Production Heterogeneity in Collective Labor Supply Models with Children

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Abstract

This paper provides a revealed preference characterization of a collective labor supply model with household production where adult members have preferences that depend on their own leisure, expenditures, and children welfare. We show that the nonparametric revealed preference restrictions allow to partially identify the impacts of parental inputs on children welfare. We propose a novel estimation strategy that exploits these restrictions and allows for production heterogeneity. In our application, we apply our methodology to Dutch data on couples with children. We find extensive heterogeneity in the production technology, decreasing returns to scale in the production of children welfare, and a positive impact of education on the effects of parental inputs on children welfare.
JEL Classification: D11, D12, D13, C51, C63.

1 Introduction

It is widely recognized that the unitary model, which assumes that household members preferences can be represented by a single household utility function, is inappropriate for analyzing household data (see e.g., Fortin and Lacroix (1997) and Browning and Chiappori (1998)). In an effort to provide a proper foundation to analyze household behavior, Chiappori (1988, 1992) suggested a collective model in which household members have distinct preferences and whose allocations are the result of a Pareto efficient bargaining process. This framework has

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proven to be empirically successful at rationalizing household decision-making (e.g., Cherchye and Vermeulen, 2008) and understanding power dynamics within the household (e.g., Cherchye, De Rock and Vermeulen, 2011).

Once the household is recognized as a collection of individuals, it opens up the possibility that the impact of a policy depends on which individual is targeted within the household. Indeed, the distribution of resources within the household has been shown to have significant impacts on children welfare as early as Thomas (1990). Motivated by this "targeted" view, Blundell, Chiappori and Meghir (2005) extended the collective framework to caring parents whereby children welfare is treated as a public good in parents utility functions and produced via parental time investment and children expenditure.[1]

The first application of Blundell, Chiappori and Meghir (2005)'s ideas was undertaken by Cherchye, De Rock and Vermeulen (2012) using a unique panel data set containing information on time use and expenditure. Despite their innovative approach, they only identify the model using differential arguments that rely on a fully parametric model, preference homogeneity, and constant returns to scale.[2] We argue that those assumptions are unavoidable when only using the cross-sectional implications of the model. However, we show that in panel data the model has additional implications that can be used to learn about the production of children welfare.

Our first contribution is to derive a revealed preference characterization of a collective labor supply model with children as proposed by Blundell, Chiappori and Meghir (2005) and Cherchye, De Rock and Vermeulen (2012).[3] The characterization builds on the original work of Cherchye, De Rock and Vermeulen (2007, 2011), but further incorporates household production.[4] Interestingly, the collective model implies optimizing behavior with respect to the production of children welfare. We show that these restrictions, along with standard revealed preference conditions, are essential to learn about the production function in the absence of auxiliary assumptions.

Our second contribution is to propose a novel estimation strategy to analyze the collective model with unrestricted preference and production heterogeneity.

[1]This modelling choice is consistent with Becker (1965)'s view of households as producing units.

[2]A study of identification in multi-member collective models by Chiappori and Ekeland (2009) also suggests that constant returns to scale is necessary in such model.

[3]See also Dunbar, Lewbel and Pendakur (2013) for a collective model with children that does not require the share of resources allocated to children to be known.

[4]See Apps and Rees (1997) and Chiappori (1997) for earlier work that incorporate household production in the collective model.

To this end, we use the framework developed by Aguiar and Kashaev (2021) which provides a tractable approach to make statistical testing and inference in partially identified models defined by shape constraints.[5] The main challenge faced in applying their framework is that collective models tend to be highly nonlinear. This poses a nontrivial computational problem as existing implementations only work well for models defined by linear constraints.[6]

We solve this practical limitation by proposing a blocked Gibbs sampler that allows direct sampling from the feasible space even when the latter does not define a polytope.[7] We observe that our methodology may prove useful for a broad range of collective models such as noncooperative models (e.g., Cherchye, Demuynck and De Rock, 2011, d'Aspremont and Dos Santos Ferreira, 2019, Cherchye et al., 2020), among others. As such, we believe our algorithm may be of independent interest.

Our third contribution is to learn about heterogeneity in the production of children welfare. Using the LISS (Longitudinal Internet Studies for the Social sciences) panel data from Cherchye, De Rock and Vermeulen (2012), we first use our novel (partial) identification strategy to learn about returns to scale. We find that households face decreasing returns to scale. Then, we recover the expected impacts of parents inputs on children welfare without relying on the constant returns to scale assumption. Finally, we show that education increases the amount of children welfare produced by an hour of time invested by parents.

The paper is organized as follows. Section 2 describes our collective model and characterizes its implications. Section 3 analyzes the empirical content of the model. Section 4 presents the empirical specification. Section 5 presents the estimation strategy. Section 6 presents the data set used in the application. Section 7 presents the empirical results. Section 8 concludes. The Appendix contains proofs that are not in the main text and our Gibbs sampler.

[5]Their framework builds on the Entropic Latent Variable Integration via Simulation (ELVIS) methodology developed by Schennach (2014). Intuitively, ELVIS can be viewed as a generalization of the method of simulated moments (McFadden, 1989; Pakes and Pollard, 1989).

[6]For example, Aguiar and Kashaev (2021) consider the collective exponential discounting model of Adams et al. (2014) but only test necessary conditions to simplify the implementation. Gauthier (2023) considers a model of price search, but assumes a quasilinear specification in the application that alleviates the computational burden.

[7]Direct sampling generally uses a Hit-and-Run algorithm that requires the feasible space to define a (convex) polytope. See Aguiar and Kashaev (2021) for an application to models defined by shape constraints and Demuynck (2021) for an application to models defined by the Generalized Axiom of Revealed Preference (GARP) as introduced by Varian (1982).

2 Household Model

This section presents the environment considered in the paper, the collective model, and its empirical implications.

2.1 Environment

We consider households with two adults ($i = 1, 2$) and children. We assume that parents care about their children and incorporate this feature in the model by treating children welfare as a public good. The preferences of each adult household member are represented by a utility function U^i that is continuous, increasing, and concave.

At every observation $t \in \mathcal{T} = \{1, 2, \ldots T\}$, adult household members spend their time on leisure l_t^i, market work b_t^i, and childcare h_t^i such that the following normalized time constraint is satisfied:

$$l_t^i + b_t^i + h_t^i = 1.$$

Parents use time spent on childcare and children expenditure (c_t) to produce children welfare. The relationship between parents inputs and children welfare is formalized through the production function

$$W_t \equiv W(h_t^1, h_t^2, c_t, \epsilon_t),$$

where $\epsilon_t \in \mathbb{R}$ represents a productivity shock. Each household member receives a wage w_t^i per unit of market work. As such, the budget constraint is given by

$$q_t + Q_t + c_t = y_t + w_t^1 b_t^1 + w_t^2 b_t^2,$$

where $q_t \in \mathbb{R}_+$ represents expenditure on private goods, $Q_t \in \mathbb{R}_+$ represents expenditure on public goods, and $y_t > 0$ represents nonlabor income.

Since private expenditure cannot be used simultaneously by both household members, it has to be split in some way between them.

Definition 1. Let D be a data set. For every observation $t \in \mathcal{T}$, we say that $q_t^i \in \mathbb{R}_+$, $i \in \{1, 2\}$, represent personalized private expenditures of each household member if $\sum_{i=1}^2 q_t^i = q_t$.

Household members get utility from their share of private expenditure such that their preferences depend on leisure, private expenditure, public expenditure, and children welfare. In what follows, we assume that private expenditure of each household member is observed to match the data available in the application. However, our results can be generalized to the case where only total private expenditure is observed.

Let $\mathcal{U}$ be the set of continuous, increasing, and concave utility functions and $\mathcal{W}$ be the set of continuous, increasing, and concave in (h_t^1, h_t^2, c_t) production functions. A household $j \in J$ is an i.i.d. draw (U_j^1, U_j^2, W_j) from $\mathcal{W}$ and a data set $D_j := \{(q_{jt}^i, Q_{jt}, c_{jt}, b_{jt}^i, h_{jt}^i, w_{jt}^i)_{i=1}^2\}_{t\in\mathcal{T}}$ is an i.i.d. draw from some distribution. To avoid overcrowding, we do not explicitly write the household subscript j on variables unless it is relevant. The next subsection formalizes the relationship between the data and the abstract notion of household through the lenses of a model.

2.2 Collective Model

We follow Chiappori (1988, 1992) and assume that household members choose an intrahousehold allocation that is Pareto efficient. This choice is motivated by the observation that Pareto efficiency is a minimal condition for optimal resource allocation (and hence, rationality) in a group setting. Hence, for every observation $t \in \mathcal{T}$, the household picks an intrahousehold allocation that solves

$$\max_{(l^1,l^2,h^1,h^2,q^1,q^2,Q,c)\in\mathbb{R}_+^2\times\mathbb{R}_{++}^2\times\mathbb{R}_+^2\times\mathbb{R}_+\times\mathbb{R}_{++}} \mu_t^1 U^1(l^1, q^1, Q, W_t) + \mu_t^2 U^2(l^2, q^2, Q, W_t), \tag{1}$$

subject to satisfying the constraints

$$\begin{gathered}(q^1 + q^2) + Q + c = y_t + w_t^1 b^1 + w_t^2 b^2 \\ l^i + b^i + h^i = 1 \ (i = 1, 2),\end{gathered}$$

where $\mu_t^i > 0$ denote the bargaining power of household member i. Note that the model makes no assumption on the underlying process by which the Pareto efficient allocation is achieved. That is, the weights μ_t^i result from some black box bargaining process that takes place within the household.[8]

We propose a natural notion of collective rationalizability based on the household maximization problem.

[8] Although Nash equilibria are not always Pareto efficient, the black box bargaining process could be a (Pareto efficient) Nash equilibrium. Indeed, since married couples effectively play a repeated game, an appeal to folk theorems provide some intuitive motivation for the idea that the Pareto efficient allocation is a (cooperative) Nash equilibrium.

Definition 2. Let D be a data set. The model (1) rationalizes the data if there exist concave utility functions U^i and a production function W concave in (h^1, h^2, c) such that the first-order conditions of the problem are satisfied.

This definition states that the model rationalizes the data if there are latent model parameters that satisfy the first-order conditions. Since household members utility functions are concave and the budget set is linear, the first-order conditions exhaust the empirical content of the model.

2.3 Characterization

This section derives restrictions on the data implied by the model. First, we define a few notions that will be useful for the characterization of the model.

Definition 3. Let D be a data set. For every observation $t \in \mathcal{T}$, we say that $\mathcal{P}_t^i \in \mathbb{R}_{++}$, $i \in \{1, 2\}$, represent personalized (or Lindahl) prices for public expenditure of each household member if $\sum_{i=1}^{2} \mathcal{P}_t^i = 1$.

Definition 4. Let D be a data set. For every observation $t \in \mathcal{T}$, we say that $P_t^i \in \mathbb{R}_{++}$, $i \in \{1, 2\}$, represent personalized (or Lindahl) prices for children welfare of each household member if $\sum_{i=1}^{2} P_t^i = P_t$.

It is worth noting that the personalized prices $(\mathcal{P}_t^i, P_t^i)$ are not observed by the econometrician. Furthermore, while the price of public expenditure $(\mathcal{P}_t)$ can safely be set to 1, the price of children welfare (P_t) is unobservable as children welfare is a nonmarket good.

We now introduce some revealed preference terminology. Let $a_{s,t}^i := w_t^i(l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i(Q_s - Q_t) + P_t^i(W_s - W_t)$ and $x_t^i := (l_t^i, q_t^i, Q_t, W_t)$. We say that x_t^i is (strictly) directly revealed preferred to x_s^i if $a_{s,t}^i$ $(<) \leq 0$. We say that x_t^i is revealed preferred to x_s^i if there exists a sequence $t_1, t_2, \ldots, t_m$ such that $a_{t_1,t}^i \leq 0$, $a_{t_2,t_1}^i \leq 0$, $\ldots$, a_{t_{m-1},t_m}^i, $a_{t_m,s}^i \leq 0$. Likewise, we say that x_t^i is strictly revealed preferred to x_s^i if one of the inequalities in the sequence is strict.

Definition 5. A household member $i \in \{1, 2\}$ satisfies the Generalized Axiom of Revealed Preference (GARP) if there exist personalized prices for public expenditure $\mathcal{P}_t^i$, personalized prices for children welfare P_t^i, and children welfare W_t such that if x_t^i is revealed preferred to x_s^i then x_s^i is not strictly directly revealed preferred to x_t^i.

The notion of revealed preference relates the ordinal value of allocations that enter preferences of each household member to their expenditure levels. In our setup, the presence of a public good (Q) implies that the expenditure of an allocation depends on unknown personalized prices. Further, in the case of the public nonmarket good (W) neither the price or the quantity is known. Finally, it is worth observing that childcare and children expenditure do not enter the definition of revealed preference as the preferences of a household member only depends on those through their impact on children welfare. The following result provides equivalent characterizations of the model.

Theorem 1. *Let D be a given data set. The following conditions are equivalent:*

(i) The household model (1) *rationalizes the data.*

(ii) There exist personalized prices for public expenditure $\mathcal{P}_t^i > 0$ such that $\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1$, personalized prices for children welfare $P_t^i > 0$, and numbers U^i, λ_t^i, $\dot{W}_{h_t^1}$, $\dot{W}_{h_t^2}$, $\dot{W}_{c_t} > 0$ such that for all $s, t \in \mathcal{T}$ and each adult member $i \in \{1, 2\}$

$$U_s^i - U_t^i \leq \lambda_t^i \left[w_t^i(l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i(Q_s - Q_t) + P_t^i(W_s - W_t) \right],$$
$$P_t \dot{W}_{h_t^1} = w_t^1, \; P_t \dot{W}_{h_t^2} = w_t^2, \; P_t \dot{W}_{c_t} = 1.$$

(iii) There exist personalized prices for public expenditure $\mathcal{P}_t^i > 0$ such that $\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1$, personalized prices for children welfare P_t^i, children welfare $W_t > 0$, and numbers $\dot{W}_{h_t^1}$, $\dot{W}_{h_t^2}$, $\dot{W}_{c_t} > 0$ such that GARP holds for each adult member $i \in \{1, 2\}$ and

$$P_t \dot{W}_{h_t^1} = w_t^1, \; P_t \dot{W}_{h_t^2} = w_t^2, \; P_t \dot{W}_{c_t} = 1.$$

Theorem 1 shows that the Afriat inequalities are equivalent to GARP and that those conditions must be satisfied for both household members. The latter implies that the household problem has an equivalent characterization in terms of a two-step procedure (Chiappori, 1988, 1992). That is, the solution of the household maximization problem can be viewed as the outcome of separate utility maximization problems for each adult in the household conditional on a distribution of nonlabor income.

It is interesting to note that neither the Afriat inequalities or GARP exhaust the empirical implications of the model. Indeed, the model further implies that the household is a profit maximizer. To see why, observe that the equality constraints in Theorem 1 (*ii*)-(*iii*) can be viewed as first-order conditions from the profit maximization problem

$$\max_{h_t^1, h_t^2, c_t} P_t W_t - w_t^1 h_t^1 - w_t^2 h_t^2 - c_t,$$

where $\dot{W}_x$ stands for the derivative of children welfare with respect to input x. As such, household members increase each input in the production of children welfare up until the point where marginal revenue equates marginal cost. Note that this profit maximization behavior is not assumed but implied by the model.

3 Empirical Content

This section shows that the collective model informatively partially identifies the production function. Intuitively, if the production function exhibited constant returns to scale, the household would make zero profit as a firm and revenue $P_t W_t$ would equate costs $w_t^1 h_t^1 + w_t^2 h_t^2 + c_t$. In this special case, revenue would be identified and the first-order conditions would recover the production function from its partial derivatives. We show that the household revenue from producing children welfare is inversely proportional to its costs when the production function is homogeneous, where the factor of proportionality is given by its return to scale. We then leverage the panel structure of the data to bound returns to scale from restrictions on household members preferences.

We consider a production function subject to Hicks-neutral productivity shocks.

Assumption 1. *The productivity shocks are Hicks-neutral such that children welfare is given by*

$$W_t = F(h_t^1, h_t^2, c_t) e^{\epsilon_t}.$$

This assumption is necessary to disentangle the impacts of productivity shocks and parental inputs on the production of children welfare. Next, we impose a mild support condition that ensures sufficient variation in inputs.

Assumption 2. *The panel distribution of wages and income $(w_t^1, w_t^2, y_t)_{t \in \mathcal{T}}$ is absolutely continuous.*

Assumption 2 is a mild regularity condition that ensures time series variation in wages and income. It rules out mass points that may arise if wages and income were constant in time, for example. In the latter case, the model would not generate variation in inputs and it would thus be impossible to identify the production function. Note that we do not rely on cross-sectional variation for identification as we allow for unrestricted heterogeneity in the production technology.

Our first result shows that, if children welfare expenditure were known, the production function would be identified.

Proposition 1. *Suppose Assumptions 1-2 hold and P_tW_t is known, then the production function is nonparameterically identified up to scale.*

Proof. The first-order conditions with respect to inputs imply that the household equates the marginal product of factors of production to their marginal cost such that

$$\begin{aligned}\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^1} e^{\epsilon_t} &= \frac{w_t^1}{P_t} \\ \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^2} e^{\epsilon_t} &= \frac{w_t^2}{P_t} \\ \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial c_t^2} e^{\epsilon_t} &= \frac{1}{P_t}.\end{aligned}$$

Divide the marginal products by W_t to obtain

$$\begin{aligned}\frac{\partial f(h_t^1, h_t^2, c_t)}{\partial h_t^1} &= \frac{w_t^1}{P_tW_t} \\ \frac{\partial f(h_t^1, h_t^2, c_t)}{\partial h_t^2} &= \frac{w_t^2}{P_tW_t} \\ \frac{\partial f(h_t^1, h_t^2, c_t)}{\partial c_t^2} &= \frac{1}{P_tW_t},\end{aligned}$$

where $f(\cdot)$ denote the log production function. Since P_tW_t is known, the marginal products are identified. Next, variation in inputs allows us to integrate each marginal product, giving the following system of partial differential equations

$$\begin{aligned}\int_{h_0^1}^{h_t^1} \frac{\partial f(h_t^1, h_t^2, c_t)}{\partial h_t^1} dh_t^1 &= f(h_t^1, h_t^2, c_t) + C(h_t^2, c_t) \\ \int_{h_0^2}^{h_t^2} \frac{\partial f(h_t^1, h_t^2, c_t)}{\partial h_t^2} dh_t^2 &= f(h_t^1, h_t^2, c_t) + C(h_t^1, c_t) \\ \int_{c_0}^{c_t} \frac{\partial f(h_t^1, h_t^2, c_t)}{\partial c_t} dc_t &= f(h_t^1, h_t^2, c_t) + C(h_t^1, h_t^2).\end{aligned}$$

These equations can be used to recover the log production function up to a constant:

$$f(h_t^1, h_t^2, c_t) = \int_{h_0^1}^{h_t^1} \frac{\partial f(h^1, h_0^2, c_0)}{\partial h_t^1} dh^1 + \int_{h_0^2}^{h_t^2} \frac{\partial f(h_t^1, h^2, c_0)}{\partial h_t^2} dh^2 + \\ + \int_{c_0}^{c_t} \frac{\partial f(h_t^1, h_t^2, c)}{\partial c_t} dc - C,$$

where C is a constant of integration. One recovers the production function up to scale after taking the exponential function. □

Proposition 1 states that, in principle, the model imposes enough structure to nonparameterically identify the production function provided the unobservable quantity $P_t W_t$ is known. Interestingly, note that identification does not require knowledge of children welfare per se. Intuitively, any scaling of children welfare is offset by a rescaling of children welfare prices, thus leaving revenue $P_t W_t$ unchanged.[9] This is reflected in Proposition 1 through the statement that the production function is identified up to scale.

The previous discussion makes clear that the identification problem amounts to obtaining restrictions on $P_t W_t$. This condition is still problematic as $P_t W_t$ is also unobservable. We argue that the class of nonparametric concave production functions is too flexible when the output is unobservable. Hence, we focus on the class of homogeneous production functions.

Assumption 3. *The production function is homogeneous of degree $RTS \in (0, 1]$.*

The importance of the homogeneity assumption is displayed in the following result.

Lemma 1. *Suppose Assumptions 1-3 hold, then $RTS \cdot P_t W_t = w_t^1 h_t^1 + w_t^2 h_t^2 + c_t$ for all $t \in \mathcal{T}$.*

Proof. From the first-order conditions of the model and the Hicks-neutrality of productivity shocks, we have

$$\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^1} e^{\epsilon_t} = \frac{w_t^1}{P_t}$$
$$\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^2} e^{\epsilon_t} = \frac{w_t^2}{P_t}$$
$$\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial c_t^2} e^{\epsilon_t} = \frac{1}{P_t}.$$

[9] This mechanism has a natural economic interpretation. Namely, it captures the idea of scarcity whereby the value of a good decreases with its abundance.

We can multiply each marginal product by its own factor of production to get

$$\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^1} h_t^1 e^{\epsilon_t} = \frac{w_t^1 h_t^1}{P_t}$$
$$\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^2} h_t^2 e^{\epsilon_t} = \frac{w_t^2 h_t^2}{P_t}$$
$$\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial c_t^2} c_t e^{\epsilon_t} = \frac{c_t}{P_t}.$$

Summing up these equations and multiplying by P_t, we obtain

$$P_t \left[\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^1} h_t^1 + \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^2} h_t^2 + \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial c_t^2} c_t \right] e^{\epsilon_t} = E_t,$$

where $E_t := w_t^1 h_t^1 + w_t^2 h_t^2 + c_t$. Since the production function is homogeneous of degree $RTS \in (0, 1]$, an application of Euler's theorem gives

$$RTS P_t W_t = E_t,$$

where we used the production function equation $W_t = F(h_t^1, h_t^2, c_t) e^{\epsilon_t}$. $\square$

Lemma 1 implies that the identification problem can be stated in terms of restrictions on returns to scale RTS rather than restrictions on $P_t W_t$ directly. Indeed, if the scalar RTS was known, we would immediately identify $P_t W_t$ via the equation $P_t W_t = RTS^{-1} E_t$. The reason this reformulation of the identification problem is useful is that $P_t W_t$ is a quantity that varies in time, whereas RTS is time invariant. It is precisely this simplification that allows us to bound $P_t W_t$ by exploiting the panel structure of the data. In what follows, we say that a lower bound on RTS is informative if it is greater than 0 and that an upper bound on RTS is informative if it is lower than 1.

Proposition 2. *Suppose Assumptions 1-3 hold. A data set can have informative lower bounds and informative upper bounds on RTS, though at possibly different values of personalized prices for public expenditure $\mathcal{P}_t^i$ and personalized prices for children welfare P_t^i.*

Proof. Let $X_{t_j,t_k}^i := w_{t_k}^i (l_{t_j}^i - l_{t_k}^i) + (q_{t_j}^i - q_{t_k}^i) + \mathcal{P}_{t_k}^i (Q_{t_j} - Q_{t_k})$ and recall that

$a^i_{t_j,t_k} = X^i_{t_j,t_k} + P^i_{t_k}(W_{t_j} - W_{t_k})$. There are three possibilities when $a^i_{t_j,t_k} \geq 0$:

$$\begin{aligned}
&\text{Case 1:} \quad X^i_{t_j,t_k} > 0 \text{ and } P^i_{t_k}(W_{t_j} - W_{t_k}) \geq 0 \\
&\text{Case 2:} \quad X^i_{t_j,t_k} > 0 \text{ and } P^i_{t_k}(W_{t_j} - W_{t_k}) \leq 0 \\
&\text{Case 3:} \quad X^i_{t_j,t_k} < 0 \text{ and } P^i_{t_k}(W_{t_j} - W_{t_k}) \geq 0.
\end{aligned}$$

Likewise, there are also three possibilities when $a^i_{t_j,t_k} \leq 0$:

$$\begin{aligned}
&\text{Case 1:} \quad X^i_{t_j,t_k} < 0 \text{ and } P^i_{t_k}(W_{t_j} - W_{t_k}) \leq 0 \\
&\text{Case 2:} \quad X^i_{t_j,t_k} < 0 \text{ and } P^i_{t_k}(W_{t_j} - W_{t_k}) \geq 0 \\
&\text{Case 3:} \quad X^i_{t_j,t_k} > 0 \text{ and } P^i_{t_k}(W_{t_j} - W_{t_k}) \leq 0.
\end{aligned}$$

By Lemma 1, we have

$$W_{t_k} = \frac{E_{t_k}}{RTSP_{t_k}} \quad \forall k.$$

Substituting this expression in $a^i_{t_j,t_k}$, one obtain bounds on RTS of the form:

$$RTS \lesseqgtr -\frac{\left(\frac{P^i_{t_k}}{P_{t_j}}E_{t_j} - \frac{P^i_{t_k}}{P_{t_k}}E_{t_k}\right)}{X^i_{t_j,t_k}}.$$

It is easy to see that Case 1 yields a negative and therefore uninformative lower bound on RTS regardless of the sign of $a^i_{t_j,t_k}$. Likewise, it is easy to see that Case 2 yields informative lower bounds on RTS regardless of the sign of $a^i_{t_j,t_k}$. Indeed, the bound is either in the interval $(0, 1]$ or the interval $(1, \infty)$. In the former case, the bound improves upon 0 and, in the latter case, the data would refute the model. Finally, it is easy to see that Case 3 yields an upper bound on RTS such that

$$RTS < -\frac{\left(\frac{P^i_{t_k}}{P_{t_j}}E_{t_j} - \frac{P^i_{t_k}}{P_{t_k}}E_{t_k}\right)}{X^i_{t_j,t_k}}.$$

Let us first consider the case where $a^i_{t_j,t_k} \geq 0$. The upper bound is less than 1 if $P^i_{t_k}(W_{t_j} - W_{t_k}) < -X^i_{t_j,t_k}$ or

$$\left(\frac{P^i_{t_k}}{P_{t_j}}E_{t_j} - \frac{P^i_{t_k}}{P_{t_k}}E_{t_k}\right) < -X^i_{t_j,t_k}.$$

Since $P^1_{t_k} + P^2_{t_k} = P_{t_k}$ for all k, it follows that $\frac{P^i_{t_k}}{P_{t_k}} \in (0, 1)$. If $\frac{P^1_{t_k}}{P_{t_k}}$ goes to zero, then $\frac{P^2_{t_k}}{P_{t_k}}$ goes to one. For household member 1, we thus get the restriction

$$\frac{P^1_{t_k}}{P_{t_j}} E_{t_j} < -X^1_{t_j,t_k}.$$

Clearly, this condition is satisfied if $\frac{P^1_{t_k}}{P_{t_j}}$ goes to zero. However, without an upper bound on $\frac{P_{t_k}}{P_{t_j}}$ (and thus, $P^1_{t_k}/P_{t_j}$) we cannot guarantee that the inequality holds for all personalized prices. For household member 2, we get the restriction

$$\frac{P^2_{t_k}}{P_{t_j}} E_{t_j} - E_{t_k} < -X^2_{t_j,t_k}.$$

Once again, it is clear that $\frac{P_{t_k}}{P_{t_j}}$ (and thus, $P^2_{t_k}/P_{t_j}$) should be bounded if the inequality is to hold for all personalized prices. Indeed, in that case one can pick E_{t_j} and E_{t_k} to be similar and $X^2_{t_j,t_k} < 0$ large enough such that the inequality holds for every latent children welfare prices satisfying the support constraint. In short, we can guarantee informative upper bounds on RTS when $a^i_{t_j,t_k} \geq 0$ if $\frac{P_{t_k}}{P_{t_j}}$ is bounded. Let us now consider the case where $a^i_{t_j,t_k} \leq 0$. The upper bound is less than 1 if $P^i_{t_k}(W_{t_j} - W_{t_k}) > -X^i_{t_j,t_k}$ or

$$\left(\frac{P^i_{t_k}}{P_{t_j}} E_{t_j} - \frac{P^i_{t_k}}{P_{t_k}} E_{t_k}\right) > -X^i_{t_j,t_k}.$$

Since $P^1_{t_k} + P^2_{t_k} = P_{t_k}$ for all k, it follows that $\frac{P^i_{t_k}}{P_{t_k}} \in (0,1)$. If $\frac{P^1_{t_k}}{P_{t_k}}$ goes to zero, then $\frac{P^2_{t_k}}{P_{t_k}}$ goes to one. For the left-hand side to remain negative, it must be that

$$\frac{P^i_{t_k}}{P_{t_j}} E_{t_j} < \frac{P^i_{t_k}}{P_{t_k}} E_{t_k} \iff \frac{P_{t_k}}{P_{t_j}} < \frac{E_{t_k}}{E_{t_j}}.$$

Letting the ratio $\frac{P^1_{t_k}}{P_{t_k}}$ go to zero for household member 1, we get the restriction

$$0^- > -X^1_{t_j,t_k}.$$

This condition can clearly be satisfied. For household member 2, we get the restriction

$$\frac{P^2_{t_k}}{P_{t_j}} E_{t_j} - E_{t_k} < -X^2_{t_j,t_k}.$$

Since $\frac{P_{t_k}}{P_{t_j}}$ is bounded, we can pick E_{t_k} small enough and $X^i_{t_j,t_k}$ large enough such that this inequality holds for all personalized prices.

Observe that $X^i_{t_j,t_k}$ depends on latent prices $\mathcal{P}^i_{t_k}$, but $\mathcal{P}^1_{t_k}+\mathcal{P}^2_{t_k}=1$ such that the data can always be picked to obtain a desired sign on $X^i_{t_j,t_k}$. Furthermore, it is possible to pick $X^1_{t_j,t_k}$ and $X^2_{t_j,t_k}$ to have different signs. Thus, our previous discussion leads to three possible scenarios depending on the data:

(i) Suppose $X^i_{t_j,t_k}<0$ and $X^i_{t_k,t_j}<0$. Without loss of generality, suppose $P^i_{t_k}(W_{t_j}-W_{t_k})\geq 0$ such that $P^i_{t_j}(W_{t_k}-W_{t_j})\leq 0$. By GARP, $a^i_{t_k,t_j}\leq 0$ implies $a^i_{t_j,t_k}\geq 0$. Thus, the pair (t_j,t_k) yields an uninformative bound, but the pair (t_k,t_j) may yield an informative bound. Precisely, since $a^i_{t_k,t_j}\geq 0$, $X^i_{t_k,t_j}<0$ and $P^i_{t_j}(W_{t_k}-W_{t_j})>0$, we fall in Case 3. There are values of personalized prices such that the upper bound is informative.

(ii) Suppose $X^i_{t_j,t_k}>0$ and $X^i_{t_k,t_j}>0$. Without loss of generality, suppose $P^i_{t_k}(W_{t_j}-W_{t_k})\geq 0$ such that $P^i_{t_j}(W_{t_k}-W_{t_j})\leq 0$. Then, the pair (t_j,t_k) gives an uninformative bound, but the pair (t_k,t_j) yields an informative lower bound if $a^i_{t_k,t_j}\geq 0$, and can yield an informative upper bound if $a^i_{t_k,t_j}\leq 0$.

(iii) Suppose $X^i_{t_j,t_k}>0$ and $X^i_{t_k,t_j}<0$, then there is no guarantee that an informative bound can arise. Indeed, it is possible to pick $P^i_{t_k}(W_{t_j}-W_{t_k})\geq 0$ and $P^i_{t_j}(W_{t_k}-W_{t_j})\leq 0$ such that both (t_j,t_k) and (t_k,t_j) fall in Case 1 for all personalized prices. This situation is only globally relevant if there are two time periods.

□

The proof of Proposition 2 informs us that, given a data set, the bounds on RTS depend on the particular solution of personalized prices and children welfare that rationalizes the data. The information of each solution can be combined to recover bounds on the average RTS and thus bounds on the average production function. Suppose we get an informative lower bound L at $(\mathcal{P}^i_t,P^i_t,W_t)_{t\in\mathcal{T}}$ and an informative upper bound U at $(\widetilde{\mathcal{P}}^i_t,\widetilde{P}^i_t,\widetilde{W}_t)_{t\in\mathcal{T}}$. Although neither solution provides both informative lower and upper bounds on RTS, the identified set for the average RTS does as it gives $\frac{1}{2}\cdot([L,1]+(0,U])=[\frac{L}{2},\frac{1+U}{2}]$.[10]

To get some intuition on the mechanism by which household member choices provide information on RTS, consider the case where $X^i_{t_j,t_k}>0$ and $X^i_{t_k,t_j}>0$.

[10]The same idea applies if we consider the identified set on the average RTS in the cross-section. Later on, we exploit this insight to obtain meaningful bounds the average production function.

Without loss of generality, suppose $P^i_{t_k}(W_{t_j} - W_{t_k}) \geq 0$ such that $P^i_{t_j}(W_{t_k} - W_{t_j}) \leq 0$. Further suppose $a^i_{t_k,t_j} \leq 0$ such that an informative upper bound is obtained from the data. Since $a^i_{t_k,t_j} \leq 0$, household members prefer the allocation $(l^i_{t_j}, q^i_{t_j}, Q_{t_j}, W_{t_j})$ over $(l^i_{t_k}, q^i_{t_k}, Q_{t_k}, W_{t_k})$ in period t_j. Since $a^i_{t_k,t_j} \leq 0$ despite $X^i_{t_k,t_j} > 0$, it must be that $(l^i_{t_j}, q^i_{t_j}, Q_{t_j}, W_{t_j})$ is preferred to $(l^i_{t_k}, q^i_{t_k}, Q_{t_k}, W_{t_k})$ because children welfare is sufficiently more enticing. Children welfare is more enticing if it gives a higher marginal utility or when $P^i_{t_j}$ is large, but this exactly occurs when RTS is not too large.

4 Empirical Specification

The previous section showed the Afriat inequalities provide restrictions that can be used to nonparameterically (partially) identify the production function. Even if identification was actually achieved, we would possibly still need many more observations than what is available in typical panel data sets. As such, this section specializes the production technology.

4.1 Cobb-Douglas Technology

In what follows, we choose to focus on a Cobb-Douglas technology.

Assumption 4. *The production function is Cobb-Douglas such that*

$$W_{jt} = (h^1_{jt})^{\alpha_{j1}}(h^2_{jt})^{\alpha_{j2}}(c_{jt})^{\alpha_{j3}}e^{\epsilon_{jt}}.$$

The Cobb-Douglas technology is a natural choice as it is homogeneous of degree $\alpha_{j1} + \alpha_{j2} + \alpha_{j3}$. Furthermore, it is easy to see that the output elasticities are given by

$$\alpha_{j1} = \frac{RTS w^1_{jt} h^1_{jt}}{w_{jt}h^1_{jt} + w_{jt}h^2_{jt} + c_{jt}}$$
$$\alpha_{j2} = \frac{RTS w^2_{jt} h^2_{jt}}{w_{jt}h^1_{jt} + w_{jt}h^2_{jt} + c_{jt}}$$
$$\alpha_{j3} = \frac{RTS c_{jt}}{w_{jt}h^1_{jt} + w_{jt}h^2_{jt} + c_{jt}}.$$

In words, the model implies that each output elasticity equates a fraction RTS of its share of total children expenditure. These shares are constant in time, regardless of changes in the shadow price of children welfare, P_{jt}. The next result warns against ignoring productivity shocks in the model.

Claim 1. *Suppose Assumptions 1-4 hold. If productivity shocks are ignored, then the data may erroneously reject the model at the true return to scale.*

Proof. In what follows, we remove the j subscript from the variables. Suppose the data are rationalized by the model at the true return to scale $RTS^0 \in (0,1]$ and the true children welfare $W_t = (h_t^1)^{\alpha_1}(h_t^2)^{\alpha_2}(c_t)^{\alpha_3}e^{\epsilon_t}$. Suppose now the econometrician ignores productivity shocks and assumes

$$\widetilde{W}_t = (h_t^1)^{\alpha_1}(h_t^2)^{\alpha_2}(c_t)^{\alpha_3}.$$

Conditional on RTS^0, the output elasticities are identified. Therefore, children welfare is also identified. From the first-order conditions of the model and Lemma 1, we have

$$\widetilde{P}_t = \frac{E_t}{RTS^0\widetilde{W}_t}.$$

Since $\widetilde{W}_t$ is identified, it follows that $\widetilde{P}_t$ is also identified. It is then obvious that the Afriat inequalities

$$U_s^i - U_t^i \leq \lambda_t^i \left[w_t^i(l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i(Q_s - Q_t) + \widetilde{P}_t^i(\widetilde{W}_s - \widetilde{W}_t)\right]$$

can be rejected by the data even if the data are consistent with the Afriat inequalities under the correct specification of the production function. $\square$

Since output elasticities are a function of returns to scale, Claim 1 implies that ignoring productivity shocks may lead to inconsistent output elasticities.[11] The next result shows that the first-order conditions of the model have empirical bite under the Cobb-Douglas specification.

Claim 2. *Suppose Assumptions 1-4 hold. The first-order conditions of the model are refutable independently of returns to scale.*

Proof. By Lemma 1, we have

$$P_{jt}W_{jt} = RTS^{-1}(w_{jt}^1 h_{jt}^1 + w_{jt}^2 h_{jt}^2 + c_{jt}) \quad \forall t \in \mathcal{T},$$

where $RTS \in (0,1]$. For the sake of simplicity, suppose there are only two time

[11] More generally, productivity shocks are useful as they may absorb omitted variables that could otherwise bias the output elasticities.

periods. As such, we have

$$\frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} = \frac{(w^1_{j2}h^1_{j2} + w^2_{j2}h^2_{j2} + c_{j2})}{(w^1_{j1}h^1_{j1} + w^2_{j1}h^2_{j1} + c_{j1})}.$$

Since output elasticities are time invariant, it must be that the following set of equations holds

$$\frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} = \frac{w^1_{j2}h^1_{j2}}{w^1_{j1}h^1_{j1}}$$
$$\frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} = \frac{w^2_{j2}h^2_{j2}}{w^2_{j1}h^2_{j1}}$$
$$\frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} = \frac{c_{j2}}{c_{j1}}.$$

Note that these equations do not depend on returns to scale. Furthermore, they can easily be violated such as with $w^1_{j2}h^1_{t2} = 1/2$ and $w^2_{j2}h^2_{t2} = c_{j2} = 1/4$. $\square$

Contrary to the fully nonparametric setup, Claim 2 shows that the first-order conditions have meaningful implications that can be tested in the data given a Cobb-Douglas specification. Still, the first-order conditions do not imply any restriction on returns to scale and thus on the output elasticities. Fortunately, Proposition 2 shows that the Afriat inequalities provide the missing source of identification to learn about the production function.

4.2 Measurement Error

Claim 2 shows the the model implies a set of overidentifying restrictions on output elasticities. Hence, any measurement error in the inputs, however small, would lead to the erroneous rejection of the model. It follows that any test of the model that does not address this issue would be dubious in our framework. For this reason, we impose mild centering conditions on measurement error. Let $m^x_t := x_t - x^\star_t$ denote the difference between the observed and true value of a variable x_t in period t.

Assumption 5. $\mathbb{E}[m^x_t] = 0$, *where* $x \in \{h^1, h^2, c\}$, $t = 1, 2, \ldots, T$.

Assumption 5 requires that observed inputs be consistent with the true inputs on average in the cross-section. Note that we do not require the distribution of measurement error to be parametric or to be identical over time. An indirect benefit of introducing measurement error in inputs is that we will be able to keep households with missing inputs in our application. Further details about the data are discussed in Section 6.

5 Testing and Estimation

This section presents the statistical framework used for testing the model and making inference on the production function.

5.1 Testing

Let $\zeta_j := \{(U^i_{jt}, \lambda^i_{jt}, \mathcal{P}^i_{jt}, P^i_{jt}, W_{jt}, \boldsymbol{\alpha}_{jt}, \omega_{jt}, m^x_{jt})_{i\in\{1,2\}, x\in\{h^1,h^2,c\}}\}_{t\in\mathcal{T}} \in Z|\mathcal{D}$ denote the set of household-specific latent variables in the model, where Z denote the support of the latent variables and $\mathcal{D}$ denote the support of the data. Our revealed preference characterization along with our moment conditions can be used to define the statistical rationalizability of a panel data set $D := \{D_j\}_{j\in\mathcal{N}}$. To this end, write the constraints of the model in the form of moment functions:

$$
\begin{aligned}
g^U_{ist}(D_j, \zeta_j) &:= \mathbb{1}\Big(U^i_{js} - U^i_{jt} \leq \lambda^i_{jt}\Big[w^i_{jt}(l^i_{js} - l^i_{jt}) + (q^i_{js} - q^i_{jt}) + \\
&\qquad + \mathcal{P}^i_{jt}(Q_{js} - Q_{jt}) + P^i_{jt}(W_{js} - W_{jt})\Big]\Big) - 1 \\
g^{\alpha_1}_t(D_j, \zeta_j) &:= \mathbb{1}\left(\alpha_{jt1} = \frac{w^1_{jt}h^1_{jt}}{P_{jt}W_{jt}}\right) - 1 \\
g^{\alpha_2}_t(D_j, \zeta_j) &:= \mathbb{1}\left(\alpha_{jt2} = \frac{w^2_{jt}h^2_{jt}}{P_{jt}W_{jt}}\right) - 1 \\
g^{\alpha_3}_t(D_j, \zeta_j) &:= \mathbb{1}\left(\alpha_{jt3} = \frac{c_{jt}}{P_{jt}W_{jt}}\right) - 1 \\
g^W_t(D_j, \zeta_j) &:= \mathbb{1}\left(W_{jt} = (h^1_{jt})^{\alpha_{j1}}(h^2_{jt})^{\alpha_{j2}}(c_{jt})^{\alpha_{j3}}e^{\epsilon_{jt}}\right) - 1 \\
g^m_{xt}(D_j, \zeta_j) &:= m^x_{jt},
\end{aligned}
$$

where $\mathbb{1}(\cdot)$ denote the indicator function and equates 1 if the expression inside the parenthesis is satisfied and 0 otherwise. The latent variables further need to satisfy their support constraints such that $q^1_{jt} + q^2_{jt} = q_{jt}$ and $\mathcal{P}^1_{jt} + \mathcal{P}^2_{jt} = 1$.

Note that we let output elasticities vary in time in the moment functions $g^{\alpha_k}_t(D_j, \zeta_j)$. This guarantees that the equations for the output elasticities can be satisfied in every period. To obtain time invariant output elasticities, we require their expected variance to be zero:

$$\mathbb{E}[\boldsymbol{g}^v(D_j,\zeta_j)] = 0,$$

where $\boldsymbol{g}^v(D_j,\zeta_j) := var(\boldsymbol{\alpha})$. Since the variance is always positive, those moment conditions are satisfied if and only if the variance is zero for all households. As such, this formulation is equivalent to directly imposing that production parameters are time invariant.[12]

In what follows, we let $\boldsymbol{g}(D_j,\zeta_j)$ denote the vector of all moment functions, $\boldsymbol{g}^{(m,v)}(D_j,\zeta_j) := (\boldsymbol{g}^m(D_j,\zeta_j)', \boldsymbol{g}^v(D_j,\zeta_j)')'$ denote the set of moment functions on measurement error and variance of output elasticities, and $\boldsymbol{g}^{-(m,v)}(D_j,\zeta_j) := (\boldsymbol{g}^U(D_j,\zeta_j)', \boldsymbol{g}^\alpha(D_j,\zeta_j)', \boldsymbol{g}^W(D_j,\zeta_j)')'$ denote its complement.

Definition 6. Under Assumptions 1-5, a data set D is statistically rationalizable if

$$\inf_{\mu \in \mathcal{M}_{\mathcal{Z}|\mathcal{D}}} \|\mathbb{E}_{\mu\times\pi_0}[\boldsymbol{g}(D,\zeta)]\| = 0,$$

where $\mathcal{M}_{Z|\mathcal{D}}$ is the set of all conditional probability distributions on $Z|\mathcal{D}$ and $\pi_0 \in \mathcal{M}_{\mathcal{X}}$ is the observed distribution of D.

In its current form, the notion of statistical rationalizability has $2T^2+T+T+T+T+3T+T$ moment conditions, including some that are discontinuous. Let d_m denote the number of moment conditions on measurement error and d_v denote the number of moment conditions on the variance of the output elasticities. The following result due to Schennach (2014) and Aguiar and Kashaev (2021) allows us to considerably reduce the complexity of the problem.

Proposition 3. *Under Assumptions 1-5, a data set D is statistically rationalizable if and only if*

$$\min_{\boldsymbol{\gamma}\in\mathbb{R}^{d_m+d_v}} \|\mathbb{E}_{\pi_0}[\bar{\boldsymbol{g}}(D;\boldsymbol{\gamma})]\| = 0,$$

where

$$\bar{g}_j(D_j;\boldsymbol{\gamma}) := \frac{\int_{\zeta_j\in Z|\mathcal{D}} \boldsymbol{g}_j^{(m,v)}(D_j,\zeta_j)\exp\Big(\boldsymbol{\gamma}'\boldsymbol{g}_j^{(m,v)}(D_j,\zeta_j)\Big)\mathbb{1}(\boldsymbol{g}_j^{-(m,v)}(D_j,\zeta_j)=0)\,d\eta(\zeta_j|D_j)}{\int_{\zeta_j\in Z|\mathcal{D}} \exp\Big(\boldsymbol{\gamma}'\boldsymbol{g}_j^{(m,v)}(D_j,\zeta_j)\Big)\mathbb{1}(\boldsymbol{g}_j^{-(m,v)}(D_j,\zeta_j)=0)\,d\eta(\zeta_j|D_j)},$$

and $\eta(\cdot|D_j)$ is an arbitrary user-specified distribution supported on $Z|\mathcal{D}$ such that $\mathbb{E}_{\pi_0}[\log\big(\mathbb{E}_\eta[\exp\big(\boldsymbol{\gamma}'\boldsymbol{g}^{(m,v)}(D,\zeta)\big)|D]\big)]$ exists and is twice continuously differentiable in $\boldsymbol{\gamma}$ for all $\boldsymbol{\gamma}\in\mathbb{R}^{d_m+d_v}$.

[12] This is formally proven in Aguiar and Kashaev (2021).

The previous result calls for some comments. First, the dimensionality of the problem is greatly reduced as it only requires finding a finite dimensional parameter $\boldsymbol{\gamma}$ rather than a distribution μ. Second, the moment conditions associated with the concavity of the utility functions, first-order conditions, and production function equations are directly imposed on each household data set such as to restrict the support of the unobservables. In particular, observe that the optimization problem no longer includes any discontinuous moment condition. Finally, it is worth noting that the result states that there is no loss in generality in averaging out the unobservables in the moment functions provided the distribution is from the exponential family.

The simplification allowed by Proposition 3 requires finding unobservables ζ_j that exactly satisfy the concavity of the utility functions, first-order conditions, and production function equations. If the constraints were linear in the unobservables, it would be possible to use a standard Hit-and-Run algorithm to directly sample them from the feasible space defined by the intersection of the inequalities and the system of equations. Unfortunately, the inequalities are highly nonlinear, therefore making this approach impossible.[13]

We resolve this pervasive issue by proposing a blocked Gibbs sampler. The idea is to break down the sampling procedure into multiple blocks, where each block fixes a subset of all unobservables. The key is to create those blocks in such a way that the inequalities are linear in the unobservables conditional on a certain subset of all unobservables. Thus, the inequalities effectively define a (conditional) convex polytope in each block. This allows for a straightforward sampling procedure that guarantees the unobservables to exactly satisfy the inequalities, first-order conditions, and production function equations. The details of the algorithm are provided in the Appendix.

5.2 Inference

One of the advantages of ELVIS is that testing and inference are quite simple even if the model is partially identified. Indeed, testing the model can be done by constructing the sample analogues of the averaged moments and by computing a test statistic that is stochastically bounded by the chi-square distribution. Inference is achieved by further adding moment conditions on parameters of

[13] In principle, it would be possible to use rejection sampling along with a mixed-integer programming (MIP) problem to draw from the feasible space. However, these types of MIP for collective models are NP-complete (Nobibon et al., 2016) so they do not scale well. Also, rejection sampling is generally slow.

interest and inverting the test statistic. Since the test statistic is stochastically bounded by the chi-square distribution, it suffices to compare the value of the test statistic against the chi-square critical value with $d_m + d_v$ ($d_m + d_v + d_\theta$) degrees of freedom for testing (inference). Importantly, the identified set is convex under mild conditions.[14]

6 Data

We conduct our empirical analysis with the Longitudinal Internet Studies for the Social Sciences (LISS) panel data. The panel consists of about 5000 households representative of the Dutch population and gathers information about panelists yearly. Since the LISS data directly include information on private expenditures within the household, an important point of departure from the model is that private expenditures q_t^i are observed.

The time use data were collected by means of survey questions about the time spent on a set of time use categories during the past seven days. Although the survey is not demanding of household members memory, the actual time allocations throughout the month are likely to differ from the ones reported at the time of the survey. Similarly, data on monthly expenditures were collected via survey questions. Additional details relating to data collection can be found in Cherchye, De Rock and Vermeulen (2012).

Since our main goal is to estimate the production function, we only consider measurement error in inputs. Nevertheless, we observe that our methodology could accommodate measurement error in other variables. Another rationale for our choice is that it ensures that the overidentifying restrictions implied by the Cobb-Douglas specification can be rationalized by the model. We refer the reader to Section 4 for details about the specification of the production function and the restrictions on measurement error.

Our empirical analysis focuses on couples with children. This restriction alone reduces the number of households to about a thousand. We further restrict our sample to those with nonmissing and nonzero wages, private expenditures, and public expenditures. For households with missing or zero data on inputs, we impute their values. Note that our treatment of measurement error explicitly handles the imperfection of our imputation. Lastly, we restrict our sample to households that are in the panel for three periods.

[14] We refer the reader to Aguiar and Kashaev (2021) for additional details about the statistical procedure.

The final sample consists of 147 couples with children observed over 3 time periods pooled from the years 2008 to 2017. While our sample size is relatively small, we note that it is comparable with Cherchye, De Rock and Vermeulen (2012) despite our restriction to panel data. Summary statistics of the sample are displayed in Table 1. Further details about the sample construction are given in the Appendix.

Table 1: Sample Summary Statistics

	Husband		Wife		Household	
	Mean	Std dev.	Mean	Std dev.	Mean	Std dev.
Age	46.24	8.13	43.91	7.42		
Wage (EUR/hour)	14.09	8.54	12.75	11.53		
Number of children					1.98	0.83
Mean age of children					12.88	6.58
Childcare (hours/week)	10.44	7.83	17.52	13.25		
Work (hours/week)	36.90	6.13	23.33	7.87		
Private expenditure (EUR/month)	379.05	355.26	412.15	357.29		
Public expenditure (EUR/month)					2749.44	7670.31
Children expenditure (EUR/month)					517.93	536.27
Total households						147

7 Results

This section recovers confidence sets on expected returns to scale and expected output elasticities. Then, it investigate how demographic characteristics such as the education level of each parent impacts the production of children welfare.

7.1 Returns to Scale

To recover the 95% confidence set on expected returns to scale, we test the model at various values for the expected returns to scale. Since the confidence set is convex, we only need to find the lower and upper bounds. We find that the 95% confidence set on expected returns to scale is $[0.225, 0.35]$. Since the confidence set is nonempty, we conclude that the model is not rejected by the data.

7.2 Output Elasticities

To recover 95% confidence sets on the expected output elasticities, we allow household-specific returns to scale to vary fully on the support $(0, 1]$. The results are reported in Figure 1.

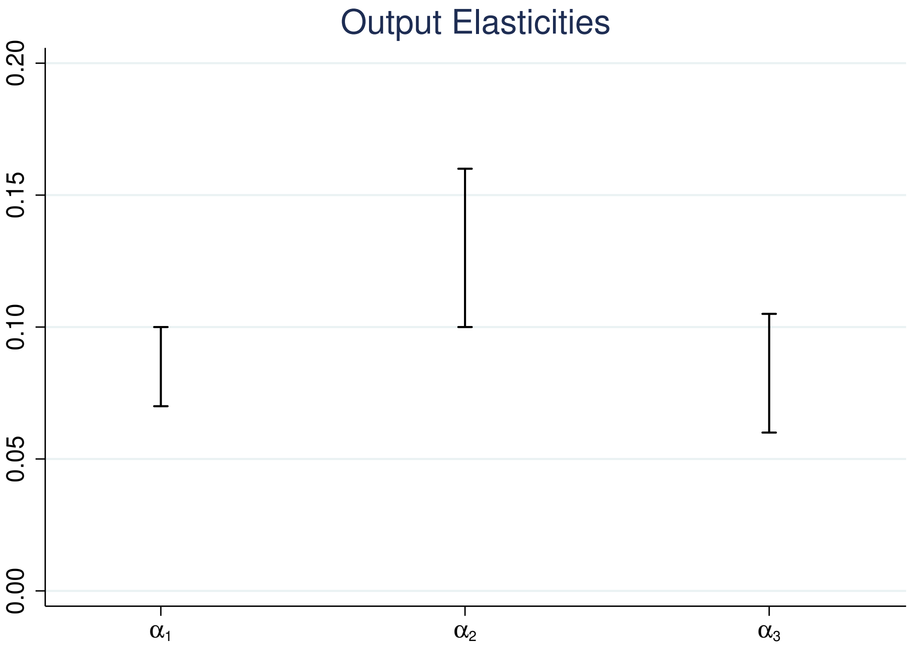

Figure 1: 95% Confidence Sets on Production Parameters

The results show that, on average, time inputs by mothers increases children welfare by more than time inputs by fathers or children expenditure.

To assess the degree of heterogeneity in returns to scale across households, we report the distribution of returns to scale obtained from the Monte Carlo Markov Chain (MCMC) used in the calculation of the average moment functions. The distribution is reported in Figure 2.

Figure 2 makes clear that there is significant heterogeneity in returns to scale and, hence, in output elasticities. Further, the vertical lines heuristically show that certain observed demographics such as education impact the production function.[15]

7.3 Heterogeneity

Figure 2 shows significant heterogeneity in the production technology across households. Accordingly, this section aims quantify the role of observables in generating those differences. Due to the small sample size, we choose to incorporate a linear regression into our model such that

$$\frac{\boldsymbol{\alpha}_2}{\boldsymbol{\alpha}_1} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{C}\boldsymbol{\delta} + \boldsymbol{\omega}, \tag{2}$$

[15]The vertical lines for average returns to scale in Figure 2 is based on the education level of the male in the household.

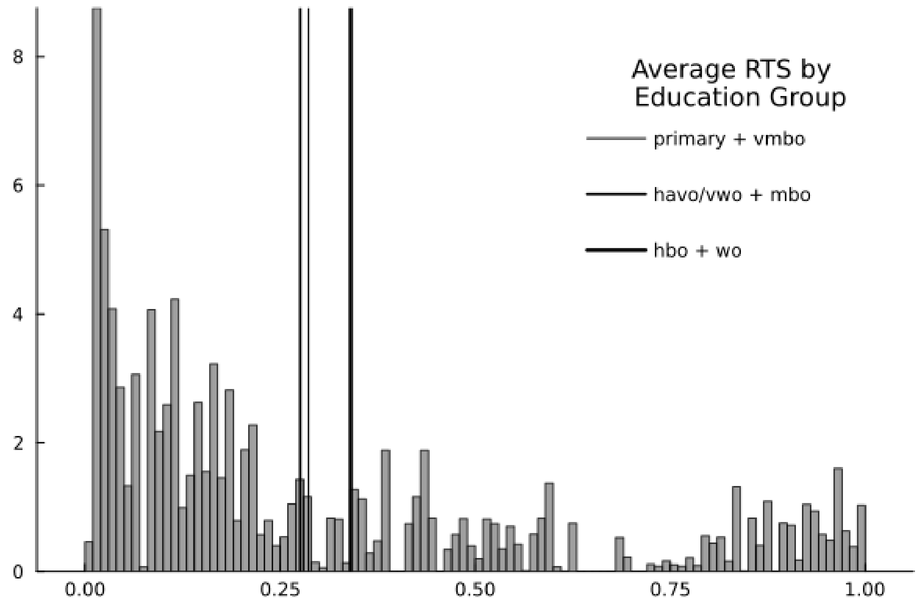

Figure 2: MCMC Distribution of Returns to Scale

where $\boldsymbol{\alpha}_k$ represents output elasticities with respect to input k, $\boldsymbol{X}$ is a set of variables of interest, $\boldsymbol{\beta}$ is a set of parameters of interest, $\boldsymbol{C}$ is a set of control variables, $\boldsymbol{\delta}$ is a set of nuisance parameters, and $\boldsymbol{\omega}$ is a random error. The parameters of this regression are unbiased and consistent if the error is mean zero conditional on the data.

Assumption 6. $\mathbb{E}[\boldsymbol{\omega}|\boldsymbol{X}, \boldsymbol{Z}] = 0$.[16]

We wish to emphasize that this regression equation is not estimated separately from the rest of the model. Rather, it is imposed as another equation restriction within the model.

We focus on the effects of education on the output elasticities and include number of children and the average age of children in the household as controls. The education of each household member is a categorical variable that reflects the type of education. The first category represents primary school and pre-vocational secondary education (VMBO). The second category represents general education that leads to higher education (HAVO and VWO) and vocational education (MBO) that can lead to higher education. The third category represents higher education (HBO, WO).

The 95% confidence sets of each parameter associated with the education level of each household member are reported in Table 2, where the effect of each

[16]In the results derived below, we imposed and tested $\mathbb{E}[\boldsymbol{X}\boldsymbol{\omega}] = 0$. We view the nonrejection of the restriction as evidence that Assumption 6 is reasonable.

education level is relative to the first category of education.

Table 2: 95% Confidence Sets on $\boldsymbol{\beta}$

	Education[a]	Father	Mother
95% confidence set	havo, vwo & mbo	$[-3.0, -1.20]$	$[0.65, 2.0]$
	hbo & wo	$[-3.6, -1.7]$	$[0.875, 2.1]$

[a] The education category "havo, vwo, & mbo" represents general education that leads to higher education and vocational education that can lead to higher education. The education category "hbo & wo" represents higher education.

The results in Table 1 show that parents with higher education levels have, on average, larger output elasticities relative to their partner compared to parents with the lowest level of education, everything else equal. Note that the coefficients are negative for fathers because the dependent variable is the mother-to-father output elasticity ratio in the regression equation (2). Also, it seems that output elasticities with respect to time inputs are monotone in education for both parents. Finally, observe that the results in Table 2 further imply that households whose parents have higher education levels have larger returns to scale on average.

8 Conclusion

This paper exploits the full empirical content of the collective model available in panel data to investigate heterogeneity in the production technology. The level of flexibility allowed forces us into the realm of partial identification. As such, we propose a novel estimation strategy that makes it possible to tractably analyze the model and to conduct valid statistcal testing and inference. In our application, we find that constant returns to scale is inconsistent with the data in our collective labor supply model with children. Furthermore, we find that education has a significant impact on the effects of parental time inputs on the production of children welfare. We view the treatment of labor nonparticipation in collective models with children as an interesting extension for future research, and we observe that our approach seems appropriate for such endeavor.

Appendix

A Sample Construction

For each household, we compute how many children live at home. We focus on households with children living at home rather than households with children as it is a precondition for household production. We drop single households and those that do not have any child living at home. Next, we remove households where one or both members have missing wages. Finally, we remove households with missing private expenditures or missing public expenditures.

We impute missing and zero values of hours worked, childcare, and children expenditure for households that remain in the sample. We treat zeros as missing because they are highly unrealistic. The imputation is a simple year average of the variable for households that satisfy our previous selection criteria. This imputation is likely to be an overestimate of the actual value for some households and an underestimate for others. Hence, it should be consistent with our moment conditions on measurement error.

Besides the imputation for some missing or zero inputs, we compute leisure of each household member as a residual according to the following equation:

$$l_t^i = 168 - 56 - b_t^i - h_t^i,$$

where l_t^i is leisure, 168 is the total number of hours in a week, 56 is the number of hours spent sleeping in a week, b_t^i is time spent working, h_t^i is time spent on childcare. Clearly, our construction of leisure may be inaccurate. We tackle this problem by allowing for measurement error in leisure. Precisely, since the time constraint requires $l_t^i + b_t^i + h_t^i = 168 - 56$ and h_t^i is mismeasured, we let true leisure be minus true childcare ($l_t^{\star i} = -h_t^{\star i}$) such that the time constraint holds at the true variables. These weekly variables are then scaled such as to obtain time inputs for the average number of days in a month. Since there are seven days in a week and a month has slightly more than 30 days on average, we multiply time inputs by 4.3.

As a last refinement of the sample, we remove households that have public expenditure equal to zero as it is highly unrealistic and drop households that are part of the LISS data for strictly less than 3 years. To obtain a balanced panel, we keep the first 3 observations of each household that is present for strictly more than 3 years. Thus, our sample is composed of households from various sets of 3 periods (e.g., 2009-2010-2012 or 2010-2012-2015). We limit ourselves to a three-year panel despite the greater empirical bite that could be obtained with additional periods to avoid any additional decrease in the sample size.

B Proofs

B.1 Proof of Theorem 1

$(i) \implies (ii)$
The household problem can be written as

$$\max_{(l^1,l^2,h^1,h^2,q^1,q^2,Q,c)\in\mathbb{R}^2_+\times\mathbb{R}^2_{++}\times\mathbb{R}^{2L}_+\times\mathbb{R}_+\times\mathbb{R}_{++}} \mu^1_t U^1(l^1,q^1,Q,W_t)+\mu^2_t U^2(l^2,q^2,Q,W_t),$$

subject to satisfying the household constraints

$$(q^1+q^2)+Q+c = w^1_t(1-l^1-h^1)+w^2_t(1-l^2-h^2).$$

The first-order conditions are given by

$$\begin{aligned}
\mu^i_t \frac{\partial U^i}{\partial l^i} &= \eta_t w^i_t \\
\mu^i_t \frac{\partial U^i}{\partial q^i} &= \eta_t. \\
\sum_i \mu^i_t \frac{\partial U^i}{\partial W_t}\cdot\frac{\partial W_t}{\partial h^1} &= \eta_t w^1_t \\
\sum_i \mu^i_t \frac{\partial U^i}{\partial W_t}\cdot\frac{\partial W_t}{\partial h^2} &= \eta_t w^2_t \\
\sum_i \mu^i_t \frac{\partial U^i}{\partial W_t}\cdot\frac{\partial W_t}{\partial c} &= \eta_t \\
\sum_i \mu^i_t \frac{\partial U^i}{\partial Q} &= \eta_t,
\end{aligned}$$

where the equalities hold for some supergradient of the utility function.[17] Next,

[17] For corner solutions, the first-order conditions may only hold with inequality. The argument does not require any substantive change to accommodate this possibility.

define

$$\lambda_t^i = \frac{\eta_t}{\mu_t^i}$$
$$P_t^i = \frac{\mu_t^i}{\eta_t}\frac{\partial U^i}{\partial W_t}$$
$$\mathcal{P}_t^i = \frac{\mu_t^i}{\eta_t}\frac{\partial U^i}{\partial Q}.$$

The first-order conditions can be rewritten as

$$\frac{\partial U^i}{\partial l^i} = \lambda_t^i w_t^i \quad (3)$$
$$\frac{\partial U^i}{\partial q^i} = \lambda_t^i. \quad (4)$$
$$(P_t^1 + P_t^2)\frac{\partial W_t}{\partial h^1} = w_t^1 \quad (5)$$
$$(P_t^1 + P_t^2)\frac{\partial W_t}{\partial h^2} = w_t^2 \quad (6)$$
$$(P_t^1 + P_t^2)\frac{\partial W_t}{\partial c} = 1 \quad (7)$$
$$\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1. \quad (8)$$

Since the children welfare function is unknown, equations (5)-(7) require the existence of numbers $\dot{W}_{h_t^1}$, $\dot{W}_{h_t^2}$, and $\dot{W}_{c_t} > 0$ such that

$$(P_t^1 + P_t^2)\dot{W}_{h_t^1} = w_t^1$$
$$(P_t^1 + P_t^2)\dot{W}_{h_t^2} = w_t^2$$
$$(P_t^1 + P_t^2)\dot{W}_{c_t} = 1.$$

Next, using the concavity of the utility functions we obtain

$$U_s^i - U_t^i \leq \left[\frac{\partial U^i}{\partial l_t^i}(l_s^i - l_t^i) + \frac{\partial U^i}{\partial q_t^i}(q_s^i - q_t^i) + \frac{\partial U^i}{\partial Q}(Q_s - Q_t) + \frac{\partial U^i}{\partial W_t}(W_s - W_t)\right],$$

where $U_t^i := U^i(l_t^i, q_t^i, Q_t, W_t)$ for all $t \in \mathcal{T}$. Substituting the derivatives of the utility function for their expressions yields

$$U_s^i - U_t^i \leq \lambda_t^i\Big[w_t^i(l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i(Q_s - Q_t) + P_t^i(W_s - W_t)\Big].$$

Putting everything together, these (in)equalities should hold for some U_t^i, $\lambda_t^i > 0$, $\mathcal{P}_t^i > 0$ such that $\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1$, $P_t^i > 0$ such that $P_t^1 + P_t^2 = P_t$, and $\dot{W}_{h_t^1}$, $\dot{W}_{h_t^2}$,

$\dot{W}_{c_t}$, $W_t > 0$, $t = 1, \ldots, T$.

$(ii) \implies (i)$

We have to show that, if Theorem 1 (ii) holds, then there exist concave utility functions and a production function concave in (h_1, h^2, c) that rationalize the data. Thus, define

$$W(h^1, h^2, c, \epsilon) := \min_{t \in \mathcal{T}} \Big\{ W_t + \frac{1}{P_t} \Big[w_t^1(h^1 - h_t^1) + w_t^2(h^2 - h_t^2) + (c - c_t) \Big] \Big\}.$$

This function is continuous, increasing, and concave in (h^1, h^2, c). Further, the supergradients of $W(h_t^1, h_t^2, c_t, \epsilon_t)$ yield equations (5)-(7). Finally, note that the children welfare production function is such that $W(h_t^1, h_t^2, c_t, \epsilon_t) \equiv W_t$ for all $t \in \mathcal{T}$. Next, let $\tau = \{t_j\}_{j=1}^m$, $m \geq 2$, $t_j \in \mathcal{T}$ denote a sequence of indices and $\mathcal{I}$ denote the set of all such indices. Define

$$\begin{aligned}
&U^i(l^i, q^i, Q, W) := \\
&\quad \min_{\tau \in \mathcal{I}} \Big\{ \lambda_{t_m}^i \Big[w_{t_m}^i (l^i - l_{t_m}^i) + (q^i - q_{t_m}^i) + \mathcal{P}_{t_m}^i (Q - Q_{t_m}) + P_{t_m}^i (W - W_{t_m}) \Big] + \\
&\quad + \sum_{j=1}^{m-1} \lambda_{t_j}^i \Big[w_{t_j}^i (l_{t_{j+1}}^i - l_{t_j}^i) + (q_{t_{j+1}}^i - q_{t_j}^i) + \mathcal{P}_{t_j}^i (Q_{t_{j+1}} - Q_{t_j}) + P_{t_j}^i (W_{t_{j+1}} - W_{t_j}) \Big] \Big\}.
\end{aligned}$$

The function is the pointwise minimum of a collection of linear functions. Thus, it is continuous, increasing, and concave. By definition of U^i, there is some sequence of indices such that

$$\begin{aligned}
&U^i(l_t^i, q_t^i, Q_t, W_t) \geq \\
&\quad \lambda_{t_m}^i \Big[w_{t_m}^i (l_t^i - l_{t_m}^i) + (q_t^i - q_{t_m}^i) + \mathcal{P}_{t_m}^i (Q_t - Q_{t_m}) + P_{t_m}^i (W_t - W_{t_m}) \Big] + \\
&\quad + \sum_{j=1}^{m-1} \lambda_{t_j}^i \Big[w_{t_j}^i (l_{t_{j+1}}^i - l_{t_j}^i) + (q_{t_{j+1}}^i - q_{t_j}^i) + \mathcal{P}_{t_j}^i (Q_{t_{j+1}} - Q_{t_j}) + P_{t_j}^i (W_{t_{j+1}} - W_{t_j}) \Big].
\end{aligned}$$

Add any allocation (l^i, q^i, Q, W) to the sequence and use the definition of U^i

once again to obtain

$$\begin{aligned}
&\lambda_t^i\Big[w_t^i(l^i - l_t^i) + (q^i - q_t^i) + \mathcal{P}_t^i(Q - Q_t) + P_t^i(W - W_t)\Big] + \\
&+ \lambda_{t_m}^i\Big[w_{t_m}^i(l_t^i - l_{t_m}^i) + (q_t^i - q_{t_m}^i) + \mathcal{P}_{t_m}^i(Q_t - Q_{t_m}) + P_{t_m}^i(W_t - W_{t_m})\Big] + \\
&+ \sum_{j=1}^{m-1} \lambda_{t_j}^i\Big[w_{t_j}^i(l_{t_{j+1}}^i - l_{t_j}^i) + (q_{t_{j+1}}^i - q_{t_j}^i) + \mathcal{P}_{t_j}^i(Q_{j+1} - Q_{t_j}) + P_{t_j}^i(W_{t_{j+1}} - W_{t_j})\Big] \\
&\geq U^i(l^i, q^i, W, Q).
\end{aligned}$$

Hence, rearranging the previous expression yields

$$U^i(l^i, q^i, Q, W) - U^i(l_t^i, q_t^i, Q_t, W_t) \leq \lambda_t^i\Big[w_t^i(l^i - l_t^i) + (q^i - q_t^i) + \mathcal{P}_t^i(Q - Q_t) + P_t^i(W - W_t)\Big].$$

Note that the two first supergradients of $U^i(l_t^i, q_t^i, Q_t, W_t)$ give the first-order conditions (3)-(4). As such, Theorem 1 (*ii*) has the same implications as the household problem (1).

(*ii*) $\implies$ (*iii*)

Let us begin by noting that the Afriat inequalities can be combined such that for all $\{t_k\}_{k=1}^m \in \mathcal{I}$ and all $i \in \{1, 2\}$

$$0 \leq \sum_{k=1}^{m} \lambda_{t_{k+1}}^i a_{t_k, t_{k+1}}^i.$$

Observe that the set of all sequences $\mathcal{I}$ can be reduced to the set of all finite sequences as any sequence that satisfies this inequality is also satisfied without cycles. For the sake of a contradiction, suppose GARP is not satisfied for some household member. Then, there exists a cycle such that $a_{t_1,t_2}^i \leq 0$, $a_{t_2,t_3}^i \leq 0$, $\ldots$, $a_{t_m,t_1}^i < 0$. Thus, it follows that

$$\lambda_{t_2}^i a_{t_1,t_2}^i + \lambda_{t_3}^i a_{t_2,t_3}^i + \cdots + \lambda_{t_1}^i a_{t_m,t_1}^i < 0,$$

a contradiction of cyclical monotonicity.

(*iii*) $\implies$ (*ii*)

Suppose that GARP holds for each household member. Then, an application of Fostel, Scarf and Todd (2004) shows the existence of the Afriat inequalities for each household member.

C Sampling from the Feasible Space: A Blocked Gibbs Sampler

This section explains how to draw latent variables that satisfy the household problem. Since private expenditures are directly observed in our application, we do not need to find such quantities in our procedure. It is quite straightforward to extend our procedure to further find private expenditures if those were not observed, however.

Let $P_t = P_t^1 + P_t^2$ and recall that the data are consistent with the model if there exist personalized prices $P_t^i > 0$, personalized prices $\mathcal{P}_t^i > 0$ such that $\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1$, numbers U^i, λ_t^i, $W_t > 0$, and true inputs $l_t^{\star i}$, $h_t^{\star i}$, $c_t^\star > 0$ such that for all $s, t \in \mathcal{T}$ and all $i \in \{1,2\}$

$$\begin{aligned} U_s^i - U_t^i &\leq \lambda_t^i \Big[w_t^i (l_s^{\star i} - l_t^{\star i}) + (q_s^i - q_t^i) + \mathcal{P}_t^i (Q_s - Q_t) + P_t^i (W_s - W_t) \Big] \\ \alpha_1 &= \frac{w_t^1 h_t^{\star 1}}{P_t W_t} \\ \alpha_2 &= \frac{w_t^2 h_t^{\star 2}}{P_t W_t} \\ \alpha_3 &= \frac{w_t^1 h_t^{\star 1}}{P_t W_t} \\ \epsilon_t &= \log(W_t) - \alpha_1 \log\big(h_t^{\star 1}\big) - \alpha_2 \log\big(h_t^{\star 2}\big) - \alpha_3 \log(c_t^\star), \end{aligned}$$

where the last equation is obtained from the natural logarithm of the production function equation. Suppose we have a solution

$$(U_t^i(r), \lambda_t^i(r), l_t^{\star i}(r), h_t^{\star i}(r), c_t^\star(r), P_t^i(r), \mathcal{P}_t^i(r))_{i \in \{1,2\}, t \in \mathcal{T}},$$

where r denote the rth solution found by some solver. We provide a feasible algorithm that guarantees the next set of latent variables to be in the feasible space conditional on the data. The algorithm works provided the feasible space is nonempty in each block and standard regularity conditions associated with Gibbs samplers hold.

Step 1: Marginal Utility of Expenditure

Let $\Lambda = [1, L]$ denote the support of λ_t^i, where L is an arbitrarily large number. Given a solution at step r, we want to find $\lambda_t^i(r+1)$ that satisfies

$$\begin{aligned} U_s^i(r) - U_t^i(r) \leq \lambda_t^i(r+1) \Big[& w_t^i (l_s^{\star i}(r) - l_t^{\star i}(r)) + (q_s^i - q_t^i) + \\ & + \mathcal{P}_t^i(r)(Q_s - Q_t) + P_t^i(r)\big(W_s(r) - W_t(r)\big) \Big]. \end{aligned}$$

For convenience, let

$$denom^i := \Big[w_t^i(l_s^{\star i} - l_t^{\star i}) + (q_s^i - q_t^i) + \mathcal{P}_t^i(r)(Q_s - Q_t) + P_t^i(r)\big(W_s(r) - W_t(r)\big) \Big].$$

It follows that

$$\lambda_t^i(r+1) \Delta \frac{U_s^i(r) - U_t^i(r)}{denom^i},$$

where $\Delta :=>$ if $denom^i > 0$ and $\Delta :=<$ otherwise. Note that each $\lambda_t^i(r+1)$ has T bounds. The greatest lower bound on $\lambda_t^i(r+1)$ is the maximum between one and the greatest lower bound. If there is no lower bound, then the greatest lower bound is one. Likewise, the least upper bound is the minimum between one and the least upper bound. If there is no upper bound, then the least upper bound is one. Draw $\lambda_t^i(r+1)$ uniformly over the support defined by the greatest lower bound and least upper bound.

Step 2: Children Welfare, Leisure, and Childcare

Conditional on the new solution $\lambda_t^i(r+1)$, we want children welfare, true leisure, and true childcare to be positive such that

$$W_t(r) + \alpha\xi(W_t) > 0 \tag{9}$$

$$h_t^{\star 1}(r) + \alpha\xi(h_t^{\star 1}) > 0 \tag{10}$$

$$h_t^{\star 2}(r) + \alpha\xi(h_t^{\star 2}) > 0 \tag{11}$$

$$l_t^{\star 1}(r) + \alpha\xi(l_t^{\star 1}) > 0 \tag{12}$$

$$l_t^{\star 2}(r) + \alpha\xi(l_t^{\star 2}) > 0. \tag{13}$$

These positivity constraints provide a set of inequality restrictions on α. Next, we must further ensure that new leisure and new childcare of each household member satisfy the normalized time constraint

$$h_t^{\star i}(r) + \alpha\xi(h_t^{\star i}) + m_t^i + l_t^{\star i}(r) + \alpha\xi(l_t^{\star i}) = 1.$$

This equation implies $\xi(h_t^{\star i}) = -\xi(l_t^{\star i})$, $i \in \{1, 2\}$. That is, the direction taken for new childcare is the opposite of the direction for new leisure.

Next, it is important to ensure new children welfare and new childcare are consistent with a positive children expenditure:

$$w_t^1(h_t^{\star 1}(r) + \alpha\xi(h_t^{\star 1})) + w_t^2(h_t^{\star 2}(r) + \alpha\xi(h_t^{\star 2})) \leq P_t(r)(W_t(r) + \alpha\xi(W_t)).$$

Rearranging, one obtains

$$\alpha \Delta \frac{P_t(r)W_t(r) - w_t^1 h_t^{\star 1}(r) - w_t^2 h_t^{\star 2}(r)}{w_t^1 \xi(h_t^{\star 1}) + w_t^2 \xi(h_t^{\star 2}) - P_t \xi(W_t)},$$

where $\Delta :=<$ if $w_t^1 \xi(h_t^1) + w_t^2 \xi(h_t^2) - P_t \xi(W_t) > 0$ and $\Delta :=>$ otherwise. Further, we must also ensure that new children welfare and new childcare are compatible with decreasing returns to scale given children expenditure:

$$w_t^1(h_t^{\star 1}(r) + \alpha \xi(h_t^{\star 1})) + w_t^2(h_t^{\star 2}(r) + \alpha \xi(h_t^{\star 2})) + c_t(r) \leq P_t(r)(W_t(r) + \alpha \xi(W_t)).$$

Rearranging, one obtains

$$\alpha \Delta \frac{P_t(r)W_t(r) - w_t^1 h_t^{\star 1}(r) - w_t^2 h_t^{\star 2}(r) - c_t(r)}{w_t^1 \xi(h_t^{\star 1}) + w_t^2 \xi(h_t^{\star 2}) - P_t \xi(W_t)},$$

where $\Delta :=<$ if $w_t^1 \xi(h_t^1) + w_t^2 \xi(h_t^2) - P_t \xi(W_t) > 0$ and $\Delta :=>$ otherwise.

Finally, we want new children welfare $W_t(r+1)$, new leisure $l_t^{\star i}(r+1)$, and new childcare $h_t^{\star i}(r+1)$ to satisfy

$$\begin{aligned} U_s^i(r) - U_t^i(r) \leq \lambda_t^i(r+1)\Big[& w_t^i(l_s^{\star i}(r) + \alpha \xi(l_s^{\star i}) - l_t^{\star i}(r) - \alpha \xi(l_t^{\star i})) + (q_s^i - q_t^i) + \\ & + \mathcal{P}_t^i(r)(Q_s - Q_t) + P_t^i(r)\big(W_s(r) + \alpha \xi(W_s) - W_t(r) - \alpha \xi(W_t)\big)\Big]. \end{aligned}$$

This inequality can be rewritten as

$$\begin{aligned} U_s^i(r) - U_t^i(r) - \lambda_t^i(r+1)\Big[& w_t^i(l_s^{\star i}(r) - l_t^{\star i}(r)) + (q_s^i - q_t^i) + \mathcal{P}_t^i(r)(Q_s - Q_t) \\ & + P_t^i(r)\big(W_s(r) - W_t(r)\big)\Big] \\ & \leq \alpha \lambda_t^i(r+1)\big(P_t^i(r)(\xi(W_s) - \xi(W_t)) + w_t^i(\xi(l_s^{\star i}) - \xi(l_t^{\star i}))\big). \end{aligned}$$

Let num^i denote the left-hand side of this inequality. Thus, we have

$$\alpha \Delta \frac{num^i}{\lambda_t^i(r)\big(P_t^i(r)(\xi(W_s) - \xi(W_t)) + \xi(w_t^i)(l_s^{\star i} - l_t^{\star i})\big)},$$

where $\Delta :=>$ if $\lambda_t^i(r)\big(P_t^i(r)(\xi(W_s) - \xi(W_t)) + w_t^i(\xi(l_s^{\star i}) - \xi(l_t^{\star i}))\big) > 0$ and $\Delta :=<$ otherwise. Draw α uniformly over its support as defined by the greatest lower bound and the least upper bound from the previous sets of inequalities. We obtain $(W_t(r+1), l_t^{\star i}(r+1), h_t^{\star i}(r+1))_{t \in \mathcal{T}}$ by picking α uniformly over its support defined by the greatest lower bound and least upper bound derived from the above inequalities.

Step 3: Utilities and Personalized Prices

We want personalized prices to be positive such that

$$\mathcal{P}_t^i(r) + \beta\xi(\mathcal{P}_t^i) > 0 \tag{14}$$

$$P_t^i(r) + \beta\xi(P_t^i) > 0. \tag{15}$$

These inequalities can be transformed to get bounds on β:

$$\beta\Delta - \frac{\mathcal{P}_t^i(r)}{\xi(\mathcal{P}_t^i)} \tag{16}$$

$$\beta\Delta - \frac{P_t^i(r)}{\xi(P_t^i)}, \tag{17}$$

where $\Delta :=>$ if $\xi(\cdot) > 0$ and $\Delta :=<$ otherwise. Next, similar to the previous step it is important to ensure new personalized prices for children welfare are consistent with a positive children expenditure:

$$w_t^1 h_t^{\star 1}(r+1) + w_t^2 h_t^{\star 2}(r+1) \leq (P_t^1(r) + \beta\xi(P_t^1) + P_t^2(r) + \beta\xi(P_t^2))W_t(r+1).$$

Rearranging, one obtains

$$\beta\Delta \frac{w_t^1 h_t^{\star 1}(r+1) + w_t^2 h_t^{\star 2}(r+1) - (P_t^1(r) + P_t^2(r))W_t(r+1)}{(\xi(P_t^1) + \xi(P_t^2))W_t(r+1)},$$

where $\Delta :=>$ if $(\xi(P_t^1)+\xi(P_t^2))W_t(r+1) > 0$ and $\Delta :=<$ otherwise. Further, we must also ensure that new personalized prices for children welfare are compatible with decreasing returns to scale given children expenditure:

$$w_t^1 h_t^{\star 1}(r+1) + w_t^2 h_t^{\star 2}(r+1) + c_t(r) \leq (P_t^1(r) + \beta\xi(P_t^1) + P_t^2(r) + \beta\xi(P_t^2))W_t(r+1).$$

Rearranging, one obtains

$$\beta\Delta \frac{w_t^1 h_t^{\star 1}(r+1) + w_t^2 h_t^{\star 2}(r+1) + c_t(r) - (P_t^1(r) + P_t^2(r))W_t(r+1)}{(\xi(P_t^1) + \xi(P_t^2))W_t(r+1)},$$

where $\Delta :=>$ if $(\xi(P_t^1) + \xi(P_t^2))W_t(r+1) > 0$ and $\Delta :=<$ otherwise.

Finally, starting with the new numbers $\lambda_t^i(r+1)$, $W_t(r+1)$, $l_t^{\star i}(r+1)$, and $h_t^{\star i}(r+1)$, we want new utilities $U_t^i(r+1)$ and new personalized prices $\mathcal{P}_t^i(r+1)$,

$P_t^i(r+1)$ to satisfy

$$
\begin{aligned}
&U_s^1(r) + \beta\xi(U_s^1) - U_t^1(r) - \beta\xi(U_t^1) \leq \\
&\qquad \lambda_t^1(r+1)\Big[w_t^1(l_s^{\star 1}(r+1) - l_t^{\star 1}(r+1)) + (q_s^1 - q_t^1) + \\
&\qquad + \big(\mathcal{P}_t^1(r) + \beta\xi(\mathcal{P}_t^1)\big)\big(Q_s - Q_t\big) + \big(P_t^1(r) + \beta\xi(P_t^1)\big)\big(W_s(r+1) - W_t(r+1)\big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
&U_s^2(r) + \beta\xi(U_s^2) - U_t^2(r) - \beta\xi(U_t^2) \leq \\
&\qquad \lambda_t^2(r+1)\Big[w_t^2(l_s^{\star 2}(r+1) - l_t^{\star 2}(r+1)) + (q_s^2 - q_t^2) + \\
&\qquad + \big(\mathcal{P}_t^2(r) + \beta\xi(\mathcal{P}_t^2)\big)\big(Q_s - Q_t\big) + \big(P_t^2(r) + \beta\xi(P_t^2)\big)\big(W_s(r+1) - W_t(r+1)\big)\Big].
\end{aligned}
$$

With some algebra, we can rewrite the inequalities for household member 1 as

$$
\begin{aligned}
&\beta\Big[\xi(U_s^1) - \xi(U_t^1) + \lambda_t^1(r+1)\big[\xi(\mathcal{P}_t^1)\big(Q_t - Q_s\big) + \xi(P_t^1)\big(W_t(r+1) - W_s(r+1)\big)\big]\Big] \\
&\qquad \leq U_t^1(r) - U_s^1(r) + \lambda_t^1(r+1)\Big[w_t^1(l_s^{\star 1}(r+1) - l_t^{\star 1}(r+1)) + (q_s^1 - q_t^1) + \\
&\qquad + \mathcal{P}_t^1(r)\big(Q_s - Q_t\big) + P_t^1(r)\big(W_s(r+1) - W_t(r+1)\big)\Big],
\end{aligned}
$$

and the inequalities for household member 2 as

$$
\begin{aligned}
&\beta\Big[\xi(U_s^2) - \xi(U_t^2) + \lambda_t^2(r+1)\big[\xi(\mathcal{P}_t^2)\big(Q_t - Q_s\big) + \xi(P_t^2)\big(W_t(r+1) - W_s(r+1)\big)\big]\Big] \\
&\qquad \leq U_t^2(r) - U_s^2(r) + \lambda_t^2(r+1)\Big[w_t^2(l_s^{\star 2}(r+1) - l_t^{\star 2}(r+1)) + (q_s^2 - q_t^2) + \\
&\qquad + \mathcal{P}_t^2(r)\big(Q_s - Q_t\big) + P_t^2(r)\big(W_s(r+1) - W_t(r+1)\big)\Big].
\end{aligned}
$$

For convenience, let

$$
\begin{aligned}
num^1 &:= U_t^1(r) - U_s^1(r) + \lambda_t^1(r+1)\Big[w_t^1(l_s^{\star 1}(r+1) - l_t^{\star 1}(r+1)) + (q_s^1 - q_t^1) + \\
&\quad + \mathcal{P}_t^1(r)\big(Q_s - Q_t\big) + P_t^1(r)\big(W_s(r+1) - W_t(r+1)\big)\Big] \\
denom^1 &:= \xi(U_s^1) - \xi(U_t^1) + \\
&\quad + \lambda_t^1(r+1)\big[\xi(\mathcal{P}_t^1)\big(Q_t - Q_s\big) + \xi(P_t^1)\big(W_t(r+1) - W_s(r+1)\big)\big]
\end{aligned}
$$

and

$$\begin{aligned}
num^2 &:= U_t^2(r) - U_s^2(r) + \lambda_t^2(r+1)\Big[w_t^2(l_s^{\star 2}(r+1) - l_t^{\star 2}(r+1)) + (q_s^2 - q_t^2) + \\
&\quad + \mathcal{P}_t^2(r)\big(Q_s - Q_t\big) + P_t^2(r)\big(W_s(r+1) - W_t(r+1)\big)\Big] \\
denom^2 &:= \xi(U_s^2) - \xi(U_t^2) + \\
&\quad + \lambda_t^2(r+1)\big[\xi(\mathcal{P}_t^2)\big(Q_t - Q_s\big) + \xi(P_t^2)\big(W_t(r+1) - W_s(r+1)\big)\big].
\end{aligned}$$

Therefore, we have

$$\beta \Delta \frac{num^i}{denom^i},$$

where $\Delta :=<$ if $denom^i > 0$ and $\Delta :=>$ otherwise. We obtain $(U_t^i(r+1), \mathcal{P}_t^i(r+1), P_t^i(r+1))_{t \in \mathcal{T}}$ by picking β uniformly over its support defined by the greatest lower bound and least upper bound derived from the above inequalities.

Step 4: Children Expenditure

We need to pick new true children expenditure that is positive such that

$$c_t^\star(r) + \kappa \xi(c_t^\star) > 0. \tag{18}$$

In addition, new children expenditure must yield decreasing returns to scale such that

$$w_t^1 h_t^{\star 1}(r+1) + w_t^2 h_t^{\star 2}(r+1) + c_t^\star(r) + \kappa \xi(c_t^\star) \leq P_t(r+1) W_t(r+1).$$

Rearranging, one gets

$$\kappa \Delta \frac{P_t(r+1)W_t(r+1) - w_t^1 h_t^{\star 1}(r+1) - w_t^2 h_t^{\star 2}(r+1) - c_t^\star(r)}{\xi(c_t^\star)},$$

where $\Delta :=<$ if $\xi(c_t^\star) > 0$ and $\Delta :=>$ otherwise. We obtain $c_t^\star(r+1)$ by picking κ uniformly over its defined by the greatest lower bound and least upper bound derived from the above inequalities.

Step 5: Production Parameters and Productivity

We are left with the task to recover output elasticities and productivity shocks. This requires no work as they are directly deduced from the first-order conditions and production function equation:

$$
\begin{aligned}
\alpha_{1,t}(r+1) &= \frac{w_t^1 h_t^{\star 1}(r+1)}{P_t(r+1)W_t(r+1)} \\
\alpha_{2,t}(r+1) &= \frac{w_t^2 h_t^{\star 2}(r+1)}{P_t(r+1)W_t(r+1)} \\
\alpha_{3,t}(r+1) &= \frac{c_t^\star(r+1)}{P_t(r+1)W_t(r+1)} \\
\epsilon_t(r+1) &= \log(W_t(r+1)) - \alpha_{1,t}(r+1)\log\big(h_t^{\star 1}(r+1)\big) \\
&\quad - \alpha_{2,t}(r+1)\log\big(h_t^{\star 2}(r+1)\big) - \alpha_{3,t}(r+1)\log(c_t^\star).
\end{aligned}
$$

This last step of the Gibbs sampler combined with the previous steps give a completely new solution to the model.

Sampling from the Feasible Space in 5 Easy Steps

Suppose an initial solution $r = 0$ to the household problem is given (e.g., by solving a mixed-integer program). Then,

1. Given r, get $(\lambda_t^i(r+1))_{t\in\mathcal{T}}$ as outlined in Step 1.
2. Given 1, get $(W_t(r+1), l_t^{\star i}(r+1), h_t^{\star i}(r+1))_{t\in\mathcal{T}}$ as outlined in Step 2.
3. Given 1-2, get $(U_t^i(r+1), \mathcal{P}_t^i(r+1), P_t^i(r+1))_{t\in\mathcal{T}}$ as outlined in Step 3.
4. Given 1-3, get $(c_t^\star(r+1))_{t\in\mathcal{T}}$ as outlined in Step 4.
5. Given 1-4, get $(\boldsymbol{\alpha}_t(r+1), \epsilon_t(r+1))_{t\in\mathcal{T}}$ as outlined in Step 5.
6. Set $r = r+1$ and repeat 1-5 until $r = R > 0$.

C.1 Miscellaneous

This subsection provides additional details about the sampling procedure.

Target Distribution

We ensure that the sampling procedure yields the desired least favorable distribution on measurement error by using a Metropolis-Hastings algorithm. Once a complete new solution is obtained from the Gibbs sampler, update the Markov chain with the appropriate acceptance ratio. Note that the acceptance ratio depends on the target distribution. In our application, the target distribution is proportional to a normal distribution:

$$d\tilde{\eta}(\cdot|x_i) \propto \exp\left(-||\boldsymbol{g}_i^{\omega}(x_i, e_i)||^2\right).$$

As pointed out by Schennach (2014), under mild regularity conditions the mean and variance of the distribution are inconsequential for the validity of by Proposition 3.

Length of the Monte Carlo Markov Chain

The Gibbs sampler generates a Markov Chain that suffers from autocorrelation. For this reason, it is good practice to only keep a subset of the R solutions, a technique known as thinning. In our application, we keep 5% of all solutions. Also, the theory of stochastic processes tells us that convergence to the stationary distribution may take some time —its existence follows by construction of the Metropolis-Hastings algorithm. Accordingly, it is good practice to leave out the first few solutions. In our application, we leave out the first 100000 solutions. We then draw another 100000 solutions from the feasible space.

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