一阶微分方程.

① 可分离变量.

1> $\underline{\dfrac{dy}{dx} = F(x,y) = f(x)g(y)}$.　　$\underline{M_1(x)M_2(y)\,dx + N_1(x)N_2(y)\,dy = 0}$

↳ $\dfrac{dy}{g(y)} = f(x)\,dx \quad (g(y)\neq 0)$　　$\dfrac{M_1(x)}{N_1(x)}dx + \dfrac{N_2(y)}{M_2(y)}dy = 0.$

2> $\boxed{\text{齐次型}}$.

Ⅰ $\underline{\dfrac{dy}{dx} = \varphi\left(\dfrac{y}{x}\right)}$.　令 $y = ux$.　$\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$.

↳ $u + x\dfrac{du}{dx} = \varphi(u) \Rightarrow \dfrac{du}{\varphi(u)-u} = \dfrac{dx}{x}$.

Ⅱ $\underline{\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}}$.

↳ 1°. $\dfrac{a_2}{a_1} \neq \dfrac{b_2}{b_1}$.　令 $x = X + h$, $y = Y + k \Rightarrow dx = dX \quad dy = dY$

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + \boxed{a_1h + b_1k + c_1}}{a_2X + b_2Y + \boxed{a_2h + b_2k + c_2}}$$

令 $\begin{cases} a_1h + b_1k + c_1 = 0 \\ a_2h + b_2k + c_2 = 0 \end{cases}$

↳ $\begin{cases} h = \\ k = \end{cases}$

即 $\dfrac{dY}{dX} = \dfrac{a_1X + b_1Y}{a_2X + b_2Y} = \varphi\left(\dfrac{Y}{X}\right)$

2°. $\dfrac{a_2}{a_1} = \dfrac{b_2}{b_1} = \lambda$.

$$\Rightarrow \frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{\lambda(a_1x+b_1y)+c_2}$$

$\boxed{\text{令 } h=a_1x+b_1y.}$ $\downarrow$

$$\frac{dh}{dx} = a_1 + b_1\frac{dy}{dx}$$

$$\hookrightarrow \frac{dh}{dx} = a + b\,\frac{h+c_1}{\lambda h+c_2} = f(x)\cdot g(h)$$

② 一阶线性微分方程.

1) 齐次型：$\frac{dy}{dx}+P(x)y=0$　通解 $y=Ce^{-\int P(x)dx}$

2) 非齐次型：$\frac{dy}{dx}+P(x)y=Q(x)$　通解 $y=e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx+C\right]$

③ 伯努利微分方程.

$$\frac{dy}{dx}+P(x)y=Q(x)y^n \quad (n\neq 0、1)$$

$$\frac{dy}{dx} = \frac{Q(x)-y^{1-n}P(x)}{y^{-n}}$$

$\hookrightarrow$ 两边同除 $y^n \Rightarrow \underline{y^{-n}\frac{dy}{dx}+P(x)y^{1-n}}=Q(x)$.　令 $z=y^{1-n}$.

$\hookrightarrow \frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$

$\hookrightarrow \frac{dz}{dx}=(1-n)y^{-n}\,\frac{Q(x)-y^{1-n}P(x)}{y^{-n}}=(1-n)\left[Q(x)-zP(x)\right]$

$\hookrightarrow \frac{dz}{dx}+(1-n)P(x)z=(1-n)Q(x)$.

高阶微分方程.

也是解.

①叠加原理：y_1, y_2 是 $y''+P(x)y'+Q(x)y=0$ 的解 $\Rightarrow$ $y=C_1y_1+C_2y_2$

②线性无关性：若 y_1, y_2 线性无关，则 $y=C_1y_1+C_2y_2$ 是方程的通解.

③二阶非齐次通解：$y''+P(x)y'+Q(x)y=f(x)$. 通解 = 齐次通解 + 特解.

可推广 n 维

④ $y_k(x)$ 是 $y''+P(x)y'+Q(x)y=f_k(x)$ 的特解.

$\hookrightarrow$ $y=\sum_{k=1}^{m} y_k$ 是 $y''+P(x)y'+Q(x)y=\sum_{k=1}^{m} f_k(x)$ 的特解.

③ 常系数的微分方程.

1) 齐次型. $\Rightarrow$ 求特征方程的特征根.

$y''+py'+qy=0 \Rightarrow r^2+pr+q=0$. 特征根 r_1, r_2.

Ⅰ $r_1 \neq r_2$ (实根) $\Leftrightarrow \Delta>0$ $\quad y=C_1e^{r_1x}+C_2e^{r_2x}$.

Ⅱ $r_1=r_2$ $\Leftrightarrow \Delta=0$ $\quad y=(C_1+C_2x)e^{rx}$

Ⅲ $r_{1,2}=\alpha\pm i\beta$ $\Leftrightarrow \Delta<0$ $\quad y=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$

II> 非齐次型.

$y''+py'+qy=f(x)$ → 齐次通解. / 特解.

① $f(x)=e^{\lambda x}P_m(x)$.　　$P_m(x)$ 为 m 次多项式.

设特解 $y^*=e^{\lambda x}Q(x)$

$$(y^*)'=e^{\lambda x}[\lambda Q(x)+Q'(x)]$$

$$(y^*)''=e^{\lambda x}[\lambda^2Q(x)+2\lambda Q'(x)+Q''(x)].$$

↳ $Q''(x)+(2\lambda+p)Q'(x)+(\lambda^2+p\lambda+q)Q(x)=P_m(x)$ ★

(i). λ 不是特征根 ($\lambda^2+p\lambda+q\neq0$). $\Rightarrow Q_m(x)$ 为 m 次多项式.

(ii). λ 是单根. $\lambda^2+p\lambda+q=0$, $2\lambda+p\neq0 \Rightarrow y^*=xQ_m(x)e^{\lambda x}$.

(iii). λ 是重根. $\lambda^2+p\lambda+q=0$, $2\lambda+p=0 \Rightarrow y^*=x^2Q_m(x)e^{\lambda x}$.

② $f(x)=e^{\lambda x}[P_l(x)\cos\omega x+\tilde{P}_n(x)\sin\omega x]$

讨论 $\lambda+i\omega$ 为特征方程的 k 重根 → 不是. $k=0$. / 是. $k=1$.

↳ 特解: $y^*=x^ke^{\lambda x}[R_m(x)\cos\omega x+\tilde{R}_m(x)\sin\omega x]$　$m=\max\{l,n\}$.