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1. a) $\bar{x} = \frac{1}{n}\sum x_i = 26$, $s^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2 = 27.14$

b) The hypothesis test is $H_0: \mu = \mu_0 = 26$, $H_1: \mu > \mu_0 = 26$. where μ is the mean of no. of blueberries Micheal is now adding

c) $t = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{26-26}{4/\sqrt{15}} = 0$

d) Since $t = 0 \not\geq Z_{0.05} = 1.645$, we do not reject H_0.

e) The sample size does not matter as $\bar{x} = \mu_0$ so the test statistic is always 0.

f) $t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{26-26}{5.21/\sqrt{15}} = 0$ Since $t = 0 \not\geq t_{0.05}(14) = 1.761$, we do not reject H_0.

2. a) $H_0: p = p_0 = 0.5$, $H_A: p \neq p_0 = 0.5$. We see that $\frac{1}{n}\sum x_i \sim N(p, \frac{p(1-p)}{n})$ where $p = \frac{35}{50} = 0.7$ by CLT as $n=50$ is large enough.

Hence, $t = \frac{p - p_0}{\sqrt{p(1-p)/n}} = 3.08 \geq Z_{0.05/2} = 1.960$. Hence, we reject H_0.

b) To reject H_0, $|t| \geq Z_{0.05/2}$ so $\frac{(h-25)/50}{\sqrt{h(50-h)/50^3}} = \frac{h-25}{\sqrt{h(50-h)/50}} = 1.960$ or $\frac{h-25}{\sqrt{h(50-h)/50}} = -1.960$ $(h-25)^2 = 1.960^2 \frac{1}{50}(50-h)h$ $h^2 - 50h + 25^2 - 1.960^2 h + 1.960^2/50\, h^2 = 0$, $h = 18.3 \approx 18$ and $h = 31.6 \approx 32$.

Hence, Micheal would reject H_0 for $h \leq 18$ or $h \geq 32$.

3. a) $\beta(\mu') = P(Z \leq -z_\beta) = P(Z \leq \frac{\bar{x} - \mu'}{\sigma/\sqrt{n}}) = P(Z \leq \frac{\bar{x} - \mu_0 + \mu_0 - \mu'}{\sigma/\sqrt{n}}) = P(Z \leq \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}) = P(Z \leq z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}) = \Phi(Z \leq z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}})$ where $\bar{x}, \sigma, \sqrt{n}$ are values that give α and β for the errors.

b) By a), $-Z_\beta = Z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}$ $\quad -(Z_\alpha + Z_\beta) = \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}$ $\quad -\frac{\sigma(Z_\alpha + Z_\beta)}{\mu_0 - \mu'} = \sqrt{n}$ $\quad n = \left(\frac{\sigma(Z_\alpha + Z_\beta)}{\mu_0 - \mu'}\right)^2$

c) $n = \left(\frac{1300(1.645 + 1.960)}{25000 - 27000}\right)^2 = 5.49 \approx 6$, round up to get minimum accuracy.

4. a) $H_0: p_A = p_B$, $H_1: p_A \neq p_B$

b) $p_A = 0.120$, $p_B = 0.119$

c) $t = 1.362$

d) Since $t = 1.362 \not\geq Z_{0.05/2} = 1.960$, we do not reject H_0 at 5% significance.

5. a) $H_0: \mu_X = \mu_Y$, $H_1: \mu_X > \mu_Y$

b) $t = \frac{\mu_X - \mu_Y}{s_p\sqrt{1/n + 1/m}} = -8.329 \not\geq t_{0.05}(19061) \approx Z_{0.05} = 1.645$. We do not reject H_0 at 5% significance

c) $t = (\mu_X - \mu_Y)/\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{n}} = -8.337 \not\geq t_{0.05}(10704) \approx Z_{0.05} = 1.645$. We do not reject H_0 at 5% significance

d) $F = 1.005 \geq 0.964 = F_{0.05}(13390, 5671)$. Hence, we reject H_0 at 5% significance

ST2132 Mathematical Statistics
Assignment 3
Due date: 14 March 2025 23:59 on Canvas
Please submit in pdf format.

Please work on the assignment by yourself only. We follow a strict rule on academic honesty.

Generative AI policy: The use of generative AI tools (e.g., ChatGPT, Gemini, etc.) is strictly prohibited for structural or conceptual questions in this course. However, these tools may be used to assist with coding tasks, provided their use is clearly documented.

1. Charles suspects that Michael has been adding more blueberries than usual to his muffins. For some reason, he has kept meticulous records, and found that Michael's muffins usually contain a number of blueberries that is normally distributed with mean $\mu_0 = 26$ and standard deviation $\sigma = 4$. However, the last 15 muffins Charles bought contained the following numbers of blueberries:

   21 22 18 29 35 20 30 32 26 23 19 29 31 28 27

   Charles is considerate, and decides that he doesn't want to alarm Michael unless his findings are significant at the $\alpha = 0.05$ level.

   (a) Compute the sample mean $\overline{x}$ and sample variance s^2.

   (b) Write down the (one-sided) hypothesis test being conducted above.

   (c) Compute the test statistic associated with the above data.

   (d) Does Charles reject H_0, the null hypothesis?

   (e) If your answer to part (d) is "yes", how small would the sample mean $\overline{x}$ need to be for Charles not to reject H_0? Alternatively, if your answer to part (d) is "no", how large would $\overline{x}$ need to be for Charles to reject H_0?

   (f) Repeat parts (c) and (d), but assume this time that Charles has no idea what σ^2 is.

2. Michael suspects that the quarter in his pocket may not be a fair coin. He flips it 50 times, and to conduct a two-sided hypothesis test at the $\alpha = 0.05$ significance level. Heads appears $h = 35$ times.

   (a) Does Michael reject H_0?

   (b) For what values of h would Michael reject H_0, while fixing $n = 50$?

3. (Sample size determination for hypothesis testing) Suppose that $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$ and σ^2 is known. We would like to test

$$H_0 : \mu = \mu_0, \quad H_1 : \mu > \mu_0.$$

   We consider the test statistic

$$T = \overline{X}$$

   and the critical region at significance level α

$$C = \left\{ \overline{x};\ \overline{x} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right\}.$$

(a) We write $\beta(\mu')$ to be the probability of type II error given the true value of $\mu = \mu'$. Prove that

$$\beta(\mu') = \Phi\left(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right),$$

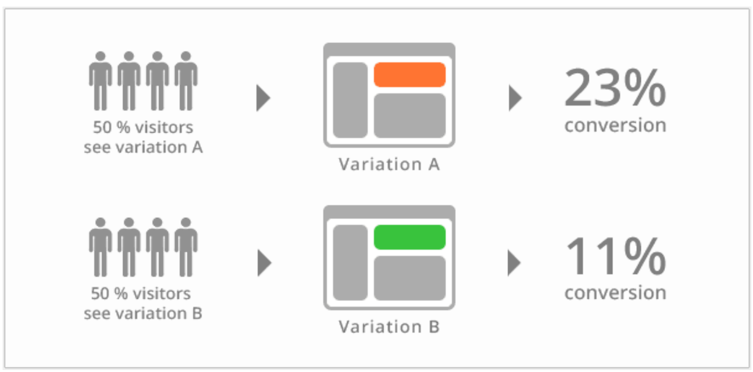
where $\Phi(z) = P(Z \leq z)$ is the cdf of $Z \sim N(0,1)$.

(b) We would like to determine the sample size n so that the probability of type I error is α and the probability of type II error is β. Using part (a), prove that

$$n = \left(\frac{\sigma\left(z_\alpha + z_\beta\right)}{\mu_0 - \mu'}\right)^2.$$

(c) Let μ denote the true average tread life of a certain type of tire. Consider testing $H_0 : \mu = 25,000$ versus $H_1 : \mu > 25,000$ from a normal population distribution with $\sigma = 1300$. Assuming the true value of μ is $\mu' = 27,000$. What is the sample size n so that the probability of type I error is $\alpha = 0.05$ and the probability of type II error is $\beta = 0.025$?

4. (A/B testing) When designing websites, people would like to design a website that maximize the click through rate (CTR), that is, the probability that an user will click on an advertisement presented on the website. This is usually done by what people call A/B testing. Suppose that we have two designs of the website, say Version A and Version B. Denote by p_A (respectively p_B) to be the proportion of users that click on an advertisement in Version A (respectively Version B) of the website. Use your favourite computing language to answer this question, and attach your code at the end of your assignment.



(a) We would like to test whether the proportion of users that click on an advertisement is different between Version A and Version B. Formulate the null hypothesis H_0 and the (two-sided) alternative hypothesis H_1.

(b) Download the dataset "AB_2024.csv" on Canvas. There are 4 columns with 294490 data points. We will only use the column "landing_page" and "converted". "landing_page" is either "A" or "B", indicating the user is landing on Version A or Version B. "converted" is either "1" or "0", with "1" meaning the user clicks on an advertisement and "0" meaning the user does not click on an advertisement.

Report $\widehat{p}_A$ and $\widehat{p}_B$, the maximum likelihood estimate of p_A and p_B respectively.

(c) Assume that the user behaviour in Version A is independent of that in Version B. Compute the test statistic of the hypothesis test.

(d) Is H_0 rejected at 5% significance level? That is, are the two versions of the website really give different CTR at 5% significance?

5. (Traffic volume continued) Recall the last question of Assignment 2. Michael lives in Minneapolis. He is interested in the traffic volume of Minneapolis. Download the dataset "traffic.csv" on Canvas. There are 9 columns and 48,204 data points. We will only use two columns for this question: "weather_main" and "traffic_volume". "weather_main" is a short description of the weather, and "traffic_volume" is the hourly traffic volume. Use your favourite computing language and attach your code at the end of your answer.

Assume the hourly traffic volume when the weather is "Rain" are $Y_1, \dots, Y_m \overset{i.i.d.}{\sim} N(\mu_Y, \sigma_Y^2)$, and the hourly traffic volume when the weather is "Clear" are $X_1, \dots, X_n \overset{i.i.d.}{\sim} N(\mu_X, \sigma_X^2)$, and X_i and Y_i are independent and $\sigma_X^2 \neq \sigma_Y^2$.

(a) Michael is interested in doing a comparison between the hourly traffic volume when the weather is "Clear" verus the hourly traffic volume when the weather is "Rain". He has the feeling that there is less traffic on average when the weather is "Rain", and would like to derive some statistical insights from the data to support his idea. Formulate the null hypothesis H_0 and the (one-sided) alternative hypothesis H_1.

(b) Assume that $\sigma_X^2 = \sigma_Y^2$. Compute the test statistic and conduct an appropriate hypothesis test at 5% significance level.

(c) Assume that $\sigma_X^2 \neq \sigma_Y^2$. Compute the test statistic and conduct an appropriate hypothesis test at 5% significance level.

(d) Conduct the F-test to test

$$H_0 : \sigma_X^2 = \sigma_Y^2, \quad H_1 : \sigma_X^2 \neq \sigma_Y^2.$$

at 5% significance level.