一. 1. $\underline{\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$

$\because \vec{N} = (-2, 3, 1) \times (3, -2, 1)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 5\bar{i} + 5\bar{j} - 5\bar{k}$$

$$-\vec{N}^0 = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

2. $\underline{0}$

$$\frac{\partial w}{\partial u_1} = -\frac{\partial f}{\partial x_{2013}} + \frac{\partial f}{\partial x_1}, \quad \frac{\partial w}{\partial u_2} = -\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}$$

$$\frac{\partial w}{\partial u_3} = -\frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3}, \quad \cdots\cdots$$

$$\frac{\partial w}{\partial u_{2012}} = -\frac{\partial f}{\partial x_{2011}} + \frac{\partial f}{\partial x_{2012}}, \quad \frac{\partial w}{\partial u_{2013}} = -\frac{\partial f}{\partial x_{2012}} + \frac{\partial f}{\partial x_{2013}}$$

3. $\underline{\frac{8}{3}\sqrt{2}\pi}$

$$原式 = 2\iint\limits_{x^2+y^2\le 2} \sqrt{2-x^2-y^2}\,dx\,dy$$

$$= 2\int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \sqrt{2-\rho^2}\cdot\rho\,d\rho$$

$$= 4\pi\left[-\frac{1}{3}(2-\rho^2)^{\frac{3}{2}}\right]_0^{\sqrt{2}}$$

$$= \frac{8}{3}\sqrt{2}\pi$$

4. $\underline{6\sqrt{6}\pi}$

Γ为$\begin{cases} x^2+y^2 = \frac{3}{2} \\ z = 2 \end{cases}$，这是一个半径为$\sqrt{\frac{3}{2}}$的圆. 于是

$$\int_\Gamma (2x^2+2y^2+3)\,ds = \int_\Gamma \left(2\cdot\frac{3}{2}+3\right) ds = \int_\Gamma 6\,ds$$

$$= 6\cdot 2\pi\cdot\frac{\sqrt{3}}{\sqrt{2}} = 6\sqrt{6}\,\pi$$

5. $\underline{\frac{\sqrt{3}}{120}}$

在 $x=0$、$y=0$、$z=0$ 上 $\oiint xyz\,dS = 0$

在 $\Sigma_1: z=1-x-y$ 上. $z_x = z_y = -1$.

$\therefore dS = \sqrt{1+z_x^2+z_y^2}\,dxdy = \sqrt{3}\,dxdy$

$$\therefore \oiint_\Sigma xyz\,dS = \iint_D xy(1-x-y)\cdot\sqrt{3}\,dxdy$$

$$= \sqrt{3}\int_0^1 dx\int_0^{1-x} xy(1-x-y)\,dy$$

$$= \sqrt{3}\int_0^1 dx\left[\frac{1}{2}x(1-x)y^2 - \frac{1}{3}xy^3\right]_0^{1-x}$$

$$= \sqrt{3}\int_0^1 \frac{1}{6}x(1-x)^3\,dx \quad \because x = 1-(1-x)$$

$$= \frac{\sqrt{3}}{6}\int_0^1 \left((1-x)^3-(1-x)^4\right) dx$$

$$= \frac{\sqrt{3}}{6}\left[-\frac{1}{4}(1-x)^4+\frac{1}{5}(1-x)^5\right]_0^1$$

$$= \frac{\sqrt{3}}{120}$$

二. 1. (C).

① 沿着 y轴 $\rightarrow (0,0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^4-y^2}{y-\sin^3 x} = \lim_{y\to 0} -y = 0$$

② 延着曲线 $y=x^4+\sin^3x \to (0,0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^4+\sin^3x}} \frac{x^4-y^2}{y-\sin^3x} = 1-\lim_{x\to 0}\frac{(x^4+\sin^3x)^2}{x^4} = 1$$

∴ 此二重极限不存在.

2. (B).

$$f_x'(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\Delta x^2\sin\frac{1}{\Delta x^2}=0$$

同理 $f_y'(0,0)=0$

于是 $\lim\limits_{\rho\to 0^+}\dfrac{\Delta f - f_x'(0,0)\Delta x - f_y'(0,0)\Delta y}{\rho}$

$$= \lim_{\rho\to 0^+}\frac{(\Delta x^3+\Delta y^3)\sin\frac{1}{\Delta x^2+\Delta y^2}}{\rho} \qquad \begin{cases}\Delta x=\rho\cos\theta\\ \Delta y=\rho\sin\theta\end{cases}$$

$$= \lim_{\rho\to 0^+}\rho^2(\sin^3\theta+\cos^3\theta)\sin\frac{1}{\rho} = 0$$

∴ 可微.

又∵ $f_x = \begin{cases} 3x^2\sin\frac{1}{x^2+y^2} - \frac{2x(x^3+y^3)}{(x^2+y^2)^2}\cos\frac{1}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

∴ f_x 与 f_y 均在 $(0,0)$ 处不连续.

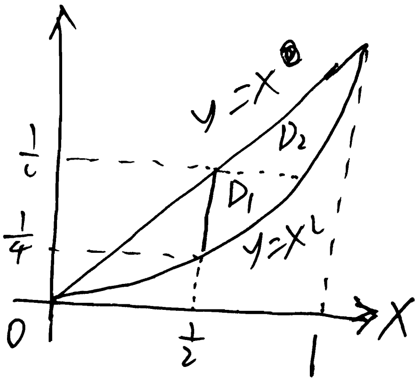
3. (C) 切点 $(1,2,4)$

$\vec{n}_1=(6x^2,-1,0)|_{x=1}=(6,-1,0)$

$\vec{n}_2=(2x,0,-1)|_{x=1}=(2,0,-1)$

$$\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 6 & -1 & 0\\ 2 & 0 & -1\end{vmatrix} = \vec{i}+6\vec{j}+2\vec{k}$$

4. (C) 5 (B)



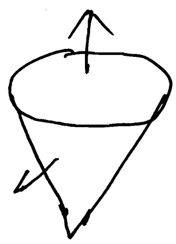
三. 1. 原式 $=\int_{\frac{1}{2}}^{1}dx\int_{x^2}^{x}e^{\frac{y}{x}}dy$

$=\int_{\frac{1}{2}}^{1}dx\left[xe^{\frac{y}{x}}\right]_{x^2}^{x}$

$=\int_{\frac{1}{2}}^{1}x(e-e^{x})dx$

$=\left[\frac{e}{2}x^2-(x-1)e^{x}\right]_{\frac{1}{2}}^{1}=\frac{3e}{8}-\frac{\sqrt{e}}{2}.$

2. 补上曲面 Σ': $z=2$, $x^2+y^2=4$. 取上侧



则 $\Sigma+\Sigma'$ 是封闭曲面, 取外侧. 所围空间闭曲域记为 Ω. 则由高斯公式.

$$\oiint_{\Sigma+\Sigma'}=\iiint_{\Omega}\left(\frac{\partial(x^2e^{-y})}{\partial x}+\frac{\partial y^2}{\partial y}+\frac{\partial z^2}{\partial z}\right)dV.$$

$$=\iiint_{\Omega}(2xe^{-y}+2y+2z)dV$$

$\because$ Ω 关于 xOz 面和 yOz 面对称,

$\therefore$ $\iiint_{\Omega}2xe^{-y}dV=\iiint_{\Omega}2y\,dV=0$

$\therefore$ $\oiint_{\Sigma+\Sigma'}=2\iiint_{\Omega}z\,dV\ \overset{\text{截面法}}{=\!=\!=}\ 2\int_0^2dz\iint_{D_z}z\,dxdy$

其中 D_z 由 $x^2+y^2=z^2$ 围成, $\therefore$

上式 $=2\int_0^2z\cdot\pi z^2dz=2\pi\left[\frac{1}{4}z^4\right]_0^2=8\pi$

而 $\iint_{\Sigma'}=\iint_{D}2^2dxdy=4\cdot\pi\cdot2^2=16\pi$

$\therefore$ $\iint_{\Sigma}=\iiint_{\Omega}-\iint_{\Sigma'}=8\pi-16\pi=-8\pi$

3. 特征方程 $r^2-2012r-2013=0$

$r_1=-1,\ r_2=2013$

通解: $y=C_1e^{-x}+C_2e^{2013x}-\frac{2014}{2013}$

$\because y(0)=C_1+C_2-\frac{2014}{2013}=\frac{2012}{2013}$

$\therefore\ C_1+C_2=2$

$\because y'(0)=-C_1+2013C_2=2012$

$\therefore\ C_1=C_2=1$

$\therefore\ y=e^{-x}+e^{2013x}-\frac{2014}{2013}$

四 1. 由积分中值定理，$\exists(u,v)\in\{(x,y)\mid x^2+y^2\le\rho^2\}$ 使得

$$\iint\limits_{x^2+y^2\le\rho^2}f(x,y)\,dxdy=f(u,v)\cdot\pi\rho^2$$

于是 $\lim\limits_{\rho\to0^+}\frac{1}{\pi\rho^3}\iint\limits_{x^2+y^2\le\rho^2}f(x,y)\,dxdy=\lim\limits_{\rho\to0^+}\frac{1}{\rho}f(u,v)$

$=\lim\limits_{\rho\to0^+}\frac{1}{\rho}\cdot(u^2+v^2)\sin\frac{1}{\sqrt{u^2+v^2}}$

$=\lim\limits_{\rho\to0^+}\rho\cdot\frac{u^2+v^2}{\rho^2}\sin\frac{1}{\sqrt{u^2+v^2}}$

$\because\left|\frac{u^2+v^2}{\rho^2}\sin\frac{1}{\sqrt{u^2+v^2}}\right|\le1$，且 $\rho\to0^+$ 为无穷小.

$\therefore$ 上式 $=0$.

2. (1) $\because\ \frac{\partial(3x^2y+8xy^2)}{\partial y}=\frac{\partial(x^3-y\varphi(x)+e^y)}{\partial x}$

$\therefore\ 3x^2+16xy=3x^2-y\varphi'(x)$

$\therefore \varphi'(x) = -16x,\ \varphi(x) = \int -16x\,dx = -8x^2 + \tilde{C}_1$

$$\therefore u(x,y) = \tilde{C}_2 + \int_{(0,0)}^{(x,y)} (3\tilde{x}^2\tilde{y} + 8\tilde{x}\tilde{y}^2)\,d\tilde{x} + (\tilde{x}^3 + (8\tilde{x}^2 - \tilde{C}_1)\tilde{y} + e^{\tilde{y}})\,d\tilde{y}$$

$$= \tilde{C}_2 + \int_{\overrightarrow{OA}} + \int_{\overrightarrow{AB}}$$

其中 $A(x,0)$, $B(x,y)$

$\overrightarrow{OA}$: $\tilde{y} = 0$, $\tilde{x}: 0 \to x$; $\overrightarrow{AB}$: $\tilde{x} = x$, $\tilde{y}: 0 \to y$

$$\therefore u(x,y) = \tilde{C}_2 + 0 + \int_0^y (x^3 + (8x^2 - \tilde{C}_1)\tilde{y} + e^{\tilde{y}})\,d\tilde{y}$$

$$= \tilde{C}_2 + \left[x^3\tilde{y} + \frac{1}{2}(8x^2 - \tilde{C}_1)\tilde{y}^2 + e^{\tilde{y}}\right]_0^y$$

$$= \tilde{C}_2 + x^3y + \frac{1}{2}(8x^2 - \tilde{C}_1)y^2 + e^y - 1$$

$$= x^3y + (4x^2 + C_1)y^2 + e^y + C_2$$

(2) 由曲线积分的基本定理,

$$\int_{\overrightarrow{L}} = u\left(\frac{\pi}{2}, 1\right) - u(0,0) = \frac{\pi^3}{8} + \pi^2 + C_1 + e - 1$$

五 1. $d = |OP| = \sqrt{x^2+y^2+z^2}$

目标函数可取为 $u = x^2+y^2+z^2$

条件: $z = x^2+y^2$、$x+y+z=1$.

辅助函数: $F(x,y,z,\lambda_1,\lambda_2) = x^2+y^2+z^2 + \lambda_1(x^2+y^2-z) + \lambda_2(x+y+z-1)$.

$F_x = 2(1+\lambda_1)x + \lambda_2 = 0$ (1)

$F_y = 2(1+\lambda_1)y + \lambda_2 = 0$ (2)

$F_z = 2z - \lambda_1 + \lambda_2 = 0$ (3)

$F_{\lambda_1} = x^2+y^2-z=0$ (4)

$F_{\lambda_2} = x+y+z-1=0$ (5)

(1)−(2)得：$2(1+\lambda_1)(x-y)=0$ (6)

若 $1+\lambda_1=0$ 即 $\lambda_1=-1$，则由(1)得 $\lambda_2=0$.

代入(3)得：$2z+1=0$，即 $z=-\frac{1}{2}$，

这与(4)矛盾. 于是 $1+\lambda_1\neq 0$. 由(6)得 $y=x$.

代入(4)得：$z=2x^2$ (7)

代入(5)得：$z=1-2x$ (8)

(7)&(8) $\Rightarrow 2x^2=1-2x \Rightarrow 2x^2+2x-1=0$

解之得：$x=y=\frac{-1\pm\sqrt{3}}{2}$，由(8)得 $z=2\mp\sqrt{3}$

此时：$u=x^2+y^2+z^2=9\mp 5\sqrt{3}$

所以点 $\left(-\frac{1+\sqrt{3}}{2}, -\frac{1+\sqrt{3}}{2}, 2+\sqrt{3}\right)$ 与原点距离最大，为 $\sqrt{9+5\sqrt{3}}$，点 $\left(\frac{-1+\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}, 2-\sqrt{3}\right)$ 与原点距离最小，为 $\sqrt{9-5\sqrt{3}}$.

2.(1) 设切点为 (x_0, y_0). 则切线为 $y-y_0=y'(x_0)(x-x_0)$.

与y轴截距为 $y_0-x_0y'(x_0)$.

由题意：$y_0-x_0y'(x_0)=\sqrt{x_0^2+y_0^2}$.

由 (x_0, y_0) 的任意性：$y-x\frac{dy}{dx}=\sqrt{x^2+y^2}$

即 $\frac{dy}{dx}=\frac{y-\sqrt{x^2+y^2}}{x}$ 这是一个齐次方程.

令 $u=\frac{y}{x}$，$\varphi(u)=u-\sqrt{1+u^2}$.

代入公式 $\int \frac{du}{\varphi(u)-u} = \int \frac{dx}{x}$ 得:

$$-\int \frac{du}{\sqrt{1+u^2}} = \int \frac{dx}{x} \Longrightarrow$$

$$-\ln(u+\sqrt{1+u^2}) = \ln x + \tilde{C}$$

即 $u+\sqrt{1+u^2} = \frac{C}{x} \quad (C = e^{-\tilde{C}})$

又 $\because u = \frac{y}{x}$, $\therefore \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = \frac{C}{x}$

$\therefore y + \sqrt{x^2+y^2} = C$

代入 $y(\frac{1}{2})=0$ 得 $C=\frac{1}{2}$，于是L的方程为

$$y+\sqrt{x^2+y^2} = \frac{1}{2}$$

六. 1. $\int p(x)dx = \int \phi'(x)dx = \phi(x)$

$$\int Q(x) e^{\int p(x)dx} dx = \int \phi'(x)\phi(x) e^{\phi(x)} dx$$

$$= \int \phi(x) e^{\phi(x)} d\phi(x) = (\phi(x)-1) e^{\phi(x)}$$

$$\therefore y = Ce^{-\phi(x)} + e^{-\phi(x)} \cdot (\phi(x)-1) e^{\phi(x)}$$

$$= Ce^{-\phi(x)} + \phi(x) - 1.$$

2. ~~原式 球面坐标~~ $\cancel{\frac{1}{\pi t^4}\int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{t} f(r) r^2 \sin\varphi \, dr}$

$= \lim\limits_{t\to 0}$

2. 利用球面坐标

$$\text{原式} = \lim_{t \to 0} \frac{1}{\pi t^4} \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^t f(r) r^2 \sin\varphi \, dr$$

$$= \lim_{t \to 0} \frac{1}{\pi t^4} \cdot 2\pi \cdot \int_0^{\pi} \sin\varphi \, d\varphi \int_0^t f(r) r^2 \, dr$$

$$= \lim_{t \to 0} \frac{2}{t^4} \left[-\cos\varphi\right]_0^{\pi} \cdot \int_0^t f(r) r^2 \, dr$$

$$= \lim_{t \to 0} \frac{4\int_0^t f(r) r^2 \, dr}{t^4} \xlongequal[\text{法则}]{\text{洛必达}} \lim_{t \to 0} \frac{4 f(t) t^2}{4 t^3}$$

$$= \lim_{t \to 0} \frac{f(t)}{t} = f'(0)$$