Dynamic Programming

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Outline

What is DP?

- Wikipedia definition: "method for solving complex problems by breaking them down into simpler subproblems"

- This definition will make sense once we see some examples
  - Actually, we'll only see problem solving examples today

Steps for Solving DP Problems

1. Define subproblems
2. Write down the recurrence that relates subproblems
3. Recognize and solve the base cases

- Each step is very important!

Outline

1-dimensional DP Example

- Problem: given n, find the number of different ways to write n as the sum of 1, 3, 4
- Example: for $n = 5$, the answer is 6

$$
\begin{aligned}
5 &= 1+1+1+1+1 \\
&= 1+1+3 \\
&= 1+3+1 \\
&= 3+1+1 \\
&= 1+4 \\
&= 4+1
\end{aligned}
$$

1-dimensional DP Example

- Define subproblems
  - Let D_n be the number of ways to write n as the sum of 1, 3, 4
- Find the recurrence
  - Consider one possible solution $n = x_1 + x_2 + \cdots + x_m$
  - If $x_m = 1$, the rest of the terms must sum to $n - 1$
  - Thus, the number of sums that end with $x_m = 1$ is equal to D_{n-1}
  - Take other cases into account ($x_m = 3$, $x_m = 4$)

1-dimensional DP Example

- Recurrence is then

$$D_n = D_{n-1} + D_{n-3} + D_{n-4}$$

- Solve the base cases
  - $D_0 = 1$
  - $D_n = 0$ for all negative n
  - Alternatively, can set: $D_0 = D_1 = D_2 = 1$, and $D_3 = 2$

- We're basically done!

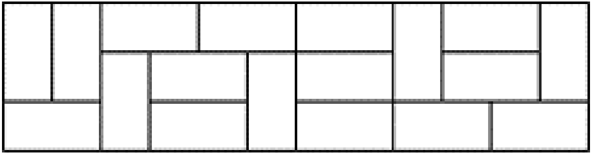
Implementation

```
D[0] = D[1] = D[2] = 1; D[3] = 2;
for(i = 4; i <= n; i++)
    D[i] = D[i-1] + D[i-3] + D[i-4];
```

- Very short!
- Extension: solving this for huge n, say $n \approx 10^{12}$
  - Recall the matrix form of Fibonacci numbers

POJ 2663: Tri Tiling

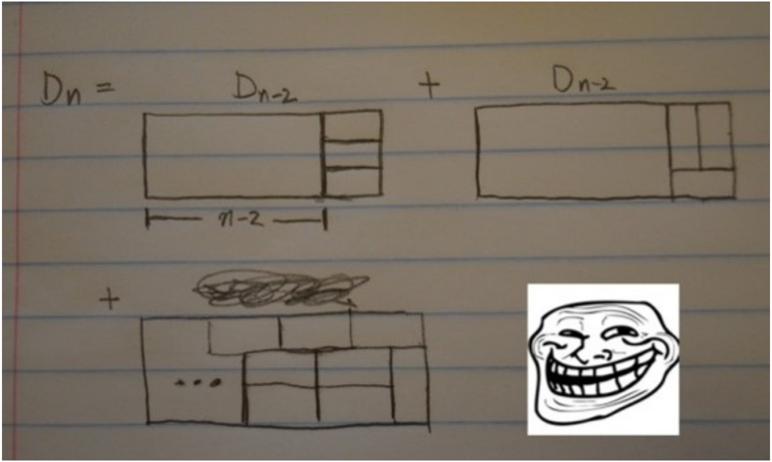
- Given n, find the number of ways to fill a $3 \times n$ board with dominoes
- Here is one possible solution for $n = 12$



POJ 2663: Tri Tiling

- Define subproblems
  - Define D_n as the number of ways to tile a $3 \times n$ board
- Find recurrence
  - Uuuhhhhh...

Troll Tiling

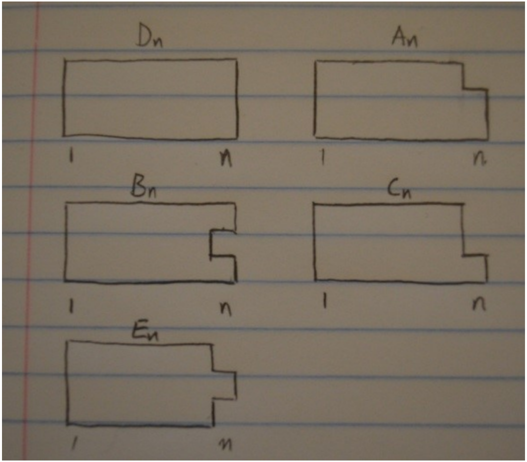


Defining Subproblems

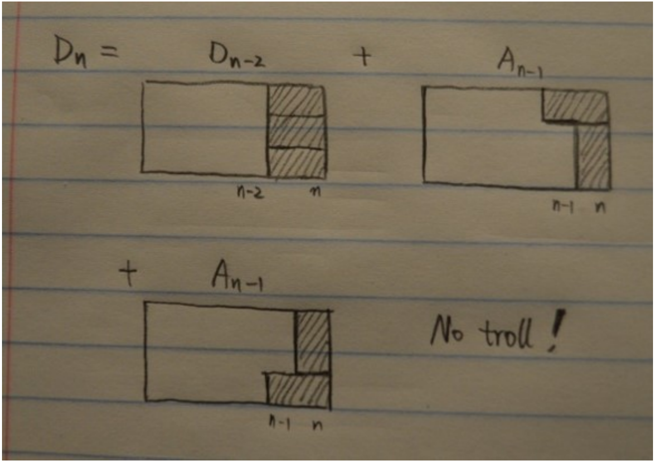
- Obviously, the previous definition didn't work very well
- D_n's don't relate in simple terms

- What if we introduce more subproblems?

Defining Subproblems



Finding Recurrences



Finding Recurrences

- Consider different ways to fill the nth column
  - And see what the remaining shape is
- Exercise:
  - Finding recurrences for A_n, B_n, C_n
  - Just for fun, why is B_n and E_n always zero?
- Extension: solving the problem for $n \times m$ grids, where n is small, say $n \leq 10$
  - How many subproblems should we consider?

Outline

2-dimensional DP Example

- Problem: given two strings x and y, find the longest common subsequence (LCS) and print its length
- Example:
  - x: `A`**`BC`**`BD`**`AB`**
  - y: **`B`**`D`**`CAB`**`C`
  - "`BCAB`" is the longest subsequence found in both sequences, so the answer is 4

Solving the LCS Problem

- Define subproblems
  - Let D_{ij} be the length of the LCS of $x_{1\dots i}$ and $y_{1\dots j}$
- Find the recurrence
  - If $x_i = y_j$, they both contribute to the LCS
    - $D_{ij} = D_{i-1,j-1} + 1$
  - Otherwise, either x_i or y_j does not contribute to the LCS, so one can be dropped
    - $D_{ij} = \max\{D_{i-1,j}, D_{i,j-1}\}$
  - Find and solve the base cases: $D_{i0} = D_{0j} = 0$

Implementation

```
for(i = 0; i <= n; i++) D[i][0] = 0;
for(j = 0; j <= m; j++) D[0][j] = 0;
for(i = 1; i <= n; i++) {
    for(j = 1; j <= m; j++) {
        if(x[i] == y[j])
            D[i][j] = D[i-1][j-1] + 1;
        else
            D[i][j] = max(D[i-1][j], D[i][j-1]);
    }
}
```

Outline

Interval DP Example

- Problem: given a string $x = x_{1 \dots n}$, find the minimum number of characters that need to be inserted to make it a palindrome

- Example:
  - x: `Ab3bd`
  - Can get "`dAb3bAd`" or "`Adb3bdA`" by inserting 2 characters (one '`d`', one '`A`')

Interval DP Example

- Define subproblems
  - Let D_{ij} be the minimum number of characters that need to be inserted to make $x_{i\dots j}$ into a palindrome
- Find the recurrence
  - Consider a shortest palindrome $y_{1\dots k}$ containing $x_{i\dots j}$
  - Either $y_1 = x_i$ or $y_k = x_j$ (why?)
  - $y_{2\dots k-1}$ is then an optimal solution for $x_{i+1\dots j}$ or $x_{i\dots j-1}$ or $x_{i+1\dots j-1}$
    - Last case possible only if $y_1 = y_k = x_i = x_j$

Interval DP Example

- Find the recurrence

$$D_{ij} = \begin{cases} 1 + \min\{D_{i+1,j}, D_{i,j-1}\} & x_i \neq x_j \\ D_{i+1,j-1} & x_i = x_j \end{cases}$$

- Find and solve the base cases: $D_{ii} = D_{i,i-1} = 0$ for all i

- The entries of D must be filled in increasing order of $j - i$

Interval DP Example

```
// fill in base cases here
for(t = 2; t <= n; t++)
    for(i = 1, j = t; j <= n; i++, j++)
        // fill in D[i][j] here
```

- Note how we use an additional variable `t` to fill the table in correct order
- And yes, for loops can work with multiple variables

An Alternate Solution

- Reverse x to get x^R
- The answer is $n - L$, where L is the length of the LCS of x and x^R

- Exercise: Think about why this works

Outline

Dynamic Programming

1-dimensional DP

2-dimensional DP

Interval DP

Tree DP

Subset DP

Tree DP Example

- Problem: given a tree, color nodes black as many as possible without coloring two adjacent nodes
- Subproblems:
  - First, we arbitrarily decide the root node r
  - B_v: the optimal solution for a subtree having v as the root, where we color v black
  - W_v: the optimal solution for a subtree having v as the root, where we don't color v
  - Answer is $\max\{B_r, W_r\}$

Tree DP Example

- Find the recurrence
  - Crucial observation: once v's color is determined, subtrees can be solved independently
  - If v is colored, its children must not be colored

$$B_v = 1 + \sum_{u \in \text{children}(v)} W_u$$

  - If v is not colored, its children can have any color

$$W_v = 1 + \sum_{u \in \text{children}(v)} \max\{B_u, W_u\}$$

- Base cases: leaf nodes

Outline

Subset DP Example

- Problem: given a weighted graph with n nodes, find the shortest path that visits every node exactly once (Traveling Salesman Problem)

- Wait, isn't this an NP-hard problem?
  - Yes, but we can solve it in $O(n^2 2^n)$ time
  - Note: brute force algorithm takes $O(n!)$ time

Subset DP Example

- Define subproblems
  - $D_{S,v}$: the length of the optimal path that visits every node in the set S exactly once and ends at v
  - There are approximately $n2^n$ subproblems
  - Answer is $\min_{v \in V} D_{V,v}$, where V is the given set of nodes
- Let's solve the base cases first
  - For each node v, $D_{\{v\},v} = 0$

Subset DP Example

- Find the recurrence
  - Consider a path that visits all nodes in S exactly once and ends at v
  - Right before arriving v, the path comes from some u in $S - \{v\}$
  - And that subpath has to be the optimal one that covers $S - \{v\}$, ending at u
  - We just try all possible candidates for u

$$D_{S,v} = \min_{u \in S - \{v\}} \left(D_{S-\{v\},u} + \text{cost}(u, v) \right)$$

Working with Subsets

- When working with subsets, it's good to have a nice representation of sets
- Idea: Use an integer to represent a set
  - Concise representation of subsets of small integers $\{0, 1, \ldots\}$
  - If the ith (least significant) digit is 1, i is in the set
  - If the ith digit is 0, i is not in the set
  - *e.g.*, $19 = \texttt{010011}_{(2)}$ in binary represent a set $\{0, 1, 4\}$

Using Bitmasks

- Union of two sets `x` and `y`: `x | y`
- Intersection: `x & y`
- Symmetric difference: `x ^ y`
- Singleton set $\{i\}$: `1 << i`
- Membership test: `x & (1 << i) != 0`

Conclusion

- Wikipedia definition: “a method for solving complex problems by breaking them down into simpler subproblems”
  - Does this make sense now?

- Remember the three steps!
  1. Defining subproblems
  2. Finding recurrences
  3. Solving the base cases