# 一.[代换积分法].

① 求 $\int \frac{\arctan\frac{1}{x}}{1+x^2}dx$.　　析：考虑 $\frac{\arctan u}{1+u^2}du = \arctan u\, d(\arctan u)$.

原式 $= \int \frac{\arctan\frac{1}{x}}{1+(\frac{1}{x})^2}\cdot d(-\frac{1}{x}) = -\int \arctan\frac{1}{x}\, d(\arctan\frac{1}{x}) = -\frac{1}{2}\arctan^2\frac{1}{x}$.

② 求 $\int \frac{dx}{\sin(2x)+2\sin x}$.　　两个奇次式，考虑凑 $\cos^2\frac{x}{2}$　　向 tan、sec 靠.

原式 $= \int \frac{dx}{2\sin x(\cos x+1)} = \frac{1}{4}\int \frac{d(\frac{x}{2})}{\sin\frac{x}{2}\cos^3\frac{x}{2}} = \frac{1}{4}\int \frac{d(\tan\frac{x}{2})}{\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1}{4}\int \frac{d(\tan\frac{x}{2})}{\tan\frac{x}{2}\cos^2\frac{x}{2}}$

$= \frac{1}{4}\int \frac{\tan^2\frac{x}{2}+1}{\tan\frac{x}{2}}d(\tan\frac{x}{2}) = \frac{1}{4}\int(\tan\frac{x}{2}+\frac{1}{\tan\frac{x}{2}})d(\tan\frac{x}{2}) = \frac{1}{8}\tan^2\frac{x}{2}+\frac{1}{4}\ln|\tan\frac{x}{2}|+C$.

③ 求 $\int \tan^3 x\,dx$.

原式 $= \int \sec^2 x\tan x\,dx - \int \tan x\,dx = \int \tan x\,d(\tan x) + \ln|\cos x|$

$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$.

小结：利用 $\begin{cases}\tan^2 x+1=\sec^2 x\\ \sec^2 x\,dx = d\tan x\end{cases}$ ⇒ 对 $\tan^n x$ 进行降次处理.

④ $\int \frac{dx}{\sin^2 x+5\cos^2 x}$

原式 $= \int \frac{dx}{\cos^2 x(\tan^2 x+5)} = \int \frac{\sec^2 x\,dx}{\tan^2 x+5} = \int \frac{d(\tan x)}{\tan^2 x+5}$

$= \frac{1}{\sqrt{5}} \arctan \frac{\tan x}{\sqrt{5}} + C.$

小结：遇分子为常数，分母为 $A\sin^2 x + B\cos^2 x$ 型，尝试把 $\frac{1}{\cos^2 x}$，

提化为 $\frac{n}{A\tan^2 x+B} d(\tan x)$ 进行求解.

## 二.【分部积分法】

① $\int \frac{xe^x}{\sqrt{e^x-2}}dx$.

先分部积分，后换元

原式 $=\int x\,d(2\sqrt{e^x-2}) = 2x\sqrt{e^x-2} - 2\int\sqrt{e^x-2}\,dx$

令 $t=\sqrt{e^x-2} \Rightarrow t^2+2=e^x \Rightarrow x=\ln(t^2+2) \Rightarrow dx=\frac{2t}{t^2+2}$.

消去根号.

$\int\sqrt{e^x-2}\,dx = \int\frac{2t^2}{t^2+2}dt = 2\int\left(1-\frac{2}{t^2+2}\right)dt$

$= 2t - \frac{4}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}} + C$.

故 原式 $= 2x\sqrt{e^x-2} - 4\sqrt{e^x-2} + 4\sqrt{2}\arctan\sqrt{\frac{e^x-2}{2}} + C$.

② $\int\frac{x^2}{1+x^2}\arctan x\,dx$.

令 $\arctan x = u \Rightarrow x=\tan u$. $dx=\sec^2 u\,du$.

"化反为正"　化简.

原式 $=\int\frac{\tan^2 u}{1+\tan^2 u}u\cdot\sec^2 u\,du = \int\frac{\sin^2 u}{\cos^2 u}\cdot\frac{\cos^2 u}{\sin^2 u+\cos^2 u}\cdot u\cdot\frac{1}{\cos^2 u}du$

出现 $\tan x$ 偶次项，转 $\sec x$.

$=\int u\tan^2 u\,du = \int u(\sec^2 u-1)\,du = \int u\,d(\tan u) - \frac{1}{2}u^2$

$= u\tan u - \int\tan u\,du - \frac{1}{2}u^2 = u\tan u + \ln|\cos u| - \frac{1}{2}u^2 + C$

$= x\arctan x + \ln\left|\cos(\arctan x)\right| - \frac{1}{2}(\arctan x)^2 + C.$

③ $\int \frac{\ln\sin x}{\sin^2 x}\,dx.$

$$\text{原式} = \int \ln\sin x \, d(-\cot x) = -\cot x \ln\sin x + \int \frac{\cos x}{\sin x}\cdot \cot x \, dx$$

$\cot^2 x$

$\csc^2 x = \cot^2 x + 1$

$$= -\cot x \ln\sin x + \int(\csc^2 x - 1)\,dx = -\cot x\ln\sin x - \cot x - x + C.$$

$$\begin{cases} \int \csc^2 x \, dx = -\cot x \\ \int \frac{1}{\sin^2 x}\,dx = -\cot x \end{cases}$$

④ 设 $f(\ln x) = \frac{\ln(1+x)}{x}$，求 $\int f(x)\,dx$.

令 $\ln x = t$，$t = e^x \Rightarrow f(x) = \frac{\ln(e^x+1)}{e^x}$

$$\int f(x)\,dx = \int \frac{\ln(e^x+1)}{e^x}\,dx = -\int \ln(e^x+1)\,d(e^{-x})$$

$$= -e^{-x}\ln(1+e^x) + \int e^{-x}\frac{e^x}{1+e^x}\,dx$$

$$\int \frac{1}{1+e^x}\,dx = \int\left(1 - \frac{e^x}{1+e^x}\right)dx$$

$$= -e^{-x}\ln(1+e^x) + x - \ln|1+e^x| + C = x - (1+e^{-x})\ln(1+e^x) + C.$$

常用添加项

⑤ 求 $\int d\int f'(x)dx$.

析： $\underline{\int dy} = y + C$.

原式 $= \int f'(x)dx + C = f(x) + C$.

↳ 求 $\int d[f(x)+2]$.

原式 $= f(x) + 2 + C_1 = f(x) + C$.

⑥ 求 $\int x^2(1-x)^{1000}dx$.

<u>高次项难以打开</u>，看作一块，打开低次项即。

令 $u = 1-x \Rightarrow dx = -du$.

原式 $= -\int (1-u)^2 \cdot u^{1000}\, du$

$= -\int (u^{1002} - 2u^{1001} + u^{1000})\, du$

$= -\frac{(1-x)^{1001}}{1001} + \frac{2}{1002}(1-x)^{1002} - \frac{(1-x)^{1003}}{1003} + C$.

⑦ $\int x(\arctan x)^2 dx$. $\left(1 - \frac{1}{1+x^2}\right)$

原式 $= \frac{1}{2}\int (\arctan x)^2 dx^2 = \frac{1}{2}x^2(\arctan x)^2 - \frac{1}{2}\int x^2 \cdot 2\arctan x \cdot \frac{1}{1+x^2}\, dx$

$$= \frac{1}{2}x^2(\arctan x)^2 - \int \arctan x\,dx + \int \arctan x\,d(\arctan x)$$

$$= \frac{1}{2}x^2(\arctan x)^2 - \left[x\arctan x + \frac{1}{2}\ln(1+x^2)\right] + \frac{1}{2}(\arctan x)^2 + C.$$

⑧ 已知 $f(x)=\begin{cases} 2(x-1), & x<1 \\ \ln(x), & x\geq 1. \end{cases}$ 求 $f(x)$ 的原函数 $F(x)$.

$$F(x)=\begin{cases} x^2-2x+C_1, & x<1. \\ x(\ln x-1)+C_2 & x\geq 1. \end{cases}$$

由 $\lim\limits_{x\to 1} F(x) = F(1) \Rightarrow \quad C_1-1 = C_2-1 \Leftrightarrow C_1=C_2.$ 注意原函数连续性.

故 $F(x)=\begin{cases} x^2-2x+C. & x<1. \\ x(\ln x-1)+C. & x\geq 1. \end{cases}$

⑨ 求 $\int \frac{\cos x}{2\sin x+\cos x}dx.$

$$\int \frac{\cos x}{2\sin x+\cos x}dx = \int \frac{A(2\sin x+\cos x) + B(2\cos x-\sin x)}{2\sin x+\cos x}dx$$

$$= \int \frac{(2A-B)\sin x + (A+2B)\cos x}{2\sin x+\cos x}\,dx$$

$$\begin{cases} 2A=B \\ 1=A+2B \end{cases} \Rightarrow \begin{cases} A=\frac{1}{5} \\ B=\frac{2}{5} \end{cases}$$

$$原式 = \int \frac{\frac{1}{5}(2\sin x+\cos x) + \frac{2}{5}(2\cos x-\sin x)}{2\sin x+\cos x}\,dx$$

$$= \frac{1}{5}x + \frac{2}{5}\ln|2\sin x+\cos x| + C$$

小结：分子/分母是 $\sin x/\cos x$ 的线性组合.

$$\int \frac{a\sin x+b\cos x}{A\sin x+B\cos x}\,dx = \int \frac{M(A\sin x+B\cos x)+N(A\cos x-B\sin x)}{A\sin x+B\cos x}\,dx$$

$$= \int \frac{(AM-BN)\sin x+(BM+AN)\cos x}{A\sin x+B\cos x}\,dx$$

$$= (AM-BN)x + (BM+AN)\ln|A\sin x+B\cos x| + C.$$

$$\begin{cases} a=AM-BN \\ b=BM+AN \end{cases} \Rightarrow \begin{cases} M=\frac{Aa+Bb}{A^2+B^2} \\ N=\frac{Ab-Ba}{A^2+B^2} \end{cases}$$

⑩ 求 $\int \frac{1}{x(x^2+1)}\,dx.$

原式 $= \int \frac{1}{(x^7+1)} \, d\ln x.$

$= \frac{1}{7}\int \frac{1}{(x^7+1)} \, d\ln x^7.$

令 $t = x^7 \Rightarrow d\ln x^7 = \frac{1}{t}dt.$

原式 $= \frac{1}{7} \frac{1}{t(t+1)} dt$

$= \frac{1}{7} \ln\left|\frac{t}{t+1}\right| + C.$

$= \frac{1}{7} \ln\left|\frac{x^7}{x^7+1}\right| + C$

任取一点 $m$.

$f(m) = q.$ $(0<q<1).$


$f'(\xi) = \frac{f(m)-f(0)}{m-0} = \frac{f(m)}{m} = \frac{q}{m}.$

$f'(\eta) = \frac{f(1)-f(m)}{1-m} = \frac{1-q}{1-m}.$

$\frac{a}{f'(\xi)} + \frac{b}{f'(\eta)} = \frac{am}{q} + \frac{b(1-m)}{1-q} = a+b.$

与 $m$ 无关.

$$\frac{am}{q}+\frac{b}{1-q}-\frac{bm}{1-q}=a+b.$$

$$\overbrace{\frac{a(1-q)-bq}{q(1-q)}}^{=0}m+\frac{b}{1-q}=a+b.$$

↳ $q=\frac{a}{b+a}\in(0,1).$

$$\frac{b}{1-\frac{a}{a+b}}=\frac{b(a+b)}{b}=(a+b),$$ ✓ 成立