一. 有理函数的积分.

①两个多项式之商称为有理函数. （有理函数理论上一定是可积的）

↳ 它的原函数是初等函数.

$$\frac{P(x)}{Q(x)}=\frac{a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_{m-1}x+b_m}$$

（$n<m$，真分式；$m\geq n$，假分式）

如果 $Q(x)=Q_1(x)\cdot Q_2(x)$. 且 $Q_1(x)$ 与 $Q_2(x)$ 没有公因式.

那么 $\dfrac{P(x)}{Q(x)}=\dfrac{P_1(x)}{Q_1(x)}+\dfrac{P_2(x)}{Q_2(x)}$.

② 对于真分式而言:

若分母有 $(x-a)^k$ 分解后: $\boxed{\dfrac{A_1}{(x-a)^k}+\dfrac{A_2}{(x-a)^{k-1}}+\cdots+\dfrac{A_k}{x-a}.}$

一次项.

若分母有 $(x^2+px+q)^k$ 其中 $p^2-4q<0$ (分母>0)

平方项.

分解后: $\boxed{\dfrac{M_1x+N_1}{(x^2+px+q)^k}+\dfrac{M_2x+N_2}{(x^2+px+q)^{k-1}}+\cdots+\dfrac{M_kx+N_k}{(x^2+px+q)}}$

Eg. $\dfrac{1}{x(x-1)^2}=\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}=\dfrac{A(x-1)^2+Bx(x-1)+Cx}{x(x-1)^2}=\dfrac{(A+B)x^2+(C-B-2A)x+A}{x(x-1)^2}$

与 x 相对: $\dfrac{A}{x}$.

与 $(x-1)^2$ 相对: $\dfrac{B}{(x-1)}+\dfrac{C}{(x-1)^2}$.

即 $\begin{cases}A+B=0\\C-B-2A=0\\A=1\end{cases}\Rightarrow\begin{cases}A=1\\B=-1\\C=1\end{cases}$

即 $\dfrac{1}{x(x-1)^2}=\dfrac{1}{x}-\dfrac{3}{x-1}-\dfrac{1}{(x-1)^2}$.

Eg. $\dfrac{1}{(1+2x)(1+x^2)} = \dfrac{A}{1+2x} + \dfrac{Bx+C}{1+x^2} = \dfrac{A(1+x^2)+(Bx+C)(1+2x)}{(1+2x)(1+x^2)}$

令$1+2x=0$，$\Rightarrow x=-\frac{1}{2}$，即 $1=\frac{5}{4}A \Rightarrow A=\frac{4}{5}$.

令$x=0$，即 $1=\frac{4}{5}+C \Rightarrow C=\frac{1}{5}$.

令$x=2$，即 $1=4+(2B+C)5 \Rightarrow (2B+C)=-\frac{3}{5} \Rightarrow B=-\frac{2}{5}$.

即 $\dfrac{1}{(1+2x)(1+x^2)} = \dfrac{4}{5(1+2x)} + \dfrac{1-2x}{5(1+x^2)}$.

$\Rightarrow \displaystyle\int \frac{A_1}{(x-a)}\,dx = A_1 \ln|x-a|$.

$\displaystyle\int \frac{A_1}{(x-a)^k}\,dx = \frac{A_1}{(k+1)} \ln(x-a)^{k+1}$

$\Rightarrow$

求：$\displaystyle\int \frac{x-2}{x^2+2x+3}\,dx$

↓拆积分?!

原式 $= \displaystyle\int \frac{\frac{1}{2}(x^2+2x+3)'-3}{x^2+2x+3}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\,dx - 3\int \frac{dx}{x^2+2x+3}$

$\displaystyle = \frac{1}{2}\ln(x^2+2x+3) - 3\int \frac{d(x+1)}{(x+1)^2+2}$. $\displaystyle\int \frac{du}{u^2+(\sqrt{2})^2}$

$$= \frac{1}{2}\ln(x^2+2x+3) - 3\frac{1}{\sqrt{2}}\cdot \arctan\frac{x+1}{\sqrt{2}} + C.$$

求 $\int \frac{x-2}{(x^2+2x+3)^2}\,dx$?? why ???

$$原式 = \int \frac{\frac{1}{2}(2x+2)-3}{(x^2+2x+3)^2}\,dx = \frac{1}{2}\underbrace{\int \frac{2x+2}{(x^2+2x+3)^2}\,dx}\; - 3\int \frac{dx}{(x^2+2x+3)^2}$$

$$\frac{-1}{u} = \frac{du}{u^2} \rightarrow$$

$$= \underbrace{\frac{1}{2}\cdot\frac{-1}{(x^2+2x+3)} - 3\int \frac{d(x+1)}{[(x+1)^2+2]^2}}$$

$$\Downarrow \qquad \frac{dt}{(t^2+2)^2}$$

令 $x+1 = t = \sqrt{2}\tan u$.

$$\boxed{dt = \sqrt{2}\sec^2 u\, du}$$

$$\int \frac{\sqrt{2}\sec^2 u}{4\sec^4 u}\,du$$

$$= \frac{\sqrt{2}}{4}\int \frac{1}{\sec^2 u}\,du$$

$$= \frac{\sqrt{2}}{4}\int \cos^2 u\,du$$

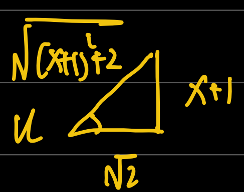
$$= \frac{\sqrt{2}}{4}\int \frac{1+\cos 2u}{2}\,du.$$

$$= \frac{\sqrt{2}}{4}\left(\frac{u}{2} + \frac{1}{4}\sin 2u\right)$$

$$= \frac{\sqrt{2}}{4}\left(\frac{u}{2} + \frac{1}{2}\sin u\cos u\right)$$

$$= \frac{\sqrt{2}}{4}\left[\frac{1}{2}\arctan\frac{x+1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{\sqrt{2}(x+1)}{(x+1)^2+2}\right]$$

$$\tan u = \frac{x+1}{\sqrt{2}}$$



求 $\int \frac{x^2-1}{x^4+1}dx$.

$u=x+\frac{1}{x}$

原式 $=\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2} = \int \frac{du}{u^2-2} = \frac{1}{2\sqrt{2}}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|+C.$

$= \frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+C.$

求 $\int \frac{x^2+1}{x^4+1}dx$.

$u=x-\frac{1}{x}$

原式 $=\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+\sqrt{2}^2} \overset{\downarrow}{=} \int \frac{du}{u^2+2} = \frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}}+C.$

求 $\int \frac{dx}{x^4+1}$.

同上：

原式 $=\frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}dx = \frac{1}{2\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}} - \frac{1}{4\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+C.$

求 $\int \frac{1}{1+e^x}dx$

令 $t=e^x$. 则 $x=\ln t$, $dx=\frac{1}{t}dt$. 做指数代换.

原式 $=\int \frac{1}{1+t}\cdot\frac{1}{t}dt = \int\left[\frac{1}{t}-\frac{1}{1+t}\right]dt = \ln t - \ln(t+1)+C.$

$= x-\ln(e^x+1)+C.$

二、三角函数有理式的积分.

① $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$　　$\cos 2x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$.　　令 $u=\tan\frac{x}{2}$.

万能置换公式.　　$\frac{x}{2}=\arctan u$.

$\sin x = \frac{2t}{1+t^2}$，$\cos t = \frac{1-t^2}{1+t^2}$，$t=\tan\frac{x}{2}$.　　$\boxed{dx = \frac{2}{1+u^2}du.}$

$\Rightarrow \int R(\sin x, \cos x)\,dx = \int R\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right)\frac{2}{1+u^2}du.$

令 $\tan x = t \Rightarrow x=\arctan t$.

$dx = \frac{1}{1+t^2}dt$.

Eg. $\int \frac{\sin x}{\cos x+\sin x}dx = \int \frac{\tan x}{1+\tan x}dx$

$= \int \frac{t}{1+t}\cdot\frac{1}{1+t^2}dt.$

//

$\frac{t}{1+t}\cdot\frac{1}{1+t^2} = \left[\frac{A}{1+t}+\frac{Bt+C}{1+t^2}\right] = \frac{A(1+t^2)+(Bt+C)(1+t)}{(1+t)(1+t^2)}.$

$\begin{cases} A=-\frac{1}{2} \\ B=\frac{1}{2} \\ C=\frac{1}{2} \end{cases}$

$\Rightarrow \int\left(\frac{-\frac{1}{2}}{1+t}+\frac{\frac{1}{2}t+\frac{1}{2}}{1+t^2}\right)dt = -\frac{1}{2}\ln|1+t| + \frac{1}{4}\ln(1+t^2)+\frac{1}{2}\arctan t + C.$

(分母)$'= \cos x - \sin x$.

$$\text{原式}=\int\left(\frac{\sin x-\cos x+\cos x+\sin x}{\cos x+\sin x}\right)\frac{1}{2}\,dx=-\frac{1}{2}\int\frac{(\cos x+\sin x)'}{\cos x+\sin x}\,dx+\frac{1}{2}+C$$

$$=-\frac{1}{2}\ln\left|\cos x+\sin x\right|+\frac{1}{2}+C$$

三、简单无理函数的积分.

$\int R(x,\sqrt[n]{ax+b})\,dx$ 　令 $t=\sqrt[n]{ax+b}$.

$\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+e}}\right)dx$ 　令 $t=\sqrt[n]{\frac{ax+b}{cx+e}}$.

Eg. $\int\frac{dx}{1+\sqrt[3]{x+2}}$. 令 $u=\sqrt[3]{x+2}$，则 $x=u^3-2$. $dx=3u^2du$.