MATH2621 Exam Sheet

Triangle Inequality: $|w+z| \le |w| + |z|$
Circle Inequality: $||w| - |z|| \le |w - z|$
Topology:

- A set S is a *region* if it looks like a 2D object, with the exception of the $\emptyset$ which is also a region, or mathematically S must be an open set with some, none, or all of its boundary points.
- A set S is *compact* if it closed and bounded.
- A set S is *polygonally path-connected* if any two points on S can be joined by a polygonal arc fully contained in S.
- A set S is *simply polygonally connected* if the interior of every simple closed polygonal arc of S is a subset of S, or S does not have any holes.
- A set S is a *domain* if is both open and polygonally path-connected.

Fractional Linear Transformation: Every 2×2 complex matrix T_M with determinant 1 can be factorised in terms of

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It can also be decomposed as $\dfrac{a}{c} - \dfrac{1}{c^2(z+d/c)}$.

Limits: A limit $\lim_{z \to z_0} f(z) = \lambda$, where $\lambda \in \mathbb{C}$ and $z_0 \in \overline{\text{Domain}(f)}$, iff $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(z) - \lambda| < \epsilon$, provided $z \in \text{Domain}(f)$ and $0 < |z - z_0| < \delta$.

- For finite limits: $z \to z_0$, then $\forall \epsilon > 0, \exists \delta > 0 : 0 < |z - z_0| < \delta \implies |f(z) - \lambda| < \epsilon$.
- $\lim_{z \to \infty} f(z) = L \iff \lim_{z \to 0} f(1/z) = L$.
- $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} 1/f(z) = 0$.
- $\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} 1/f(1/z) = 0$.

Continuity: A complex function f is continuous at z_0 iff $\lim_{z \to z_0} f(z) = f(z_0)$. If for every $\epsilon > 0, \exists \delta > 0 : 0 < |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$, then f is continuous.
Theorem (Continuous functions): The functions cf, $f + g$, $|f|$, $\bar{f}$, $\text{Re}(f)$, $\text{Im}(f)$ and fg are continuous, and f/g where $g(z) \neq 0$ and $f \circ g$ are continuous too.
Theorem (Min-max theorem): If a set S is compact and f is a continuous complex function, then $\exists z_0 : f(z_0) = \max\{|f(z)| : z \in S\}$.
Differentiability: A complex function f is differentiable at z_0 iff $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$, or equivalently $\lim_{h \to 0} \dfrac{f(z_0 + h) - f(z_0)}{h}$.
Theorem (Cauchy Riemann equations): Suppose $f(x+iy) = u(x,y) + iv(x,y)$, then f is differentiable at z_0 if u_x, u_y, v_x, v_y exist and are continuous and $u_x = v_y$, $u_y = -v_x$. If f is differentiable at z_0, then $f'(x_0 + iy_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.
Theorem (Cauchy Riemann polar equations): Suppose $f = u + iv$, and $z_0 = r_0 e^{i\theta_0}$, then $u_\theta = -r_0 v_r$ and $v_\theta = r_0 u_r$, further $f'(z_0) = e^{-i\theta_0}(u_r + iv_r)$.
Harmonic: A function $u(x,y)$ is harmonic if $u_{xx} + u_{yy} = 0$. If a complex function $f = u + iv$ is differentiable, then u, v are harmonic conjugates.

Power Series: A power series is of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where the centre is z_0, and the radius of convergence is the largest $\rho \in \mathbb{R}^+$ where it converges if $|z - z_0| < \rho$.

Ratio Test: $\rho = \lim_{n \to \infty} \dfrac{|a_n|}{|a_{n+1}|}$. Root Test: $\rho = \lim_{n \to \infty} \dfrac{1}{|a_n|^{1/n}}$.

Stirling's Formula: $\lim_{n \to \infty} \dfrac{n!}{(2\pi)^{1/2} e^{-n} n^{n+1/2} = 1}$.
Trigonometric and Hyperbolic Functions:

- $\exp(z) = \exp(z + 2\pi k)$, $k \in \mathbb{Z}$.
- $\cosh(z) = \dfrac{e^z + e^{-z}}{2}$, $\sinh(z) = \dfrac{e^z - e^{-z}}{2}$.
- $\cosh(-z) = \cosh(z)$, $\sinh(-z) = -\sinh(z)$.
- $\cosh(z) = \cosh(z + 2\pi ik$, $\sinh(z) = \sinh(z + 2\pi ik)$, $k \in \mathbb{Z}$
- $\cosh(z+w) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w)$.
- $\sinh(z+w) = \sinh(z)\cosh(w) + \cosh(z)\sinh(w)$.
- $\sinh(iy) = i\sin(y)$, $\sin(iy) = i\sinh(y)$.
- $\cosh(iy) = \cos(y)$, $\cos(iy) = \cosh(y)$.
- $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$, $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2}$.

Principal Value:

- $\text{Log}(z) = \ln|z| + i\text{Arg}(z)$, $\text{Arg}(z) \in (-\pi, \pi]$ on $\mathbb{C} \setminus \{0\}$
- $z^\alpha = \exp(\alpha \text{Log}(z))$.
- $\text{PV}\sinh^{-1}(w) = \text{Log}(w + \text{PV}(w^2+1)^{1/2})$.
- $\text{PV}\cosh^{-1}(w) = \text{Log}(w + \text{PV}(w+1)^{1/2}\text{PV}(w-1)^{1/2}$.
- $\text{PV}z^{1/n} = \exp\left(\dfrac{\text{Log}(z)}{n}\right) = |z|^{1/n} e^{i\text{Arg}(z)/n}$.

Contours: The length of a piecewise smooth curve $\gamma : [a,b] \to \mathbb{C}$ is $\text{Length}(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

Contour Line Integral: $\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt$.

ML Lemma: $\left|\int_\gamma f(z)\, dz\right| \le ML$, where L is the length of γ, and $|f(z)| \le M$ for all $z \in \text{Range}(\gamma)$.
Theorem (Cauchy-Goursat theorem): Suppose $f \in H(\Omega)$, where Γ is a closed contour in Ω, a simply connected domain, then $\int_\Gamma f(z)\, dz = 0$.
Theorem (Cauchy's generalised integral formula): Suppose $f \in H(\Omega)$ where Ω is a simply connected domain, and $w \in \Gamma$ is a simple closed contour in Ω, then $f^{(n)}(w) = \dfrac{n!}{2\pi i} \int_\Gamma \dfrac{f(z)}{(z-w)^{n+1}}\, dz$.
Theorem (Liouville's theorem): If f is bounded and entire, then f is constant.
Theorem (Morera's theorem): If Ω is a domain that the function $f : \Omega \in \mathbb{C}$ is continuous and that $\int_\Gamma f(z)\, dz = 0$ whenever Γ is a closed contour in Ω, then f is holomorphic in Ω.
Series: A Taylor series centered about $z = z_0$ for a function f is given by $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(z_0)}{n!}(z - z_0)^n$. Common Taylor series include:

- $\sin(z) = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^{2n+1}}{(2n+1)!}$.
- $\cos(z) = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^{2n}}{(2n)!}$.
- $\text{Log}(1+z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \cdots = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} z^n}{n}$, $z \in (-1, 1]$
- $\dfrac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n$, $|z| \le 1$.

Theorem (Laurent): Suppose that A is the annulus $z \in \mathbb{C} : R_1 < |z - z_0| < R_2$ and $R_1 < r < R_2$. If $f \in H(A)$, then $\forall w \in A$, $f(w) = \displaystyle\sum_{n=0}^{\infty} c_n (w - z_0)^n$, where $c_n = \dfrac{1}{2\pi i} \displaystyle\int_{\partial B(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$.

Singularities: Consider the Laurent series of f on the punctured ball $B(z_0, r)$, with coefficients c_n.

- z_0 is a removeable singularity if there are no negative powers, specifically if $\lim_{z \to z_0} f(z)$ exists, it is removeable.
- z_0 is a pole of order M, where M is the amount of negative power, specifically if $\lim_{z \to z_0} (z - z_0)^k f(z)$ exists for $k = n$, but not $k = 0, 1, \ldots, n - 1$, then it is a pole of order n.
- z_0 is an essential singularity if it has an infinite amount of negative power, specifically if $\lim_{z \to z_0} (z - z_0)^k f(z)$ does not exist for any k, then it is an essential singularity.

Residues: Suppose a function f has a Laurent series $\displaystyle\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$, the residue is defined as $\mathrm{Res}(f, z_0) = c_{-1}$.

Theorem (Cauchy's residue theorem): Suppose that Γ is a simple closed contour with standard orientation in a domain Ω and $f \in H(\Omega)$. If f contains isolated singularities $z_1, z_2, \ldots, z_K$ inside $\mathrm{Int}(\Gamma)$, then $\displaystyle\int_\Gamma f(z)\, dz = 2\pi i \sum_{k=1}^{K} \mathrm{Res}(f, k)$.

Suppose f has an isolated singularity z_0, then the residue is

- *Case 1*: If the singularity is removeable, $\mathrm{Res}(f, z_0) = 0$.
- *Case 2*: If the singularity is a pole of order N at z_0, then $\mathrm{Res}(f, z_0) = \dfrac{1}{(N-1)!} \lim_{z \to z_0} \dfrac{d^{N-1}}{dz^{N-1}} (z - z_0)^N f(z)$.
- *Case 3*: If the singularity is essential at z_0, then we find the Laurent series and extract $\mathrm{Res}(f, z_0) = c_{-1}$.

Formula (p/q' formula): Suppose that $f(z) = \dfrac{p(z)}{q(z)}$ in Ω and $p(z_0) \neq 0$ while $q(z_0) = 0$. If z_0 is a simple pole of $q(z)$, then $\mathrm{Res}(f, z_0) = \dfrac{p(z_0)}{q'(z_0)}$.

Lemma (Jordan's Lemma): Suppose that f is continuous in the upper semi-circle arc, and Γ_R is the upper half of the circle with center 0 and radius R, and that $|f(z)| \leq M_R$ where $\lim_{R \to \infty} M_R = 0$. Then for any $\xi > 0$, $\lim_{R \to \infty} \displaystyle\int_{\Gamma_R} e^{i\xi z} f(z)\, dz = 0$.

Fourier Transform: The set of all locally integrable functions $f : \mathbb{R} \to \mathbb{C}$ such that $\displaystyle\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$ is denoted by $L^1(\mathbb{R})$. Given $f \in L^{-1}(\mathbb{R})$, the Fourier transform of f is the function $\hat{f}(\xi) = \displaystyle\int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx = \lim_{R \to \infty} \int_{[-R, R]} f(x) e^{-ix\xi}\, dx$.

Lemma (Riemann-Lebesgue lemma): If $f \in L^1(\mathbb{R})$, then the function $\hat{f}$ is bounded and continuous, and vanishes at infinity.

Theorem (Inverse Fourier transform): If $f, \hat{f} \in \mathcal{M}(\mathbb{R})$, the set of continuous bounded functions f such that $|f|$ is integrable, then the inversion formula is given by $f(x) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi}\, dx$.

Proofs of Theorems:

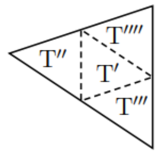
Proof: If $f = u + iv$, and f is holomorphic on an open subset, then u, v are harmonic. Prove using $u_{xx} + u_{yy} = 0$ with Cauchy Riemann equations and vector calculus.

Proof: The ML Lemma is proved with $\left| \displaystyle\int_\Gamma f(z)\, dz \right| \leq \int_\Gamma |f(z)|\, dz$

Proof: Independence of contour is proved using two contours with the same initial and final point, then apply Cauchy-Goursat.

Proof: Existence of primitives states there exists a function F for all contours Γ such that $\displaystyle\int_\Gamma f(z)\, dz = F(q) - F(p)$. This is proved by letting $F(p)$ be the contour integral of $f(z)$ with initial point b

Figure 1: *
Cauchy Goursat Proof



to final point p. Then consider the contour $\Gamma \sqcup \Gamma_q^* \sqcup \Gamma_p$, where Γ_q and Γ_p have base points b, then we get $\int_\Gamma f(z)\, dz = F(q) - F(p)$. Then take $\left| \dfrac{F(q) - F(p)}{q - p} - f(p) \right| \to 0$ as p and q are close, using $f(p) = \dfrac{1}{q - p} \displaystyle\int_\Delta f(p)\, dz$ to show $F'(p) = f(p)$.

Proof: Cauchy-Goursat theorem is proved by subdividing a triangle in Figure 1, and inductively show that one of the subdivided triangles is greater than $|I|/4$, repeating we get $\left| \displaystyle\int_{\partial T_n} f(z)\, dz \right| \geq \dfrac{|I|}{4^n}$. Now $\mathrm{Length}(\partial T_n) = 2^{-n} \mathrm{Length}(\partial T_0)$, $|z - z_0| \leq \dfrac{1}{2} \mathrm{Length}(\partial T_n)$ $= 2^{-n-1} \mathrm{Length}(\partial T_0)$. Write $f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z)$, $|I| \leq 4^n |\int_{\partial T_n} f(z)\, dz| \leq 4^n \max\{\frac{|E(z)|}{|z - z_0|} |z - z_0| : z \in \partial T_n\} \mathrm{Len}(\partial T_n)$.

Proof: Cauchy's integral formula is proved by making a contour $\partial \Upsilon = \Gamma \sqcup \Gamma_\epsilon$ where Γ_ϵ is the ball around the singularity. Then by Cauchy-Goursat we have $\displaystyle\int_\Gamma \frac{f(z)}{z - w}\, dz = \int_{\Gamma_\epsilon^*} \frac{f(z)}{z - w}\, dz$, then paramet Γ_ϵ^* as $\gamma_\epsilon^*(\theta) = w + \epsilon e^{i\theta}$, $\theta \in [0, 2\pi]$ and take the limit as $\epsilon \to 0$.

Proof: Cauchy's generalised integral formula is proved similarly by using Γ_r, a circle at z_0 and radius r, therefore $\displaystyle\int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$ is equivalent to $\displaystyle\int_{\Gamma_r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$. Using the Cauchy integral formula $f(w) = \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_r} \frac{f(z)}{(z - z_0)} \frac{1}{(1 - (w - z_0)/(z - z_0))}\, dz$, then $|w - z_0| < |z - z_0| = r$, so $f(w) = \dfrac{1}{2\pi i} \displaystyle\sum_{n=0}^{\infty} \int_{\Gamma_r} \frac{f(z)}{(z - z_0)} \frac{(w - z_0)^n}{(z - z_0)^n}\, dz$, then by independence of contour, $\Gamma = \Gamma_r$.

Proof: Liouville's theorem is proved using the power series $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$, where $f^{(n)}(0) = \dfrac{n!}{2\pi i} \displaystyle\int_{\Gamma_R} \frac{f(z)}{z^{n+1}}\, dz$, where Γ_R is the circle at 0, radius R since f is entire. Using the ML Lemma, $|f^{(n)}(0)| = \dfrac{n! C}{R^n}$ where $|f(z)| \leq C$, and $f^{(n)}(0) = 0$ for $n \geq 1$.

Proof: Morera's theorem is proved by using the existence of primitives, define $F(w)$ to be the contour integral from a fixed b to w, then $F'(w) = f(w)$ and so F is holomorphic, so we can turn F into a power series, and so $f(z) = F'(z)$ is holomorphic in $B(z_0, r) \in \Omega$.

Proof: Analytical continuation lemma, if $f(z) = 0$ where $z \in B(z_1, r_1) \subset B(w, R)$, then $f(z) = 0$ for $z \in B(w, R)$, we can prove this with a finite sequence of open balls $B(z_1, r_1) \subset B(z_2, r_2) \subset \ldots$ where the centre of $B(z_{j+1}, r_{j+1}) \in B(z_j, r_j)$, then induction.

Proof: Laurent theorem is proved by considering the annulus $\Omega = A(z_0, r_1, r_2)$ then let $\Omega_2 = \Omega \setminus \{\overline{B}(w, r)\}$, with a ball on w, then by Cauchy-Goursat, $\dfrac{1}{2\pi i} \displaystyle\int_{\partial \Omega_2} \frac{f(z)}{z - w}\, dz = 0$. Then break the integral into its 3 boundaries, apply the Cauchy integral formula to the $\partial B(w, r)$ boundary, then if z lines on $\partial B(z_0, r_2)$ then $\displaystyle\int_{\partial B(z_0, r_2)} \frac{f(z)}{z - w}\, dz = \sum_{n=0}^{\infty} (w - z_0)^n \int_{\partial B(z_0, r_2)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$. Similarly we can get the negative coefficients when z lies on $\partial B(z_0, r_1)$.

Proof: The p/q' formula is proved by using the fact that z_0 is a simple pole, so $\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) \dfrac{p(z)}{q(z)}$.

Proof: Cauchy's residue theorem is proved by taking balls $B(z_k, \epsilon)$ on every singularity, where the balls are disjoint, then define $\Upsilon = \mathrm{Int}(\Gamma) \setminus (\bigcup_{k=1}^{K} \overline{B}(z_k, \epsilon))$, then use Laurent's theorem to get $\displaystyle\int_\Gamma f(z)\, dz + \sum_{k=1}^{K} \int_{\partial B(z_k, \epsilon)} f(z)\, dz = 2\pi i \sum_{k=1}^{K} \mathrm{Res}(f, z_k)$.

Proof: Riemann-Lebesgue Lemma is proved by showing $|\hat{f}(\xi)| < \infty$ and $\lim_{\xi \to \xi_0} \hat{f}(\xi) = \hat{f}(\xi_0)$ by exchanging limits and integrals.