线性判别分析

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线性辨别分析算法(Linear Discriminant Analysis, LDA)

- 线性判别分析算法（LDA）是一个有监督学习算法。
- 假设特征都是连续数值、服从高斯分布，而且个类别特征高斯分布的协方差矩阵一样。
- LDA是一个线性分类器。
- LDA将样本输入投影到一个具有最大分类能力的一维空间中。

线性辨别分析算法(Linear Discriminant Analysis, LDA)

- 给定一个样本输入向量 $\boldsymbol{x} = [x_1 \quad x_2 \quad \cdots \quad x_m]^T$
- 假设在标签类别给定前提下输入向量服从高斯分布，而且每个类别特征的高斯分布的协方差矩阵相同

$$p(\boldsymbol{x}|y=0) = \frac{1}{(2\pi)^{m/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_0)}$$

$$p(\boldsymbol{x}|y=1) = \frac{1}{(2\pi)^{m/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)}$$

线性辨别分析算法(Linear Discriminant Analysis, LDA)

根据贝叶斯准则

$$p(y=0|\boldsymbol{x})=\frac{p(y=0)p(\boldsymbol{x}|y=0)}{p(\boldsymbol{x})}=\frac{p(y=0)}{p(\boldsymbol{x})}\frac{1}{(2\pi)^{m/2}|\boldsymbol{\Sigma}|^{1/2}}e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_0)}$$

$$p(y=1|\boldsymbol{x})=\frac{p(y=1)p(\boldsymbol{x}|y=1)}{p(\boldsymbol{x})}=\frac{p(y=1)}{p(\boldsymbol{x})}\frac{1}{(2\pi)^{m/2}|\boldsymbol{\Sigma}|^{1/2}}e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)}$$

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$$p(y=1|\boldsymbol{x})=\frac{p(y=1)p(\boldsymbol{x}|y=1)}{p(\boldsymbol{x})}=\frac{p(y=1)}{p(\boldsymbol{x})}\frac{1}{(2\pi)^{m/2}|\boldsymbol{\Sigma}|^{1/2}}e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)}$$

- 如果$p(y=1|\boldsymbol{x})>p(y=0|\boldsymbol{x})$，预测类别为1；否则，预测类别为0.

线性辨别算法(Linear Discriminant Analysis, LDA)

$$p(y=1|\boldsymbol{x}) \lesseqgtr p(y=0|x) \iff \frac{p(y=1|x)}{p(y=0|x)} \lesseqgtr 1 \iff \log\left(\frac{p(y=1|x)}{p(y=0|x)}\right) \lesseqgtr 0$$

$$\frac{p(y=1|\boldsymbol{x})}{p(y=0|\boldsymbol{x})} = \frac{p(y=1)p(\boldsymbol{x}|y=1)/p(\boldsymbol{x})}{p(y=0)p(\boldsymbol{x}|y=0)/p(\boldsymbol{x})} = \frac{p(y=1)p(\boldsymbol{x}|y=1)}{p(y=0)p(\boldsymbol{x}|y=0)}$$

$$= \frac{p(y=1)}{p(y=0)} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)+\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_0)}$$

$$\log\left(\frac{p(y=1|\boldsymbol{x})}{p(y=0|\boldsymbol{x})}\right) = \log\left(\frac{p(y=1)}{p(y=0)}\right) - \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1) + \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_0)$$

$$= \log\left(\frac{p(y=1)}{p(y=0)}\right) - \frac{1}{2}\boldsymbol{\mu}_1{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1 + \frac{1}{2}\boldsymbol{\mu}_0{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0 + \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1 - \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0$$

线性辨别算法(Linear Discriminant Analysis, LDA)

$$\log\left(\frac{p(y=1|\boldsymbol{x})}{p(y=0|\boldsymbol{x})}\right)=\log\left(\frac{p(y=1)}{p(y=0)}\right)-\frac{1}{2}\boldsymbol{\mu}_1{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1+\frac{1}{2}\boldsymbol{\mu}_0{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0+\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1-\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0$$

$$\log\left(\frac{p(y=1|\boldsymbol{x})}{p(y=0|\boldsymbol{x})}\right)\lesseqgtr 0 \Longleftrightarrow \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)\lesseqgtr\log\left(\frac{p(y=0)}{p(y=1)}\right)+\frac{1}{2}(\boldsymbol{\mu}_1{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1-\boldsymbol{\mu}_0{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0)$$

$$\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)>\log\left(\frac{p(y=0)}{p(y=1)}\right)+\frac{1}{2}(\boldsymbol{\mu}_1{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1-\boldsymbol{\mu}_0{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0) \Rightarrow \hat{y}=1$$

$$\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)<\log\left(\frac{p(y=0)}{p(y=1)}\right)+\frac{1}{2}(\boldsymbol{\mu}_1{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1-\boldsymbol{\mu}_0{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0) \Rightarrow \hat{y}=0$$

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$$\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) \lessgtr \log\left(\frac{p(y=0)}{p(y=1)}\right)+\frac{1}{2}(\boldsymbol{\mu}_1{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1-\boldsymbol{\mu}_0{}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0)$$

$$\boldsymbol{x}^T\boldsymbol{\omega} \lessgtr t$$

线性辨别算法(Linear Discriminant Analysis, LDA)

$$\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) > \log\left(\frac{p(y=0)}{p(y=1)}\right)+\frac{1}{2}({\boldsymbol{\mu}_1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1-{\boldsymbol{\mu}_0}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0) \Rightarrow \hat{y}=1$$

$$\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) < \log\left(\frac{p(y=0)}{p(y=1)}\right)+\frac{1}{2}({\boldsymbol{\mu}_1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1-{\boldsymbol{\mu}_0}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0) \Rightarrow \hat{y}=0$$

$$\boldsymbol{x}^T\boldsymbol{\omega} \lesseqgtr t$$