一. 定积分的换元法.

①定理1: 1) $f(x)$ 在 $[a,b]$ 上连续.

2) $x=\varphi(t)$ 在 $[\alpha,\beta]$ 上是单调且有连续导数.

3) 当 t 在 $[\alpha,\beta]$ 上变化时, $x=\varphi(t)$ 在 $[a,b]$ 上变化, 且 $\varphi(\alpha)=a$, $\varphi(\beta)=b$.

则有: $\boxed{\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt} = F[\varphi(t)]\Big|_\alpha^\beta$

Eg. 计算 $\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}dx \ (a>0)$.

$x=a \Rightarrow t=\frac{\pi}{2}$,

令 $x=a\sin t$, $dx=a\cos t\,dt$.

$x=0 \Rightarrow t=0$.

原式 $= \int_0^{\frac{\pi}{2}} \frac{a\cos t}{a\sin t+a\cos t}dt = \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t}dt$

$= \underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1+\frac{\cos t-\sin t}{\cos t+\sin t}\right)}\,dt = \frac{1}{2}\cdot\frac{\pi}{2}+\frac{1}{2}\cdot\left[\ln|\cos t+\sin t|\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{4}+\frac{1}{2}\cdot(\ln 1-\ln 1)$

$= \frac{\pi}{4}$.

Eg. 若 $f(x)\in C[0,1]$, 证明:

(1). $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$.

令 $x=\frac{\pi}{2}-t$，　$x=\frac{\pi}{2}$，$t=0$，$x=0$，$t=\frac{\pi}{2}$．　$dx=-dt$．

$\sin x=\sin(\frac{\pi}{2}-t)=\cos t$，$\cos x=\cos(\frac{\pi}{2}-t)=-\sin t$．

$$\int_0^{\frac{\pi}{2}} f(\sin x)dx=-\int_{\frac{\pi}{2}}^{0} f(\cos t)dt=\int_0^{\frac{\pi}{2}} f(\cos t)dt.$$

故 $\int_0^{\frac{\pi}{2}} f(\sin x)dx=\int_0^{\frac{\pi}{2}} f(\cos x)dx$．

$$\Rightarrow \boxed{I_n=\int_0^{\frac{\pi}{2}} \sin^n x\,dx=\int_0^{\frac{\pi}{2}} \cos^n x\,dx}$$

例．$\int_0^{\pi} x\cdot f(\sin x)dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$．

令 $x=\pi-t$，　$x=\pi$，$t=0$，$x=0$，$t=\pi$．　$dx=-dt$．

$\sin x=\sin(\pi-t)=\sin t$．

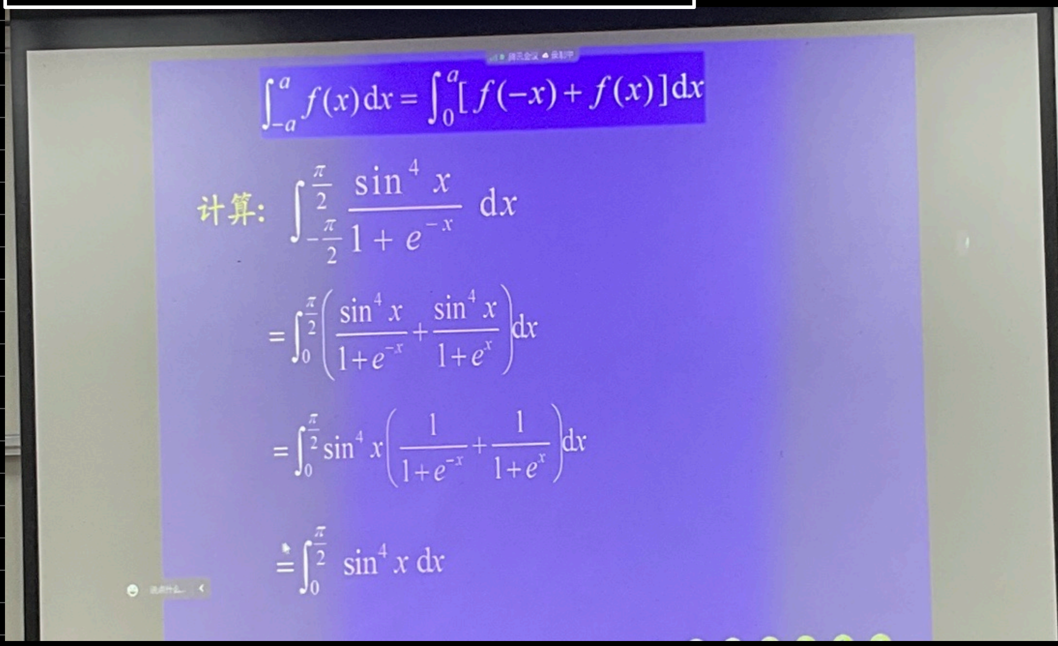
$$左边=-\int_{\pi}^{0}(\pi-t)f(\sin t)dt=\int_0^{\pi}(\pi-t)f(\sin t)dt$$

$$=\int_0^{\pi}\pi f(\sin x)dx-\int_0^{\pi}x\cdot f(\sin x)dx$$

故 $\int_0^{\pi} x\cdot f(\sin x)dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$．

$$\Rightarrow \boxed{I_n = \int_0^{\pi} x\cdot f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx.}$$

$$\boxed{\int_{-a}^{a} f(x)\,dx = \int_0^{a} [f(-x)+f(x)]\,dx,}$$ 无须奇偶性.



化简关于原点对称的式子.

周期函数.

$$\int_T^{a+T} f(x)\,dx = \int_0^{T} f(x)\,dx = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx.$$

$$\int_x^{x+T} f(t)\,dt = \Phi(x).$$ $\equiv C$ (为常数!) 个乱取!

$$\Phi'(x) = f(x+T) - f(x) = 0. \Rightarrow C.$$

二、定积分的分部积分法.

② 定理2：设函数 $u(x), v(x)$ 在区间 $[a,b]$ 上具有连续导数，则有

$$\boxed{\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du}$$

Eg. $\int_0^{\frac{1}{2}} \arcsin x\,dx = x\arcsin x\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x\cdot\frac{1}{\sqrt{1-x^2}}\,dx.$

$= \frac{1}{2}\arcsin\frac{1}{2} + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,d(1-x^2).$

$= \frac{\pi}{12} + \sqrt{1-x^2}\Big|_0^{\frac{1}{2}}.$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$

Eg. 证明：$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx.$ $I_0 = 1$

$I_1 = \frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_{\frac{\pi}{2}}^{0} \sin^n(\frac{\pi}{2}-t)\,d(\frac{\pi}{2}-t) = \int_0^{\frac{\pi}{2}} \cos^n t\,dt = \int_0^{\frac{\pi}{2}} \cos^n x\,dx.$

$$I_n = \begin{cases} \frac{n-1}{n}, \frac{n-3}{n-2}, \frac{n-5}{n-4} \cdots \frac{3}{4}, \frac{1}{2}, \frac{\pi}{2}. & n\text{为正偶数} \\ \frac{n-1}{n}, \frac{n-3}{n-2}, \frac{n-5}{n-4} \cdots \frac{4}{5}, \frac{2}{3}, & n\text{为大于1的正奇数}. \end{cases}$$

↳ $$I_n = \begin{cases} \dfrac{(n-1)!!}{n!!} \cdot \dfrac{\pi}{2} & (n\text{为正偶数}) \\ \dfrac{(n-1)!!}{n!!} & (n\text{为大于1的正奇数}) \end{cases}$$