一. 1. $(0,-1,-2)$

$$\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=\vec{a}\times\vec{b}-\vec{c}\times\vec{b}=(\vec{a}-\vec{c})\times\vec{b}$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 2 & -1\\ 0 & -2 & 1\end{vmatrix}=-\vec{j}-2\vec{k}$$

2. $3dx+4\ln 2\,dy$

$$\frac{\partial z}{\partial x}=(2y+1)(x^2+x)^{2y}(2x+1),\quad \left.\frac{\partial z}{\partial x}\right|_{(1,0)}=3$$

$$\frac{\partial z}{\partial y}=2(x^2+x)^{2y+1}\ln(x^2+x),\quad \left.\frac{\partial z}{\partial y}\right|_{(1,0)}=4\ln 2$$

3. $\frac{x-1}{1}=\frac{y-1}{-1}=\frac{z+2}{0}$

$$\vec{n}_1=(1,1,1),\quad \vec{n}_2=(2x,2y,2z)|_{(1,1,-2)}=(2,2,-4)$$

$$\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 1 & 1\\ 2 & 2 & -4\end{vmatrix}=-6\vec{i}+6\vec{j}$$

于是切向量可取为 $\vec{T}=-\frac{1}{6}(\vec{n}_1\times\vec{n}_2)=(1,-1,0)$

4. $z=xy+x^3+e^y+1$

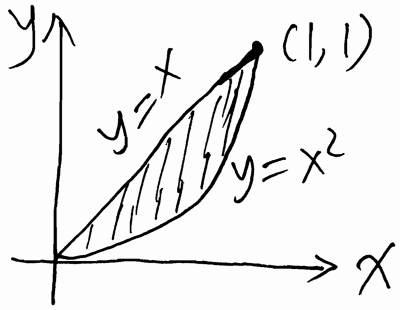
$$z_x=y+3x^2\Rightarrow z=\int(y+3x^2)dx=xy+x^3+\varphi(y)$$

$$z_y=x+\varphi'(y)=x+e^y\Rightarrow\varphi'(y)=e^y$$

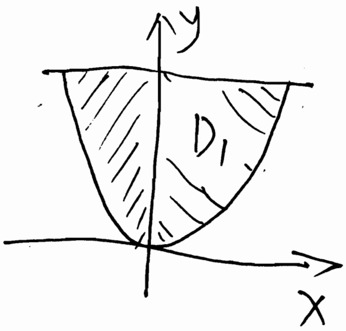
$$\Rightarrow\varphi(y)=\int e^y dy=e^y+C$$

于是 $z=xy+x^3+e^y+C$. 再由 $z(0,0)=2$ 得 $C=1$.

5. $\int_0^1 dy \int_y^{\sqrt{y}} f(x,y)\,dx$



6. $\frac{1}{2}$



注意到D关于y轴对称，于是

原式 $= 2\iint\limits_{D_1} x\,dxdy$

$= 2\int_0^1 dx \int_{x^2}^1 x\,dy$

$= 2\int_0^1 x(1-x^2)\,dx = 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{2}$

7. $\frac{\pi}{5}$

利用平行截面法. 过z轴上点z且与z轴垂直的平面与Ω的截面D_z由$x^2+y^2=z^2$围成. 这是一个半径为z的圆, 其面积为πz^2. 于是

$\iiint\limits_{\Omega} z^2\,dxdydz = \int_0^1 dz \iint\limits_{D_z} z^2\,dxdy = \int_0^1 z^2\cdot\pi z^2\,dz$

$= \left[\frac{\pi}{5}z^5\right]_0^1 = \frac{\pi}{5}$

8. $\frac{2\pi}{3}(3\sqrt{3}-1)$

曲面方程为 $z=\frac{1}{2}(x^2+y^2)$. 它与 $z=1$ 的交线在 xOy 面上的投影为 $x^2+y^2=2$, 其围成的区域记为 D.

$\because z_x = x,\ z_y = y$

$\therefore dS = \sqrt{1+z_x^2+z_y^2}\,dxdy = \sqrt{1+x^2+y^2}\,dxdy$

$\therefore$ 面积 $= \iint\limits_D \sqrt{1+x^2+y^2}\,dxdy$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \sqrt{1+\rho^2}\cdot\rho\, d\rho$$

$$= 2\pi\left[\frac{1}{3}(1+\rho^2)^{\frac{3}{2}}\right]_0^{\sqrt{2}}$$

$$= \frac{2\pi}{3}(3\sqrt{3}-1)$$

二. 1. 令 $F(x,y,z) = z^5 - xz^4 + yz^3 + 1$. 则

$F_x = -z^4,\quad F_y = z^3,\quad F_z = 5z^4 - 4xz^3 + 3yz^2$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{z^4}{5z^4-4xz^3+3yz^2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z^3}{5z^4-4xz^3+3yz^2}$$

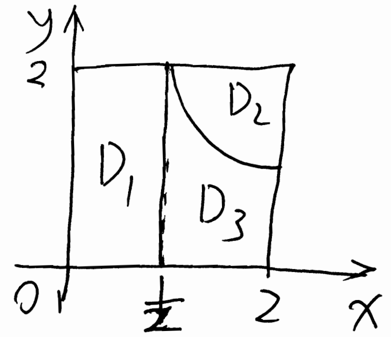
$\because z|_{(0,0)} = -1$, $\therefore \frac{\partial z}{\partial x}\Big|_{(0,0)} = \frac{1}{5}$, $\frac{\partial z}{\partial y}\Big|_{(0,0)} = \frac{1}{5}$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{z^2}{5z^2-4xz+3y}\right)$$

$$=\frac{2z\cdot\frac{\partial z}{\partial y}(5z^2-4xz+3y)-z^2\left(10z\frac{\partial z}{\partial y}-4x\frac{\partial z}{\partial y}+3\right)}{(5z^2-4xz+3y)^2}$$

$$\therefore \left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(0,0)}=-\frac{3}{25}$$

2. 用曲线 $x=\frac{1}{2}$ 与 $y=\frac{1}{x}$ 将 D 分为三部分



在 $D_1\cup D_3$ 上，$xy\le 1$

在 D_2 上，$xy\ge 1$.

于是

$$原式=\iint\limits_{D_1\cup D_3}dxdy+\iint\limits_{D_2}xy\,dxdy$$

$$=1+\int_{\frac{1}{2}}^{2}\frac{1}{x}dx+\int_{\frac{1}{2}}^{2}dx\int_{\frac{1}{x}}^{2}xy\,dy$$

$$=1+[\ln x]_{\frac{1}{2}}^{2}+\int_{\frac{1}{2}}^{2}dx\left[\frac{1}{2}xy^2\right]_{\frac{1}{x}}^{2}$$

$$=1+2\ln 2+\int_{\frac{1}{2}}^{2}\left(2x-\frac{1}{2x}\right)dx$$

$$=1+2\ln 2+\left[x^2-\frac{1}{2}\ln x\right]_{\frac{1}{2}}^{2}$$

$$=1+2\ln 2+\left(\frac{15}{4}-\ln 2\right)=\frac{19}{4}+\ln 2$$

三 1. $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}\cdot y$

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$\because \left|\frac{x^2}{x^2+y^2}\right| \le 1$，且 $y\to 0$，

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

$\therefore f(x,y)$ 在 $(0,0)$ 处连续.

$$f_x(0,0) = \left.\frac{d f(x,0)}{dx}\right|_{x=0} = 0$$

$$f_y(0,0) = \left.\frac{d f(0,y)}{dy}\right|_{y=0} = 0$$

下面考虑可微性. 记 $\Delta x = x-0 = x$,

$\Delta y = y-0 = y$，$\rho = \sqrt{\Delta x^2+\Delta y^2} = \sqrt{x^2+y^2}$

$$\Delta f|_{(0,0)} = f(\Delta x, \Delta y) - f(0,0) = f(x,y)$$

$$= \frac{x^2 y}{x^2+y^2}$$

则 $\lim\limits_{\rho\to 0^+} \frac{\Delta f|_{(0,0)} - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\rho}$

$$= \lim_{\rho\to 0^+} \frac{x^2 y}{x^2+y^2} \Big/ \sqrt{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{(x^2+y^2)^{\frac{3}{2}}}$$

沿着 x 轴 $\to (0,0)$ 时，上述极限为 0

沿着 $y=x \to (0,0)$ 时，上述极限为

$$\lim_{x\to 0} \frac{x^2\cdot x}{(2x^2)^{\frac{3}{2}}} = \frac{1}{2^{\frac{3}{2}}} \ne 0.$$

$\therefore$ 二重极限不存在，于是在 $(0,0)$ 处不可微.

2 利用球面坐标，则

$$\pi=\frac{A}{r^4}\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^{r}\dot{r}\cdot\dot{r}^2\sin\varphi\,d\dot{r}$$

$$=\frac{A}{r^4}\cdot 2\pi\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{r}\dot{r}^3\,d\dot{r}$$

$$=\frac{2\pi A}{r^4}\left[-\cos\varphi\right]_0^{\pi}\cdot\left[\frac{1}{4}\dot{r}^4\right]_0^{r}$$

$$=\frac{2\pi A}{r^4}\cdot 2\cdot\frac{1}{4}r^4=\pi A$$

$\therefore A=1$

四 1. $d=|OP|=\sqrt{x^2+y^2+z^2}$

目标函数可取为 $u=x^2+y^2+z^2$

条件 $z=x^2+y^2$ 与 $x+y+z=1$.

辅助函数：$F(x,y,z,\lambda_1,\lambda_2)=x^2+y^2+z^2$

$+\lambda_1(x^2+y^2-z)+\lambda_2(x+y+z-1)$.

$$\begin{cases}F_x=2(1+\lambda_1)x+\lambda_2=0 & (1)\\ F_y=2(1+\lambda_1)y+\lambda_2=0 & (2)\\ F_z=2z-\lambda_1+\lambda_2=0 & (3)\\ F_{\lambda_1}=x^2+y^2-z=0 & (4)\\ F_{\lambda_2}=x+y+z-1=0 & (5)\end{cases}$$

(1)−(2)得：$2(1+\lambda_1)(x-y)=0$ (6)

若 $1+\lambda_1=0$，即 $\lambda_1=-1$. 则由(1)&(2)得 $\lambda_2=0$。代入(3)得：$2z+1=0$，即 $z=-\frac{1}{2}$.

这与(4)矛盾. 于是 $1+\lambda_1\neq 0$. 由(6)：$y=x$.

代入(4)得：$z=2x^2$ (7)

代入(5)得：$z=1-2x$ (8)

由(7)&(8)得：$2x^2=1-2x \Longrightarrow 2x^2+2x-1=0$

解之得：$x=y=\frac{-1\pm\sqrt{3}}{2}$

由(8) $z=2\mp\sqrt{3}$.

此时：$u=x^2+y^2+z^2=9\mp 5\sqrt{3}$

于是点 $\left(-\frac{1+\sqrt{3}}{2}, -\frac{1+\sqrt{3}}{2}, 2+\sqrt{3}\right)$ 与原点距离达到最大 $\sqrt{9+5\sqrt{3}}$.

五 1. $\begin{cases} u=x-2y \\ v=x+3y \end{cases} \Longrightarrow \begin{cases} x=\frac{3}{5}u+\frac{2}{5}v \\ y=-\frac{1}{5}u+\frac{1}{5}v \end{cases}$

记 $f_1'=\frac{\partial z}{\partial x}=f_x'$，$f_2'=\frac{\partial z}{\partial y}=f_y'$，则

$$\frac{\partial z}{\partial u}=\frac{\partial f}{\partial u}=\frac{3}{5}f_1'-\frac{1}{5}f_2'$$

$$\frac{\partial z}{\partial v}=\frac{\partial f}{\partial v}=\frac{2}{5}f_1'+\frac{1}{5}f_2'$$

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial v}\left(\frac{3}{5}f_1' - \frac{1}{5}f_2'\right)$$

$$= \frac{3}{5}\frac{\partial f_1'}{\partial v} - \frac{1}{5}\frac{\partial f_2'}{\partial v}$$

$$= \frac{3}{5}\left(\frac{2}{5}(f_1')_1' + \frac{1}{5}(f_1')_2'\right)$$

$$- \frac{1}{5}\left(\frac{2}{5}(f_2')_1' + \frac{1}{5}(f_2')_2'\right)$$

$$= \frac{1}{25}\left(6f_{11}'' + f_{12}'' - f_{22}''\right)$$

$$= \frac{1}{25}\left(6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2}\right)$$

2. $$\iiint_{\Omega} f(x^2+y^2+z^2)\,dx\,dy\,dz \xrightarrow{\text{球面坐标}}$$

$$\int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{t} f(r^2)\cdot r^2 \sin\varphi \, dr$$

$$= 2\pi \int_0^{\pi} \sin\varphi \, d\varphi \int_0^{t} f(r^2) r^2 \, dr$$

$$= 4\pi \int_0^{t} r^2 f(r^2)\, dr = 4\pi\, \varphi(t)$$

$$\iint_D f(x^2+y^2)\,dx\,dy \xrightarrow{\text{极坐标}} \int_0^{2\pi} d\theta \int_0^{t} f(\rho^2)\rho\, d\rho$$

$$= 2\pi \int_0^{t} r f(r^2)\, dr = 2\pi\, \psi(t)$$

则 $F(t) = \dfrac{2\varphi(t)}{\psi(t)}$

$$F'(t) = \frac{2W(t)}{\psi^2(t)}，其中$$

$$\begin{aligned} W(t) &= \varphi'(t)\psi(t) - \varphi(t)\cdot\psi'(t) \\ &= t^2 f(t^2)\int_0^t r f(r^2)\,dr \\ &\qquad - t f(t^2)\int_0^t r^2 f(r^2)\,dr \\ &= t f(t^2)\left(\int_0^t t r f(r^2)\,dr - \int_0^t r^2 f(r^2)\,dr\right) \\ &= t f(t^2)\int_0^t r(t-r) f(r^2)\,dr \end{aligned}$$

注意到当 $0<r<t$ 时，$r(t-r)f(r^2)>0$

$\therefore W(t)>0$. 即 $F'(t)=\frac{2W(t)}{\psi^2(t)}>0$.

$\therefore F(t)$ 在 $(0,+\infty)$ 上 $\nearrow$