小测验 3 不定积分 答案

姓名：________________序号：________

1. $\int \frac{x^{11}}{x^8-4x^4-12}dx$ **(6′)**

解：

$$原式=\frac{1}{4}\int \frac{x^8 d(x^4)}{x^8-4x^4-12} \overset{t=x^4}{\Longleftrightarrow} \frac{1}{4}\int \frac{t^2}{t^2-4t-12}dt=\frac{1}{4}\int \frac{t^2-4t-12+4t+12}{t^2-4t-12}dt$$

$$=\frac{1}{4}\left(\int dt+\int \frac{4t+12}{(t-6)(t+2)}dt\right)=\frac{1}{4}t+\frac{1}{4}\int \left(\frac{\frac{9}{2}}{t-6}-\frac{\frac{1}{2}}{t+2}\right)dt$$

$$=\frac{1}{4}t+\frac{9}{8}\ln|t-6|-\frac{1}{8}\ln|t+2|+C=\frac{1}{4}x^4+\frac{1}{8}\ln\left|\frac{(x^4-6)^9}{x^4+2}\right|+C.$$

2. $\int e^{sinx}\cdot\frac{xcos^3x-sinx}{cos^2x}dx$ **(6′)**

解：

$$原式=\int x\cdot e^{sinx}cosxdx-\int e^{sinx}\cdot\frac{sinx}{cos^2x}dx=\int xd\left(e^{sinx}\right)+\int e^{sinx}\cdot\frac{1}{cos^2x}d(cosx)$$

$$=xe^{sinx}-\int e^{sinx}dx-\int e^{sinx}d\left(\frac{1}{cosx}\right)$$

$$=xe^{sinx}-\int e^{sinx}dx-e^{sinx}\cdot\frac{1}{cosx}+\int \frac{1}{cosx}d\left(e^{sinx}\right)$$

$$=xe^{sinx}-\int e^{sinx}dx-e^{sinx}\cdot\frac{1}{cosx}+\int e^{sinx}dx=xe^{sinx}-\frac{e^{sinx}}{cosx}+C.$$

3. $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ **(6′)**

解：

设
$$x=sint\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right),$$

则
$$dx=costdt,\sqrt{1-x^2}=cost,$$

$$\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}=\int \frac{dt}{1+sin^2t}=\int \frac{dt}{2sin^2t+cos^2t}=\int \frac{dt}{cos^2t(2tan^2t+1)}$$

$$=\frac{1}{\sqrt{2}}\int \frac{d(\sqrt{2}tant)}{(\sqrt{2}tant)^2+1}=\frac{1}{\sqrt{2}}\arctan(\sqrt{2}tant)+C=\frac{1}{\sqrt{2}}\arctan\frac{\sqrt{2}x}{\sqrt{1-x^2}}+C.$$

4. $\int \frac{sinxcosx}{sinx+cosx}dx$(**6′**)

解：

$$原式=\int \frac{1}{2}\cdot\frac{2sinxcosx+1-1}{sinx+cosx}dx=\frac{1}{2}\int \frac{(sinx+cosx)^2}{sinx+cosx}dx-\frac{1}{2}\int \frac{1}{sinx+cosx}dx$$

$$=\frac{1}{2}\int (sinx+cosx)dx-\frac{1}{2}\int \frac{d\left(x+\frac{\pi}{4}\right)}{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}$$

$$=\frac{1}{2}(sinx-cosx)-\frac{1}{2\sqrt{2}}\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|+C.$$

5. $\int \max\{1,x^2\}dx$ (**8′**)

解： 当$|x|\le 1$时，

$$\int \max\{1,x^2\}dx=\int dx=x+C_1,$$

当$x>1$时，

$$\int \max\{1,x^2\}dx=\int x^2dx=\frac{1}{3}x^3+C_2,$$

当$x<-1$时，

$$\int \max\{1,x^2\}dx=\int x^2dx=\frac{1}{3}x^3+C_3,$$

设$F(x)$是满足$F(1)=1$的原函数，则上面的讨论知

$$F(x)=\begin{cases}x+C_1,-1\le x\le 1,\\ \frac{1}{3}x^3+C_2,x>1,\\ \frac{1}{3}x^3+C_3,x<-1,\end{cases}$$

其中C_1,C_2,C_3为常数.由于

$$1=F(1)=\lim_{x\to 1+0}F(x),$$

有

$$1=1+C_1=\frac{1}{3}+C_2,$$

故$C_1=0,C_2=\frac{2}{3}$;

又$F(-1)=\lim_{x\to -1-0}F(x)$,

$$-1=-\frac{1}{3}+C_3,$$

故$C_3=-\frac{2}{3}$.

从而

$$F(x)=\begin{cases} x,-1\le x\le 1,\\ \dfrac{1}{3}x^3+\dfrac{2}{3},x>1,\\ \dfrac{1}{3}x^3-\dfrac{2}{3},x<-1. \end{cases}$$

因此

$$\int \max\{1,x^2\}dx=F(x)+C=\begin{cases} x+C, & |x|\le 1,\\ \dfrac{1}{3}x^3+\dfrac{2}{3}sgnx+C, & |x|\le 1. \end{cases}$$

6. 设$f(x^2-1)=ln\dfrac{x^2}{x^2-2}$，且$f[\varphi(x)]=lnx$，求$\int \varphi(x)dx$.**(8′)**

解：

设$x^2-1=t$，则$x^2=t+1$，代入$f(x^2-1)=ln\dfrac{x^2}{x^2-2}$中得$f(t)=ln\dfrac{t+1}{t-1}$.

因此

$$f\big(\varphi(x)\big)=ln\frac{\varphi(x)+1}{\varphi(x)-1}=lnx.$$

再由

$$\frac{\varphi(x)+1}{\varphi(x)-1}=lnx$$

解得

$$\varphi(x)=\frac{x+1}{x-1},$$

于是

$$\int \varphi(x)dx=\int\frac{x+1}{x-1}dx=\int\left(\frac{x-1+2}{x-1}\right)dx=\int\left(1+\frac{2}{x-1}\right)dx=x+2\ln|x-1|+C.$$