Red-Black Trees

交通大學 資訊工程系

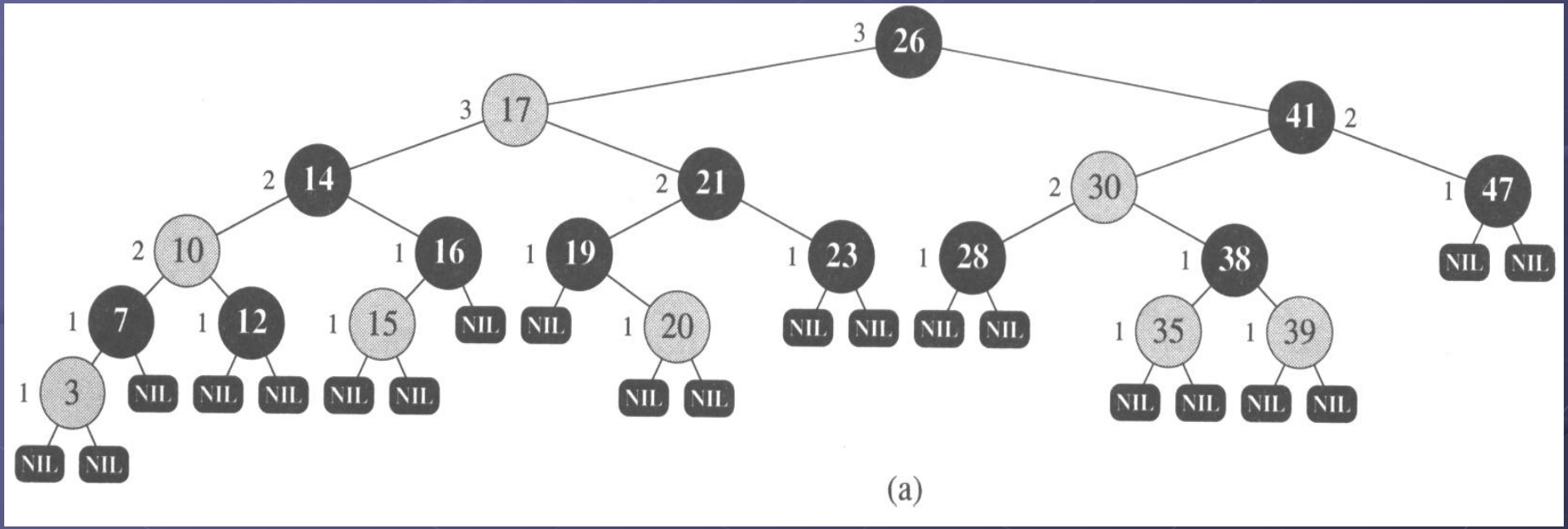
Properties of Red-Black Trees

- **Red-black tree**
  - **A binary search tree with one extra bit of storage per node: its color, which can be either RED or BLACK.**
  - **One of many search-tree schemes that are "balanced" in order to guarantee that basic dynamic-set operations take $O(lg\ n)$ time in the worse case.**
    - **By constraining the way nodes can be colored on any path from the root to a leaf, red-black trees ensure that no such path is more than *twice* as long as any other.**

Properties of Red-Black Trees

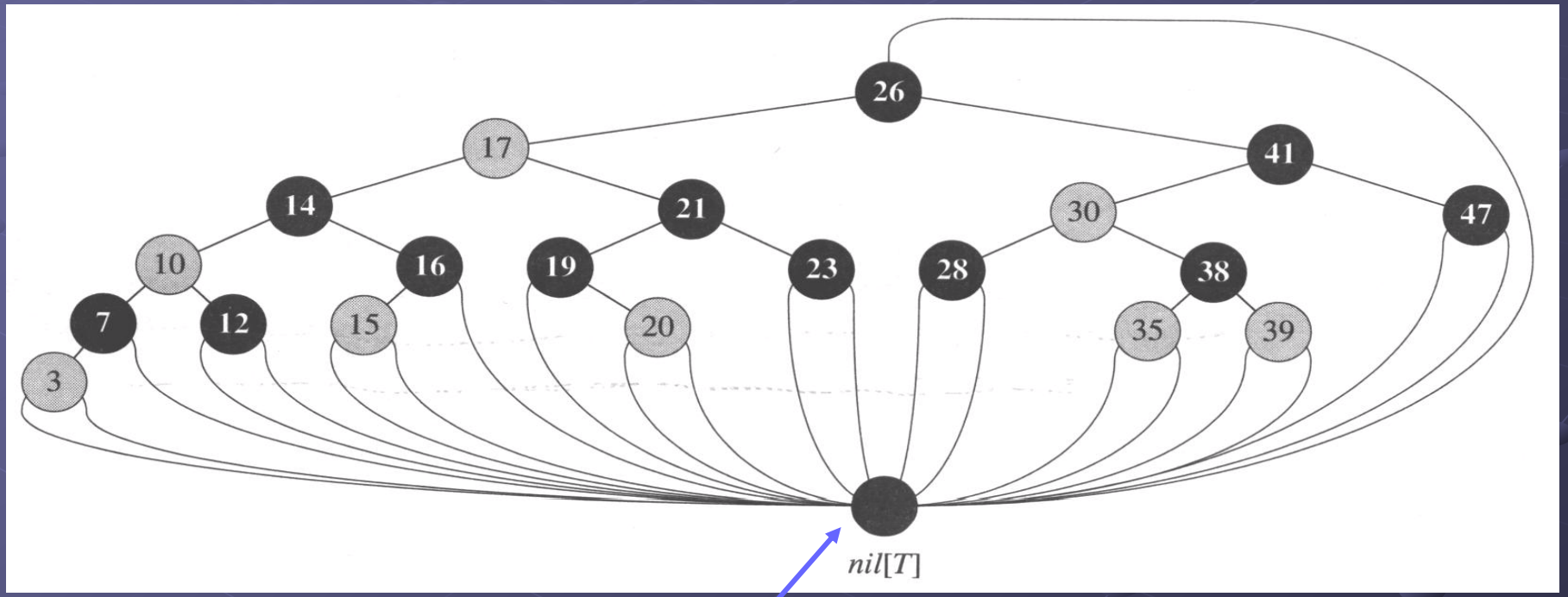
- **Red-black properties:**
  - **Every node is either red or black.**
  - **The root is black.**
  - **Every leaf (NIL) is black.**
  - **If a node is red, then both its children are black.**
  - **For each node, all paths from the node to descendant leaves contain the same number of black nodes.**
- ***Black-height* of a node, bh(x)**
  - **The number of black nodes on any path from, but not including, a node x down to a leaf.**

Properties of Red-Black Trees



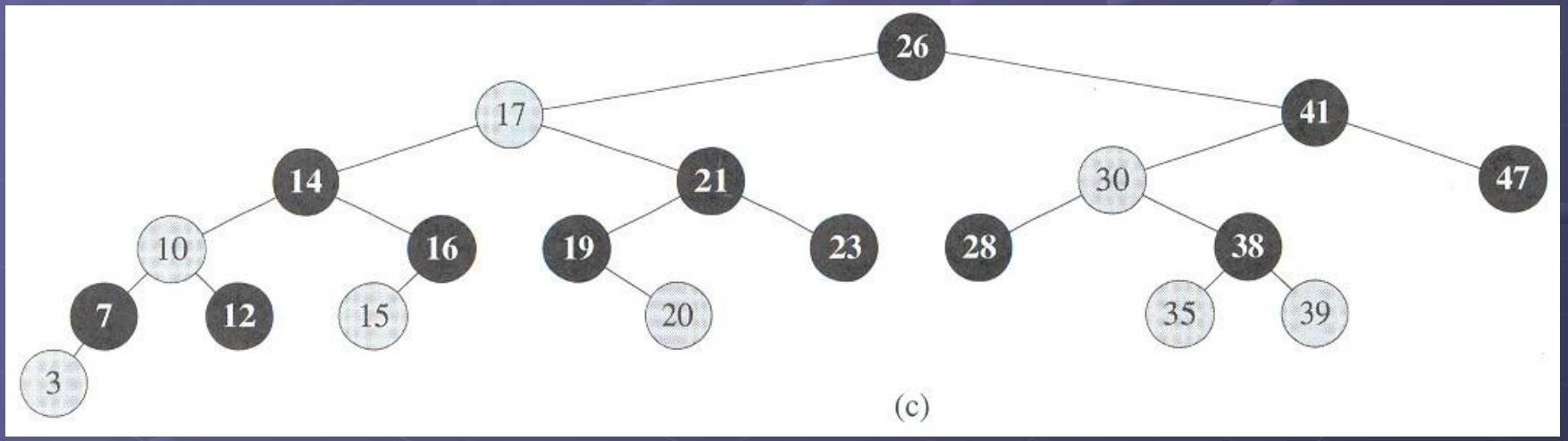
(a)

Properties of Red-Black Trees



Sentinel

Properties of Red-Black Trees

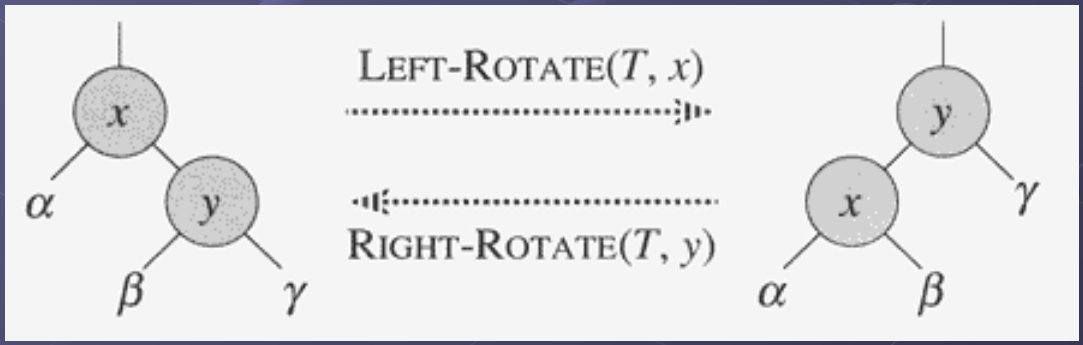


Properties of Red-Black Trees

- **The black-height of a red-black tree**
  - **To be the black-height of its root.**
- ***Lemma 13.1***
  - **A red-black tree with n internal nodes has height at most $2lg(n + 1)$.**
  - **Proof:**
    - **First showing the subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal nodes.**
    - **By induction on the height of x.**
    - **The black-height of the root must be at least $h/2$; thus, $n \geq 2^{h/2} - 1$.**

Rotations

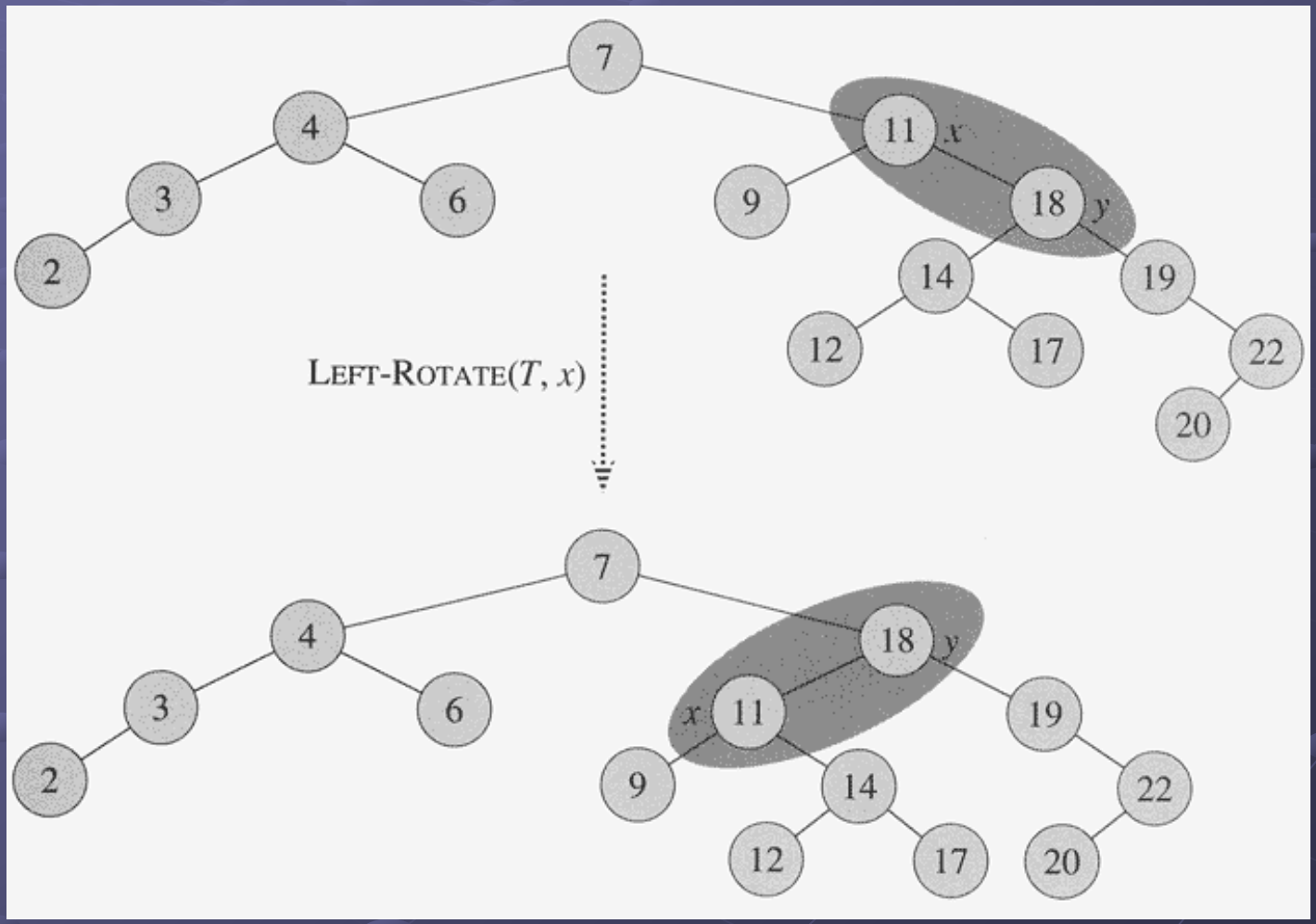
- **Rotation**
  - **A local operation preserves the binary-search-tree property.**
  - **Change the pointer structure.**
  - ***Left rotations***
  - ***Right rotations***



Rotations

```
LEFT-ROTATE(T, x)
 1  y = x.right                    // set y
 2  x.right = y.left               // turn y's left subtree into x's right subtree
 3  if y.left ≠ T.nil
 4      y.left.p = x
 5  y.p = x.p                      // link x's parent to y
 6  if x.p == T.nil
 7      T.root = y
 8  elseif x == x.p.left
 9      x.p.left = y
10  else x.p.right = y
11  y.left = x                     // put x on y's left
12  x.p = y
```

Rotations


7
4
11
x
3
6
9
18
y
2
14
19
12
17
22
20
LEFT-ROTATE(T, x)
7
4
18
y
3
6
x
11
19
2
9
14
22
12
17
20

Insertion

- **RB-INSERT procedure**
  - **Insert node *z* into the tree *T* as if it were an ordinary binary search tree, and then color *z* red.**

```
1   y = T.nil
2   x = T.root
3   while x ≠ T.nil
4       y = x
5       if z.key < x.key
6           x = x.left
7       else x = x.right
8   z.p = y
9   if y == T.nil
10      T.root = z
11  elseif z.key < y.key
12      y.left = z
13  else y.right = z
14  z.left = T.nil
15  z.right = T.nil
16  z.color = RED
17  RB-INSERT-FIXUP(T, z)
```

Insertion

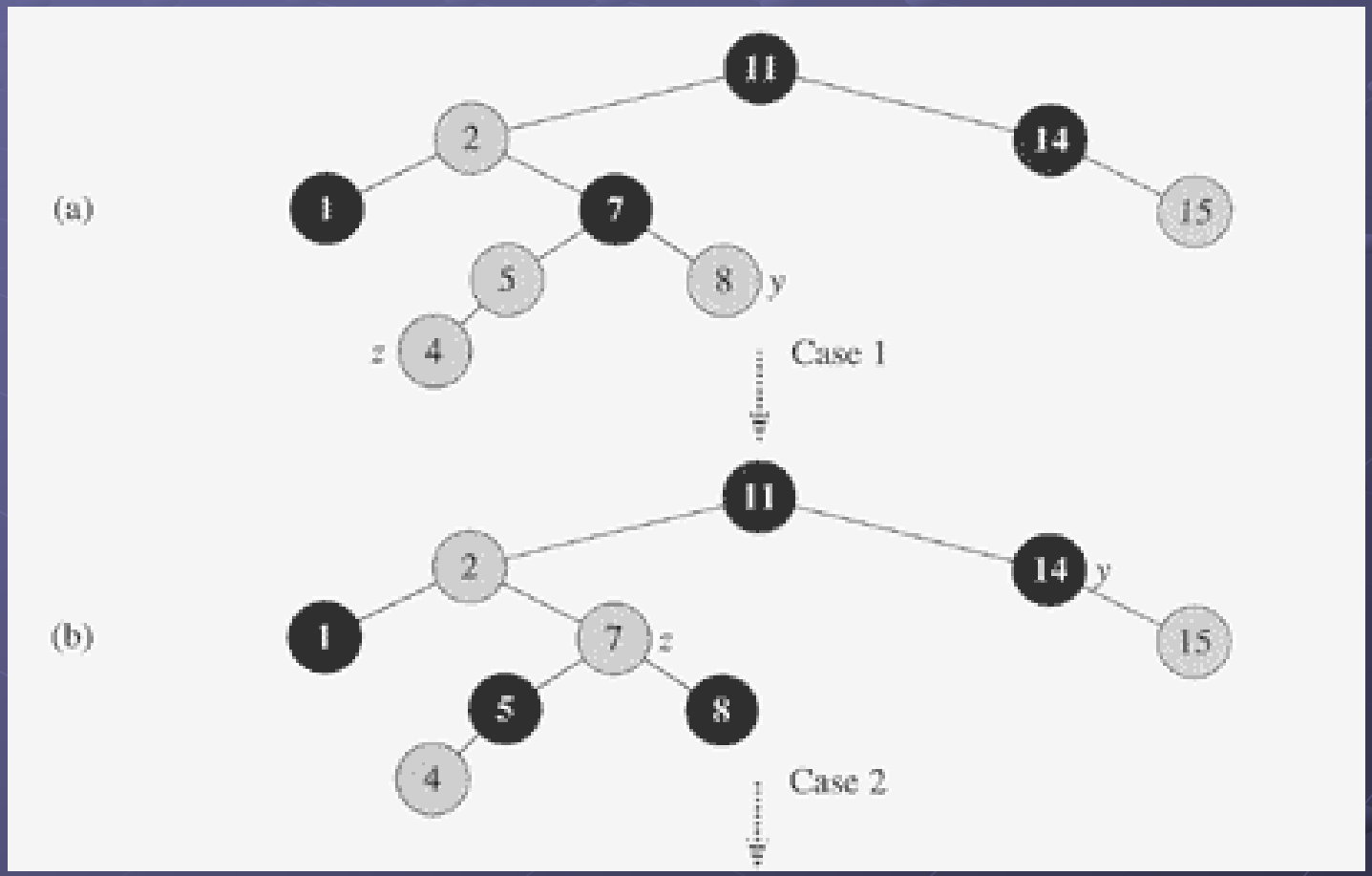
- **RB-INSERT-FIXUP procedure**

```
RB-INSERT-FIXUP(T, z)
 1  while z.p.color == RED
 2      if z.p == z.p.p.left
 3          y = z.p.p.right
 4          if y.color == RED
 5              z.p.color = BLACK                  // case 1
 6              y.color = BLACK                    // case 1
 7              z.p.p.color = RED                  // case 1
 8              z = z.p.p                          // case 1
```

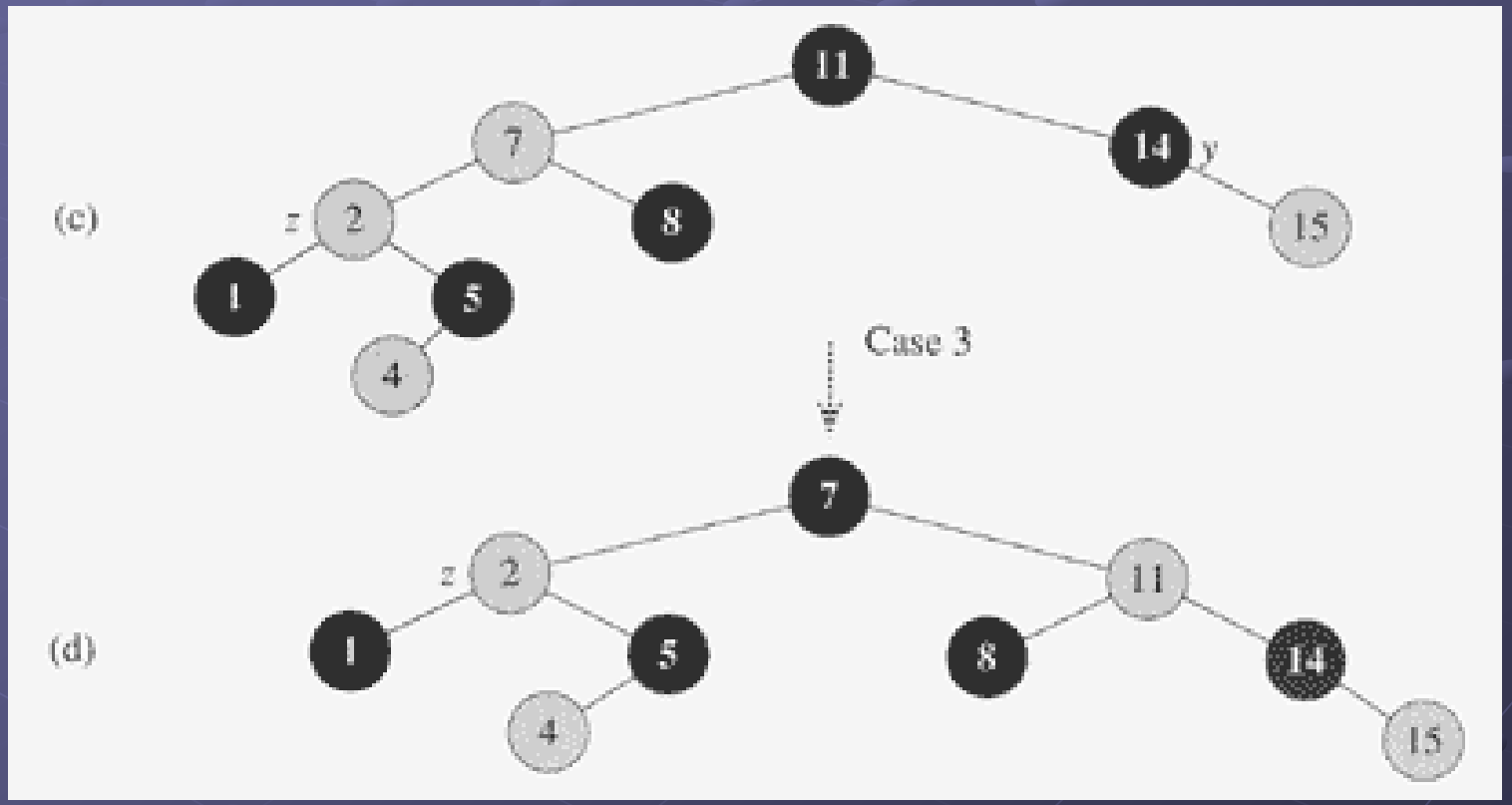
Insertion

```
 9          else if z == z.p.right
10                  z = z.p                          // case 2
11                  LEFT-ROTATE(T, z)                // case 2
12              z.p.color = BLACK                    // case 3
13              z.p.p.color = RED                    // case 3
14              RIGHT-ROTATE(T, z.p.p)               // case 3
15      else (same as then clause
                with "right" and "left" exchanged)
16  T.root.color = BLACK
```

Insertion

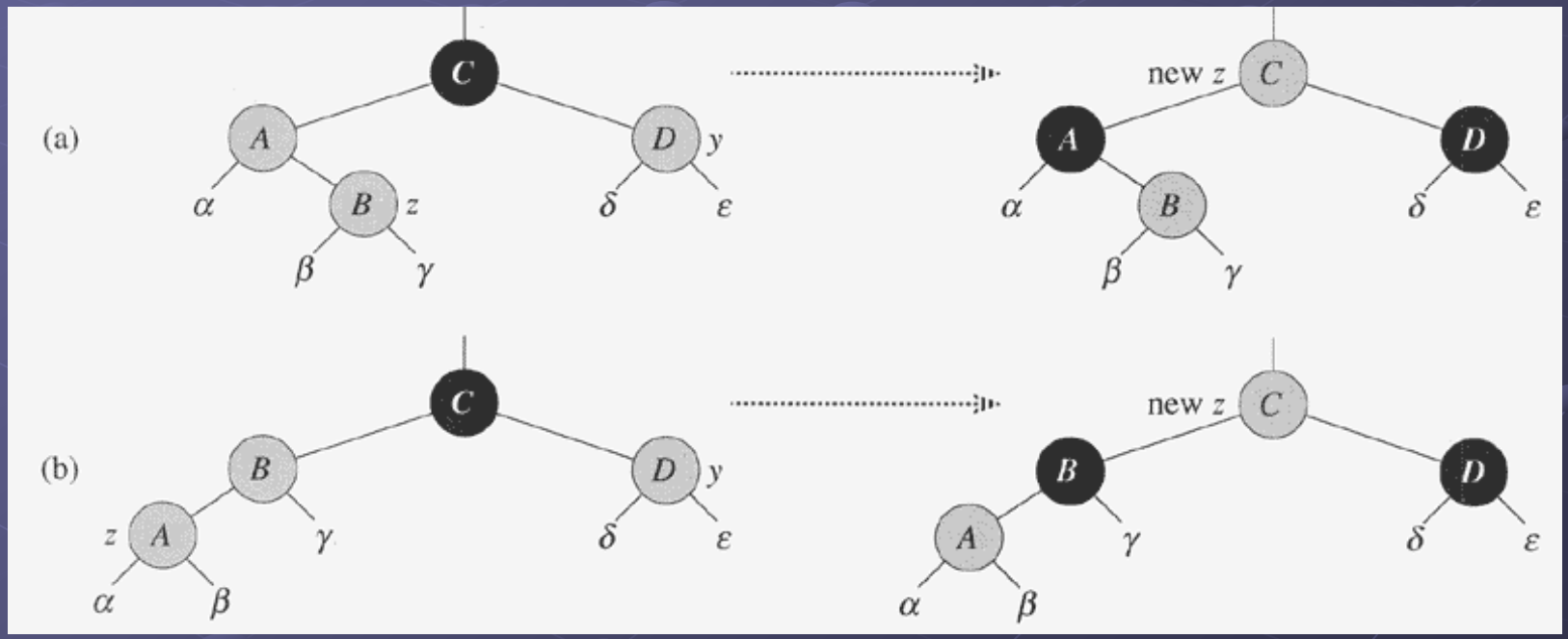


Insertion



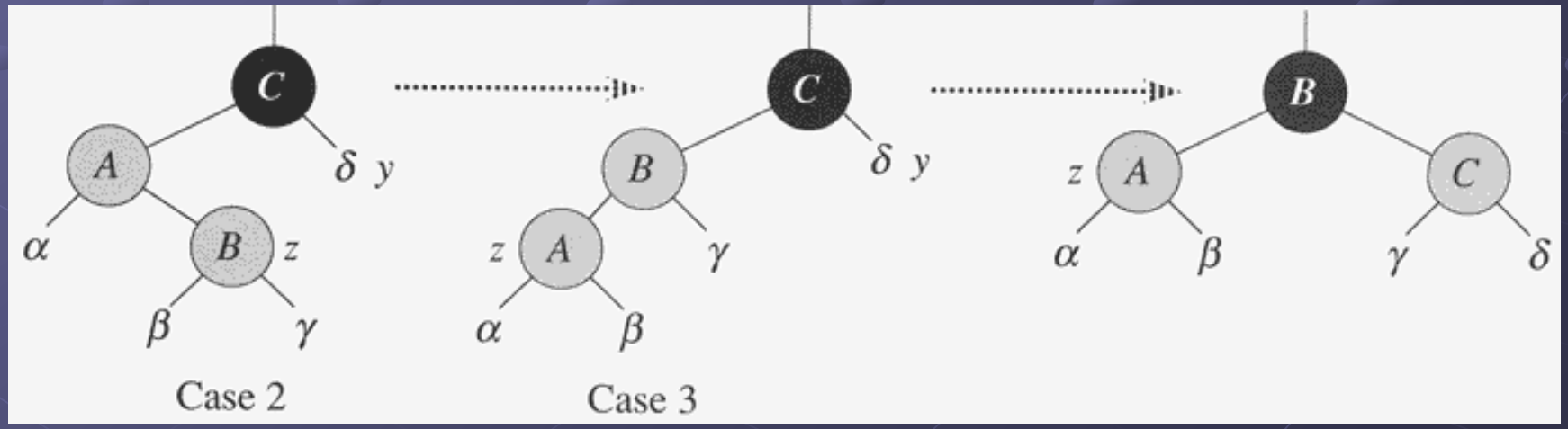
Insertion

- **Case 1:**
  - ***z*'s uncle *y* is red.**



Insertion

- **Case 2:**
  - **z's uncle y is black and z is a right child.**
- **Case 3:**
  - **z's uncle y is black and z is a left child.**



Deletion

- **RB Transplant**

```
RB-TRANSPLANT(T, u, v)
1   if u.p == T.nil
2       T.root = v
3   elseif u == u.p.left
4       u.p.left = v
5   else u.p.right = v
6   v.p = u.p
```

Deletion

- **RB-DELETE procedure**
  - **Takes time O(lg *n*).**

```
RB-DELETE(T, z)
 1   y = z
 2   y-original-color = y.color
 3   if z.left == T.nil
 4         x = z.right
 5         RB-TRANSPLANT(T, z, z.right)
 6   elseif z.right == T.nil
 7         x = z.left
 8         RB-TRANSPLANT(T, z, z.left)
 9   else y = TREE-MINIMUM(z.right)
10         y-original-color = y.color
```

Deletion

```
11        x = y.right
12        if y.p == z
13            x.p = y
14        else RB-TRANSPLANT(T, y, y.right)
15            y.right = z.right
16            y.right.p = y
17        RB-TRANSPLANT(T, z, y)
18        y.left = z.left
19        y.left.p = y
20        y.color = z.color
21    if y-original-color == BLACK
22        RB-DELETE-FIXUP(T, x)
```

Deletion

- If the spliced-out node *y* in RB-DELETE is *black*, three problems may arise:
  - If *y* had been the root and a red child of *y* becomes the new root. (violate property 2)
  - If both *x* and *x.p* were red. (violate property 4)
  - *y*'s removal cause any path that previously contained *y* to have one fewer black node. (violate property 5)

Deletion

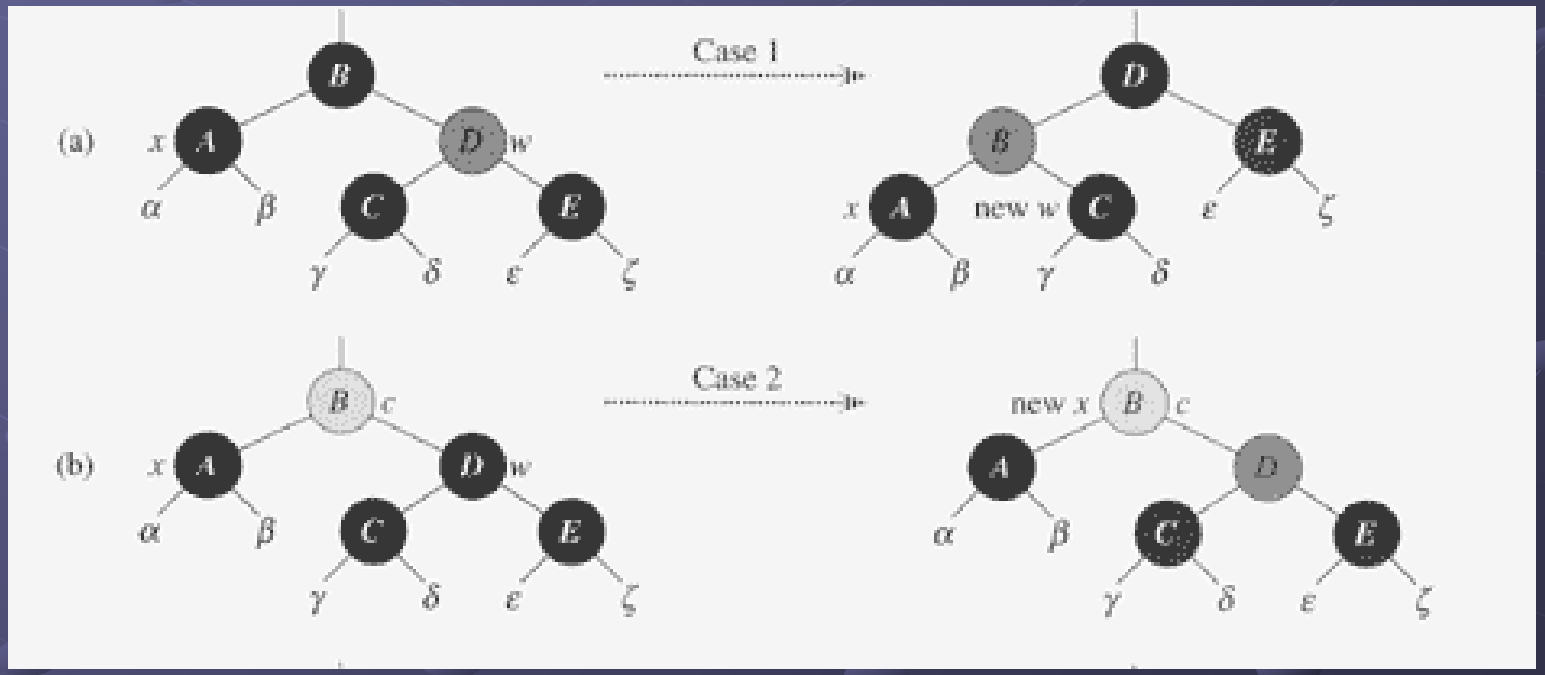
- RB-DELETE-FIXUP procedure
  - Restores the RB-tree properties when y.*color* is *black*.
- If *y* is *red*, the red-black properties still hold when *y* is spliced out:
  - No black-heights in the tree have changed.
  - No red nodes have been made adjacent, and
  - Since *y* could not have been the root if it was red, the root remains black.
- The goal of the while loop
  - Move the extra black up the tree.

Deletion

```
RB-DELETE-FIXUP(T, x)
 1  while x ≠ T.root and x.color == BLACK
 2      if x == x.p.left
 3          w = x.p.right
 4          if w.color == RED
 5              w.color = BLACK                          // case 1
 6              x.p.color = RED                          // case 1
 7              LEFT-ROTATE(T, x.p)                      // case 1
 8              w = x.p.right                            // case 1
 9          if w.left.color == BLACK and w.right.color == BLACK
10              w.color = RED                            // case 2
11              x = x.p                                  // case 2
```

Deletion

- Case 1: *x*'s sibling *w* is red.
- Case 2: *x*'s sibling *w* is black, and both of *w*'s children are black.



Deletion

```
12          else if w.right.color == BLACK
13                  w.left.color = BLACK                    // case 3
14                  w.color = RED                           // case 3
15                  RIGHT-ROTATE(T, w)                      // case 3
16                  w = x.p.right                           // case 3
17              w.color = x.p.color                         // case 4
18              x.p.color = BLACK                           // case 4
19              w.right.color = BLACK                       // case 4
20              LEFT-ROTATE(T, x.p)                         // case 4
21              x = T.root                                  // case 4
22      else (same as then clause with "right" and "left" exchanged)
23  x.color = BLACK
```

Deletion

- Case 3: *x*'s sibling *w* is black, and *w*'s left child is red, and *w*'s right child is black.
- Case 4: *x*'s sibling *w* is black, and *w*'s right child is red.