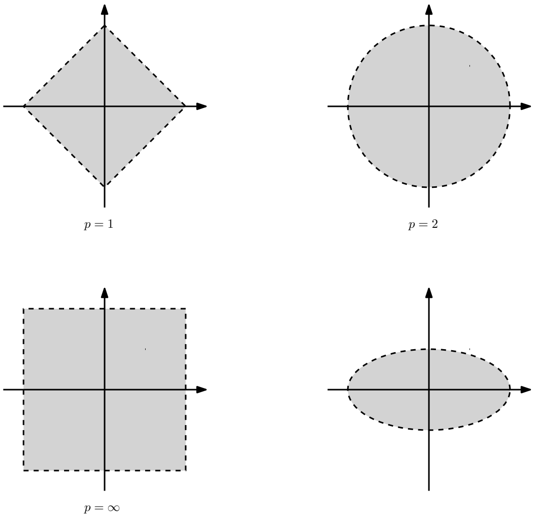
    and for $p = \infty$ as

$$\|x\|_\infty := \max_{i=1,\ldots,n} |x_i|.$$

  - **energy norm:** For $A \in \mathbb{R}^{n \times n}$ positive definite, the energy norm of $x \in \mathbb{R}^n$ is defined as

$$\|x\|_A := \sqrt{x^T A x}.$$

Looking at the open unit ball $B_1(0) := \left\{ x \in \mathbb{R}^n : \ \|x\| < 1 \right\}$ helps to get a feel for these norms. For example in the case of $n = 2$:



Open unit ball for the various norms.
The lower right entry is the energy norm for $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0.5 \end{smallmatrix}\right)$.

2.2 Topology in metric spaces

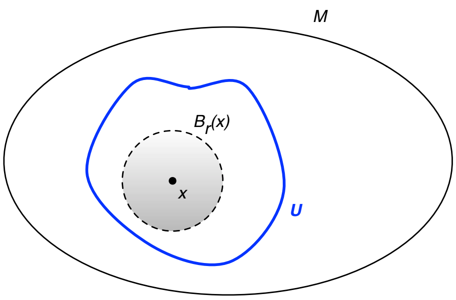
Definition (Open ball, neighborhood)**.** *Let* (M, d) *be a metric space. For* $x \in M$ *and* $r > 0$

$$B_r(x) := \big\{ y \in M : \; d(x, y) < r \big\}$$

is called the ***open ball*** *around* x *with radius* r*.*

A set $U \subset M$ *is called* ***neighborhood*** *of* $x \in M$*, if there exists* $\varepsilon > 0$ *such that*

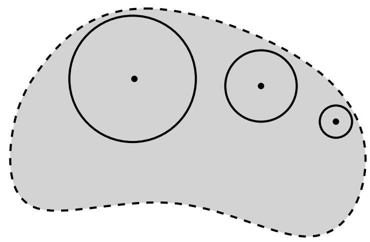
$$x \in B_\varepsilon(x) \subset U.$$



Definition (Open, closed sets)**.** *Let* (M, d) *be a metric space. A subset* $U \subset M$ *is called* ***open****, if* U *is a neighborhood of every* $x \in U$*, that is:*

$$\forall x \in U \quad \exists \varepsilon > 0 : \qquad x \in B_\varepsilon(x) \subset U.$$

A subset $A \subset M$ *is called* ***closed****, if the complement* $A^c = M \setminus A$ *is open.*



Remarks: Let (M,d) be a metric space.

- The set $\mathcal{T}$ of all open subsets of (M,d) is called **topology** on M.
- $\emptyset$ and M are both open **and** closed!
- If the subsets U, V of M are open / closed, then $U \cup V$, $U \cap V$ are open / closed.
- If a set $U \subset M$ is open or closed depends both on M and d. For example $[0,1]$ is not open in $(\mathbb{R}, |\cdot|)$, but it is open in $([0,1], |\cdot|)$. It will turn out however, that all norms of $\mathbb{R}^n$ induce the same topology.

Examples: Let (M,d) be a metric space.

- $\{x\}$ for $x \in M$ is closed.
- Let $x \in M$, $r > 0$.
  - $B_r(x) := \{y \in M :\ d(x,y) < r\}$ is open (**open ball** around x with radius r).
  - $K_r(x) := \{y \in M :\ d(x,y) \le r\}$ is closed (**closed ball** around x with radius r).

  Special case $(\mathbb{R}, |\cdot|)$: intervals $(a,b) \subset \mathbb{R}$ are open, intervals $[a,b] \subset \mathbb{R}$ are closed.

Definition (Boundary)**.** *Let (M,d) be a metric space, $A \subset M$ a subset. $x \in M$ is called a* ***boundary point*** *of A, if for all $\varepsilon > 0$ $B_\varepsilon(x)$ contains both a point of A and $M \setminus A$. The set*

$$\partial A := \{x \in M :\ x \text{ boundary point of } A\}$$

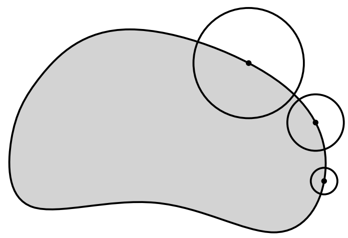
is called ***boundary*** *of A. The set*

$$int(A) := A \setminus \partial A$$

is called ***interior*** *of A, and the set*

$$\overline{A} := A \cup \partial A$$

is called the ***closure*** *of A.*



Remarks: Let (M, d) be a metric space, $A \subset M$.

- $\mathrm{int}(A)$ is open, $\overline{A}$ is closed, ∂A is closed.
- $\mathrm{int}(A) \subset A \subset \overline{A}$.
- $\partial A = \overline{A} \setminus \mathrm{int}(A)$.
- A closed $\iff \quad A = \overline{A}$.

Examples: Let (M, d) be a metric space.

- Let $x \in M$, $r > 0$.
  - $S_r(x) := \big\{y \in M : \ d(x, y) = r\big\}$ is closed (**sphere** around x with radius r).
  - $S_r(x) = \partial K_r(x) = \partial B_r(x) = K_r(x) \setminus B_r(x)$.
- Special case $(\mathbb{R}, |\cdot|)$:
  Let $(a, b) \subset A \subset [a, b] \subset \mathbb{R}$, then $\partial A = \{a, b\}$, $\mathrm{int}(A) = (a, b)$ and $\overline{A} = [a, b]$.
  Also $\partial\mathbb{Q} = \overline{\mathbb{Q}} = \mathbb{R}$, $\mathrm{int}(\mathbb{Q}) = \emptyset$, but $\overline{\mathbb{Q}} \neq \overline{\mathrm{int}(\mathbb{Q})}$.

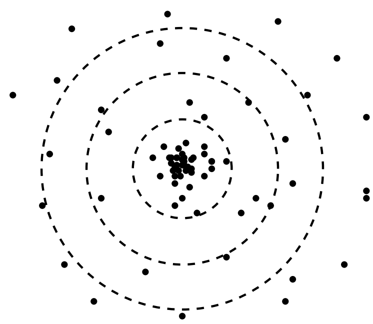
2.3 Convergence and compactness

Definition (Convergence)**.** *Let (M, d) be a metric space. A sequence $(x_k)_{k\in\mathbb{N}} \subset M$ is called **convergent** to a **limit** $a \in M$, if for every neighborhood U of a there exists a $N \in \mathbb{N}$ such that $x_k \in U$ $\forall k \geq N$. We write in short*

$$\lim_{k\to\infty} x_k = a \qquad \text{or} \qquad x_k \overset{k\to\infty}{\longrightarrow} a.$$

Equivalently we have:

$$\begin{aligned} x_k \overset{k\to\infty}{\longrightarrow} a \quad &\iff \quad \lim_{k\to\infty} d(x_k, a) = 0 \\ &\iff \quad \forall \varepsilon > 0 \ \exists M \in \mathbb{N} \text{ such that } d(x_k, a) < \varepsilon \ \forall k \geq M. \end{aligned}$$



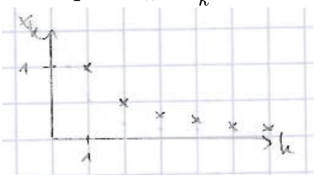
Remarks: Let (M, d) be a metric space.

- If $(x_k)_{k\in\mathbb{N}}$ is a convergent sequence in M, the limit $\lim_{k\to\infty} x_k$ is uniquely defined, and every subsequence of (x_k) converges to it.

- Let $A \subset M$. Then:

$$A \text{ closed} \iff \text{every convergent sequence } (x_k) \subset A \text{ has } \lim_{k\to\infty} x_k \in A$$

- Example: $x_k = \frac{1}{k}$ for $k \in \mathbb{N}$. Then $\lim_{k\to\infty} x_k = 0$.



Definition. *Let (M, d) be a metric space. A sequence $(x_k)_{k\in\mathbb{N}} \subset M$ is called **Cauchy sequence**, if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon\ \forall n, m \geq N$.*

*(M, d) is called **complete**, if every Cauchy sequence converges.*

Remark: every convergent sequence is a Cauchy sequence.

Examples:

- $(\mathbb{R}, |\cdot|)$ is complete.
- $(\mathbb{Q}, |\cdot|)$ is not complete, as for example $x_1 := 1$, $x_{k+1} := \frac{x_k}{2} + \frac{1}{x_k}$ has the limit $\lim_{k\to\infty} x_k = \sqrt{2} \notin \mathbb{Q}$.
- A normed linear space $(V, \|\cdot\|)$ which is complete is also called **Banach space**.
- A linear space V with a scalar product $\langle\cdot,\cdot\rangle$ induces a metric via the induced norm $\|x\| := \sqrt{\langle x, x\rangle}$ for $x \in V$. If V is complete it is also called **Hilbert space**.

$$\begin{array}{ccc}
\text{linear space with } \langle\cdot,\cdot\rangle & \overset{\text{complete}}{\longrightarrow} & \text{Hilbert space} \\
\downarrow & & \downarrow \\
\text{linear space with } \|\cdot\| & \overset{\text{complete}}{\longrightarrow} & \text{Banach space} \\
\downarrow & & \downarrow \\
\text{metric space with } d(\cdot,\cdot) & \overset{\text{complete}}{\longrightarrow} & \text{complete metric space}
\end{array}$$

Banach/Hilbert space examples:

- $(\mathbb{R}^n, \|\cdot\|_p)$ is a Banach space. For $p = 2$ we have $\|x\|_2 = \sqrt{\langle x, x\rangle}$, hence $(\mathbb{R}^n, \|\cdot\|_2)$ is a Hilbert space.
- For X set and $p \geq 1$

$$\mathcal{L}^p(X) := \left\{ f : X \to \mathbb{R} : \int |f(x)|^p \, dx < \infty \right\}$$

is a linear space using Lebesgue integration.

$$\|f\|_p := \left(\int |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

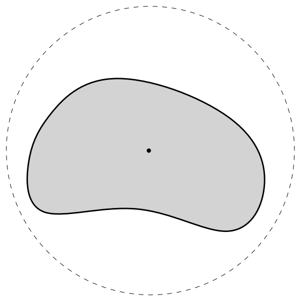
however is not a norm, just a semi–norm, as there exist functions $f \neq 0$ with $\|f\|_p = 0$ (for example with $X = \mathbb{R}$: $f(x) = 0$ for $x \neq 0$, and $f(0) = 1$). Using

$$f \text{ equivalent to } g \qquad \Longleftrightarrow \qquad \|f - g\|_p = 0,$$

we can define $L^p(X)$ as the equivalence classes of this equivalence relation. $\big(L^p(X), \|\cdot\|_p\big)$ is then a Banach space, for $p = 2$ it is also a Hilbert space.

Definition (Bounded set)**.** *Let (M, d) be a metric space. A subset $U \subset M$ is called **bounded** if $\sup_{x,y \in U} d(x, y) < \infty$. Alternatively:*

$$U \text{ is bounded} \qquad \Longleftrightarrow \qquad \exists \varepsilon > 0, x \in U \quad \text{such that} \quad U \subset B_\varepsilon(x).$$



Definition (Compactness)**.** *Let (M, d) be a metric space. (M, d) is called **compact**, if every sequence $(x_k)_{k \in \mathbb{N}} \subset M$ has a convergent subsequence.*
*A subset $U \subset M$ is called **compact**, if every sequence $(x_k)_{k \in \mathbb{N}} \subset U$ has a convergent subsequence with a limit in U.*

Theorem (Bolzano–Weierstraß)**.** *Consider $(\mathbb{R}^n, \|\cdot\|_\infty)$ and $U \subset \mathbb{R}^n$. Then*

$$U \text{ compact} \quad \Longleftrightarrow \quad U \text{ bounded and closed in } \mathbb{R}^n.$$

Examples:

- $[a, b] \subset \mathbb{R}$ is compact in $(\mathbb{R}, |\cdot|)$.
- $K_r(x)$ is compact in $\mathbb{R}^n$ (for $x \in \mathbb{R}^n$, $r > 0$).

Theorem (Equivalence of norms in $\mathbb{R}^n$)**.** *For every norm $\|\cdot\|$ on $\mathbb{R}^n$ there exist $c_1, c_2 > 0$ with*

$$c_1\|x\|_\infty \leq \|x\| \leq c_2\|x\|_\infty \qquad \forall x \in \mathbb{R}^n.$$

In other words: For $\mathbb{R}^n$, properties like convergence, Cauchy sequence, completeness, open, closed, bounded, compact, boundary, interior, closure do not depend on the norm inducing the metric! This is in fact true for any finite dimensional normed linear space!

Remark (Peculiarities of infinite dimensional spaces). The reason why one has to emphasize the fact whether a space is finite dimensional or not, is that a lot of concepts and theorems which hold true in finite dimensional spaces cannot be carried over to infinite dimensional

spaces. A famous example is the theorem of Bolzano-Weierstraß:
Let us consider the space of all bounded sequences

$$\ell^\infty = \left\{x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R}, \|x\|_\infty < \infty\right\},$$

where

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| \,.$$

The closed unit ball in ℓ^∞, i.e.,

$$K_1(0) = \left\{x = (x_n)_{n\in\mathbb{N}} : \|x\|_\infty \leq 1\right\} \subset \ell^\infty$$

is obviously bounded and closed. However, considering the sequence (of sequences) $(x^k)_{k\in\mathbb{N}} \subset B$ which is given by:

$x^1 = (1, 0, 0, 0, \ldots)$, $x^2 = (0, 1, 0, 0, \ldots)$, $x^3 = (0, 0, 1, 0, \ldots)$, $x^4 = (0, 0, 0, 1, \ldots)$, etc.

one realizes that there exists no converging subsequence of this sequence. Hence, B cannot be compact!
It is actually even worse. It can be shown that it is possible to construct such sequences, i.e. sequences with no convergent subsequence, in *any* infinite dimensional normed space!