CS 6501 Natural Language Processing

Sequence Labeling (II)

Yangfeng Ji

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Department of Computer Science
University of Virginia

UNIVERSITY of VIRGINIA | ENGINEERING

Overview

1. Conditional Random Fields
2. Inference
3. Parameter Estimation

POS Tagging

Example

`Rain and wind from Florence will pop up on Tuesday`

POS tagging

$\texttt{Rain}_{\texttt{NN}}$ $\texttt{and}_{\texttt{CC}}$ $\texttt{wind}_{\texttt{NN}}$ $\texttt{from}_{\texttt{IN}}$ $\texttt{Florence}_{\texttt{NNP}}$ $\texttt{will}_{\texttt{MD}}$ $\texttt{pop}_{\texttt{VB}}$ $\texttt{up}_{\texttt{RP}}$ $\texttt{on}_{\texttt{IN}}$ $\texttt{Tuesday}_{\texttt{NNP}}$

- NN: Noun, singular or mass
- NNP: Proper noun, singular
- IN: Preposition or subordinating conjunction
- CC: Coordinating conjunction
- MD: Modal
- VB: Verb, base form
- RP: Particle

Sequence Labeling

Example

[Atlantis]$_{\text{MSIC}}$ touched down at [Kennedy Space Center]$_{\text{LOC}}$

Atlantis	touched	down	at	Kennedy	Space	Center	.
B_{MSIC}	O	O	O	B_{LOC}	I_{LOC}	I_{LOC}	O

Tag set

- B: beginning
- I: inside
- O: outside

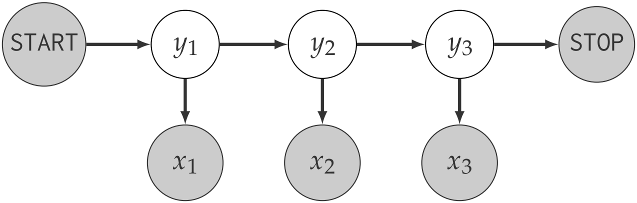
Category

- Person
- Location
- Organization
- Msic

Hidden Markov Models

$$P(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1} \left\{ P(y_i | y_{i-1}) P(x_i | y_i) \right\} \quad (1)$$

Graphical model



- $\boldsymbol{x}$: observation (e.g., sentences)
- $\boldsymbol{y}$: hidden variables (e.g., POS sequences)

Generative Models

$$P(\boldsymbol{x}, \boldsymbol{y}) = P(\boldsymbol{x}|\boldsymbol{y}) \cdot P(\boldsymbol{y}) \tag{2}$$

Generative Models

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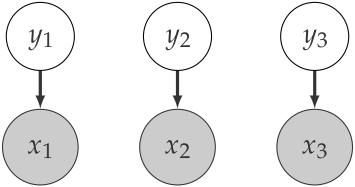
Factorization

$$P(\boldsymbol{x}|\boldsymbol{y}) = \prod_{i=1} \underbrace{P(x_i|y_i)}_{\text{Emission probability}} \tag{3}$$

Graphical model



Discriminative Models: Logistic Regression

$$P(\boldsymbol{y}|\boldsymbol{x}) = \frac{\exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}))}{\sum_{\boldsymbol{y}' \in \mathcal{Y}^T} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}'))} \tag{4}$$

- $\boldsymbol{y} \in \mathcal{Y}^T$ is a sequence, where T is the sequence length
- $|\mathcal{Y}^T| = K^T$ is a function of T, where K is the size of the label set

Question

How to build a discriminative sequential model?

Conditional Random Fields

Logistic Regression

$$P(\boldsymbol{y}|\boldsymbol{x}) = \frac{\exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}))}{\sum_{\boldsymbol{y}' \in \mathcal{Y}^T} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}'))} \tag{5}$$

Huge $\mathcal{Y}^T$ causes the problems on

- decoding $\arg\max_{\boldsymbol{y}'} P(\boldsymbol{y}'|\boldsymbol{x})$

Logistic Regression

$$P(\boldsymbol{y}|\boldsymbol{x}) = \frac{\exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}))}{\underbrace{\sum_{\boldsymbol{y}' \in \mathcal{Y}^T} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}'))}_{\text{partition function } Z}} \tag{5}$$

Huge $\mathcal{Y}^T$ causes the problems on

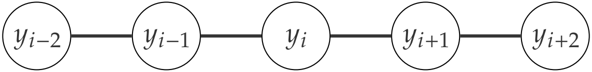
- decoding $\arg\max_{\boldsymbol{y}'} P(\boldsymbol{y}'|\boldsymbol{x})$
- computing the partition function

Markov Property

Global feature function:

$$f(x, y) \tag{6}$$

Markov assumption:



- Conditional independence
- Factorization over cliques

Decomposition of $f(x, y)$

$$f(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{T} \underbrace{f_i(\boldsymbol{x}, y_{i-1}, y_i)}_{\text{local feature function}} \tag{7}$$

- $f_i(\boldsymbol{x}, y_{i-1}, y_i)$ captures the transition from y_i to y_i
- i: the position to be tagged
- $y_i \in \mathcal{Y}$: POS tag at position i
- $y_{i-1} \in \mathcal{Y}$: POS tag at position $i - 1$
- $\boldsymbol{x}$: the entire sentence

Local Feature Function: Example

- standard features

[Lafferty et al., 2001]

Local Feature Function: Example

- standard features
- whether a spelling begins with upper case letter,
  - `IBM`, `Virginia`: PROPER NOUNS

[Lafferty et al., 2001]

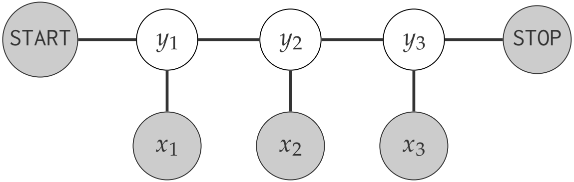
Local Feature Function: Example

- standard features
- whether a spelling begins with upper case letter,
  - `IBM`, `Virginia`: PROPER NOUNS
- whether it ends in one of the following suffixes:
  - `-ies` e.g., `parties`: PROPER NOUN, PLURAL
  - `-ly` e.g., `extremely`, `loudly`: ADVERB
  - `-ing` e.g., : VERB, GERUND OR PRESENT PARTICIPLE
  - ...

[Lafferty et al., 2001]

Graphical Model Representation

Conditional Random Fields:



Graphical Model Representation

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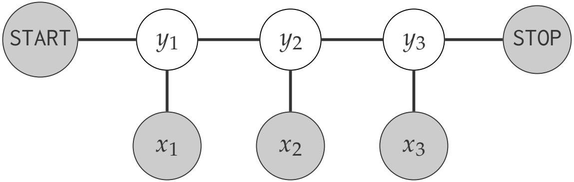
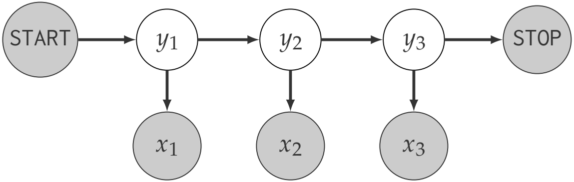


Hidden Markov Models:



CRF vs HMM

HMM factorization

$$P(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1} \left\{ P(y_i | y_{i-1}) P(x_i | y_i) \right\} \tag{8}$$

Let

- $\psi(y_1, y_2) = P(y_1)P(y_2|y_1)P(x_1|y_1)P(x_2|y_2)$
- $\psi(y_{i-1}, y_i) = P(y_i|y_{i-1})P(x_i|y_i), \forall i > 2$

CRF vs HMM

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- $\psi(y_{i-1}, y_i) = P(y_i|y_{i-1})P(x_i|y_i), \forall i > 2$

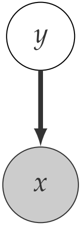
$$P(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{T} \psi(y_{i-1}, y_i) \tag{9}$$

with $Z = 1$

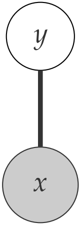
Directed vs. Undirected Graphical Models

- Undirected GMs allow more flexibile definitions
- Directed GMs can always be converted into an undirected GMs

$$P(x, y) = P(x|y)P(y)$$



$$P(x, y) = \psi(x, y)/Z$$

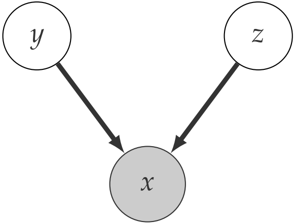


[Bishop, 2006, Chap. 8]

Directed vs. Undirected Graphical Models (II)

- Directed GMs can give more concise definitions

$$P(x, y, z) = P(x|y, z)P(y)P(z)$$



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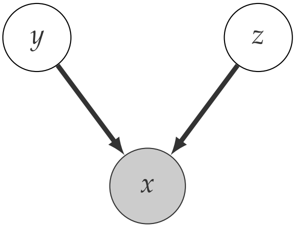
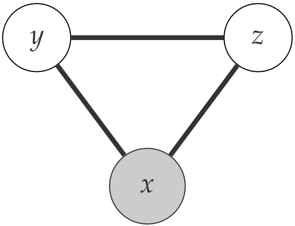


$$P(x, y, z) = \psi(x, y, z)/Z$$



[Bishop, 2006, Chap. 8]

Inference

Decode $P(\boldsymbol{y}|\boldsymbol{x})$

$$f(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{T} f_i(\boldsymbol{x}, y_{i-1}, y_i) \tag{10}$$

Decode $P(\boldsymbol{y}|\boldsymbol{x})$

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{T} \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \tag{10}$$

$$\begin{aligned}
\arg\max_{\boldsymbol{y} \in \mathcal{Y}^T} P(\boldsymbol{y}|\boldsymbol{x}) &= \arg\max_{\boldsymbol{y} \in \mathcal{Y}^T} \frac{\exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}))}{\sum_{\boldsymbol{y}' \in \mathcal{Y}^T} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}'))} \\
&= \arg\max_{\boldsymbol{y} \in \mathcal{Y}^T} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}))
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\end{aligned}$$

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\arg\max_{\boldsymbol{y}\in\mathcal{Y}^T} P(\boldsymbol{y}|\boldsymbol{x}) &= \arg\max_{\boldsymbol{y}\in\mathcal{Y}^T} \frac{\exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}))}{\sum_{\boldsymbol{y}'\in\mathcal{Y}^T} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}'))} \\
&= \arg\max_{\boldsymbol{y}\in\mathcal{Y}^T} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})) \\
&= \arg\max_{\boldsymbol{y}\in\mathcal{Y}^T} \boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \\
&= \arg\max_{\boldsymbol{y}\in\mathcal{Y}^T} \boldsymbol{\theta}^\top \sum_{i=1}^{T} \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i)
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Decode $P(\boldsymbol{y}|\boldsymbol{x})$

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&= \arg\max_{\boldsymbol{y} \in \mathcal{Y}^T} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})) \\
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&= \arg\max_{\boldsymbol{y} \in \mathcal{Y}^T} \boldsymbol{\theta}^\top \sum_{i=1}^{T} \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \\
&= \arg\max_{\boldsymbol{y} \in \mathcal{Y}^T} \sum_{i=1}^{T} \boldsymbol{\theta}^\top \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i)
\end{aligned}$$

Factorization

Factorize $\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$ with respect to timestep i

$$\sum_{i=1}^{T} \boldsymbol{\theta}^\top \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) = \underbrace{\sum_{j \le i-1} \boldsymbol{\theta}^\top \boldsymbol{f}_j(\boldsymbol{x}, y_{j-1}, y_j)}_{\text{past}} + \underbrace{\boldsymbol{\theta}^\top \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i)}_{\text{present}} + \underbrace{\sum_{k \ge i+1} \boldsymbol{\theta}^\top \boldsymbol{f}_k(\boldsymbol{x}, y_{k-1}, y_k)}_{\text{future}} \tag{11}$$

Viterbi Algorithm

$$s_i(k, k') = \boldsymbol{\theta}^\top \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1} = k', y_i = k)$$

Algorithm 11 The Viterbi algorithm. Each $s_m(k, k')$ is a local score for tag $y_m = k$ and $y_{m-1} = k'$.

for $k \in \{0, \ldots K\}$ **do**
  $v_1(k) = s_1(k, \Diamond)$
for $m \in \{2, \ldots, M\}$ **do**
  for $k \in \{0, \ldots, K\}$ **do**
    $v_m(k) = \max_{k'} s_m(k, k') + v_{m-1}(k')$
    $b_m(k) = \operatorname{argmax}_{k'} s_m(k, k') + v_{m-1}(k')$
$y_M = \operatorname{argmax}_k s_{M+1}(\blacklozenge, k) + v_M(k)$
for $m \in \{M-1, \ldots 1\}$ **do**
  $y_m = b_m(y_{m+1})$
return $\boldsymbol{y}_{1:M}$

[Eisenstein, 2018]

Parameter Estimation

Parameter Estimation: Logistic regression

When label y is still a single component

$$\frac{\partial \log P(y|\boldsymbol{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \boldsymbol{f}\left(\boldsymbol{x}, y\right) - \mathbb{E}_{Y|X}[\boldsymbol{f}\left(\boldsymbol{x}, y\right)] \tag{12}$$

where

$$\mathbb{E}_{Y|X}[\boldsymbol{f}\left(\boldsymbol{x}, y\right)] = \sum_{y \in \mathcal{Y}} \left\{ P(y|\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x}, y) \right\} \tag{13}$$

Parameter Estimation: CRFs

When label $\boldsymbol{y}$ is a sequence

$$\frac{\partial \log P(\boldsymbol{y}|\boldsymbol{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) - \mathbb{E}_{Y|X}[\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})] \tag{14}$$

where

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{T} \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \tag{15}$$

and

$$\mathbb{E}_{Y|X}[\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})] = \sum_{\boldsymbol{y}\in\mathcal{Y}^T} \left\{ P(\boldsymbol{y}|\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) \right\} \tag{16}$$

Expectation

$$
\begin{aligned}
\mathbb{E}_{Y|X}[\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})] &= \sum_{\boldsymbol{y}\in\mathcal{Y}^T} \left\{ P(\boldsymbol{y}|\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) \right\} \\
&= \sum_{\boldsymbol{y}\in\mathcal{Y}^T} \left\{ P(\boldsymbol{y}|\boldsymbol{x}) \sum_{i=1}^{T} \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \right\} \\
&= \sum_{\boldsymbol{y}\in\mathcal{Y}^T} \sum_{i=1}^{T} \left\{ P(\boldsymbol{y}|\boldsymbol{x}) \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \right\} \\
&= \sum_{i=1}^{T} \sum_{\boldsymbol{y}\in\mathcal{Y}^T} \left\{ P(\boldsymbol{y}|\boldsymbol{x}) \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \right\} \\
&= \sum_{i=1}^{T} \sum_{y_{i-1}\in\mathcal{Y}; y_i\in\mathcal{Y}} \left\{ P(y_{i-1}, y_i|\boldsymbol{x}) \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \right\}
\end{aligned}
$$

Expectation

$$
\begin{aligned}
\mathbb{E}_{Y|X}[\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})] &= \sum_{\boldsymbol{y} \in \mathcal{Y}^T} \left\{ P(\boldsymbol{y}|\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \right\} \\
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&= \sum_{i=1}^{T} \sum_{y_{i-1} \in \mathcal{Y}; y_i \in \mathcal{Y}} \left\{ P(y_{i-1}, y_i|\boldsymbol{x}) \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \right\}
\end{aligned}
$$

Basic Operation

$$P(y_{i-1}, y_i | \boldsymbol{x}) = \frac{\sum_{\boldsymbol{y} \setminus \{y_{i-1}, y_i\}} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}))}{\sum_{\boldsymbol{y}} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{s}, \boldsymbol{y}))} \tag{17}$$

Basic operation

$$\sum_{\tilde{\boldsymbol{y}}} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{s}, \tilde{\boldsymbol{y}})) \tag{18}$$

where $\tilde{\boldsymbol{y}}$ could be

- $\tilde{\boldsymbol{y}} = \boldsymbol{y} \setminus \{y_{i-1}, y_i\}$
- $\tilde{\boldsymbol{y}} = \boldsymbol{y}$

$\psi_i(y_{i-1}, y_i)$

$$
\begin{aligned}
\psi(\boldsymbol{y}) &= \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})) \\
&= \exp(\sum_{i=1}^{T} \boldsymbol{\theta}^\top \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \\
&= \prod_{i=1}^{T} \psi_i(y_{i-1}, y_i)
\end{aligned}
$$

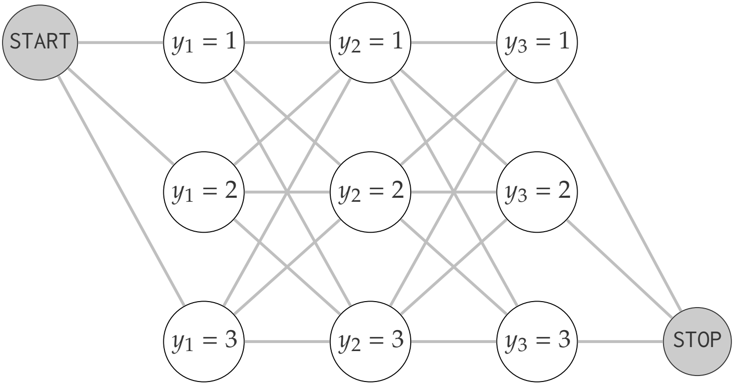
where

$$
\psi_i(y_{i-1}, y_i) = \exp(\boldsymbol{\theta}^\top \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i)) \tag{19}
$$

What We Need?

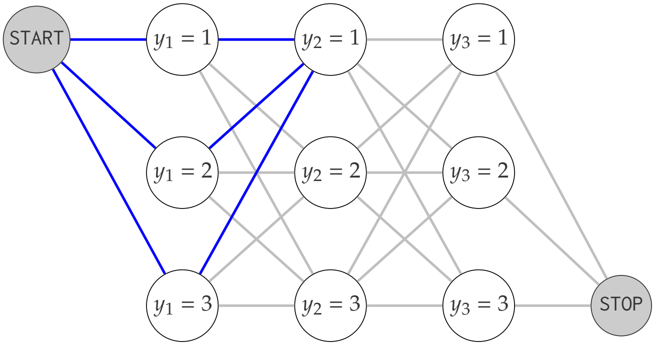
$$
\begin{aligned}
P(y_{i-1}, y_i | \boldsymbol{x}) &= \frac{\sum_{\boldsymbol{y} \setminus \{y_{i-1}, y_i\}} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}))}{\sum_{\boldsymbol{y}} \exp(\boldsymbol{\theta}^\top \boldsymbol{f}(\boldsymbol{s}, \boldsymbol{y}))} \\
&= \frac{\sum_{\boldsymbol{y} \setminus \{y_{i-1}, y_i\}} \prod_{i=1}^{T} \psi_i(y_{i-1}, y_i)}{\sum_{\boldsymbol{y}} \prod_{i=1}^{T} \psi_i(y_{i-1}, y_i)}
\end{aligned}
$$

Forward-Backward Algorithm: Basic idea



Forward-Backward Algorithm: Forward term

$$P(y_2 = 1, y_3 = 1) \propto \sum_{y_1} \psi_1(\textsc{start}, y_1)\psi_2(y_1, y_2 = 1) \cdot \psi_3(y_2 = 1, y_3 = 1) \tag{20}$$



Forward-Backward Algorithm: Forward term

Forward term

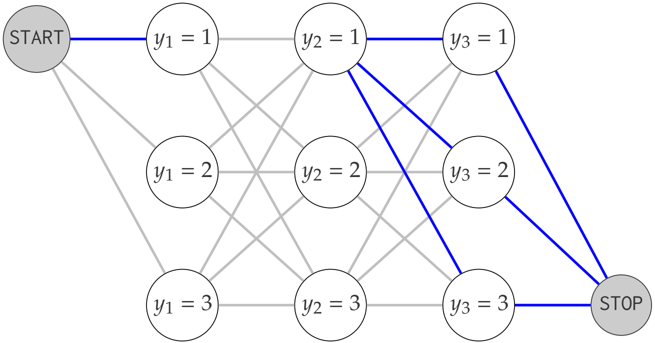
$$\alpha_i(y) = \sum_{y' \in \mathcal{Y}} \alpha_{i-1}(y')\psi_i(y', y) \tag{21}$$

Base case

$$\alpha_1(y) = \psi(\textsc{start}, y) \tag{22}$$

Forward-Backward Algorithm: Back term

$$P(y_1 = 1, y_2 = 1) \propto \sum_{y_3} \psi_1(\textsc{start}, y_1)\psi_2(y_1 = 1, y_2 = 1) \cdot \psi_3(y_2 = 1, y_3) \tag{23}$$



Forward-Backward Algorithm: Backward term

Backward term

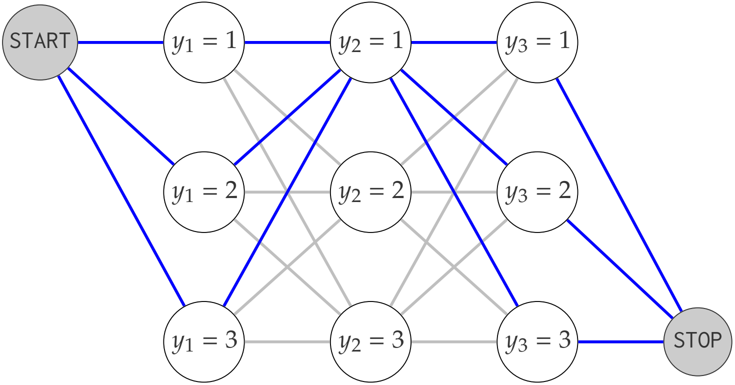
$$\beta_i(y) = \sum_{y' \in \mathcal{Y}} \psi_{i+1}(y, y')\beta_{i+1}(y') \tag{24}$$

Base case

$$\beta_T(y) = 1 \tag{25}$$

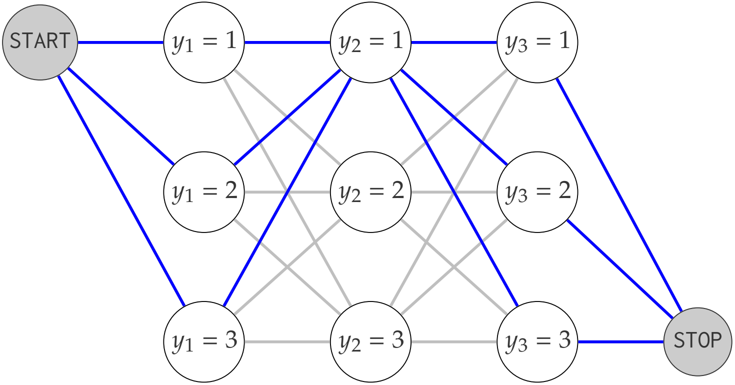
Test

What we compute here?



Test

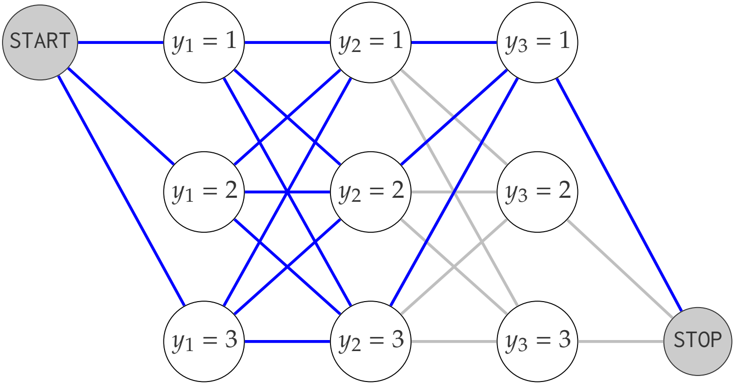
What we compute here?



Answer: $P(y_2 = 1)$

Normalization Term Z

$$Z = \sum_{\boldsymbol{y} \in \mathcal{Y}^T} \psi(\boldsymbol{y}) = \sum_{y \in \mathcal{Y}} \alpha_T(y)\beta_T(y) = \sum_{y \in \mathcal{Y}} \alpha_T(y) \qquad (26)$$



Expectation

$$
\begin{aligned}
\mathbb{E}_{Y|X}[\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})] &= \sum_{\boldsymbol{y} \in \mathcal{Y}^T} \left\{ P(\boldsymbol{y}|\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \right\} \\
&= \sum_{\boldsymbol{y} \in \mathcal{Y}^T} \left\{ P(\boldsymbol{y}|\boldsymbol{x}) \sum_{i=1}^{T} \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \right\} \\
&= \sum_{\boldsymbol{y} \in \mathcal{Y}^T} \sum_{i=1}^{T} \left\{ P(\boldsymbol{y}|\boldsymbol{x}) \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \right\} \\
&= \sum_{i=1}^{T} \sum_{\boldsymbol{y} \in \mathcal{Y}^T} \left\{ P(\boldsymbol{y}|\boldsymbol{x}) \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \right\} \\
&= \sum_{i=1}^{T} \sum_{y_{i-1} \in \mathcal{Y}; y_i \in \mathcal{Y}} \left\{ P(y_{i-1}, y_i|\boldsymbol{x}) \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \right\}
\end{aligned}
$$

Parameter Estimation: CRFs

$$\frac{\partial \log P(\boldsymbol{y}|\boldsymbol{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) - \mathbb{E}_{Y|X}[\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})] \tag{27}$$

where

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{T} \boldsymbol{f}_i(\boldsymbol{x}, y_{i-1}, y_i) \tag{28}$$

and

$$\mathbb{E}_{Y|X}[\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})] = \sum_{\boldsymbol{y}\in\mathcal{Y}^T} \left\{ P(\boldsymbol{y}|\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) \right\} \tag{29}$$

Summary

1. Conditional Random Fields
   1.1 Logistic Regression
   1.2 Decomposition and CRF Fromulation
   1.3 Directed vs. Undirected Graphical Models
2. Inference
   2.1 Viterbi Algorithm
3. Parameter Estimation
   3.1 Gradient based Learning
   3.2 Forward-Backward Algorithm

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