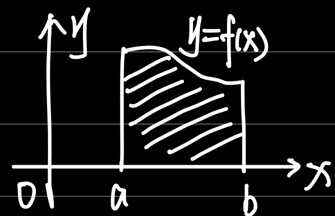
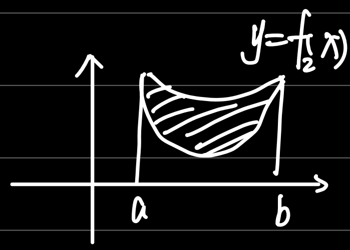
(一). 平面图形的面积.

① 直角坐标系的情形.





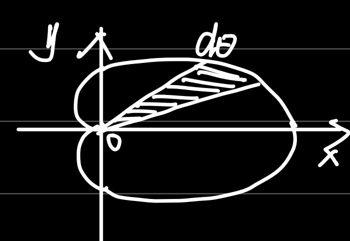
$$A=\int_a^b f(x)\,dx \qquad A=\int_a^b [f_2(x)-f_1(x)]\,dx$$

② 极坐标系情形. 设曲线 $r=\varphi(\theta)$ 且 $\varphi(\theta)\geq 0$.

面积元素: $\boxed{dA=\frac{1}{2}[\varphi(\theta)]^2 d\theta.}$

曲边扇形面积 $\boxed{A=\int_\alpha^\beta \frac{1}{2}[\varphi(\theta)]^2 d\theta.}$

例. 求心形线 $r=a(1+\cos\theta)$ 所围平面图形的面积 $(a>0)$.



$$dA=\frac{1}{2}[a^2(1+\cos\theta)^2]\,d\theta.$$

$$A=2\cdot\frac{1}{2}a^2\int_0^\pi (1+\cos\theta)^2\,d\theta$$

$$=a^2\int_0^\pi (\overset{\frac{\cos2\theta+1}{2}}{\cos^2\theta}+2\cos\theta+1)\,d\theta$$

$$=a^2\int_0^\pi \left(\frac{\cos2\theta}{2}+2\cos\theta+\frac{3}{2}\right)d\theta$$

$$=a^2\left[\frac{1}{4}\sin2\theta+2\sin\theta+\frac{3}{2}\theta\right]_0^\pi$$

曲线面积关于x轴对称.

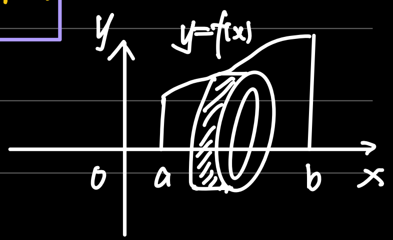
$= \frac{3}{2}\pi a^2$.

(二). 体积.

① 旋转体的体积.

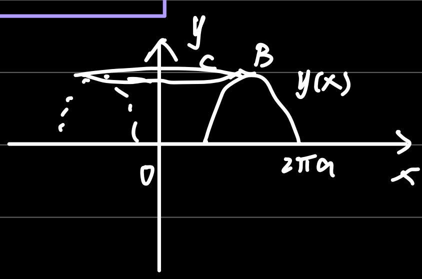
1) 旋转体由 $y=f(x)$, $x=a$, $x=b$ 围成的曲边绕 x 轴转一圈.

$$\boxed{dV = \pi[f(x)]^2 dx \Rightarrow V = \int_a^b \pi[f(x)]^2 dx.}$$

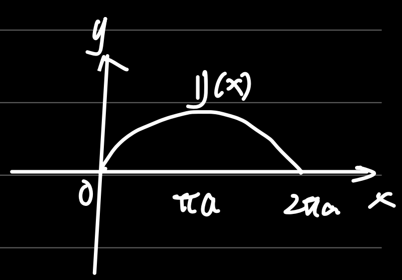


2) 旋转体由 $y=f(x)$, $x=a$, $x=b$ 围成的曲边绕 y 轴转一圈.

☆ $$\boxed{dV_y = 2\pi x|f(x)|dx \Rightarrow V_y = 2\pi\int_a^b x|f(x)|dx.}$$



Eg. 求摆线 $x=a(t-\sin t)$, $y=a(1-\cos t)$ 的一拱与 $y=0$ 所围成的图形分别绕 x 轴、y 轴的旋转体体积.



$$V_x = \int_0^{2\pi a} \pi y^2(x)\,dx$$

$$= \pi\int_0^{2\pi a} a^2(1-\cos t)^2 \cdot a(1-\cos t)\,dt$$

$$=\pi a^3\int_0^{2\pi}\left(1-3\cos t+\overset{\frac{3}{2}\cos 2t+\frac{3}{2}}{3\cos^2 t}-\cos^3 t\right)dt.$$

$$=\pi a^3\left\{\left[\frac{5}{2}t-3\sin t\right]_0^{2\pi}+\left[\frac{3}{4}\sin 2t\right]_0^{2\pi}-\int_0^{2\pi}(1-\sin^2 t)\,d\sin t\right\}$$

$$=\pi a^3\left\{5\pi+0-\left[\sin t-\frac{1}{3}\sin^3 t\right]_0^{2\pi}\right\}$$

$$=5\pi^2 a^3.$$

$$V_y=2\pi\int_0^{2\pi a}x|f(x)|dx=2\pi\int_0^{2\pi}a(t-\sin t)\cdot a(1-\cos t)\,d[a(t-\sin t)]$$

$$=2\pi a^3\int_0^{2\pi}\underbrace{(t-\sin t)\cdot(1-\cos t)^2}_{(t-\sin t)(1-2\cos t+\cos^2 t)}dt$$

$$=2\pi a^3\int_0^{2\pi}\left(t-2t\cos t+t\cos^2 t-\sin t+\overset{\sin 2t}{\underline{2\sin t\cos t}}-\sin t\cos^2 t\right)dt.$$

$$=2\pi a^3\left\{\underbrace{\left[\frac{1}{2}t^2\right]_0^{2\pi}}_{2\pi^2}+\underbrace{2\left[t\sin t\right]_0^{2\pi}}_{0}-\underbrace{2\int_0^{2\pi}\sin t}_{0}+\underbrace{\frac{1}{4}\left[t^2\right]_0^{2\pi}}_{\pi^2}+\underbrace{\frac{1}{4}\left[t\sin 2t\right]_0^{2\pi}}_{0}-\underbrace{\frac{1}{4}\int_0^{2\pi}\sin 2t}_{0}\right.$$

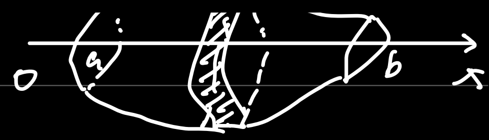
$$\left.+\underbrace{\left[\cos t\right]_0^{2\pi}}_{0}-\underbrace{\left[\frac{1}{2}\cos 2t\right]_0^{2\pi}}_{0}+\underbrace{\left[\frac{1}{3}\cos^3 t\right]_0^{2\pi}}_{0}\right\}$$

$$=6\pi^3 a^3.$$

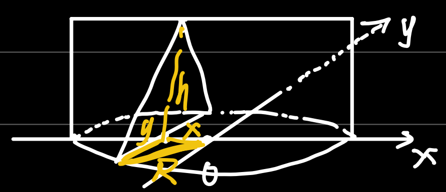
② 平面截面面积为已知的立体体积.

$A(x)$ 表示过点 x 且垂直于 x 轴的截面面积.

$$\boxed{dV = A(x)dx \Rightarrow V = \int_a^b A(x)dx}$$



Eg. 求以半径R的圆为底、平行且等于底圆直径的线段为顶，高为h的正劈锥体的体积.



截面为等腰三角形：

$$A(x) = h \cdot y = h \cdot \sqrt{R^2 - x^2}.$$

$$\text{故 } V = h\int_{-R}^{R} \sqrt{R^2-x^2}\,dx = \frac{1}{2}\pi R^2 h.$$

(二) 平面曲线弧长.

① 直角坐标系.

小切线长：$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+y'^2}\,dx.$

$$\boxed{\text{弧长元素：} ds = \sqrt{1+y'^2}\,dx.}$$

$\Rightarrow$ 弧长：$s = \int_a^b \sqrt{1+y'^2}\,dx.$

② 参数方程情形.

$$\boxed{\text{弧长元素：} ds = \sqrt{\varphi'^2(t) + \psi'^2(t)}\,dt.}$$

弧长：$s = \int_\alpha^\beta \sqrt{\varphi'^2(t) + \psi'^2(t)}\,dt.$

③ 极坐标情形.

由 $\begin{cases} x = r(\theta)\cos\theta \\ y = r(\theta)\sin\theta \end{cases}$ $\quad$ $dx = [r'(\theta)\cos\theta - r(\theta)\sin\theta]\,d\theta$, $\quad dy = [r'(\theta)\sin\theta + r(\theta)\cos\theta]\,d\theta$ $\quad(\alpha \le \theta \le \beta)$

弧长元素: $ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{r'^2(\theta)(\cos^2\theta+\sin^2\theta) + r^2(\theta)(\sin^2\theta+\cos^2\theta)}\,d\theta$

$$\Rightarrow \quad \boxed{S = \int_\alpha^\beta \sqrt{r^2\theta + r'^2(\theta)}\,d\theta}$$

Eg. 求极坐标系下曲线 $r = a(\sin\frac{\theta}{3})^3$ $(a>0,\ 0\le\theta\le 3\pi)$ 的长.

由 $r'(\theta) = a\,3(\sin\frac{\theta}{3})^2 \cdot \cos\frac{\theta}{3} \cdot \frac{1}{3} = a(\sin\frac{\theta}{3})^2\cos\frac{\theta}{3}$.

故 $ds = \sqrt{r^2\theta + r'^2(\theta)}\,d\theta = \sqrt{a^2(\sin\frac{\theta}{3})^6 + a^2(\sin\frac{\theta}{3})^4(\cos\frac{\theta}{3})^2}\,d\theta$

$$= \left|a(\sin\tfrac{\theta}{3})\right| d\theta.$$

故 $S = \int_0^{3\pi} \left|a(\sin\frac{\theta}{3})^2\right| d\theta$

$$= a\int_0^{3\pi}\left(\frac{1}{2} - \frac{1}{2}\cos\frac{2}{3}\theta\right)d\theta$$

$$= a\left[\frac{3}{2}\pi - \frac{3}{2}\int_0^{3\pi}\cos\frac{2}{3}\theta\ d(\tfrac{2}{3}\theta)\right] \qquad -\frac{3}{2}\left[\sin\frac{2}{3}\theta\right]_0^{3\pi}$$

$$= \frac{3}{2}\pi a.$$