一. 1. $dz = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{x\,dy - y\,dx}{x^2} = \frac{x\,dy - y\,dx}{x^2+y^2}$

$dz\Big|_{\substack{x=1\\y=2}} = \frac{1}{5}dy - \frac{2}{5}dx = \boxed{-\frac{2}{5}dx + \frac{1}{5}dy}$

2. $\text{grad}\, z = \left((2x+x^2y)e^{xy},\ x^3e^{xy}\right)$

$\text{grad}\, z\Big|_{(1,1)} = (3e, e)$

$\left|\text{grad}\, z\Big|_{(1,1)}\right| = \boxed{\sqrt{10}\, e}$

3. $F(x,y,z) = xz + y^2\ln x - yz^2$

$F_x = z + \frac{y^2}{x}$, $\quad F_z = x - 2yz$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{z+\frac{y^2}{x}}{x-2yz} = -\frac{xz+y^2}{x(x-2yz)}$

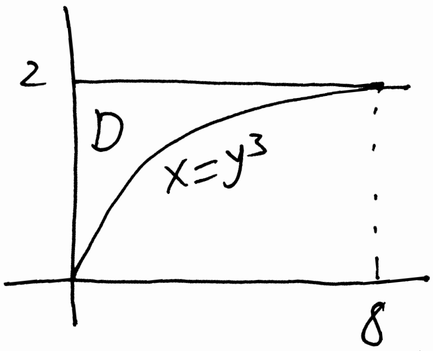
$\frac{\partial z}{\partial x}\Big|_{(1,-2,0)} = \boxed{-4}$

4. $D: x^2+y^2 \le 4 \quad (R=2)$.

$z = 1-x-y \qquad z'_x = -1, \quad z'_y = -1, \quad dS = \sqrt{3}\,dx\,dy$

$A = \iint\limits_D \sqrt{3}\,dx\,dy = \sqrt{3}\cdot\pi\cdot 2^2 = 4\sqrt{3}\,\pi$

5.



$$D: \begin{array}{l} 0 \le x \le 8 \\ \sqrt[3]{x} \le y \le 2 \end{array}$$

$$原式 = \int_0^2 dy \int_0^{y^3} \frac{1}{1+y^4} dy = \int_0^2 \frac{y^3}{1+y^4} dy$$

$$= \left[\frac{1}{4} \ln(1+y^4) \right]_0^2 = \boxed{\frac{\ln 17}{4}}$$

二. 1. C　　2. D

3. B　Ω关于 xOy 与 yOz 面均对称.
所以被积函数需关于 x 与 z 均为偶函数.

4. C　$(x'_t, y'_t, z'_t) = (1, 4t, 9t^2)$, $\vec{T} = (1, 4, 9)$
$P = (1, 2, 3)$

5. A　$f(x,y) = xy + 2y - \ln x - 2\ln y$

$$f_x = y - \frac{1}{x} \qquad f_y = x + 2 - \frac{2}{y}$$

令 $f_x = f_y = 0$ 得 $(x,y) = (2, \frac{1}{2})$

$$f_{xx} = \frac{1}{x^2} \qquad f_{xy} = 1, \qquad f_{yy} = \frac{2}{y^2}$$

$A = \frac{1}{4}$, $B = 1$, $C = 8$　　$AC - B^2 = 1$

三. 1. $\frac{\partial f}{\partial x} = y f_1' + \frac{1}{y} f_2'$, $\frac{\partial f}{\partial y} = x f_1' - \frac{x}{y^2} f_2'$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(y f_1' + \frac{1}{y} f_2'\right)$$

$$= f_1' + y \frac{\partial f_1'}{\partial y} - \frac{1}{y^2} f_2' + \frac{1}{y} \frac{\partial f_2'}{\partial y}$$

$$= f_1' - \frac{1}{y^2} f_2' + y\left(x (f_1')_1' - \frac{x}{y^2} (f_1')_2'\right)$$

$$+ \frac{1}{y}\left(x (f_2')_1' - \frac{x}{y^2} (f_2')_2'\right)$$

$$= f_1' - \frac{1}{y^2} f_2' + xy f_{11}'' - \frac{x}{y^3} f_{22}''$$

$$\frac{\partial g}{\partial x} = -\frac{y}{x^2} g'\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 g}{\partial x \partial y} = -\frac{1}{x^2} g'\left(\frac{y}{x}\right) - \frac{y}{x^2} g''\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$= -\frac{1}{x^2} g'\left(\frac{y}{x}\right) - \frac{y}{x^3} g''\left(\frac{y}{x}\right)$$

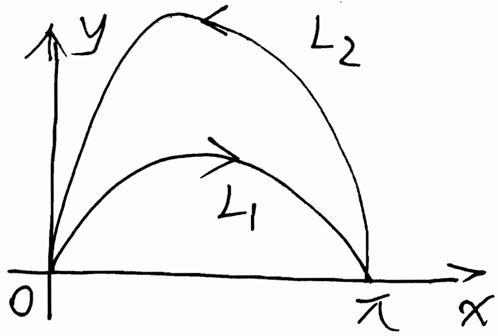
$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} = y f_1' + \frac{1}{y} f_2' - \frac{y}{x^2} g'$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 g}{\partial x \partial y}$$

$$= f_1' - \frac{1}{y^2} f_2' + xy f_{11}'' - \frac{x}{y^3} f_{22}''$$

$$- \frac{1}{x^2} g' - \frac{y}{x^3} g''$$

2.



$L = L_1 + L_2$

$L_1: y = \sin x \quad x: 0 \to \pi$

$L_2: y = 2\sin x \quad x: \pi \to 0$

方法1. $\oint_L = \oint_{L_1} + \oint_{L_2}$

$$= \int_0^{\pi} (1 + \sin^2 x + x \sin x \cos x)\, dx$$

$$- \int_0^{\pi} (1 + 4\sin^2 x + x \cdot 2\sin x \cdot 2\cos x)\, dx$$

$$= -3 \int_0^{\pi} (\sin^2 x + x \sin x \cos x)\, dx$$

$$\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

$$\int x \sin x \cos x\, dx = \int x \sin x\, d\sin x$$

$$= \frac{1}{2}\int x\, d\sin^2 x = \frac{1}{2} x \sin^2 x - \frac{1}{2}\int \sin^2 x\, dx$$

$$= \frac{1}{2} x \sin^2 x - \frac{1}{4}x + \frac{1}{8}\sin 2x + C$$

$$\therefore 原式 = -3\left[\frac{1}{4}x - \frac{1}{8}\sin 2x + \frac{1}{2}x\sin^2 x\right]_0^{\pi}$$

$$= -\frac{3}{4}\pi$$

方法2. 利用格林公式.

$$P = 1+y^2 \qquad Q = xy$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y - 2y = -y$$

$$\oint_L P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma$$

$$= \iint_D -y\,d\sigma = \int_0^{\pi} dx \int_{\sin x}^{2\sin x} -y\,dy$$

$$= \int_0^{\pi} dx \left[-\frac{1}{2}y^2\right]_{\sin x}^{2\sin x} = -\frac{3}{2}\int_0^{\pi} \sin^2 x\,dx$$

$$= -\frac{3}{2}\int_0^{\pi} \frac{1-\cos 2x}{2}\,dx = -\frac{3}{2}\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi}$$

$$= -\frac{3}{4}\pi$$

3. $P=x^2 \quad Q=y^2 \quad R=z^2$

方法1：Σ由两部分组成：1) 侧面Σ_1：$z=\sqrt{x^2+y^2}$，方向为下侧；2) 顶面Σ_2：$z=h$，方向为上侧。Σ_1与Σ_2在xOy面上的投影均为D：$x^2+y^2\leq h^2$.

(1) 计算 $\iint\limits_{\Sigma_1}$. $\because \Sigma_1: z=\sqrt{x^2+y^2}$,

$$\therefore z_x'=\frac{x}{\sqrt{x^2+y^2}}, \quad z_y'=\frac{y}{\sqrt{x^2+y^2}}$$

$$\iint\limits_{\Sigma_1}=-\iint\limits_{D}(-Pz_x'-Qz_y'+R)\,dxdy$$

$$=\iint\limits_{D}\left(\frac{x^3}{\sqrt{x^2+y^2}}+\frac{y^3}{\sqrt{x^2+y^2}}-(x^2+y^2)\right)dxdy$$

注意到D关于x轴与y轴均对称，而$\frac{x}{\sqrt{x^2+y^2}}$是x的奇函数，$\frac{y^3}{\sqrt{x^2+y^2}}$是$y$的奇函数，所以$\iint\limits_{D}\frac{x^3}{\sqrt{x^2+y^2}}dxdy=\iint\limits_{D}\frac{y^3}{\sqrt{x^2+y^3}}dxdy=0$

$$\therefore \iint\limits_{\Sigma_1}=-\iint\limits_{D}(x^2+y^2)\,dxdy \ \underline{\underline{\text{极坐标}}}$$

$$-\int_0^{2\pi}d\theta\int_0^h\rho^2\cdot\rho\,d\rho=-2\pi\cdot\frac{1}{4}h^4=-\frac{1}{2}\pi h^4.$$

(2) 计算 $\iint\limits_{\Sigma_2}$，$\because \Sigma_2: z=h$，$\therefore z_x'=z_y'=0$

$$\iint\limits_{\Sigma_2} = \iint\limits_{D} (-Pz_x' - Qz_y' + R)\,dxdy$$

$$= \iint\limits_{D} h^2\,dxdy = h^2\cdot\pi h^2 = \pi h^4$$

综合 (1) & (2)，我们有

$$\oiint\limits_{\Sigma} = \iint\limits_{\Sigma_1} + \iint\limits_{\Sigma_2} = -\frac{1}{2}\pi h^4 + \pi h^4 = \frac{1}{2}\pi h^4.$$

方法2. 利用高斯公式.

$$\oiint\limits_{\Sigma} = \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dv$$

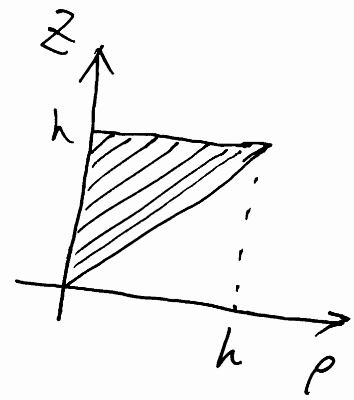
$$= \iiint\limits_{\Omega} (2x+2y+2z)\,dv$$

注意到 Ω 关于 xOz 与 yOz 平面均对称.

所以 $\iiint\limits_{\Omega} 2x\,dv = \iiint\limits_{\Omega} 2y\,dv = 0$.

$$\therefore \oiint\limits_{\Sigma} = 2\iiint\limits_{\Omega} z\,dv$$

下面利用柱面坐标计算上述三重积分，Ω 可表示为 $\rho \le z \le h$.



$$\because \oiint_{\Sigma} = 2\int_0^{2\pi} d\theta \int_0^h d\rho \int_\rho^h z\cdot\rho\, dz$$

$$= 4\pi \int_0^h d\rho \left[\frac{1}{2}\rho z^2\right]_\rho^h$$

$$= 2\pi \int_0^h \rho(h^2-\rho^2)\, d\rho$$

$$= 2\pi \left[\frac{1}{2}\rho^2 h^2 - \frac{1}{4}\rho^4\right]_0^h$$

$$= 2\pi\cdot\frac{1}{4}h^4 = \frac{1}{2}\pi h^4 .$$

四 1. 令 $P(x,y) = (e^x + 2f(x))y$

$Q(x,y) = -f(x)$. 因 $\int_L Pdx + Qdy$ 与路径无关，所以 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$，即

$$e^x + 2f(x) = -f'(x) \Rightarrow f'(x) + 2f(x) = -e^x$$

令 $p(x) = 2$，$q(x) = -e^x$，则 $\int p(x)dx = 2x$

$$\int q(x)e^{\int p(x)dx}dx = \int -e^x\cdot e^{2x}dx = -\frac{1}{3}e^{3x}.$$

于是微分方程的通解为：

$$f(x) = Ce^{-2x} + e^{-2x}\cdot\left(-\frac{1}{3}e^{3x}\right)$$

$$= Ce^{-2x} - \frac{1}{3}e^x$$

由 $f(0)=0$ 得 $C=\frac{1}{3}$，于是 $f(x) = \frac{1}{3}(e^{-2x} - e^x)$

代入得：$P=\frac{1}{3}(2e^{-2x}+e^{x})y$

$Q=\frac{1}{3}(e^{x}-e^{-2x})$

设二元函数 $u(x,y)$ 满足 $du=P\,dx+Q\,dy$

则 $\frac{\partial u}{\partial x}=P$，∴

$u=\frac{1}{3}\int(2e^{-2x}+e^{x})y\,dx$

$=\frac{1}{3}(e^{x}-e^{-2x})y+\varphi(y)$

由 $\frac{\partial u}{\partial y}=Q$ 得 $\varphi'(y)=0$ 于是 $\varphi(y)$ 为一常数.

可取 $\varphi(y)=0$，$u=\frac{1}{3}(e^{x}-e^{-2x})y$. 于是.

$$\int_{(0,0)}^{(1,1)}P\,dx+Q\,dy=u(1,1)-u(0,0)$$

$$=\frac{1}{3}(e-e^{-2})$$

2. 令 $F = z^2 y - x z^3 - 1$. 则 $F'_x = -z^3$,

$F'_z = 2zy - 3xz^2$, 于是

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} = -\frac{-z^3}{2zy-3xz^2} = \frac{z^2}{2y-3xz}$$

$$\frac{\partial u}{\partial x} = 3\cos(y+3z)\cdot\frac{\partial z}{\partial x}$$

$$= 3\cos(y+3z)\cdot\frac{z^2}{2y-3xz}$$

在 $z^2y - xz^3 - 1 = 0$ 中令 $\begin{cases} x=1 \\ y=0 \end{cases}$ 得 $z(1,0) = -1$

于是

$$\left.\frac{\partial u}{\partial x}\right|_{\substack{x=1\\y=0}} = 3\cos(y+3z)\cdot\left.\frac{z^2}{2y-3xz}\right|_{\substack{x=1\\y=0\\z=-1}}$$

$$= \cos 3$$

五 1. 两曲面的交线为 $\begin{cases} x^2+y^2=1 \\ z=1 \end{cases}$, 它在 xOy 面上的投影围成的闭区域为 $D: x^2+y^2 \le 1$. 于是

$$V = \iint_D \left(1+\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) d\sigma$$

$$\underline{\underline{\text{极坐标}}} \int_0^{2\pi} d\theta \int_0^1 \left(1+\sqrt{1-\rho^2} - \rho\right)\cdot\rho\, d\rho$$

$$= 2\pi\left[-\frac{1}{3}(1-\rho^2)^{\frac{3}{2}} + \frac{1}{2}\rho^2 - \frac{1}{3}\rho^3\right]_0^1$$

$$= \pi$$

解法2: 体积 = 半球 + 圆锥体 $= \frac{1}{2}\cdot\frac{4}{3}\pi + \frac{1}{3}\pi = \pi$.

2. 令 $F(x, y, z, \lambda_1, \lambda_2) = 2x + y + 3z$

$+ \lambda_1(x^2 + y^2 - 2) + \lambda_2(x + z - 1)$. 则

$$\begin{cases} F'_x = 2 + 2\lambda_1 x + \lambda_2 = 0 & (1) \\ F'_y = 1 + 2\lambda_1 y = 0 & (2) \\ F'_z = 3 + \lambda_2 = 0 & (3) \\ F'_{\lambda_1} = x^2 + y^2 - 2 = 0 & (4) \\ F'_{\lambda_2} = x + z - 1 = 0 & (5) \end{cases}$$

由 (3) 得：$\lambda_2 = -3$，代入 (1) 得

$$2\lambda_1 x - 1 = 0 \qquad (6)$$

由 (2) & (6) 得：$y = -x \left(= -\frac{1}{2\lambda_1}\right) \qquad (7)$

代入 (4) 得 $2x^2 - 2 = 0$，即 $x = \pm 1$.

再由 (5) & (7) 可得两驻点：$(1, -1, 0)$，$(-1, 1, 2)$

又因 $f(1, -1, 0) = 1$，$f(-1, 1, 2) = 5$

∴ 最大值为 5，最小值为 1.

六 1. (1) $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ ①

$$W'(x) = y_1'(x)y_2'(x) + y_1(x)\cdot y_2''(x) - y_1''(x)y_2(x) - y_1'(x)y_2'(x)$$

$$= y_1(x)y_2''(x) - y_1''(x)y_2(x) \quad ②$$

由 ① & ② 得:

$$\frac{dW}{dx} + p(x)W = W'(x) + p(x)W(x)$$

$$= y_1(x)\left(y_2''(x) + p(x)y_2'(x)\right) - \left(y_1''(x) + p(x)y_1'(x)\right)y_2(x)$$

$$= y_1(x)\left(-q(x)y_2(x)\right) - \left(-q(x)y_1(x)\right)y_2(x)$$

$$= 0$$

(2) 令 $F(x) = \int_{x_0}^{x} p(t)\,dt$，则 $\int p(x)\,dx = F(x) + C$.

$$\frac{dW}{dx} + p(x)W = 0 \Rightarrow \frac{dW}{W} = -p(x)\,dx$$

$$\int \frac{dW}{W} = \int -p(x)\,dx \Rightarrow \ln|W| = -F(x) + C_1$$

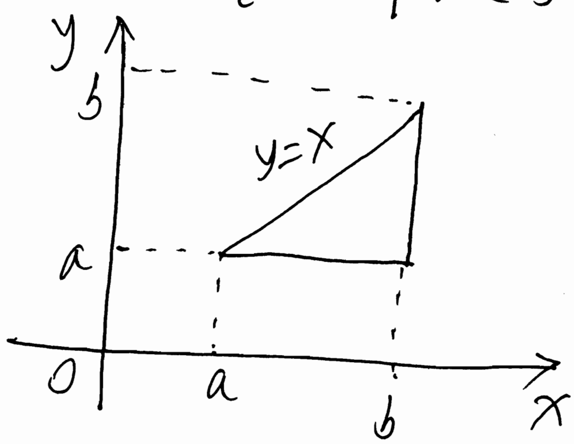
$$\Rightarrow W = Ce^{-F(x)} \quad (C = \pm e^{C_1}).$$

令 $x = x_0$ 得: $C = W(x_0)$ $(\because F(x_0) = 0)$.

于是 $W(x) = W(x_0)e^{-F(x)} = W(x_0)e^{-\int_{x_0}^{x} p(t)\,dt}$

2. $\int_a^b dx \int_a^x (x-y)^{n-2} f(y)\,dy = \iint\limits_D (x-y)^{n-2} f(y)\,dy$,

其中 $D = \{(x,y) \mid a \le y \le x,\ a \le x \le b\}$.



于是

$$上式 = \int_a^b dy \int_y^b (x-y)^{n-2} f(y)\,dx$$

$$= \int_a^b dy \left[\frac{1}{n-1} (x-y)^{n-1} f(y) \right]_y^b$$

$$= \int_a^b \frac{1}{n-1} (b-y)^{n-1} f(y)\,dy$$

$$= \frac{1}{n-1} \int_a^b (b-y)^{n-1} f(y)\,dy$$