一. 1. (A)

$$f(0^+)=\lim_{x\to0^+}\frac{\sqrt{a+4x}-1}{x}=\begin{cases}\infty & 当 a\neq 1 时\\ 2 & 当 a=1 时,\end{cases}$$

$$f(0^-)=b$$

2. (C)

$$f(0^+)=\lim_{x\to0^+}x^2\sin x=0$$

$$f(0^-)=\lim_{x\to0^-}\frac{1-\cos x}{x}=\lim_{x\to0^-}\frac{\frac12x^2}{x}=\lim_{x\to0^-}\frac12x=0$$

$$f'_+(0)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{x^2\sin x-0}{x-0}$$

$$=\lim_{x\to0^+}x\sin x=0$$

$$f'_-(0)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{\frac{1-\cos x}{x}-0}{x-0}$$

$$=\lim_{x\to0^-}\frac{1-\cos x}{x^2}=\frac12$$

3. (A)

4. (C)

$$\left|(-1)^n\left(1-\cos\frac{\alpha}{n}\right)\right|=1-\cos\frac{\alpha}{n}\sim\frac{\alpha}{2}\cdot\frac{1}{n^2}$$

5. (B)

记 $I=\int_1^{+\infty}f(x)dx$，则 $f(x)=e^{-x}+\frac{2I}{x^2}$

$$I=\int_1^{+\infty}\left(e^{-x}+\frac{2I}{x^2}\right)dx=\left[-e^{-x}-\frac{2I}{x}\right]_1^{+\infty}=e^{-1}+2I$$

$\therefore I = -e^{-1}$, $f(x) = e^{-x} - \frac{2e^{-1}}{x^2}$

$\therefore f(1) = -e^{-1}$

二. 1. $\underline{y = 2x+2}$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{2x^2+1}{x(x-1)} = 2$$

$$b = \lim_{x\to\infty}(f(x) - ax) = \lim_{x\to\infty}\left(\frac{2x^2+1}{x-1} - 2x\right)$$

$$= \lim_{x\to\infty} \frac{2x+1}{x-1} = 2$$

2. $\underline{\frac{\sin t - t\cos t}{4t^3}}$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = -\sin t$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = -\frac{\sin t}{2t} = -\frac{1}{2}\cdot\frac{\sin t}{t}$$

$$\frac{dy'}{dt} = -\frac{1}{2}\cdot\frac{(\cos t)\cdot t - (\sin t)\cdot 1}{t^2} = \frac{\sin t - t\cos t}{2t^2}$$

$$\frac{d^2y^2}{dx^2} = \frac{dy'}{dt}\Big/\frac{dx}{dt} = \frac{\sin t - t\cos t}{4t^3}$$

3. $\underline{8}$

$$原式 = \int_{-2}^{2} x^2 \sin x\,dx + \int_{-2}^{2} x^2|x|\,dx$$

$$= 0 + 2\int_0^2 x^3\,dx = 2\left[\frac{1}{4}x^4\right]_0^2 = 8.$$

4. $\frac{64}{15}\pi$

$\boxed{3}$

$$V = V_1 - V_2 = \int_0^2 \pi(2x)^2 dx - \int_0^2 \pi(x^2)^2 dx$$

$$= \pi\int_0^2 (4x^2 - x^4)dx = \pi\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$

$$= \frac{64}{15}\pi.$$

5. $\frac{1}{2e} - \frac{1}{2}$

首先 $f(1)=0$, $f'(x) = e^{-x^2}$. 于是

$$\int_0^1 f(x)\,dx \overset{\text{分部积分}}{=\!=\!=} [xf(x)]_0^1 - \int_0^1 x\,df(x)$$

$$= -\int_0^1 xe^{-x^2}dx = \left[\frac{1}{2}e^{-x^2}\right]_0^1 = \frac{1}{2e} - \frac{1}{2}.$$

三 1. 首先当 $x\to 0$ 时,

$$\sqrt{1+x\sin x} - 1 \sim \frac{1}{2}x\sin x \sim \frac{1}{2}x^2.$$

于是

$$\text{原式} = \lim_{x\to 0}\frac{e^x - \sin x - 1}{\frac{1}{2}x^2} \overset{\text{洛必达}}{\underset{\text{法则}}{=\!=\!=}} \lim_{x\to 0}\frac{e^x - \cos x}{x}$$

$$\overset{\text{洛必达}}{\underset{\text{法则}}{=\!=\!=}} \lim_{x\to 0}\frac{e^x + \sin x}{1} = 1.$$

$\boxed{4}$

2. 因 $x^2+6x+10=(x+3)^2+1$

所以 $$\int \frac{1}{x^2+6x+10}dx = \arctan(x+3)+C.$$

又令 $x=\sin t$, $t\in(-\frac{\pi}{2},\frac{\pi}{2})$. 则

$\sqrt{1-x^2}=\cos t$, $dx=\cos t\,dt$. 于是

$$\int\frac{x^2}{\sqrt{1-x^2}}dx=\int\frac{\sin^2 t}{\cos t}\cdot\cos t\,dt=\int\sin^2 t\,dt$$

$$=\int\frac{1-\cos(2t)}{2}dt=\frac{1}{2}t-\frac{1}{4}\sin(2t)+C$$

$$=\frac{1}{2}t-\frac{1}{2}\sin t\cos t+C$$

$$=\frac{1}{2}\arcsin x-\frac{1}{2}x\sqrt{1-x^2}+C$$

综上所述,

原式 $=\arctan(x+3)+\frac{1}{2}\arcsin x-\frac{1}{2}x\sqrt{1-x^2}+C$.

3. $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_0^{\pi}x\,dx=\frac{\pi}{2}$,

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}x\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\frac{1}{n}x\sin nx+\frac{1}{n^2}\cos nx\right]_0^{\pi}=\frac{(-1)^n-1}{n^2\pi}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{\pi}x\sin nx\,dx$$

$$=\frac{1}{\pi}\left[-\frac{1}{n}x\cos nx+\frac{1}{n^2}\sin nx\right]_0^{\pi}=\frac{(-1)^{n+1}}{n}$$

于是 $f(x)$ 的傅立叶级数为

$$\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{n^2\pi}\cos nx+\frac{(-1)^{n-1}}{n}\sin nx\right).$$

其中系数 $a_{2n+1}=-\frac{2}{(2n+1)^2\pi}$

四 1.
$$\left.\begin{array}{l} f(0^-)=f(0)=c \\ f(0^+)=\lim\limits_{x\to 0^+}\frac{e^x-1}{x}=1 \end{array}\right\}\Rightarrow c=1$$

$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x^2+bx+c-c}{x}$$
$$=\lim_{x\to 0^-}(x+b)=b$$

$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{\frac{e^x-1}{x}-1}{x}$$
$$=\lim_{x\to 0^+}\frac{e^x-1-x}{x^2}\overset{\text{洛比达}}{\underset{\text{法则}}{=\!=}}\lim_{x\to 0^+}\frac{e^x-1}{2x}=\frac{1}{2}$$

$\because f(x)$ 在 $x=0$ 处可导，$\therefore b=\frac{1}{2}$.

记 $g(x)=f(x)-\log_b x=\frac{e^x-1}{x}+\frac{\ln x}{\ln 2}$

则 $g'(x)=\frac{(x-1)e^x+1}{x^2}+\frac{1}{x\ln 2}=\frac{h(x)}{x^2}+\frac{1}{x\ln 2}$

其中 $h(x)=(x-1)e^x+1$.

当 $x>0$ 时，$h'(x)=xe^x>0$，所以 $h(x)$ 在 $[0,+\infty)$ 上严格单调增加. 于是当 $x>0$ 时，$h(x)>h(0)=0$，所以

$$g'(x)=\frac{h(x)}{x^2}+\frac{1}{x\ln 2}>0$$

即 $g(x)$ 在 $(0,+\infty)$ 上严格单调增加.

$\because \lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} \frac{e^x-1}{x} + \lim\limits_{x\to 0^+} \frac{\ln x}{\ln 2} = 1-\infty = -\infty$.

$\lim\limits_{x\to +\infty} g(x) = \lim\limits_{x\to +\infty} \left(\frac{e^x-1}{x} + \frac{\ln x}{\ln 2}\right) = +\infty$

$\therefore g(x)$ 在 $(0,+\infty)$ 上有唯一零点，即方程

$f(x) = \log_2 x$ 有唯一解.

2. 记 $f(x) = \sum\limits_{n=1}^{\infty} n^2 x^n = x \sum\limits_{n=1}^{\infty} (n x^n)' = x g'(x)$,

这里 $g(x) = \sum\limits_{n=1}^{\infty} n x^n = x \sum\limits_{n=1}^{\infty} (x^n)' = x\left(\sum\limits_{n=1} x^n\right)'$

$= x\left(\frac{1}{1-x} - 1\right)' = \frac{x}{(1-x)^2}$

$g'(x) = \frac{1+x}{(1-x)^3}$ $\quad \therefore f(x) = x g'(x) = \frac{x(1+x)}{(1-x)^3}$

$\sum\limits_{n=1}^{\infty} \frac{n^2}{2^n} = f\left(\frac{1}{2}\right) = 6$

五 1. 对方程两端关于 x 求导得:

$6y^2y' - 4yy' + 2(y + xy') - 2x = 0$

$\Rightarrow y' = \frac{x-y}{3y^2-2y+x}$

$\Rightarrow y'' = \frac{(1-y')(3y^2-2y+x) - (x-y)(6yy'-2y'+1)}{(3y^2-2y+x)^2}$

令 $y'=0$ 得: $x-y=0$

联立

$$\begin{cases} x-y=0 \\ 2y^3-2y^2+2xy-x^2-1=0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=1 \end{cases}$$

可得唯一驻点 $(1,1)$. 又因

$$y''\Big|_{\substack{x=1\\y=1\\y'=0}} = \frac{1}{2} > 0$$

∴ 该驻点是极小值点.

2. (1) $dF = \gamma dA = \rho g x \cdot a dx = \rho g a\, x dx$

$$F = \int_0^a \rho g a\, x dx = \rho g a \left[\frac{1}{2}x^2\right]_0^a = \frac{1}{2}\rho g a^3$$

(2) $dV = \frac{\sqrt{3}}{4} a^2 dx$

$$dW = \rho dV g \cdot x = \frac{\sqrt{3}}{4}\rho g a^2 x dx$$

$$W = \int_0^a \frac{\sqrt{3}}{4}\rho g a^2 x dx = \frac{\sqrt{3}}{4}\rho g a^2 \left[\frac{1}{2}x^2\right]_0^a$$

$$= \frac{\sqrt{3}}{8}\rho g a^4$$

六 1. 略

2. (1) 令 $g(x)=f(x)-x$. 则因 $g(0)=g(1)=0$, 所以 $\exists \xi \in (0,1)$ 使得 $g'(\xi)=0$, 即 $f'(\xi)=1$.

(2) 因 $f(x)$ 是奇函数, 所以 $f'(x)$ 是偶函数. 于是 $f'(-\xi) = f'(\xi) = 1$.

令 $h(x) = e^x(f'(x)-1)$. 则 $h(-\xi)=h(\xi)=0$.

$\boxed{8}$

附加题：$f'(x)=\frac{x^2(x-3)}{(x-1)^3}$，$f''(x)=\frac{6x}{(x-1)^4}$.

(1) 单调增区间 $(-\infty,1)$，$[3,+\infty)$

单调减区间 $(1,3]$，极小值点 3，极小值 $f(3)=\frac{27}{4}$

(2) 凹区间 $[0,1)$，$(1,+\infty)$；凸区间 $(-\infty,0]$.

拐点 $(0,0)$

(3)

$$a=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^2}{(x-1)^2}=1$$

$$b=\lim_{x\to\infty}(f(x)-ax)=\lim_{x\to\infty}\left(\frac{x^3}{(x-1)^2}-x\right)$$

$$=\lim_{x\to\infty}\frac{2x^2-x}{(x-1)^2}=2$$

渐近线：$y=x+2$.