2016-2017 春微积分（I）-2 半期考试参考答案

一、填空题（每小题 4 分，共 20 分）

1. 曲线 $\begin{cases} y = 2x^2 \\ z = 0 \end{cases}$ 绕 y 轴旋转一周所成的曲面方程为 $\underline{y = 2x^2 + 2z^2}$.

2. 设 $z = x^y (x > 0;\ x \neq 1)$,则 $dz = \underline{yx^{y-1}dx + x^y \ln x dy}$.

3. 改变二次积分的积分顺序 $\int_0^1 dy \int_y^{3y} f(x, y)\, dx = \underline{\int_0^1 dx \int_{x/3}^{x} f(x, y)\, dy + \int_1^3 dx \int_{x/3}^{1} f(x, y)\, dy}$.

4. 函数 $f(x, y) = x^2 y$ 在点 $(1, 1)$ 处方向导数的最大值为 $\underline{\sqrt{5}}$.

5. 曲线 $\begin{cases} z = xy \\ x + y + z = 3 \end{cases}$ 上点 $(1, 1, 1)$ 处的切线方程为 $\underline{\dfrac{x-1}{1} = \dfrac{y-1}{-1} = \dfrac{z-1}{0}}$.

二、解答题（每小题 10 分，共 60 分）

1. 设 $z = z(x),\ y = y(x)$ 由方程 $z = f(y, z + x)$ 及 $x + y + z = 1$ 确定，求 $\dfrac{dz}{dx}, \dfrac{dy}{dx}$.

解：对该两方程两边同时对 x 求导，得 $\begin{cases} \dfrac{dz}{dx} = f_1' \dfrac{dy}{dx} + f_2' (\dfrac{dz}{dx} + 1) \\ 1 + \dfrac{dy}{dx} + \dfrac{dz}{dx} = 0 \end{cases}$，即

$$\begin{cases} f_1' \dfrac{dy}{dx} + (f_2' - 1)\dfrac{dz}{dx} = -f_2' \\ \dfrac{dy}{dx} + \dfrac{dz}{dx} = -1 \end{cases}$$

解之得 $\dfrac{dy}{dx} = \dfrac{\begin{vmatrix} -f_2' & f_2' - 1 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} f_1' & f_2' - 1 \\ 1 & 1 \end{vmatrix}} = \dfrac{-1}{f_1' - f_2' + 1}$, $\quad \dfrac{dz}{dx} = \dfrac{\begin{vmatrix} f_1' & -f_2' \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} f_1' & f_2' - 1 \\ 1 & 1 \end{vmatrix}} = \dfrac{-f_1' + f_2'}{f_1' - f_2' + 1}$.

2. 求由曲面 $z = x^2 + 2y^2$ 及 $z = 6 - 2x^2 - y^2$ 所围成的立体的体积.

解：由题意可知该立体的投影区域 $D_{xy}: x^2 + y^2 \leq 2.$ 故体积

$$V = \iint_{D_{xy}} (6 - 3x^2 - 3y^2) dxdy = 12\pi - 3\int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r^2 \cdot r\, dx = 6\pi.$$

3. 求极限 $\lim\limits_{r \to 0} \dfrac{1}{r^3} \iiint\limits_{\Omega_r} \ln(4 + x^2 + y^4)\, dv$，其中 Ω_r: $x^2 + y^2 + z^2 \leq r^2$.

解：由积分中值定理，$\lim\limits_{r \to 0} \dfrac{1}{r^3} \iiint\limits_{\Omega_r} \ln(4 + x^2 + y^4)\, dv = \lim\limits_{r \to 0} \dfrac{\ln(4 + \xi_r{}^2 + \eta_r{}^4)}{r^3} \cdot \dfrac{4}{3}\pi r^3$

$$= \lim_{(\xi_r, \eta_r) \to (0,0)} \frac{4}{3}\pi \ln(4 + \xi_r{}^2 + \eta_r{}^4) = \frac{8}{3}\pi \ln 2.$$

4. 求曲面 $x^2 + y^2 + z^2 = 6$ 上一点的切平面，使其垂直于直线 $\begin{cases} x - y - z = 2 \\ x + z = 2 \end{cases}$.

解：直线$\begin{cases} x-y-z=2 \\ x+z=2 \end{cases}$的方向矢量可取为$\vec{l}=\begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{vmatrix}=(-1,-2,1).$

设切点为(x_0,y_0,z_0)，则切平面的法矢量为$\vec{n}=(2x_0,2y_0,2z_0)$，由题意有$\begin{cases} \dfrac{2x_0}{-1}=\dfrac{2y_0}{-2}=\dfrac{2z_0}{1} \\ x_0^2+y_0^2+z_0^2=6 \end{cases}$.

解之得$(x_0,y_0,z_0)=(1,2,-1)$或$(x_0,y_0,z_0)=(-1,-2,1)$. 由点法式得切平面方程为：
$x+2y-z\pm 6=0.$

5. 设闭区域 $D: x^2+y^2\le y, x\ge 0$，$f(x,y)$ 为 D 上的连续函数，且 $f(x,y)=\sqrt{1-x^2-y^2}-\dfrac{4}{\pi}\iint\limits_D f(u,v)dudv$，求 $f(x,y)$.

解：等式 $f(x,y)=\sqrt{1-x^2-y^2}-\dfrac{4}{\pi}\iint\limits_D f(u,v)dudv$ 两边同时在区域D上积分得，

$$\iint\limits_D f(x,y)\,dxdy=\iint\limits_D\sqrt{1-x^2-y^2}\,dxdy-\iint\limits_D[\frac{8}{\pi}\iint\limits_D f(u,v)dudv]dxdy$$
$$=\int_0^{\frac{\pi}{2}}d\theta\int_0^{\sin\theta}\sqrt{1-r^2}\cdot r\,dr-\iint\limits_D f(u,v)dudv\cdot\iint\limits_D\frac{4}{\pi}dxdy$$
$$=\frac{1}{3}\int_0^{\frac{\pi}{2}}(1-\cos^3\theta)d\theta-\frac{1}{2}\iint\limits_D f(u,v)\,dudv$$
$$=\frac{1}{3}(\frac{\pi}{2}-\frac{2}{3})-\iint\limits_D f(u,v)\,dudv$$

所以$\iint\limits_D f(x,y)\,dxdy=\dfrac{2}{9}(\dfrac{\pi}{2}-\dfrac{2}{3})$，故 $f(x,y)=\sqrt{1-x^2-y^2}-\dfrac{8}{9\pi}(\dfrac{\pi}{2}-\dfrac{2}{3}).$

6. 计算三重积分$\iiint\limits_\Omega(x^2+x^3y^3+y^2)dxdydz$，其中$\Omega$由$2z=x^2+y^2$，$z=1$，$z=2$ 围成.

解：由对称性，$\iiint\limits_\Omega x^3y^3\,dxdydz=0$，所以

$$\iiint\limits_\Omega(x^2+x^3y^3+y^2)dxdydz=\iiint\limits_\Omega(x^2+y^2)dxdydz$$
$$=\int_1^2dz\iint_{D_z}(x^2+y^2)dxdy$$
$$=\int_1^2dz\int_0^{2\pi}d\theta\int_0^{\sqrt{2z}}r^2\cdot r\,dr$$
$$=2\pi\int_1^2 z^2dz=\frac{14}{3}\pi$$

三、证明题（每小题 10 分，共 20 分）

1. 证明极限 $\lim\limits_{\substack{x\to1\\y\to0}}(x+y)^{\frac{1}{\sin(x-1)}}$ 不存在.

证明：因为 $\lim\limits_{\substack{x\to1\\y\to0}}(x+y)^{\frac{1}{\sin(x-1)}}=\lim\limits_{\substack{x\to1\\y\to0}}e^{\frac{\ln(x+y)}{\sin(x-1)}}=\lim\limits_{\substack{x\to1\\y\to0}}e^{\frac{x+y-1}{x-1}}=\lim\limits_{\substack{x\to1\\y\to0}}e^{1+\frac{y}{x-1}}$,取 $y=k(x-1)$，则

$$\lim_{\substack{x\to 1\\ y=k(x-1)}} e^{1+\frac{y}{x-1}} = \lim_{x\to 1} e^{1+\frac{k(x-1)}{x-1}} = e^{1+k}.$$ 故极限不存在.

2. 设 $z = f(x, y)$ 在有界闭区域 D 上具有二阶连续偏导数，且 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0, \frac{\partial^2 z}{\partial x\partial y}\neq 0.$ 证明：z 的最值只能在边界上取到.

证明:若区域 D 内有驻点 (x, y) 下证其不可能为极值点，故最值只能在边界上取得. 令

$$A=\frac{\partial^2 z}{\partial x^2}(x, y), B=\frac{\partial^2 z}{\partial x\partial y}(x, y), C=\frac{\partial^2 z}{\partial y^2}(x, y)$$

则 $B^2 - AC = (\frac{\partial^2 z}{\partial x\partial y}(x, y))^2 - \frac{\partial^2 z}{\partial x^2}(x, y)\cdot\frac{\partial^2 z}{\partial y^2}(x, y) > 0$，所以 (x, y) 不可能为极值点.