一. 1. πa^3　　2. $\frac{11}{5}$　　3. $x^3+x^2y-xy^2=C$

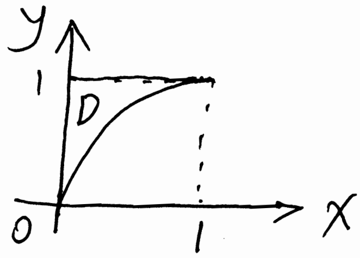
4. $\int_0^1 dy\int_{y-1}^{\sqrt{1-y^2}} f(x,y)\,dx$　　5. $\frac{\pi}{6}(5\sqrt{5}-1)$

二. C　A　D　B　C

三 1. 解法1. $I=\iint\limits_D \frac{1}{\sqrt{x}}e^{-y^2}\,dxdy$

这里 D: $0\le x\le 1$

$\sqrt{x}\le y\le 1$



$\therefore I=\int_0^1 dy\int_0^{y^2}\frac{1}{\sqrt{x}}e^{-y^2}\,dx$

$=\int_0^1 dy\left[2e^{-y^2}\sqrt{x}\right]_0^{y^2}=\int_0^1 2ye^{-y^2}\,dy$

$=\left[-e^{-y^2}\right]_0^1=1-e^{-1}$

解法2. $\varphi'(x)=-e^{-(\sqrt{x})^2}\cdot\frac{1}{2}x^{-\frac{1}{2}}=-\frac{1}{2\sqrt{x}}e^{-x}$

$\therefore I=2\int_0^1\varphi(x)\,d\sqrt{x}$ 分部积分

$\left[2\varphi(x)\sqrt{x}\right]_0^1-2\int_0^1\sqrt{x}\,d\varphi(x)$

$=2\varphi(1)-2\int_0^1\sqrt{x}\left(-\frac{1}{2\sqrt{x}}e^{-x}\right)dx$

$=\int_0^1 e^{-x}\,dx=1-e^{-1}$

2. 记 $P=e^x-x^2y$，$Q=xy^2-\sin y$，

则 $\frac{\partial P}{\partial y}=-x^2$，$\frac{\partial Q}{\partial x}=y^2$. 于是由格林公式得：此曲线积分等于

$$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\iint_D(x^2+y^2)\,dxdy$$

$$\xlongequal{极坐标}\int_0^{2\pi}d\theta\int_0^a\rho^2\cdot\rho\,d\rho$$

$$=2\pi\cdot\frac{1}{4}a^4=\frac{1}{2}\pi a^4$$

3. $P=2x+3y$，$Q=0$，$R=y+z$

解法1. 原式 $=\iint_D(-Pz'_x-Qz'_y+R)\,dxdy$

$$=\iint_D\left(-(2x+3y)\cdot 2x-0\cdot 2y+y+(x^2+y^2)\right)dxdy$$

$$=\iint_D(-3x^2+y^2-6xy+y)\,dxdy$$

$\because$ $-6xy+y$ 是 y 的奇函数，D关于x轴对称.

$\therefore$ $\iint_D(-6xy+y)\,dxdy=0$

于是 原式 $=\iint_D(-3x^2+y^2)\,dxdy$

$$\underline{\underline{\text{极坐标}}}\ \int_0^{2\pi} d\theta \int_0^1 (-3\rho^2\cos^2\theta + \rho^2\sin^2\theta)\cdot\rho\, d\rho$$

$$= \frac{1}{4}\int_0^{2\pi}(-3\cos^2\theta + \sin^2\theta)\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}(-2\cos 2\theta - 1)\, d\theta$$

$$= \frac{1}{4}\Big[-\sin 2\theta - \theta\Big]_0^{2\pi} = -\frac{1}{2}\pi$$

解法2. 记 $\Sigma_1: z=1,\ x^2+y^2\le 1$，方向为上侧. 则

由高斯公式

$$\oiint\limits_{\Sigma^- + \Sigma_1} = \iiint\limits_{\Omega}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dv$$

$$= \iiint\limits_{\Omega}(2+0+1)\, dv = 3\iiint\limits_{\Omega} dv$$

$$\underline{\underline{\text{平行截面法}}}\ 3\int_0^1 dz \iint\limits_{x^2+y^2\le z} dx\,dy$$

$$= 3\int_0^1 \pi z\, dz = 3\pi\left[\frac{1}{2}z^2\right]_0^1 = \frac{3}{2}\pi$$

而 $\displaystyle\iint\limits_{\Sigma_1} = \iint\limits_{D}(y+1)\, dx\,dy = \iint\limits_{D} dx\,dy = \pi$

于是 $\displaystyle\iint\limits_{\Sigma} = -\oiint\limits_{\Sigma^-+\Sigma_1} + \iint\limits_{\Sigma_1} = -\frac{1}{2}\pi$.

四 1.(1) $f'_x(0,0)=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$

$f'_y(0,0)=\lim\limits_{\Delta y\to 0}\dfrac{f(0,\Delta y)-f(0,0)}{\Delta y}=0$

∴ 偏导数均存在.

(2.) $\Delta z=f(\Delta x,\Delta y)-f(0,0)=\sqrt[3]{\Delta x\Delta y}$

$dz=f'_x(0,0)\cdot\Delta x+f'_y(0,0)\cdot\Delta y=0$

考虑二重极限

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta z-dz}{\rho}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\sqrt[3]{\Delta x\Delta y}}{\sqrt{\Delta x^2+\Delta y^2}}$$

当沿着 x 轴趋于 $(0,0)$ 时,

$$\lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \Delta y=0}}\frac{\sqrt[3]{\Delta x\Delta y}}{\sqrt{\Delta x^2+\Delta y^2}}=0$$

当沿着直线 $y=x$ 趋于 $(0,0)$ 时.

$$\lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \Delta y=\Delta x}}\frac{\sqrt[3]{\Delta x\Delta y}}{\sqrt{\Delta x^2+\Delta y^2}}=\lim_{\Delta x\to 0}\frac{1}{\sqrt[3]{\Delta x}}\to\infty$$

于是此二重极限不存在, 所以函数 $f(x,y)$ 在原点处不可微分.

2. $f''(x)=x+f(x)$，令 $y=f(x)$ 得：$y''-y=x$

特征方程为 $r^2-1=0$，$r_{1,2}=\pm 1$.

齐次方程的通解为 $y=C_1e^x+C_2e^{-x}$.

非齐次方程的一个特解可设为 $y^*=ax+b$

于是 $y^{*''}-y^*=-ax-b=x$

即 $\begin{cases} a=-1 \\ b=0 \end{cases}$ $\therefore y^*=-x$

即原方程通解为 $f(x)=y=C_1e^x+C_2e^{-x}-x$

由 $f(0)=1$，$f'(0)=g(0)=2$ 得 $\begin{cases} C_1+C_2=1 \\ C_1-C_2=2 \end{cases}$

即 $\begin{cases} C_1=\frac{3}{2} \\ C_2=-\frac{1}{2} \end{cases}$ 于是 $f(x)=\frac{3}{2}e^x-\frac{1}{2}e^{-x}-x$.

五 1. 目标函数：$f(x,y,z)=xyz$，条件 $x^2+y^2+z^2=3$.

构造函数 $L(x,y,z,\lambda)=xyz+\lambda(x^2+y^2+z^2-3)$

$$\text{令}\begin{cases} L_x = yz+2\lambda x=0 \\ L_y = xz+2\lambda y=0 \\ L_z = xy+2\lambda z=0 \\ L_\lambda = x^2+y^2+z^2-3=0 \end{cases}$$

若 $\lambda=0$，则 $y=0$ 或 $z=0$. 此时 $f(x,y,z)=0$

现设 $\lambda\neq0$. 则 $x^2=y^2=z^2=-\frac{1}{2\lambda}\cdot xyz$

于是 $x^2=y^2=z^2=1$，即 $f(x,y,z)=\pm1$

比较 0, ±1 得 $f_{min}=-1$，$f_{max}=1$.

2. 记 $F(x,y,z)=x^2+y^2+z^2-14$. 因 $F'_x=2x$, $F'_y=2y$, $F'_z=2z$. 所以点 $(1,2,3)$ 处的法向量为 $\vec{n}=(2,4,6)$. 于是切平面方程为

$$2(x-1)+4(y-2)+6(z-3)=0$$

即 $x+2y+3z=14$. 由此可得 $z=\frac{1}{3}(14-x-2y)$.

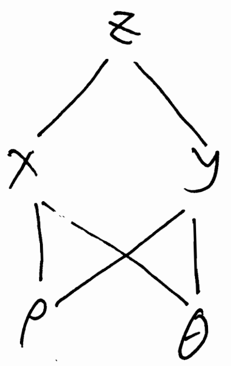
$$V=\iint_D \frac{1}{3}(14-x-2y)\,dxdy$$

$$=\frac{14}{3}\iint_D dxdy-\iint_D (x+2y)\,dxdy.$$

因 D 关于 x 轴和 y 轴均对称，所以 $\iint_D (x+2y)\,dxdy=0$.

于是 $V=\frac{14}{3}\iint_D dxdy=\frac{14}{3}\cdot\pi\cdot 2^2=\frac{56}{3}\pi$

六 1.



$$\frac{\partial z}{\partial \rho}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \rho}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \rho}$$

$$=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$=\rho\left(-\frac{\partial z}{\partial x}\sin\theta+\frac{\partial z}{\partial y}\cos\theta\right)$$

于是 $\left(\frac{\partial z}{\partial \rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

$$=\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)^2+\left(-\frac{\partial z}{\partial x}\sin\theta+\frac{\partial z}{\partial y}\cos\theta\right)^2$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

2. (1) 过 $(0,0,z)$ 处平行于 xOy 平面的截面为:

D_z : $\dfrac{x^2}{a_z^2}+\dfrac{y^2}{b_z^2}\le 1$, 这里

$a_z := a\sqrt{1-\dfrac{z^2}{c^2}}$, $b_z = b\sqrt{1-\dfrac{z^2}{c^2}}$

D_z 的面积 $A_z = \pi a_z b_z = \pi ab\left(1-\dfrac{z^2}{c^2}\right)$.

于是

$$\iiint_{\Omega} f(z)\,dz = \int_{-c}^{c} dz \iint_{D_z} f(z)\,dx\,dy$$

$$= \int_{-c}^{c} \pi ab\left(1-\frac{z^2}{c^2}\right) f(z)\,dz$$

$$= \pi ab \int_{-c}^{c} \left(1-\frac{z^2}{c^2}\right) f(z)\,dz$$

(2)

$$V = \pi ab \int_{-c}^{c}\left(1-\frac{z^2}{c^2}\right) dz$$

$$= \pi ab\left[z-\frac{z^3}{3c^2}\right]_{-c}^{c}$$

$$= \frac{4}{3}\pi abc$$