

Production Heterogeneity in Collective Labor Supply Models with Children

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Abstract

Children welfare is at the center of many welfare reforms such as cash transfers to families and training programs to parents. A key goal for policy-makers is to evaluate the costs and benefits of such reforms. The main challenge lies in that the outcome of interest, children welfare, is unobservable. To address this issue, we consider a collective labor supply model with children where adult members have preferences over their own leisure, expenditures, and children welfare. We show that the model nonparametrically partially identifies the impacts of parental inputs on children welfare in panel data. We then propose a novel estimation strategy that accommodates measurement error and can be used to efficiently construct valid confidence sets. Using Dutch data on couples with children, we investigate the structure of the expected production technology and how it varies with household characteristics. We find that the production of children welfare is characterized by decreasing returns to scale and large heterogeneity across household types. In particular, we find that children from disadvantaged households, whose parents have low education levels and are not homeowners, are significantly worse off. Our results highlight the importance for welfare reforms to include policies targeted at improving children home environment.

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1 Introduction

It is widely recognized that the unitary model, which assumes that household members preferences can be represented by a single household utility function, is inappropriate for

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analyzing household data (see e.g., [Fortin and Lacroix \(1997\)](#) and [Browning and Chiappori \(1998\)](#)). In an effort to provide a proper foundation to analyze household behavior, [Chiappori \(1988, 1992\)](#) suggested a collective model in which household members have distinct preferences and whose allocations are the result of a Pareto efficient bargaining process. This framework has proven to be empirically successful at rationalizing household decision-making (e.g., [Cherchye and Vermeulen, 2008](#)) and understanding power dynamics within the household (e.g., [Cherchye, De Rock and Vermeulen, 2011](#)).

The collective model was later extended to household production by [Apps and Rees \(1997\)](#) and [Chiappori \(1997\)](#).¹ Both papers show that the distribution of resources within the household, known as the sharing rule, cannot be identified when the output from production is unobservable such as with children welfare. However, they show that the sharing rule can be identified up to a function of wages when the production technology exhibits constant returns to scale. Perhaps because of the serious identification problem that arises with home production, the literature has since maintained the constant returns to scale assumption ([Blundell, Chiappori and Meghir, 2005](#); [Chiappori and Ekeland, 2009](#); [Cherchye, De Rock and Vermeulen, 2012](#); [Hubner, 2023](#)).

Once the household is recognized as a collection of individuals, it becomes possible to understand the impacts of a policy targeted at a specific individual. For example, the distribution of resources within the household has been shown to have significant impacts on children welfare as early as [Thomas \(1990\)](#). Motivated by this “targeted” view, [Blundell, Chiappori and Meghir \(2005\)](#) extended the collective framework to caring parents where children welfare is treated as a public good in parents preferences and produced via time investment and children expenditure. Their ideas were then brought to the data by [Cherchye, De Rock and Vermeulen \(2012\)](#) using a unique panel data set containing information on time use and expenditures.

As [Apps and Rees \(1997\)](#) acknowledged, however, there is a need to verify the empirical validity of the assumptions required for identification. Indeed, it is now well-recognized that children welfare is crucial in weighing the costs and benefits of social programs.² For example, policy-makers may be interested in determining optimal cash assistance for single mothers such as to increase children skills development ([Mullins, 2022](#)), cash transfers to poor families such as to improve children life outcomes ([Aizer et al., 2016](#)), and parenting programs for disadvantaged households such as to narrow children achievement gaps ([Waldfogel and Washbrook, 2011](#)).³ In our framework, CRS would suggest that households are equally effective at developing children skills and

¹In a similar model to the one of [Chiappori \(1988\)](#), [Apps and Rees \(1988\)](#) already include home production to analyze the effects of taxation on welfare.

²See [Aizer, Hoynes and Lleras-Muney \(2022\)](#) for a review of social programs in the United States and the importance of considering children welfare.

³See [Kalil \(2014\)](#) for a review of the literature on the importance of parenting on children development and [Shah and Gennetian \(2024\)](#) for a recent survey of the literature on the impacts of cash transfers on families with children.

could overestimate the benefits of cash transfers.

The empirical literature has largely investigated these questions by using measurable quantities such as cognitive and noncognitive skills of children.⁴ Although this approach has been fruitful, it requires rich data sets whose measure of skill is often noisy (Cunha and Heckman, 2008; Cunha, Heckman and Schennach, 2010). In contrast, the collective approach proposed by Blundell, Chiappori and Meghir (2005) leaves children welfare as an unobservable, thus facilitating policy evaluations. We believe the collective model is a natural framework that can complement costs and benefits analyses of programs directed at families, but we are also sympathetic to Apps and Rees (1997)’s reservations regarding conditions for identification.

Our first contribution is to address the plausibility of CRS within the collective model with children. Consistent with previous work, we show that the production function is nonparametrically identified under CRS in cross-sectional data, and non-identified when CRS is relaxed. Nevertheless, we show that the production function can be nonparametrically partially identified in panel data when it exhibits decreasing returns to scale, even with unrestricted heterogeneity. Indeed, the panel structure of the data gives additional implications to shape constraints on the production technology and preferences. Since household members have preferences over children welfare, those constraints provide restrictions on the production function.

Our partial identification argument uses nonparametric revealed preference conditions implied by the collective labor supply model with children.⁵ The revealed preference analysis of the collective model was developed in the original work of Cherchye, De Rock and Vermeulen (2007, 2011). We extend their former characterization by further incorporating household production.⁶ Interestingly, we show that the collective model implies profit maximizing behavior in the production of children welfare, thus giving rise to a two-step argument. In a first step, we show that restrictions on the distribution of returns to scale eliminate production functions from the set of profit-maximizing production functions. In a second step, we show that revealed preference conditions can restrict the distribution of returns to scale.

Our second contribution is to propose a novel estimation strategy to analyze the collective model with unrestricted preference and production heterogeneity. To this end, we use the framework developed by Aguiar and Kashaev (2021) which provides a tractable approach to make statistical testing and inference in partially identified models defined by shape constraints.⁷ The main challenge faced in applying their framework is that

⁴See Caucutt et al. (2020) for a recent example that does not requires data on skills.

⁵See also Dunbar, Lewbel and Pendakur (2013) for a collective model with children that does not require the share of resources allocated to children to be known.

⁶See Varian (1984) for early work on a revealed preference analysis of production.

⁷Their framework builds on the Entropic Latent Variable Integration via Simulation (ELVIS) methodology developed by Schennach (2014). Intuitively, ELVIS can be viewed as a generalization of the method of simulated moments (McFadden, 1989; Pakes and Pollard, 1989).

collective models tend to be highly nonlinear. This poses a nontrivial computational problem as existing implementations only work well for models defined by linear constraints.⁸ The reason is that the method involves integrating out the set of solutions of the model, but nonlinearities make it hard to sample from the feasible space.

We solve this practical limitation by proposing a blocked Gibbs sampler that allows direct sampling from the feasible space even when the latter does not define a polytope.⁹ Intuitively, the idea is to break down the feasible space into conditional convex polytopes for which we can obtain closed-form bounds on the support of the latent variables. We observe that our methodology may prove useful for other collective models (Cherchye, Demuynck and De Rock, 2011, d’Aspremont and Dos Santos Ferreira, 2019, Cherchye et al., 2020) and, more generally, other models that contain nonlinear constraints.

Our third contribution is to empirically investigate the impacts of parental inputs on children welfare in households with children. To do so, we use the Longitudinal Internet Studies for the Social sciences (LISS) panel data from Cherchye, De Rock and Vermeulen (2012). We first investigate the average impacts of parental inputs on children welfare. We find that a doubling of all inputs increases children welfare by about 35% on average, hence providing empirical evidence against the CRS assumption. More precisely, we find that a 10% increase in time spent on childcare increases children welfare by about 1.1% for fathers and 1.13% for mothers, while a 10% increase in children expenditure increases children welfare by about 1.0%.

Second, we investigate how household characteristics impacts the average production technology. We find that higher education levels increase the impacts of parental time inputs on children welfare by about 6% for mothers and about 2% for fathers. Further, we find that the education level of the mother has a positive spillover effect of about 3% on the impacts of time inputs by fathers. Interestingly, for households that are not homeowners, the impacts of parental time inputs is about 7% and 11% lower for mothers and fathers, respectively. Taken together, our results show that children from disadvantaged households, whose parents often have lower education levels and are not homeowners, are significantly worse off.

The approach taken in this paper relates to the research program initiated by Blundell, Browning and Crawford (2003, 2007, 2008). In this series of papers, the authors show how to combine revealed preference restrictions with additional information (e.g., expansion paths) to improve bounds on cost-of-living indices and demand responses. We

⁸For example, Aguiar and Kashaev (2021) consider the collective exponential discounting model of Adams et al. (2014) but only test necessary conditions to simplify the implementation. Gauthier (2023) considers a model of price search, but assumes a quasilinear specification in the application that alleviates the computational burden.

⁹Direct sampling generally uses a Hit-and-Run algorithm that requires the feasible space to define a (convex) polytope. See Aguiar and Kashaev (2021) for an application to models defined by shape constraints and Demuynck (2021) for an application to models defined by the Generalized Axiom of Revealed Preference (GARP) as introduced by Varian (1982).

also exploit revealed preference restrictions, but use them to learn about the production of children welfare within a collective model. We take advantage of recent developments in the partial identification literature to impose these restrictions in a cohesive statistical framework. Moreover, our method allows for unrestricted individual heterogeneity, an important feature for our application.

Our econometric framework views individuals within the household as random draws from a fixed utility distribution. Given the panel structure of the data, this amounts to a random utility model where a household preferences are drawn in the first period and kept constant. Hence, the approach relates to the burgeoning nonparametric random utility literature started by [Kitamura and Stoye \(2018\)](#).¹⁰ Contrary to our method, their approach considers a unitary model but only requires data on cross-sectional distributions of choices. Subsequent work include [Hubner \(2020\)](#), [Deb et al. \(2023\)](#), [Kashaev et al. \(2023\)](#), [Lazzati, Quah and Shirai \(2023\)](#), and [Tebaldi, Torgovitsky and Yang \(2023\)](#).

The paper is organized as follows. Section 2 describes our collective model and characterizes its implications. Section 3 studies identification in the model. Section 4 presents the empirical specification. Section 5 presents the estimation strategy. Section 6 presents the data set used in the application. Section 7 presents the empirical results. Section 8 concludes. The Appendix contains proofs that are not in the main text and our Gibbs sampler.

2 Household Model

This section presents the environment considered in the paper, the collective model, and its empirical implications.

2.1 Environment

We consider households with two adults ($i = 1, 2$) and children. We assume that parents care about their children and incorporate this feature in the model by treating children welfare as a public good. The preferences of each adult household member are represented by a utility function U^i that is continuous, increasing, and concave.

At every observation $t \in \mathcal{T} = \{1, 2, \dots, T\}$, adult household members spend their time on leisure l_t^i , market work b_t^i , and childcare h_t^i such that the following time constraint is satisfied:

$$l_t^i + b_t^i + h_t^i = \tau,$$

where τ is the total amount of time available in a time period. Parents use time spent on childcare and children expenditure (c_t) to produce children welfare. The relation-

¹⁰The theoretical ideas were put forth by [McFadden and Richter \(1990\)](#) and [McFadden \(2005\)](#).

ship between parents inputs and children welfare is formalized through the production function

$$W_t \equiv F(h_t^1, h_t^2, c_t)e^{\epsilon_t},$$

where $\epsilon_t \in \mathbb{R}$ represents a productivity shock. Each household member receives a wage w_t^i per unit of market work. As such, the budget constraint is given by

$$q_t + Q_t + c_t = y_t + w_t^1 b_t^1 + w_t^2 b_t^2,$$

where $q_t \in \mathbb{R}_+$ represents expenditure on private goods, $Q_t \in \mathbb{R}_+$ represents expenditure on public goods, and $y_t > 0$ represents nonlabor income.

Since private expenditure cannot be used simultaneously by both household members, it has to be split in some way between them.

Definition 1. Let D be a data set. For every observation $t \in \mathcal{T}$, we say that $q_t^i \in \mathbb{R}_+$, $i \in \{1, 2\}$, represent personalized private expenditures of each household member if $\sum_{i=1}^2 q_t^i = q_t$.

Household members get utility from their share of private expenditure such that their preferences depend on leisure, private expenditure, public expenditure, and children welfare. In what follows, we assume that private expenditure of each household member is observed to match the data available in the application. However, our results can be generalized to the case where only total private expenditure is observed.

Let \mathcal{U} be the set of continuous, increasing, and concave utility functions and \mathcal{W} be the set of continuous, increasing, and concave in (h_t^1, h_t^2, c_t) production functions. A household $j \in \mathcal{J}$ is an i.i.d. draw (U_j^1, U_j^2, W_j) from \mathcal{W} and a data set $D_j := \{(q_{jt}^i, Q_{jt}, c_{jt}, b_{jt}^i, h_{jt}^i, w_{jt}^i)_{i=1}^2\}_{t \in \mathcal{T}}$ is an i.i.d. draw from some distribution. To avoid overcrowding, we do not explicitly write the household subscript j on variables unless it is relevant. The next subsection formalizes the relationship between the data and the abstract notion of household through the lenses of a model.

2.2 Collective Model

We follow [Chiappori \(1988, 1992\)](#) and assume that household members choose an intrahousehold allocation that is Pareto efficient. This choice is motivated by the observation that Pareto efficiency is a minimal condition for optimal resource allocation (and hence, rationality) in a group setting. Hence, for every observation $t \in \mathcal{T}$, the household picks an intrahousehold allocation that solves

$$\max_{(l^1, l^2, h^1, h^2, q^1, q^2, Q, c) \in \mathbb{R}_+^2 \times \mathbb{R}_{++}^2 \times \mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathbb{R}_{++}} \mu_t^1 U^1(l^1, q^1, Q, W_t) + \mu_t^2 U^2(l^2, q^2, Q, W_t), \quad (1)$$

subject to satisfying the constraints

$$(q^1 + q^2) + Q + c = y_t + w_t^1 b^1 + w_t^2 b^2$$

$$l^i + b^i + h^i = \tau \quad (i = 1, 2),$$

where $\mu_t^i > 0$ denote the bargaining power of household member i . Note that the model makes no assumption on the underlying process by which the Pareto efficient allocation is achieved. That is, the weights μ_t^i result from some black box bargaining process that takes place within the household.¹¹

We propose a natural notion of collective rationalizability based on the household maximization problem.

Definition 2. Let D be a data set. The model (1) rationalizes the data if there exist concave utility functions U^i , a concave production function F , and productivity shocks ϵ_t such that the first-order conditions of the problem are satisfied.

This definition states that the model rationalizes the data if there are latent model parameters that satisfy the first-order conditions. Since household members utility functions are concave and the budget set is linear, the first-order conditions exhaust the empirical content of the model.

2.3 Characterization

This section derives restrictions on the data implied by the model. First, we define a few notions that will be useful for the characterization of the model.

Definition 3. Let D be a data set. For every observation $t \in \mathcal{T}$, we say that $\mathcal{P}_t^i \in \mathbb{R}_{++}$, $i \in \{1, 2\}$, represent personalized (or Lindahl) prices for public expenditure of each household member if $\sum_{i=1}^2 \mathcal{P}_t^i = 1$.

Definition 4. Let D be a data set. For every observation $t \in \mathcal{T}$, we say that $P_t^i \in \mathbb{R}_{++}$, $i \in \{1, 2\}$, represent personalized (or Lindahl) prices for children welfare of each household member if $\sum_{i=1}^2 P_t^i = P_t$.

It is worth noting that the personalized prices (\mathcal{P}_t^i, P_t^i) are not observed by the econometrician. Furthermore, while the price of public expenditure (\mathcal{P}_t) can safely be set to 1, the price of children welfare (P_t) is unobservable as children welfare is a nonmarket good.

We now introduce some revealed preference terminology. Let $a_{st}^i := w_t^i(l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i(Q_s - Q_t) + P_t^i(W_s - W_t)$ and $x_t^i := (l_t^i, q_t^i, Q_t, W_t)$. We say that x_t^i is

¹¹Although Nash equilibria are not always Pareto efficient, the black box bargaining process could be a (Pareto efficient) Nash equilibrium. Indeed, since married couples effectively play a repeated game, an appeal to folk theorems provide some intuitive motivation for the idea that the Pareto efficient allocation is a (cooperative) Nash equilibrium.

(strictly) directly revealed preferred to x_s^i if $a_{st}^i (<) \leq 0$. We say that x_t^i is revealed preferred to x_s^i if there exists a sequence t_1, t_2, \dots, t_m such that $a_{t_1 t}^i \leq 0, a_{t_2 t_1}^i \leq 0, \dots, a_{t_{m-1} t_m}^i, a_{t_m s}^i \leq 0$. Likewise, we say that x_t^i is strictly revealed preferred to x_s^i if one of the inequalities in the sequence is strict.

Definition 5. A household member $i \in \{1, 2\}$ satisfies the Generalized Axiom of Revealed Preference (GARP) if there exist personalized prices for public expenditure \mathcal{P}_t^i , personalized prices for children welfare P_t^i , and children welfare W_t such that if x_t^i is revealed preferred to x_s^i then x_s^i is not strictly directly revealed preferred to x_t^i .

The notion of revealed preference relates the ordinal value of allocations that enter preferences of each household member to their expenditure levels. In our setup, the presence of a public good (Q) implies that the expenditure of an allocation depends on unknown personalized prices. Further, in the case of the public nonmarket good (W) neither the price or the quantity is known. Finally, it is worth observing that childcare and children expenditure do not enter the definition of revealed preference as the preferences of a household member only depends on those through their impact on children welfare.

Next, we introduce a profit maximization condition for the production of children welfare.

Definition 6. A household satisfies profit maximization if there exist personalized prices for children welfare P_t , a production function F , and productivity shocks ϵ_t such that $P_t F(h_t^1, h_t^2, c_t) e^{\epsilon_t} - w_t^1 h_t^1 - w_t^2 h_t^2 - c_t \geq P_t F(h^1, h^2, c) e^{\epsilon_t} - w_t^1 h^1 - w_t^2 h^2 - c$ for all inputs (h^1, h^2, c) and all $t \in \mathcal{T}$.

In what follows, we make use of an analogous condition as GARP but for profit maximization.

Definition 7. A household satisfies the Generalized Axiom of Profit Maximization (GAPM) if there exist personalized prices for children welfare $P_t > 0$, numbers $F_t > 0$, and productivity shocks ϵ_t such that $P_t F_t e^{\epsilon_t} - w_t^1 h_t^1 - w_t^2 h_t^2 - c_t \geq P_t F_s e^{\epsilon_t} - w_t^1 h_s^1 - w_t^2 h_s^2 - c_s$ for all $s, t \in \mathcal{T}$.

It is possible to show that profit maximization is equivalent to GAPM (Varian, 1984). This equivalence is standard and we therefore take it for granted for the sake of brevity. The following result provides equivalent characterizations of the model.

Theorem 1. *Let D be a given data set. The following conditions are equivalent:*

- (i) *The household model (1) rationalizes the data.*
- (ii) *There exist personalized prices for public expenditure $\mathcal{P}_t^i > 0$ such that $\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1$, personalized prices for children welfare $P_t^i > 0$, numbers $U^i, \lambda_t^i, W_t, F_t > 0$ and*

productivity shocks ϵ_t such that for all $s, t \in \mathcal{T}$ and each adult member $i \in \{1, 2\}$

$$U_s^i - U_t^i \leq \lambda_t^i [w_t^i(l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i(Q_s - Q_t) + P_t^i(W_s - W_t)],$$

$$F_s - F_t \leq \frac{w_t^1}{P_t e^{\epsilon_t}}(h_s^1 - h_t^1) + \frac{w_t^2}{P_t e^{\epsilon_t}}(h_s^2 - h_t^2) + \frac{1}{P_t e^{\epsilon_t}}(c_s - c_t),$$

where $W_t = F_t e^{\epsilon_t}$ for all $t \in \mathcal{T}$.

- (iii) There exist personalized prices for public expenditure $\mathcal{P}_t^i > 0$ such that $\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1$, personalized prices for children welfare $P_t^i > 0$, numbers $F_t > 0$, and productivity shocks ϵ_t with $W_t = F_t e^{\epsilon_t}$, such that GARP holds for each adult member $i \in \{1, 2\}$ and GAPM holds.

Theorem 1 shows that the Afriat inequalities are equivalent to GARP and that those conditions must be satisfied for both household members. The latter implies that the household problem has an equivalent characterization in terms of a two-step procedure (Chiappori, 1988, 1992). That is, the solution of the household maximization problem can be viewed as the outcome of separate utility maximization problems for each adult in the household conditional on a distribution of nonlabor income.

It is interesting to note that neither the Afriat inequalities or GARP exhaust the empirical implications of the model. Indeed, the model further implies that the household satisfies GAPM, a necessary and sufficient condition for profit maximization. As such, household members increase each input in the production of children welfare up until the point where marginal revenue equates marginal cost. Note that this profit maximizing behavior is not assumed but implied by the model.

3 Empirical Content

This section shows that the collective model informatively partially identifies the production function. Intuitively, if the production function exhibited constant returns to scale, the household would make zero profit as a firm and revenue $P_t W_t$ would equate costs $w_t^1 h_t^1 + w_t^2 h_t^2 + c_t$. In this special case, revenue would be identified and the first-order conditions would recover the production function from its partial derivatives. We show that the household revenue from producing children welfare is inversely proportional to its costs when the production function is homogeneous, where the factor of proportionality is given by its return to scale. We then leverage the panel structure of the data to bound returns to scale from shape constraints on the production function and preferences.

3.1 Point Identification

In what follows, we explicitly write the household subscript $j = 1, 2, \dots, J$ on variables and formalize the assumption that production functions are subject to Hicks-neutral productivity shocks.

Assumption 1. *The productivity shocks are Hicks-neutral such that children welfare is given by*

$$W_{jt} = F_j(h_{jt}^1, h_{jt}^2, c_{jt})e^{\epsilon_{jt}} \iff \log(W_{jt}) = f_j(h_{jt}^1, h_{jt}^2, c_{jt}) + \epsilon_{jt},$$

where f_j denote the natural logarithm of the production function and is assumed differentiable.

The assumption of Hicks-neutral productivity shocks is necessary to disentangle the impacts of productivity shocks and parental inputs on the production of children welfare. The differentiability of the log production function is a technical condition that is useful for the identification argument.

Next, we suppose the production function is from the class of homogeneous production functions.

Assumption 2. *The production function is homogeneous of degree $RTS_j \in (0, 1]$.*

The usefulness of the homogeneity assumption is motivated by the following result.

Lemma 1. *Suppose Assumptions 1-2 hold, then $RTS_j P_{jt} W_{jt} = w_{jt}^1 h_{jt}^1 + w_{jt}^2 h_{jt}^2 + c_{jt}$ for all $t \in \mathcal{T}$.*

Lemma 1 implies that the identification problem can be stated in terms of restrictions on RTS_j rather than restrictions on $P_{jt} W_{jt}$ directly. This is useful because we may have better economic intuition on returns to scale exhibited by the production function than on the value of children welfare. For example, the literature typically assumes that the production functions exhibits constant returns to scale (CRS), a special case of our class of production functions.

Finally, we impose a mild regularity condition that ensures sufficient variation in inputs in the cross-section.

Assumption 3. *The cross-sectional distribution of inputs $(h_{jt}^1, h_{jt}^2, c_{jt})_{t \in \mathcal{T}}$ has full support and is absolutely continuous.*

The requirement that the distribution of inputs spans its full support is necessary to identify the whole production function. For practical purposes, it is generally sufficient to identify the production function over the support of the data. In that case, the full support condition can be relaxed without any harm.

Our first result shows that, if returns to scale were known, the expected log production function would be identified.

Proposition 1. *Suppose Assumptions 1-3 hold and RTS_j is known, then the expected log production function is nonparametrically identified up to scale.*

Proof. The first-order conditions with respect to inputs imply that the household equates the marginal product of factors of production to their marginal costs such that

$$\begin{aligned}\frac{\partial F_j(h_{jt}^1, h_{jt}^2, c_{jt})}{\partial h_{jt}^1} e^{\epsilon_{jt}} &= \frac{w_{jt}^1}{P_t} \\ \frac{\partial F_j(h_{jt}^1, h_{jt}^2, c_{jt})}{\partial h_{jt}^2} e^{\epsilon_{jt}} &= \frac{w_{jt}^2}{P_{jt}} \\ \frac{\partial F_j(h_{jt}^1, h_{jt}^2, c_{jt})}{\partial c_{jt}^2} e^{\epsilon_{jt}} &= \frac{1}{P_{jt}}.\end{aligned}$$

Divide the marginal products by W_{jt} to obtain

$$\begin{aligned}\frac{\partial f_j(h_{jt}^1, h_{jt}^2, c_{jt})}{\partial h_{jt}^1} &= \frac{w_{jt}^1}{P_{jt}W_{jt}} \\ \frac{\partial f_j(h_{jt}^1, h_{jt}^2, c_{jt})}{\partial h_{jt}^2} &= \frac{w_{jt}^2}{P_{jt}W_{jt}} \\ \frac{\partial f_j(h_{jt}^1, h_{jt}^2, c_{jt})}{\partial c_{jt}^2} &= \frac{1}{P_{jt}W_{jt}},\end{aligned}$$

where f_j denote the log production function. Taking the expectation gives

$$\begin{aligned}\frac{\partial \mathbb{E}[f(h_t^1, h_t^2, c_t)]}{\partial h_t^1} &= \mathbb{E}\left[\frac{w_t^1}{P_t W_t}\right] \\ \frac{\partial \mathbb{E}[f(h_t^1, h_t^2, c_t)]}{\partial h_t^2} &= \mathbb{E}\left[\frac{w_t^2}{P_t W_t}\right] \\ \frac{\partial \mathbb{E}[f(h_t^1, h_t^2, c_t)]}{\partial c_t^2} &= \mathbb{E}\left[\frac{1}{P_t W_t}\right],\end{aligned}$$

where we interchanged the partial derivative and integral.¹² By Lemma 1, RTS_j identifies $P_{jt}W_{jt}$ such that the expected marginal products are also identified. Next, variation in inputs in the cross-section allows us to integrate each marginal product, giving the

¹²This is possible if f_j is dominated by a function whose integral is finite, a natural assumption for a production function.

following system of partial differential equations

$$\begin{aligned}\int_{h_0^1}^{h_t^1} \frac{\partial \mathbb{E}[f(h_t^1, h_t^2, c_t)]}{\partial h_t^1} dh_t^1 &= \mathbb{E}[f(h_t^1, h_t^2, c_t)] + C(h_t^2, c_t) \\ \int_{h_0^2}^{h_t^2} \frac{\partial \mathbb{E}[f(h_t^1, h_t^2, c_t)]}{\partial h_t^2} dh_t^2 &= \mathbb{E}[f(h_t^1, h_t^2, c_t)] + C(h_t^1, c_t) \\ \int_{c_0}^{c_t} \frac{\partial \mathbb{E}[f(h_t^1, h_t^2, c_t)]}{\partial c_t} dc_t &= \mathbb{E}[f(h_t^1, h_t^2, c_t)] + C(h_t^1, h_t^2).\end{aligned}$$

These equations can be used to recover the expected log production function up to a constant:

$$\begin{aligned}\mathbb{E}[f(h_t^1, h_t^2, c_t)] &= \int_{h_0^1}^{h_t^1} \frac{\partial \mathbb{E}[f(h^1, h_0^2, c_0)]}{\partial h_t^1} dh^1 + \int_{h_0^2}^{h_t^2} \frac{\partial \mathbb{E}[f(h_t^1, h^2, c_0)]}{\partial h_t^2} dh^2 + \\ &\quad + \int_{c_0}^{c_t} \frac{\partial \mathbb{E}[f(h_t^1, h_t^2, c)]}{\partial c_t} dc - C,\end{aligned}$$

where C is a constant of integration. Hence, the expected log production function is identified over the support of the data. \square

Proposition 1 states that, in principle, the model imposes enough structure to non-parametrically identify the expected log production function provided returns to scale are known. In particular, the identification strategy could be used to nonparametrically estimate the expected log production function over the support of the data under constant returns to scale. Furthermore, observe that our previous argument would identify the production function if it were identical across households as we could simply take the exponential function of the log production function. Instead, with heterogeneity we obtain a lower bound on the expected production function by Jensen's inequality. Interestingly, note that identification does not require knowledge of children welfare per se. This is because any scaling of children welfare is offset by a rescaling of children welfare prices, thus leaving revenue $P_{jt}W_{jt}$ unchanged.¹³

3.2 Partial Identification

The previous section showed that the expected log production function depends both on the data and the distribution of returns to scale. This section shows that revealed preference conditions provide an additional source of identification absent an assumption on returns to scale.

Let $\zeta_j := \{(\mathcal{P}_{jt}^i, P_{jt}^i, W_{jt}, F_{jt}, RTS_j, \epsilon_{jt})_{i \in \{1,2\}}\}_{t \in \mathcal{T}} \in Z|\mathcal{D}$ denote the set of household-specific latent variables that enter in the definition of GAPM and GARP, where Z denote

¹³This mechanism has a natural economic interpretation. Namely, it captures the idea of scarcity whereby the value of a good decreases with its abundance.

the support of the latent variables and \mathcal{D} denote the support of the data. Further, let $\mathcal{P}_{Z|\mathcal{D}}$ denote the set of distributions of the latent variables conditional on the data and π_0 denote the distribution of the data. Our previous discussion motivates the following definition of the identified set:

$$\Theta_0 = \left\{ \mathbb{E}[f(h_t^1, h_t^2, c_t)] : \exists \mu \in \mathcal{P}_{Z|\mathcal{D}} \text{ such that } D_j \text{ is rationalized by the model for all } j \text{ and Assumptions 1-3 hold} \right\}.$$

In words, the identified set corresponds to the set of expected log production functions that arises for every distribution of the unobservables that rationalizes the data. Observe that Proposition 1 implies that each distribution of returns to scale maps to an expected log production function given the data. Hence, the identified set rules out some expected log production functions provided the model restricts the expected return to scale to a strict subset of its support. The following result shows that GAPM (nontrivially) partially identifies the expected log production function.

Proposition 2. *Suppose Assumptions 1-3 hold, then GAPM is refutable and the identified set may be nontrivial.*

Proof. GAPM implies that for all $s, t \in \mathcal{T}$, the following inequality holds

$$F_{js} - F_{jt} \leq \frac{1}{P_{jt}e^{\epsilon_{jt}}} \left[w_{jt}^1(h_{js}^1 - h_{jt}^1) + w_{jt}^2(h_{js}^1 - h_{jt}^2) + (c_{js} - c_{jt}) \right].$$

Dividing by F_{jt} on both sides, we obtain

$$\frac{F_{js}}{F_{jt}} \leq 1 + \frac{1}{P_{jt}W_{jt}} \left[w_{jt}^1(h_{js}^1 - h_{jt}^1) + w_{jt}^2(h_{js}^1 - h_{jt}^2) + (c_{js} - c_{jt}) \right],$$

where we used the equality $W_{jt} = F_{jt}e^{\epsilon_{jt}}$. Using Lemma 1, we can rewrite the inequality as

$$\frac{F_{js}}{F_{jt}} \leq 1 + \frac{RTS_j}{E_{jt}} \left[w_{jt}^1(h_{js}^1 - h_{jt}^1) + w_{jt}^2(h_{js}^1 - h_{jt}^2) + (c_{js} - c_{jt}) \right].$$

Observe that the left-hand side is always strictly positive. Further, note that wages and inputs can be such that $\frac{RTS_j}{E_{jt}}w_{jt}^1(h_{js}^1 - h_{jt}^1) + w_{jt}^2(h_{js}^1 - h_{jt}^2) + (c_{js} - c_{jt}) < 0$. For example, $w_{jt}^1 = h_{jt}^1 = c_{jt} = 10$, $w_{jt}^2 = h_{jt}^2 = 5$ and $w_{js}^1 = h_{js}^1 = c_{js} = 5$, $w_{js}^2 = h_{js}^2 = 10$ makes it negative for both $s, t \in \mathcal{T}$. Hence, for any $RTS_j \in (0, 1]$ the following inequalities must hold simultaneously

$$\frac{F_{js}}{F_{jt}} < 1; \quad \frac{F_{jt}}{F_{js}} < 1.$$

Note that this implies $F_{js} < F_{jt}$ and $F_{jt} < F_{js}$, a contradiction. Next, consider wages and inputs $w_{jt}^1 = 2$, $w_{jt}^2 = 4$, $h_{jt}^1 = h_{jt}^2 = 15$, $c_{jt} = 10$ and $w_{js}^1 = w_{js}^2 = 2$, $h_{js}^1 = 5$,

$h_{js}^2 = 10$, $c_{js} = 20$. Plugging those numbers in GAPM yields

$$\begin{aligned}\frac{F_{js}}{F_{jt}} &\leq 1 - 0.3RTS_j \\ \frac{F_{jt}}{F_{js}} &\leq 1 + 0.4RTS_j.\end{aligned}$$

It is easy to see that these inequalities are only satisfied when $RTS_j \leq \frac{5}{6}$. In other words, GAPM may give upper bounds on returns to scale. Since this is true for every household, the support of the expected return to scale can also be restricted. By Proposition 1, it follows that the identified set is nontrivial. \square

The careful reader will have noticed that the proof of Proposition 2 only shows that GAPM can provide upper bounds on returns to scale. Thus, one may wonder whether GARP can provide an additional source of identification. The following result gives a negative answer.

Proposition 3. *Suppose Assumptions 1-3 hold, then GARP is not refutable and does not provide restrictions on the expected log production function.*

Proof. We want to show that GARP imposes no restrictions on returns to scale. In other words, we wish to show that there exist personalized prices for public expenditure \mathcal{P}_{jt}^i , personalized prices for children welfare P_{jt}^i , and children welfare W_{jt} such that any data set is consistent with every decreasing return to scale. Recall that $a_{jst}^i = w_{jt}^i(l_{js}^i - l_{jt}^i) + (q_{js}^i - q_{jt}^i) + \mathcal{P}_{jt}^i(Q_{js} - Q_{jt}) + P_{jt}^i(W_{js} - W_{jt})$ and let $X_{jst}^i(\mathcal{P}_{jt}^i) := w_{jt}^i(l_{js}^i - l_{jt}^i) + (q_{js}^i - q_{jt}^i) + \mathcal{P}_{jt}^i(Q_{js} - Q_{jt})$. Lemma 1 implies $P_{jt}^i(W_{js} - W_{jt}) = RTS_j^{-1}(E_{js}\frac{P_{jt}^i}{P_{js}} - E_{jt}\frac{P_{jt}^i}{P_{jt}})$, where $E_{jt} := w_{jt}^1 h_{jt}^1 + w_{jt}^2 h_{jt}^2 + c_{jt}$. Fix $RTS_j \in (0, 1]$, \mathcal{P}_{jt}^i to any number that satisfies $\mathcal{P}_{jt}^1 + \mathcal{P}_{jt}^2 = 1$, and $P_{jt}^1 = P_{jt}^2 = 0.5P_{jt}$ for all $t \in \mathcal{T}$. Next, pick $P_{jT} = 1$ and successively choose

$$P_{jt} > \max_{i \in \{1,2\}, t' > t} \left\{ -\frac{E_{jt}P_{jt'}}{2RTS_j X_{jtt'}^i - E_{jt'}}, 1 \right\} \quad t = T-1, T-2, \dots, 1.$$

Observe that $a_{jtt'}^i > 0$ for every $t = 1, 2, \dots, T-1$ and every $t' > t$. Hence, $a_{jtt'}^i < 0$ may only arise if $t' > t$, but then our construction ensures there cannot be a violation of GARP for either household member. Since the choice of RTS_j was arbitrary, GARP imposes no additional restrictions on returns to scale and, as such, no restrictions on the expected log production function. \square

Proposition 3 implies that the model does not give lower bounds on returns to scale. From Proposition 1, it follows that we cannot obtain lower bounds on the slope

of the expected log production function.¹⁴ This implies, for example, that we cannot reject that the expected log production function is a constant function. The proof of Proposition 3 reveals that the problem is the unboundedness of personalized prices for children welfare. The next result shows that GARP can bound returns to scale if the support of personalized prices for children welfare is compact.

Proposition 4. *Suppose Assumptions 1-3 hold and the support of personalized prices for children welfare is compact, then GARP may restrict the support of the expected return to scale such that the identified set is nontrivial.*

Proof. If the support of personalized prices for children welfare is compact, then the ratio of prices across time periods is bounded. That is, $\frac{P_{jt}}{P_{js}} \in [\underline{P}, \overline{P}]$ for some $0 < \underline{P} \leq \overline{P}$. We need to show that GARP can provide lower and upper bounds on returns to scale under such support constraint. For the sake of concreteness, we assume that the support constraint is given by $\frac{P_{jt}}{P_{js}} \in [\frac{9}{10}, 1]$. The argument we lay out would apply for a larger support constraint given appropriate scaling of the data. It is worth noting, however, that a larger support constraint reduces the empirical content of GARP. That is, the type of data required to obtain restrictions on returns to scale becomes increasingly “atypical” as the support is relaxed.

(Upper bound) Let $w_{jt}^i = 10$, $w_{js}^i = 5$, $l_{jt}^i = 20$, $l_{js}^i = 10$, $h_{jt}^i = 2$, $h_{js}^i = 4$, $q_{jt} = 40$, $q_{js} = 100$, $c_{jt} = 10$, $c_{js} = 20$ and, for simplicity, $Q_{js} = Q_{jt}$. The latter is not crucial for our argument since $\mathcal{P}_{jt}^i \in (0, 1)$. Note that $X_{jst}^i = -40$, $X_{jts}^i = -10$, $E_{jt} = 60$, and $E_{js} = 50$. Further note that $a_{jst}^1 < 0$ if and only if $a_{jst}^2 < 0$ such that if either holds then $a_{jst} := a_{jst}^1 + a_{jst}^2 < 0$. Hence, if GARP holds for each household member GARP must also hold for the aggregate revealed preference relation a_{jst} . Note that $a_{jst} = -80 + P_{jt}(W_{js} - W_{jt})$ and $a_{jts} = -20 + P_{js}(W_{jt} - W_{js})$. By Lemma 1, we can express personalized prices for children welfare as

$$W_{jt} = \frac{E_{jt}}{RTS_j P_{jt}}.$$

Hence, we can write the aggregate revealed preference relation as

$$\begin{aligned} a_{jst} &= -80 + RTS_j^{-1} \left(E_{js} \frac{P_{jt}}{P_{js}} - E_{jt} \right) \\ a_{jts} &= -20 + RTS_j^{-1} \left(E_{jt} \frac{P_{js}}{P_{jt}} - E_{js} \right). \end{aligned}$$

Observe that $a_{jst} < 0$ for all $RTS_j \in (0, 1]$. Thus, GARP may only be satisfied if $a_{jts} > 0$, which happens when $RTS_j < 5/6$.

¹⁴It is possible to show that the result also applies with a parametric production function since productivity shocks are unrestricted.

(Lower bound) Let $w_{jt}^i = 8$, $w_{js}^i = 5$, $l_{jt}^i = 20$, $l_{js}^i = 10$, $h_{jt}^i = 3$, $h_{js}^i = 2$, $q_{jt} = 40$, $q_{js} = 150$, $c_{jt} = 40$, $c_{js} = 10$ and, for simplicity, $Q_{js} = Q_{jt}$. Note that $X_{jst}^i = 30$, $X_{jts}^i = -60$, $E_{jt} = 88$, and $E_{js} = 30$. Similar to before, we can write the aggregate revealed preference relation as

$$\begin{aligned} a_{jst} &= 60 + RTS_j^{-1} \left(30 \frac{P_{jt}}{P_{js}} - 88 \right) \\ a_{jts} &= -120 + RTS_j^{-1} \left(88 \frac{P_{js}}{P_{jt}} - 30 \right). \end{aligned}$$

Observe that $a_{jts} < 0$ for all $RTS_j \in (0, 1]$. Thus, GARP may only be satisfied if $a_{jst} > 0$, which happens when $RTS > \frac{41}{45}$.

For the sake of simplicity, suppose every household has the same data set. Then, the support of $\mathbb{E}[RTS]$ is a strict subset of $(0, 1]$ such that the identified set is nontrivial. \square

To gain some intuition on the mechanism by which household members' choices provide information on RTS , consider the case where $X_{t_j, t_k}^i > 0$ and $X_{t_k, t_j}^i > 0$. Without loss of generality, suppose $P_{t_k}^i(W_{t_j} - W_{t_k}) \geq 0$ such that $P_{t_j}^i(W_{t_k} - W_{t_j}) \leq 0$. Further suppose $a_{t_k, t_j}^i \leq 0$ such that an informative upper bound is obtained from the data. Since $a_{t_k, t_j}^i \leq 0$, household members prefer the allocation $(l_{t_j}^i, q_{t_j}^i, Q_{t_j}, W_{t_j})$ over $(l_{t_k}^i, q_{t_k}^i, Q_{t_k}, W_{t_k})$ in period t_j . Since $a_{t_k, t_j}^i \leq 0$ despite $X_{t_k, t_j}^i > 0$, it must be that $(l_{t_j}^i, q_{t_j}^i, Q_{t_j}, W_{t_j})$ is preferred to $(l_{t_k}^i, q_{t_k}^i, Q_{t_k}, W_{t_k})$ because children welfare is sufficiently more enticing. Children welfare is more enticing if it gives a higher marginal utility or when $P_{t_j}^i$ is large, but this exactly occurs when RTS is not too large.

Remark. In practice, it may be necessary to bound the support of the latent variables such as in our own implementation. In such case, GARP naturally provides meaningful though possibly mild restrictions on the production technology. As Proposition 4 makes clear, the analyst can obtain stronger restrictions on the production technology if he has prior knowledge about the support of the latent variables.

4 Empirical Specification

The previous section showed that the model nonparametrically partially identifies the production function. To improve the interpretability of our empirical results, we now specialize the production function to a Cobb-Douglas technology.

Assumption 4. *The production function is Cobb-Douglas such that*

$$W_{jt} = (h_{jt}^1)^{\alpha_{j1}} (h_{jt}^2)^{\alpha_{j2}} (c_{jt})^{\alpha_{j3}} e^{\epsilon_{jt}}.$$

The Cobb-Douglas technology is a natural choice as it is homogeneous of degree $RTS = \alpha_{j1} + \alpha_{j2} + \alpha_{j3}$. Furthermore, it is easy to see that the output elasticities are

given by

$$\begin{aligned}\alpha_{j1} &= \frac{RTS_j w_{jt}^1 h_{jt}^1}{w_{jt} h_{jt}^1 + w_{jt} h_{jt}^2 + c_{jt}} \\ \alpha_{j2} &= \frac{RTS_j w_{jt}^2 h_{jt}^2}{w_{jt} h_{jt}^1 + w_{jt} h_{jt}^2 + c_{jt}} \\ \alpha_{j3} &= \frac{RTS_j c_{jt}}{w_{jt} h_{jt}^1 + w_{jt} h_{jt}^2 + c_{jt}}.\end{aligned}$$

In words, the model implies that each output elasticity equates a fraction RTS of its share of total children expenditure. These shares are constant in time, regardless of changes in the shadow price of children welfare, P_{jt} . The next result warns against ignoring productivity shocks in the model.

Claim 1. *Suppose Assumptions 1-4 hold. If productivity shocks are ignored, then the data may erroneously reject the model at the true return to scale.*

Proof. In what follows, we remove the j subscript from the variables. Suppose the data are rationalized by the model at the true return to scale $RTS^0 \in (0, 1]$ and the true children welfare $W_t = (h_t^1)^{\alpha_1} (h_t^2)^{\alpha_2} (c_t)^{\alpha_3} e^{\epsilon_t}$. Suppose now the econometrician ignores productivity shocks and assumes

$$\widetilde{W}_t = (h_t^1)^{\alpha_1} (h_t^2)^{\alpha_2} (c_t)^{\alpha_3}.$$

Conditional on RTS^0 , the output elasticities are identified. Therefore, children welfare is also identified. From the first-order conditions of the model and Lemma 1, we have

$$\widetilde{P}_t = \frac{E_t}{RTS^0 \widetilde{W}_t}.$$

Since \widetilde{W}_t is identified, it follows that \widetilde{P}_t is also identified. It is then obvious that the Afriat inequalities

$$U_s^i - U_t^i \leq \lambda_t^i \left[w_t^i (l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i (Q_s - Q_t) + \widetilde{P}_t^i (\widetilde{W}_s - \widetilde{W}_t) \right]$$

can be rejected by the data even if the data are consistent with the Afriat inequalities under the correct specification of the production function. \square

Since output elasticities are a function of returns to scale, Claim 1 implies that ignoring productivity shocks may lead to inconsistent output elasticities.¹⁵ The next result shows that the first-order conditions of the model have empirical bite under the Cobb-Douglas specification.

¹⁵More generally, productivity shocks are useful as they may absorb omitted variables that could otherwise bias the output elasticities.

Claim 2. *Suppose Assumptions 1-4 hold. The first-order conditions of the model are refutable independently of returns to scale.*

Proof. By Lemma 1, we have

$$P_{jt}W_{jt} = RTS_j^{-1}(w_{jt}^1h_{jt}^1 + w_{jt}^2h_{jt}^2 + c_{jt}) \quad \forall t \in \mathcal{T},$$

where $RTS \in (0, 1]$. For the sake of simplicity, suppose there are only two time periods. As such, we have

$$\frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} = \frac{(w_{j2}^1h_{j2}^1 + w_{j2}^2h_{j2}^2 + c_{j2})}{(w_{j1}^1h_{j1}^1 + w_{j1}^2h_{j1}^2 + c_{j1})}.$$

Since output elasticities are time invariant, it must be that the following set of equations holds

$$\begin{aligned} \frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} &= \frac{w_{j2}^1h_{j2}^1}{w_{j1}^1h_{j1}^1} \\ \frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} &= \frac{w_{j2}^2h_{j2}^2}{w_{j1}^2h_{j1}^2} \\ \frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} &= \frac{c_{j2}}{c_{j1}}. \end{aligned}$$

Note that these equations do not depend on returns to scale. Furthermore, they can easily be violated such as with $w_{j2}^1h_{j2}^1 = 1/2$ and $w_{j2}^2h_{j2}^2 = c_{j2} = 1/4$. \square

Claim 2 shows that the first-order conditions have meaningful implications that can be tested in the data given a Cobb-Douglas specification. Our previous nonparametric results further guarantee that returns to scale and thus the output elasticities are restricted.

4.1 Measurement Error

Claim 2 shows the the model implies a set of overidentifying restrictions on output elasticities. Hence, any measurement error in the inputs, however small, would lead to the erroneous rejection of the model. It follows that any test of the model that does not address this issue would be dubious in our framework. For this reason, we impose mild centering conditions on measurement error. Let $m_t^x := x_t - x_t^*$ denote the difference between the observed and true value of a variable x_t in period t .

Assumption 5. $\mathbb{E}[m_t^x] = 0$, where $x \in \{h^1, h^2, c\}$, $t = 1, 2, \dots, T$.

Assumption 5 requires that observed inputs be consistent with the true inputs on average in the cross-section. Note that we do not require the distribution of measurement

error to be parametric or to be identical over time. An indirect benefit of introducing measurement error in inputs is that we will be able to keep households with missing inputs in our application. Further details about the data are discussed in Section 6.

5 Testing and Estimation

This section presents the statistical framework used for testing the model and making inference on the production function.

5.1 Testing

Let $\zeta_j := \{(U_{jt}^i, \lambda_{jt}^i, \mathcal{P}_{jt}^i, P_{jt}^i, W_{jt}, \alpha_{jt}, \omega_{jt}, m_{jt}^x)_{i \in \{1,2\}, x \in \{h^1, h^2, c\}}\}_{t \in \mathcal{T}} \in Z|\mathcal{D}$ denote the set of household-specific latent variables in the model, where Z denote the support of the latent variables and \mathcal{D} denote the support of the data. Our revealed preference characterization along with our moment conditions can be used to define the statistical rationalizability of a panel data set $D := \{D_j\}_{j \in \mathcal{N}}$. To this end, write the constraints of the model in the form of moment functions:

$$\begin{aligned} g_{ist}^U(D_j, \zeta_j) &:= \mathbb{1} \left(U_{js}^i - U_{jt}^i \leq \lambda_{jt}^i \left[w_{jt}^i (l_{js}^i - l_{jt}^i) + (q_{js}^i - q_{jt}^i) + \right. \right. \\ &\quad \left. \left. + \mathcal{P}_{jt}^i (Q_{js} - Q_{jt}) + P_{jt}^i (W_{js} - W_{jt}) \right] \right) - 1 \\ g_t^{\alpha_1}(D_j, \zeta_j) &:= \mathbb{1} \left(\alpha_{jt1} = \frac{w_{jt}^1 h_{jt}^1}{P_{jt} W_{jt}} \right) - 1 \\ g_t^{\alpha_2}(D_j, \zeta_j) &:= \mathbb{1} \left(\alpha_{jt2} = \frac{w_{jt}^2 h_{jt}^2}{P_{jt} W_{jt}} \right) - 1 \\ g_t^{\alpha_3}(D_j, \zeta_j) &:= \mathbb{1} \left(\alpha_{jt3} = \frac{c_{jt}}{P_{jt} W_{jt}} \right) - 1 \\ g_t^W(D_j, \zeta_j) &:= \mathbb{1} \left(W_{jt} = (h_{jt}^1)^{\alpha_{j1}} (h_{jt}^2)^{\alpha_{j2}} (c_{jt})^{\alpha_{j3}} e^{\epsilon_{jt}} \right) - 1 \\ g_{xt}^m(D_j, \zeta_j) &:= m_{jt}^x, \end{aligned}$$

where $\mathbb{1}(\cdot)$ denote the indicator function and equates 1 if the expression inside the parenthesis is satisfied and 0 otherwise. The latent variables further need to satisfy their support constraints such that $q_{jt}^1 + q_{jt}^2 = q_{jt}$ and $\mathcal{P}_{jt}^1 + \mathcal{P}_{jt}^2 = 1$.

Note that we let output elasticities vary in time in the moment functions $g_t^{\alpha_k}(D_j, \zeta_j)$. This guarantees that the equations for the output elasticities can be satisfied in every period. To obtain time invariant output elasticities, we require their expected variance

to be zero:

$$\mathbb{E}[\mathbf{g}^v(D_j, \zeta_j)] = 0,$$

where $\mathbf{g}^v(D_j, \zeta_j) := \text{var}(\boldsymbol{\alpha})$. Since the variance is always positive, those moment conditions are satisfied if and only if the variance is zero for all households. As such, this formulation is equivalent to directly imposing that production parameters are time invariant.¹⁶

In what follows, we let $\mathbf{g}(D_j, \zeta_j)$ denote the vector of all moment functions, $\mathbf{g}^{(m,v)}(D_j, \zeta_j) := (\mathbf{g}^m(D_j, \zeta_j)', \mathbf{g}^v(D_j, \zeta_j)')$ denote the set of moment functions on measurement error and variance of output elasticities, and $\mathbf{g}^{-(m,v)}(D_j, \zeta_j) := (\mathbf{g}^U(D_j, \zeta_j)', \mathbf{g}^\alpha(D_j, \zeta_j)', \mathbf{g}^W(D_j, \zeta_j)')$ denote its complement.

Definition 8. Under Assumptions 1-5, a data set D is statistically rationalizable if

$$\inf_{\mu \in \mathcal{M}_{Z|\mathcal{D}}} \|\mathbb{E}_{\mu \times \pi_0}[\mathbf{g}(D, \zeta)]\| = 0,$$

where $\mathcal{M}_{Z|\mathcal{D}}$ is the set of all conditional probability distributions on $Z|\mathcal{D}$ and $\pi_0 \in \mathcal{M}_{\mathcal{X}}$ is the observed distribution of D .

In its current form, the notion of statistical rationalizability has $2T^2 + T + T + T + T + 3T + T$ moment conditions, including some that are discontinuous. Let d_m denote the number of moment conditions on measurement error and d_v denote the number of moment conditions on the variance of the output elasticities. The following result due to Schennach (2014) and Aguiar and Kashaev (2021) allows us to considerably reduce the complexity of the problem.

Proposition 5. Under Assumptions 1-5, a data set D is statistically rationalizable if and only if

$$\min_{\boldsymbol{\gamma} \in \mathbb{R}^{d_m + d_v}} \|\mathbb{E}_{\pi_0}[\bar{\mathbf{g}}(D; \boldsymbol{\gamma})]\| = 0,$$

where

$$\bar{\mathbf{g}}_j(D_j; \boldsymbol{\gamma}) := \frac{\int_{\zeta_j \in Z|\mathcal{D}} \mathbf{g}_j^{(m,v)}(D_j, \zeta_j) \exp\left(\boldsymbol{\gamma}' \mathbf{g}_j^{(m,v)}(D_j, \zeta_j)\right) \mathbb{1}(\mathbf{g}_j^{-(m,v)}(D_j, \zeta_j) = 0) d\eta(\zeta_j|D_j)}{\int_{\zeta_j \in Z|\mathcal{D}} \exp\left(\boldsymbol{\gamma}' \mathbf{g}_j^{(m,v)}(D_j, \zeta_j)\right) \mathbb{1}(\mathbf{g}_j^{-(m,v)}(D_j, \zeta_j) = 0) d\eta(\zeta_j|D_j)},$$

and $\eta(\cdot|D_j)$ is an arbitrary user-specified distribution supported on $Z|\mathcal{D}$ such that $\mathbb{E}_{\pi_0}[\log(\mathbb{E}_\eta[\exp(\boldsymbol{\gamma}' \mathbf{g}^{(m,v)}(D, \zeta))|D])]$ exists and is twice continuously differentiable in $\boldsymbol{\gamma}$ for all $\boldsymbol{\gamma} \in \mathbb{R}^{d_m + d_v}$.

The previous result calls for some comments. First, the dimensionality of the problem is greatly reduced as it only requires finding a finite dimensional parameter $\boldsymbol{\gamma}$ rather than a distribution μ . Second, the moment conditions associated with the concavity

¹⁶This is formally proven in Aguiar and Kashaev (2021).

of the utility functions, first-order conditions, and production function equations are directly imposed on each household data set such as to restrict the support of the unobservables. In particular, observe that the optimization problem no longer includes any discontinuous moment condition. Finally, it is worth noting that the result states that there is no loss in generality in averaging out the unobservables in the moment functions provided the distribution is from the exponential family.

The simplification allowed by Proposition 5 requires finding unobservables ζ_j that exactly satisfy the concavity of the utility functions, first-order conditions, and production function equations. If the constraints were linear in the unobservables, it would be possible to use a standard Hit-and-Run algorithm to directly sample them from the feasible space defined by the intersection of the inequalities and the system of equations. Unfortunately, the inequalities are highly nonlinear, therefore making this approach impossible.¹⁷

We resolve this pervasive issue by proposing a blocked Gibbs sampler. The idea is to break down the sampling procedure into multiple blocks, where each block takes a subset of all unobservables as given. The key is to create those blocks in such a way that the inequalities are linear in the unobservables conditional on a certain subset of all unobservables. Thus, the inequalities effectively define a (conditional) convex polytope in each block. This allows for a straightforward sampling procedure that guarantees the unobservables to exactly satisfy the inequalities, first-order conditions, and production function equations. The details of the algorithm are provided in the Appendix.

5.2 Inference

One of the advantages of ELVIS is that testing and inference are quite simple even if the model is partially identified. Indeed, testing the model can be done by constructing the sample analogues of the averaged moments and by computing a test statistic that is stochastically bounded by the chi-square distribution. Inference is achieved by further adding moment conditions on parameters of interest and inverting the test statistic. Since the test statistic is stochastically bounded by the chi-square distribution, it suffices to compare the value of the test statistic against the chi-square critical value with $d_m + d_v$ ($d_m + d_v + d_\theta$) degrees of freedom for testing (inference). Importantly, the identified set is convex under mild conditions.¹⁸

¹⁷In principle, it would be possible to use rejection sampling along with a mixed-integer programming (MIP) problem to draw from the feasible space. However, these types of MIP for collective models are NP-complete (Nobibon et al., 2016) so they do not scale well. Also, rejection sampling is generally slow.

¹⁸We refer the reader to Aguiar and Kashaev (2021) for additional details about the statistical procedure.

6 Data

We conduct our empirical analysis with the Longitudinal Internet Studies for the Social Sciences (LISS) panel data. The panel consists of about 5000 households representative of the Dutch population and gathers information about panelists yearly. Since the LISS data directly include information on private expenditures within the household, an important point of departure from the model is that private expenditures q_t^i are observed.

The time use data were collected by means of survey questions about the time spent on a set of time use categories during the past seven days. Although the survey is not demanding of household members memory, the actual time allocations throughout the month are likely to differ from the ones reported at the time of the survey. Similarly, data on monthly expenditures were collected via survey questions. Additional details relating to data collection can be found in [Cherchye, De Rock and Vermeulen \(2012\)](#).

Since our main goal is to estimate the production function, we only consider measurement error in inputs. Nevertheless, we observe that our methodology could accommodate measurement error in other variables. Another rationale for our choice is that it ensures that the overidentifying restrictions implied by the Cobb-Douglas specification can be rationalized by the model. We refer the reader to Section 4 for details about the specification of the production function and the restrictions on measurement error.

Our empirical analysis focuses on couples with children. This restriction alone reduces the number of households to about a thousand. We further restrict the set of observations with nonmissing and nonzero wages, private expenditures, and public expenditures. For households with missing or zero data on inputs, we impute their values. Note that our treatment of measurement error explicitly handles the imperfection of our imputation. Lastly, we restrict our sample to households that are in the panel for three periods.

The final sample consists of 132 couples with children observed over 3 time periods pooled from the years 2008 to 2017. While our sample size is relatively small, we note that it is comparable with [Cherchye, De Rock and Vermeulen \(2012\)](#) despite our restriction to panel data. Summary statistics of the sample are displayed in Table 1. Further details about the sample construction are given in the Appendix.

Table 1: Sample Summary Statistics

	Husband		Wife		Household	
	Mean	Std dev.	Mean	Std dev.	Mean	Std dev.
Age	46.25	7.99	44.08	7.28		
Wage (EUR/hour)	13.68	6.65	12.95	12.00		
Number of children					2.00	0.83
Mean age of children					13.15	6.36
Childcare (hours/week)	10.41	7.89	16.92	13.14		
Work (hours/week)	37.50	6.03	23.30	8.27		
Private expenditure (EUR/month)	362.00	229.23	395.81	323.30		
Public expenditure (EUR/month)					2164.85	836.12
Children expenditure (EUR/month)					512.52	512.90
Total households						132

7 Empirical Results

This section recovers confidence sets on expected returns to scale and expected output elasticities. Then, it investigate how the production function changes with demographics.

7.1 Expected Parameters

We begin our empirical analysis by recovering the 95% confidence set on expected returns to scale. Since the confidence set is convex, we only need to find the lower and upper bound on the expected return to scale. We find that the 95% confidence set on expected returns to scale is $[0.270, 0.405]$. Since the confidence set is nonempty, it follows that the model is not rejected by the data. Hence, we continue our analysis and recover 95% confidence sets on expected output elasticities. The results are reported in Figure 1.

The figure show that, on average, time inputs by mothers increases children welfare by more than time inputs by fathers or children expenditure. More precisely, a 10% increase in childcare by the mother increases children welfare by about 14% while a 10% increase in childcare by the father increases children welfare by about 10%. The impacts of children expenditure appears lowest with a 10% increase in children expenditure increasing children welfare by about 9%.

7.2 Heterogeneity

This section investigates how the expected production technology varies with household characteristics. Due to the small sample size, we choose to the analysis using linear regressions. First, we analyze heterogeneity in returns to scale through the regression

$$RTS = X\beta + \omega, \quad (2)$$

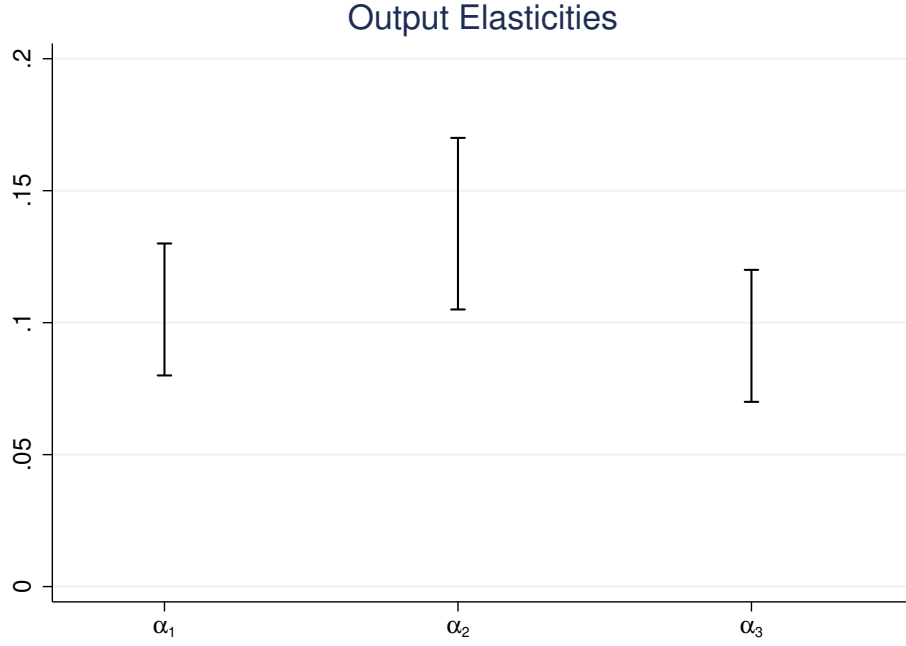


Figure 1: 95% Confidence Sets on Production Parameters

where $\mathbf{RTS} = \alpha_1 + \alpha_2 + \alpha_3$ represents returns to scale, \mathbf{X} is a set of covariates, and ω is a random error. Likewise, we analyze heterogeneity in output elasticities through the regressions

$$\alpha_k = \mathbf{X}\tilde{\beta}_k + \tilde{\omega}_k, \quad (3)$$

where α_k represents output elasticities with respect to input k , \mathbf{X} is a set of covariates, and $\tilde{\omega}_k$ is a random error. We assume the errors are uncorrelated and mean zero conditional on the data.¹⁹ We wish to emphasize that these regression equations are not estimated separately from the rest of the model, but rather imposed as additional equation restrictions within the model. That is, we make inference on the expected parameters of a regression by adding a moment function such as

$$g^\beta(D_j, \zeta_j) := \mathbf{1}(\mathbf{RTS}_j = \mathbf{X}_j\beta + \omega_j).$$

The 95% confidence sets on the expected coefficients of the regressions are reported in Table 2.

The first four rows of Table 2 capture the impacts of education on the production technology, where education is a categorical variable that reflects the type of education in the Netherlands. The first category represents primary school and pre-vocational secondary education (VMBO) and is used as the base in the regression. The second

¹⁹In the results below, we impose and test $\mathbb{E}[X_k\omega] = 0$, where k denote the k th covariate in the regression. We view the nonrejection of the restriction as support for the conditional mean zero assumption.

Table 2: 95% Confidence Sets on Regression Coefficients

Independent Variable ^a	Dependent Variable			
	$\alpha_1 + \alpha_2 + \alpha_3$	α_1	α_2	α_3
havo, vwo & mbo (Father)	$[-0.08, 0.05]$	$[0.02, 0.04]$	$[-0.02, 0.00]$	$[-0.09, 0.03]$
hbo & wo (Father)	$[-0.12, 0.03]$	$[0.00, 0.02]$	$[0.04, 0.07]$	$[-0.10, 0.05]$
havo, vwo & mbo (Mother)	$[0.06, 0.20]$	$[0.02, 0.05]$	$[0.05, 0.08]$	$[-0.05, 0.13]$
hbo & wo (Mother)	$[0.02, 0.15]$	$[0.01, 0.03]$	$[0.04, 0.07]$	$[-0.075, 0.10]$
#Children	$[-0.06, -0.01]$	$[-0.03, 0.02]$	$[-0.03, -0.01]$	$[-0.03, 0.04]$
Age Children	$[0.00, 0.01]$	$[-0.03, 0.04]$	$[0.00, 0.00]$	$[0.00, 0.01]$
Dwelling	$[-0.35, -0.10]$	$[-0.08, -0.06]$	$[-0.12, -0.10]$	$[-0.15, 0.10]$

^a The education category “havo, vwo, & mbo” represents general education that leads to higher education and vocational education that can lead to higher education. The education category “hbo & wo” represents higher education. #children is the number of children in the household. Age Children is the average age of children in the household. Dwelling is an indicator that takes value 0 if the household rents and 1 if it owns a house.

category represents general education that leads to higher education (HAVO and VWO) and vocational education (MBO) that can lead to higher education. The third category represents higher education (HBO, WO).

Table 2 shows that a higher education level increases the expected output elasticity by about 0.02 for fathers and 0.06 for mothers. Further, there is a positive spillover effect whereby higher education of one household member increases the expected output elasticity of the other member, with the exception of fathers with havo, vwo & mbo education. Interestingly, the education level of fathers does not appear to increase expected returns to scale. In contrast, the education level of mothers increases expected returns to scale by about 0.10. Finally, we cannot infer the impacts of education on the expected output elasticity with respect to children expenditure as the confidence sets are large and include zero.

Table 2 also shows that the number of children in the household has a small but negative impact on expected returns to scale and expected output elasticities with respect to time inputs. More interestingly, we find that the dwelling of the household has a large negative impacts on the production of children welfare, where the results are relative to owning a house. For example, households that rent have expected output elasticities with respect to time inputs lower by about 0.07 and 0.11 for fathers and mothers, respectively. Further, with 95% confidence, expected returns to scale decrease by at least 0.10 up to 0.35 for households that do not own a house.

8 Conclusion

This paper considers a collective model of labor supply with children to analyze the costs and benefits of welfare reforms directed at families. We show that conditions for point identification in the collective model such as constant returns to scale are rejected by the data. Consequently, costs and benefits analyses based on those conditions could lead to

misguided policy recommendations. To address this issue, we provide a nonparametric revealed preference characterization of the model and show that the expected production technology can be nonparametrically partially identified. We then provide a novel and efficient estimation strategy for the collective model that allows for decreasing returns to scale and unrestricted heterogeneity. In our application, we use our methodology to unveil the impacts of parental inputs on children welfare. First, we find that the production of children welfare is characterized by decreasing returns to scale. Second, we find that children benefits more from mothers time inputs than fathers time inputs or children expenditure. Third, we find that children living in a disadvantaged household defined as one that does not have home ownership nor parents with high education are significantly worse off. Our findings highlight the importance of policies targeted at improving children home environment in addition to cash transfers.

Appendix

A Sample Construction

For each household, we compute how many children live at home and focus on households with children living at home. We drop observations that pertain to single households and those that do not have any child living at home. Next, we remove observations for which one or both members have zero or missing wages. In this way, we avoid erroneously assuming a household member works when (s)he is not and vice versa. Furthermore, we remove observations where hours worked is missing to improve the quality of our imputation of leisure. Finally, we remove observations for which there is zero or missing private or public expenditures, as well as observations for which private or public expenditures are greater than the household gross income.

We impute zero and missing values of childcare and children expenditure for observations that remain in the sample after the previous selection criteria. The imputation is a simple year average of the variable. This imputation is likely to be an overestimate of the actual value for some households and an underestimate for others. Hence, it should be consistent with our moment conditions on measurement error.

Besides the imputation for some zero or missing inputs, we compute leisure of each household member as a residual according to the following equation:

$$l_t^i = 168 - 56 - b_t^i - h_t^i,$$

where l_t^i is leisure, 168 is the total number of hours in a week, 56 is the number of hours spent sleeping in a week, b_t^i is time spent working, and h_t^i is time spent on childcare. We drop households where implied leisure is negative as this may reflect a major problem with time data. There was only one household where this occurred in our sample. Clearly, our construction of leisure may still be inaccurate. We tackle this problem by allowing for measurement error in leisure. Precisely, since the time constraint requires $l_t^i + b_t^i + h_t^i = 168 - 56$ and h_t^i is mismeasured, we let true leisure be minus true childcare ($l_t^{*i} = -h_t^{*i}$) such that the time constraint holds at the true variables. These weekly variables are then scaled such as to obtain time inputs for the average number of days in a month. Since there are seven days in a week and a month has slightly more than 30 days on average, we multiply time inputs by 4.3.

As a last refinement of the sample, we remove households that are part of the LISS data for strictly less than 3 years. To obtain a balanced panel, we keep the first 3 observations of each household that is present for strictly more than 3 years. Thus, our sample is composed of households from various sets of 3 periods (e.g., 2009-2010-2012 or 2010-2012-2015). We limit ourselves to a three-year panel despite the greater empirical

bite that could be obtained with additional periods to avoid any additional decrease in the sample size. Lastly, we remove households with missing demographic information as it is necessary for the linear regression in Section 7.

B Proofs

B.1 Proof of Theorem 1

(i) \implies (ii)

The household problem can be written as

$$\max_{(l^1, l^2, h^1, h^2, q^1, q^2, Q, c) \in \mathbb{R}_+^2 \times \mathbb{R}_{++}^2 \times \mathbb{R}_+^{2L} \times \mathbb{R}_+ \times \mathbb{R}_{++}} \mu_t^1 U^1(l^1, q^1, Q, W_t) + \mu_t^2 U^2(l^2, q^2, Q, W_t),$$

subject to satisfying the household constraints

$$(q^1 + q^2) + Q + c = w_t^1(1 - l^1 - h^1) + w_t^2(1 - l^2 - h^2).$$

The first-order conditions are given by

$$\begin{aligned} \mu_t^i \frac{\partial U^i}{\partial l^i} &= \eta_t w_t^i \\ \mu_t^i \frac{\partial U^i}{\partial q^i} &= \eta_t. \\ \sum_i \mu_t^i \frac{\partial U^i}{\partial W_t} \cdot \frac{\partial W_t}{\partial h^1} &= \eta_t w_t^1 \\ \sum_i \mu_t^i \frac{\partial U^i}{\partial W_t} \cdot \frac{\partial W_t}{\partial h^2} &= \eta_t w_t^2 \\ \sum_i \mu_t^i \frac{\partial U^i}{\partial W_t} \cdot \frac{\partial W_t}{\partial c} &= \eta_t \\ \sum_i \mu_t^i \frac{\partial U^i}{\partial Q} &= \eta_t, \end{aligned}$$

where the equalities hold for some supergradient of the utility function.²⁰ Next, define

$$\begin{aligned} \lambda_t^i &= \frac{\eta_t}{\mu_t^i} \\ P_t^i &= \frac{\mu_t^i}{\eta_t} \frac{\partial U^i}{\partial W_t} \\ \mathcal{P}_t^i &= \frac{\mu_t^i}{\eta_t} \frac{\partial U^i}{\partial Q}. \end{aligned}$$

²⁰For corner solutions, the first-order conditions may only hold with inequality. The argument does not require any substantive change to accommodate this possibility.

The first-order conditions can be rewritten as

$$\frac{\partial U^i}{\partial l^i} = \lambda_t^i w_t^i \quad (4)$$

$$\frac{\partial U^i}{\partial q^i} = \lambda_t^i. \quad (5)$$

$$(P_t^1 + P_t^2) \frac{\partial W_t}{\partial h^1} = w_t^1 \quad (6)$$

$$(P_t^1 + P_t^2) \frac{\partial W_t}{\partial h^2} = w_t^2 \quad (7)$$

$$(P_t^1 + P_t^2) \frac{\partial W_t}{\partial c} = 1 \quad (8)$$

$$\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1. \quad (9)$$

Using the concavity of the utility functions we obtain

$$U_s^i - U_t^i \leq \left[\frac{\partial U^i}{\partial l_t^i} (l_s^i - l_t^i) + \frac{\partial U^i}{\partial q_t^i} (q_s^i - q_t^i) + \frac{\partial U^i}{\partial Q} (Q_s - Q_t) + \frac{\partial U^i}{\partial W_t} (W_s - W_t) \right],$$

where $U_t^i := U^i(l_t^i, q_t^i, Q_t, W_t)$ for all $t \in \mathcal{T}$. Substituting the derivatives of the utility function for their expressions yields

$$U_s^i - U_t^i \leq \lambda_t^i \left[w_t^i (l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i (Q_s - Q_t) + P_t^i (W_s - W_t) \right].$$

Next, using the concavity of the production function we obtain

$$F_s - F_t \leq \frac{\partial F}{\partial h_t^1} (h_s^1 - h_t^1) + \frac{\partial F}{\partial h_t^2} (h_s^2 - h_t^2) + \frac{\partial F}{\partial c_t} (c_s - c_t),$$

where $F_t := F(h_t^1, h_t^2, c_t)$ for all $t \in \mathcal{T}$. Substituting the derivatives of the production function from equations (6)-(8) yields

$$F_s - F_t \leq \frac{w_t^1}{P_t e^{\epsilon_t}} (h_s^1 - h_t^1) + \frac{w_t^2}{P_t e^{\epsilon_t}} (h_s^2 - h_t^2) + \frac{1}{P_t e^{\epsilon_t}} (c_s - c_t).$$

Putting everything together, these inequalities should hold for some $U_t^i, \lambda_t^i > 0, \mathcal{P}_t^i > 0$ such that $\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1, P_t^i > 0$ such that $P_t^1 + P_t^2 = P_t, W_t, F_t > 0$ and ϵ_t such that $W_t = F_t e^{\epsilon_t}, t = 1, \dots, T$.

(ii) \implies (i)

We have to show that, if Theorem 1 (ii) holds, then there exist concave utility functions and a concave production function that rationalize the data. Thus, let $\tau = \{t_j\}_{j=1}^m$, $m \geq 2$, $t_j \in \mathcal{T}$ denote a sequence of indices and \mathcal{I} denote the set of all such indices.

Define

$$U^i(l^i, q^i, Q, W) := \min_{\tau \in \mathcal{I}} \left\{ \lambda_{t_m}^i \left[w_{t_m}^i (l^i - l_{t_m}^i) + (q^i - q_{t_m}^i) + \mathcal{P}_{t_m}^i(Q - Q_{t_m}) + P_{t_m}^i(W - W_{t_m}) \right] + \sum_{j=1}^{m-1} \lambda_{t_j}^i \left[w_{t_j}^i (l_{t_{j+1}}^i - l_{t_j}^i) + (q_{t_{j+1}}^i - q_{t_j}^i) + \mathcal{P}_{t_j}^i(Q_{t_{j+1}} - Q_{t_j}) + P_{t_j}^i(W_{t_{j+1}} - W_{t_j}) \right] \right\}.$$

The function is the pointwise minimum of a collection of linear functions. Thus, it is continuous, increasing, and concave. By definition of U^i , there is some sequence of indices such that

$$U^i(l_t^i, q_t^i, Q_t, W_t) \geq \lambda_{t_m}^i \left[w_{t_m}^i (l_t^i - l_{t_m}^i) + (q_t^i - q_{t_m}^i) + \mathcal{P}_{t_m}^i(Q_t - Q_{t_m}) + P_{t_m}^i(W_t - W_{t_m}) \right] + \sum_{j=1}^{m-1} \lambda_{t_j}^i \left[w_{t_j}^i (l_{t_{j+1}}^i - l_{t_j}^i) + (q_{t_{j+1}}^i - q_{t_j}^i) + \mathcal{P}_{t_j}^i(Q_{t_{j+1}} - Q_{t_j}) + P_{t_j}^i(W_{t_{j+1}} - W_{t_j}) \right].$$

Add any allocation (l^i, q^i, Q, W) to the sequence and use the definition of U^i once again to obtain

$$\begin{aligned} & \lambda_t^i \left[w_t^i (l^i - l_t^i) + (q^i - q_t^i) + \mathcal{P}_t^i(Q - Q_t) + P_t^i(W - W_t) \right] + \\ & + \lambda_{t_m}^i \left[w_{t_m}^i (l_t^i - l_{t_m}^i) + (q_t^i - q_{t_m}^i) + \mathcal{P}_{t_m}^i(Q_t - Q_{t_m}) + P_{t_m}^i(W_t - W_{t_m}) \right] + \\ & + \sum_{j=1}^{m-1} \lambda_{t_j}^i \left[w_{t_j}^i (l_{t_{j+1}}^i - l_{t_j}^i) + (q_{t_{j+1}}^i - q_{t_j}^i) + \mathcal{P}_{t_j}^i(Q_{t_{j+1}} - Q_{t_j}) + P_{t_j}^i(W_{t_{j+1}} - W_{t_j}) \right] \\ & \geq U^i(l^i, q^i, W, Q). \end{aligned}$$

Hence, rearranging the previous expression yields

$$U^i(l^i, q^i, Q, W) - U^i(l_t^i, q_t^i, Q_t, W_t) \leq \lambda_t^i \left[w_t^i (l^i - l_t^i) + (q^i - q_t^i) + \mathcal{P}_t^i(Q - Q_t) + P_t^i(W - W_t) \right].$$

Note that the two first supergradients of $U^i(l_t^i, q_t^i, Q_t, W_t)$ give the first-order conditions (4)-(5). Next, define

$$\begin{aligned} F(h^1, h^2, c) &:= \min_{\tau \in \mathcal{I}} \left\{ \frac{1}{P_t e^{\epsilon t}} \left[w_t^1 (h^1 - h_t^1) + w_t^2 (h^2 - h_t^2) + (c - c_t) \right] + \right. \\ & \left. + \sum_{j=1}^{m-1} \left[\frac{1}{P_{t_j} e^{\epsilon t_j}} \left[w_{t_j}^1 (h_{t_{j+1}}^1 - h_{t_j}^1) + w_{t_j}^2 (h_{t_{j+1}}^2 - h_{t_j}^2) + (c_{t_{j+1}} - c_{t_j}) \right] \right] \right\}. \end{aligned}$$

This function is continuous, increasing, and concave in (h^1, h^2, c) . By an identical argu-

ment as before, we obtain

$$F(h^1, h^2, c) - F(h_t^1, h_t^2, c_t) \leq \frac{1}{P_t e^{\epsilon_t}} \left[w_t^1 (h^1 - h_t^1) + w_t^2 (h^2 - h_t^2) + P_t^i (c - c_t) \right].$$

Hence, the supergradients of $F(h_t^1, h_t^2, c_t)$ yield equations (6)-(8) and we conclude that Theorem 1 (ii) has the same implications as the household problem (1).

(ii) \implies (iii)

Let us begin by noting that the Afriat inequalities can be combined such that for all $\{t_k\}_{k=1}^m \in \mathcal{I}$ and all $i \in \{1, 2\}$

$$0 \leq \sum_{k=1}^m \lambda_{t_{k+1}}^i a_{t_k, t_{k+1}}^i.$$

Observe that the set of all sequences \mathcal{I} can be reduced to the set of all finite sequences as any sequence that satisfies this inequality is also satisfied without cycles. For the sake of a contradiction, suppose GARP is not satisfied for some household member. Then, there exists a cycle such that $a_{t_1, t_2}^i \leq 0$, $a_{t_2, t_3}^i \leq 0$, \dots , $a_{t_m, t_1}^i < 0$. Thus, it follows that

$$\lambda_{t_2}^i a_{t_1, t_2}^i + \lambda_{t_3}^i a_{t_2, t_3}^i + \dots + \lambda_{t_1}^i a_{t_m, t_1}^i < 0,$$

a contradiction of cyclical monotonicity. Next, wish to show that GAPM holds. Observe that the inequalities for the production function can be rearranged as

$$P_t F_t e^{\epsilon_t} + w_t^1 h_t^1 + w_t^2 h_t^2 + c_t \leq P_t F_s e^{\epsilon_t} + w_t^1 h_s^1 + w_t^2 h_s^2 + c_s \quad \forall s, t \in \mathcal{T},$$

where we further have $W_t = F_t e^{\epsilon_t}$ by assumption.

(iii) \implies (ii)

Suppose that GARP holds for each household member. Then, an application of [Fostel, Scarf and Todd \(2004\)](#) shows the existence of the Afriat inequalities for each household member. Furthermore, rearranging the inequalities in GAPM yields the desired inequalities.

B.2 Proof of Lemma 1

Proof. From the first-order conditions of the model and the Hicks-neutrality of productivity shocks, we have

$$\begin{aligned}\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^1} e^{\epsilon_t} &= \frac{w_t^1}{P_t} \\ \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^2} e^{\epsilon_t} &= \frac{w_t^2}{P_t} \\ \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial c_t^2} e^{\epsilon_t} &= \frac{1}{P_t}.\end{aligned}$$

We can multiply each marginal product by its own factor of production to get

$$\begin{aligned}\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^1} h_t^1 e^{\epsilon_t} &= \frac{w_t^1 h_t^1}{P_t} \\ \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^2} h_t^2 e^{\epsilon_t} &= \frac{w_t^2 h_t^2}{P_t} \\ \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial c_t^2} c_t e^{\epsilon_t} &= \frac{c_t}{P_t}.\end{aligned}$$

Summing up these equations and multiplying by P_t , we obtain

$$P_t \left[\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^1} h_t^1 + \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^2} h_t^2 + \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial c_t^2} c_t \right] e^{\epsilon_t} = E_t,$$

where $E_t := w_t^1 h_t^1 + w_t^2 h_t^2 + c_t$. Since the production function is homogeneous of degree $RTS \in (0, 1]$, an application of Euler's theorem gives

$$RTS P_t W_t = E_t,$$

where we used the production function equation $W_t = F(h_t^1, h_t^2, c_t) e^{\epsilon_t}$. □

C Sampling from the Feasible Space: A Blocked Gibbs Sampler

This section explains how to draw latent variables that satisfy the household problem. Since private expenditures are directly observed in our application, we do not need to find such quantities in our procedure. It is quite straightforward to extend our procedure to further find private expenditures if those were not observed, however.

Let $P_t = P_t^1 + P_t^2$ and recall that the data are consistent with the model if there exist personalized prices $P_t^i > 0$, personalized prices $\mathcal{P}_t^i > 0$ such that $\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1$, numbers U^i , λ_t^i , $W_t > 0$, and true inputs l_t^{*i} , h_t^{*i} , $c_t^* > 0$ such that for all $s, t \in \mathcal{T}$ and

all $i \in \{1, 2\}$

$$\begin{aligned}
U_s^i - U_t^i &\leq \lambda_t^i \left[w_t^i (l_s^{\star i} - l_t^{\star i}) + (q_s^i - q_t^i) + \mathcal{P}_t^i (Q_s - Q_t) + P_t^i (W_s - W_t) \right] \\
\alpha_1 &= \frac{w_t^1 h_t^{\star 1}}{P_t W_t} \\
\alpha_2 &= \frac{w_t^2 h_t^{\star 2}}{P_t W_t} \\
\alpha_3 &= \frac{w_t^1 h_t^{\star 1}}{P_t W_t} \\
\epsilon_t &= \log(W_t) - \alpha_1 \log(h_t^{\star 1}) - \alpha_2 \log(h_t^{\star 2}) - \alpha_3 \log(c_t^*),
\end{aligned}$$

where the last equation is obtained from the natural logarithm of the production function equation. Suppose we have a solution

$$(U_t^i(r), \lambda_t^i(r), l_t^{\star i}(r), h_t^{\star i}(r), c_t^*(r), P_t^i(r), \mathcal{P}_t^i(r))_{i \in \{1, 2\}, t \in \mathcal{T}},$$

where r denote the r th solution found by some solver. We provide a feasible algorithm that guarantees the next set of latent variables to be in the feasible space conditional on the data. The algorithm works provided the feasible space is nonempty in each block and standard regularity conditions associated with Gibbs samplers hold.

Intuitively, the idea is to recognize that it is difficult to uniformly sample all latent variables at once from the feasible space because the feasible space is complicated. Fortunately, we can break down the feasible space into conditional convex polytopes for which we can obtain closed-form bounds on the support of the latent variables. It thus becomes straightforward to uniformly sample a set of latent variables from each conditional convex polytope.

Step 1: Marginal Utility of Expenditure

Let $\Lambda = [1, L]$ denote the support of λ_t^i , where L is an arbitrarily large number. Given a solution at step r , we want to find $\lambda_t^i(r+1)$ that satisfies

$$\begin{aligned}
U_s^i(r) - U_t^i(r) &\leq \lambda_t^i(r+1) \left[w_t^i (l_s^{\star i}(r) - l_t^{\star i}(r)) + (q_s^i - q_t^i) + \right. \\
&\quad \left. + \mathcal{P}_t^i(r) (Q_s - Q_t) + P_t^i(r) (W_s(r) - W_t(r)) \right].
\end{aligned}$$

For convenience, let

$$denom^i := \left[w_t^i (l_s^{\star i} - l_t^{\star i}) + (q_s^i - q_t^i) + \mathcal{P}_t^i(r) (Q_s - Q_t) + P_t^i(r) (W_s(r) - W_t(r)) \right].$$

It follows that

$$\lambda_t^i(r+1)\Delta \frac{U_s^i(r) - U_t^i(r)}{\text{denom}^i},$$

where $\Delta :=>$ if $\text{denom}^i > 0$ and $\Delta :=<$ otherwise. Note that each $\lambda_t^i(r+1)$ has T bounds. The greatest lower bound on $\lambda_t^i(r+1)$ is the maximum between one and the greatest lower bound. If there is no lower bound, then the greatest lower bound is one. Likewise, the least upper bound is the minimum between one and the least upper bound. If there is no upper bound, then the least upper bound is one. Draw $\lambda_t^i(r+1)$ uniformly over the support defined by the greatest lower bound and least upper bound.

Step 2: Children Welfare, Leisure, and Childcare

Conditional on the new solution $\lambda_t^i(r+1)$, we want children welfare, true leisure, and true childcare to be positive such that

$$W_t(r) + \alpha\xi(W_t) > 0 \tag{10}$$

$$h_t^{\star 1}(r) + \alpha\xi(h_t^{\star 1}) > 0 \tag{11}$$

$$h_t^{\star 2}(r) + \alpha\xi(h_t^{\star 2}) > 0 \tag{12}$$

$$l_t^{\star 1}(r) + \alpha\xi(l_t^{\star 1}) > 0 \tag{13}$$

$$l_t^{\star 2}(r) + \alpha\xi(l_t^{\star 2}) > 0. \tag{14}$$

These positivity constraints provide a set of inequality restrictions on α . Next, we must further ensure that new leisure and new childcare of each household member satisfy the normalized time constraint

$$h_t^{\star i}(r) + \alpha\xi(h_t^{\star i}) + m_t^i + l_t^{\star i}(r) + \alpha\xi(l_t^{\star i}) = 1.$$

This equation implies $\xi(h_t^{\star i}) = -\xi(l_t^{\star i})$, $i \in \{1, 2\}$. That is, the direction taken for new childcare is the opposite of the direction for new leisure.

Next, it is important to ensure new children welfare and new childcare are consistent with a positive children expenditure:

$$w_t^1(h_t^{\star 1}(r) + \alpha\xi(h_t^{\star 1})) + w_t^2(h_t^{\star 2}(r) + \alpha\xi(h_t^{\star 2})) \leq P_t(r)(W_t(r) + \alpha\xi(W_t)).$$

Rearranging, one obtains

$$\alpha\Delta \frac{P_t(r)W_t(r) - w_t^1 h_t^{\star 1}(r) - w_t^2 h_t^{\star 2}(r)}{w_t^1 \xi(h_t^{\star 1}) + w_t^2 \xi(h_t^{\star 2}) - P_t \xi(W_t)},$$

where $\Delta :=<$ if $w_t^1 \xi(h_t^{\star 1}) + w_t^2 \xi(h_t^{\star 2}) - P_t \xi(W_t) > 0$ and $\Delta :=>$ otherwise. Further, we must also ensure that new children welfare and new childcare are compatible with

decreasing returns to scale given children expenditure:

$$w_t^1(h_t^{\star 1}(r) + \alpha\xi(h_t^{\star 1})) + w_t^2(h_t^{\star 2}(r) + \alpha\xi(h_t^{\star 2})) + c_t(r) \leq P_t(r)(W_t(r) + \alpha\xi(W_t)).$$

Rearranging, one obtains

$$\alpha\Delta \frac{P_t(r)W_t(r) - w_t^1 h_t^{\star 1}(r) - w_t^2 h_t^{\star 2}(r) - c_t(r)}{w_t^1 \xi(h_t^{\star 1}) + w_t^2 \xi(h_t^{\star 2}) - P_t \xi(W_t)},$$

where $\Delta := <$ if $w_t^1 \xi(h_t^{\star 1}) + w_t^2 \xi(h_t^{\star 2}) - P_t \xi(W_t) > 0$ and $\Delta := >$ otherwise.

Finally, we want new children welfare $W_t(r+1)$, new leisure $l_t^{\star i}(r+1)$, and new childcare $h_t^{\star i}(r+1)$ to satisfy

$$\begin{aligned} U_s^i(r) - U_t^i(r) &\leq \lambda_t^i(r+1) \left[w_t^i(l_s^{\star i}(r) + \alpha\xi(l_s^{\star i}) - l_t^{\star i}(r) - \alpha\xi(l_t^{\star i})) + (q_s^i - q_t^i) + \right. \\ &\quad \left. + \mathcal{P}_t^i(r)(Q_s - Q_t) + P_t^i(r)(W_s(r) + \alpha\xi(W_s) - W_t(r) - \alpha\xi(W_t)) \right]. \end{aligned}$$

This inequality can be rewritten as

$$\begin{aligned} U_s^i(r) - U_t^i(r) - \lambda_t^i(r+1) &\left[w_t^i(l_s^{\star i}(r) - l_t^{\star i}(r)) + (q_s^i - q_t^i) + \mathcal{P}_t^i(r)(Q_s - Q_t) \right. \\ &\quad \left. + P_t^i(r)(W_s(r) - W_t(r)) \right] \\ &\leq \alpha\lambda_t^i(r+1) (P_t^i(r)(\xi(W_s) - \xi(W_t)) + w_t^i(\xi(l_s^{\star i}) - \xi(l_t^{\star i}))). \end{aligned}$$

Let num^i denote the left-hand side of this inequality. Thus, we have

$$\alpha\Delta \frac{num^i}{\lambda_t^i(r) (P_t^i(r)(\xi(W_s) - \xi(W_t)) + \xi(w_t^i)(l_s^{\star i} - l_t^{\star i}))},$$

where $\Delta := >$ if $\lambda_t^i(r) (P_t^i(r)(\xi(W_s) - \xi(W_t)) + w_t^i(\xi(l_s^{\star i}) - \xi(l_t^{\star i}))) > 0$ and $\Delta := <$ otherwise. Draw α uniformly over its support as defined by the greatest lower bound and the least upper bound from the previous sets of inequalities. We obtain $(W_t(r+1), l_t^{\star i}(r+1), h_t^{\star i}(r+1))_{t \in \mathcal{T}}$ by picking α uniformly over its support defined by the greatest lower bound and least upper bound derived from the above inequalities.

Step 3: Utilities and Personalized Prices

We want personalized prices to be positive such that

$$\mathcal{P}_t^i(r) + \beta\xi(\mathcal{P}_t^i) > 0 \tag{15}$$

$$P_t^i(r) + \beta\xi(P_t^i) > 0. \tag{16}$$

These inequalities can be transformed to get bounds on β :

$$\beta\Delta - \frac{\mathcal{P}_t^i(r)}{\xi(\mathcal{P}_t^i)} \quad (17)$$

$$\beta\Delta - \frac{P_t^i(r)}{\xi(P_t^i)}, \quad (18)$$

where $\Delta :=>$ if $\xi(\cdot) > 0$ and $\Delta :=<$ otherwise. Next, similar to the previous step it is important to ensure new personalized prices for children welfare are consistent with a positive children expenditure:

$$w_t^1 h_t^{*1}(r+1) + w_t^2 h_t^{*2}(r+1) \leq (P_t^1(r) + \beta\xi(P_t^1) + P_t^2(r) + \beta\xi(P_t^2))W_t(r+1).$$

Rearranging, one obtains

$$\beta\Delta \frac{w_t^1 h_t^{*1}(r+1) + w_t^2 h_t^{*2}(r+1) - (P_t^1(r) + P_t^2(r))W_t(r+1)}{(\xi(P_t^1) + \xi(P_t^2))W_t(r+1)},$$

where $\Delta :=>$ if $(\xi(P_t^1) + \xi(P_t^2))W_t(r+1) > 0$ and $\Delta :=<$ otherwise. Further, we must also ensure that new personalized prices for children welfare are compatible with decreasing returns to scale given children expenditure:

$$w_t^1 h_t^{*1}(r+1) + w_t^2 h_t^{*2}(r+1) + c_t(r) \leq (P_t^1(r) + \beta\xi(P_t^1) + P_t^2(r) + \beta\xi(P_t^2))W_t(r+1).$$

Rearranging, one obtains

$$\beta\Delta \frac{w_t^1 h_t^{*1}(r+1) + w_t^2 h_t^{*2}(r+1) + c_t(r) - (P_t^1(r) + P_t^2(r))W_t(r+1)}{(\xi(P_t^1) + \xi(P_t^2))W_t(r+1)},$$

where $\Delta :=>$ if $(\xi(P_t^1) + \xi(P_t^2))W_t(r+1) > 0$ and $\Delta :=<$ otherwise.

Finally, starting with the new numbers $\lambda_t^i(r+1)$, $W_t(r+1)$, $l_t^{*i}(r+1)$, and $h_t^{*i}(r+1)$, we want new utilities $U_t^i(r+1)$ and new personalized prices $\mathcal{P}_t^i(r+1)$, $P_t^i(r+1)$ to satisfy

$$\begin{aligned} U_s^1(r) + \beta\xi(U_s^1) - U_t^1(r) - \beta\xi(U_t^1) \leq \\ \lambda_t^1(r+1) \left[w_t^1 (l_s^{*1}(r+1) - l_t^{*1}(r+1)) + (q_s^1 - q_t^1) + \right. \\ \left. + (\mathcal{P}_t^1(r) + \beta\xi(\mathcal{P}_t^1))(Q_s - Q_t) + (P_t^1(r) + \beta\xi(P_t^1))(W_s(r+1) - W_t(r+1)) \right], \end{aligned}$$

and

$$\begin{aligned} U_s^2(r) + \beta\xi(U_s^2) - U_t^2(r) - \beta\xi(U_t^2) \leq \\ \lambda_t^2(r+1) \left[w_t^2 (l_s^{*2}(r+1) - l_t^{*2}(r+1)) + (q_s^2 - q_t^2) + \right. \\ \left. + (\mathcal{P}_t^2(r) + \beta\xi(\mathcal{P}_t^2))(Q_s - Q_t) + (P_t^2(r) + \beta\xi(P_t^2))(W_s(r+1) - W_t(r+1)) \right]. \end{aligned}$$

With some algebra, we can rewrite the inequalities for household member 1 as

$$\begin{aligned} & \beta \left[\xi(U_s^1) - \xi(U_t^1) + \lambda_t^1(r+1) [\xi(\mathcal{P}_t^1)(Q_t - Q_s) + \xi(P_t^1)(W_t(r+1) - W_s(r+1))] \right] \\ & \leq U_t^1(r) - U_s^1(r) + \lambda_t^1(r+1) \left[w_t^1(l_s^{\star 1}(r+1) - l_t^{\star 1}(r+1)) + (q_s^1 - q_t^1) + \right. \\ & \quad \left. + \mathcal{P}_t^1(r)(Q_s - Q_t) + P_t^1(r)(W_s(r+1) - W_t(r+1)) \right], \end{aligned}$$

and the inequalities for household member 2 as

$$\begin{aligned} & \beta \left[\xi(U_s^2) - \xi(U_t^2) + \lambda_t^2(r+1) [\xi(\mathcal{P}_t^2)(Q_t - Q_s) + \xi(P_t^2)(W_t(r+1) - W_s(r+1))] \right] \\ & \leq U_t^2(r) - U_s^2(r) + \lambda_t^2(r+1) \left[w_t^2(l_s^{\star 2}(r+1) - l_t^{\star 2}(r+1)) + (q_s^2 - q_t^2) + \right. \\ & \quad \left. + \mathcal{P}_t^2(r)(Q_s - Q_t) + P_t^2(r)(W_s(r+1) - W_t(r+1)) \right]. \end{aligned}$$

For convenience, let

$$\begin{aligned} num^1 &:= U_t^1(r) - U_s^1(r) + \lambda_t^1(r+1) \left[w_t^1(l_s^{\star 1}(r+1) - l_t^{\star 1}(r+1)) + (q_s^1 - q_t^1) + \right. \\ & \quad \left. + \mathcal{P}_t^1(r)(Q_s - Q_t) + P_t^1(r)(W_s(r+1) - W_t(r+1)) \right] \\ denom^1 &:= \xi(U_s^1) - \xi(U_t^1) + \\ & \quad + \lambda_t^1(r+1) [\xi(\mathcal{P}_t^1)(Q_t - Q_s) + \xi(P_t^1)(W_t(r+1) - W_s(r+1))] \end{aligned}$$

and

$$\begin{aligned} num^2 &:= U_t^2(r) - U_s^2(r) + \lambda_t^2(r+1) \left[w_t^2(l_s^{\star 2}(r+1) - l_t^{\star 2}(r+1)) + (q_s^2 - q_t^2) + \right. \\ & \quad \left. + \mathcal{P}_t^2(r)(Q_s - Q_t) + P_t^2(r)(W_s(r+1) - W_t(r+1)) \right] \\ denom^2 &:= \xi(U_s^2) - \xi(U_t^2) + \\ & \quad + \lambda_t^2(r+1) [\xi(\mathcal{P}_t^2)(Q_t - Q_s) + \xi(P_t^2)(W_t(r+1) - W_s(r+1))] \end{aligned}$$

Therefore, we have

$$\beta \Delta \frac{num^i}{denom^i},$$

where $\Delta := <$ if $denom^i > 0$ and $\Delta := >$ otherwise. We obtain $(U_t^i(r+1), \mathcal{P}_t^i(r+1), P_t^i(r+1))_{t \in \mathcal{T}}$ by picking β uniformly over its support defined by the greatest lower bound and least upper bound derived from the above inequalities.

Step 4: Children Expenditure

We need to pick new true children expenditure that is positive such that

$$c_t^*(r) + \kappa \xi(c_t^*) > 0. \quad (19)$$

In addition, new children expenditure must yield decreasing returns to scale such that

$$w_t^1 h_t^{\star 1}(r+1) + w_t^2 h_t^{\star 2}(r+1) + c_t^{\star}(r) + \kappa \xi(c_t^{\star}) \leq P_t(r+1)W_t(r+1).$$

Rearranging, one gets

$$\kappa \Delta \frac{P_t(r+1)W_t(r+1) - w_t^1 h_t^{\star 1}(r+1) - w_t^2 h_t^{\star 2}(r+1) - c_t^{\star}(r)}{\xi(c_t^{\star})},$$

where $\Delta := <$ if $\xi(c_t^{\star}) > 0$ and $\Delta := >$ otherwise. We obtain $c_t^{\star}(r+1)$ by picking κ uniformly over its defined by the greatest lower bound and least upper bound derived from the above inequalities.

Step 5: Production Parameters and Productivity

We are left with the task to recover output elasticities and productivity shocks. This requires no work as they are directly deduced from the first-order conditions and production function equation:

$$\begin{aligned} \alpha_{1,t}(r+1) &= \frac{w_t^1 h_t^{\star 1}(r+1)}{P_t(r+1)W_t(r+1)} \\ \alpha_{2,t}(r+1) &= \frac{w_t^2 h_t^{\star 2}(r+1)}{P_t(r+1)W_t(r+1)} \\ \alpha_{3,t}(r+1) &= \frac{c_t^{\star}(r+1)}{P_t(r+1)W_t(r+1)} \\ \epsilon_t(r+1) &= \log(W_t(r+1)) - \alpha_{1,t}(r+1) \log(h_t^{\star 1}(r+1)) \\ &\quad - \alpha_{2,t}(r+1) \log(h_t^{\star 2}(r+1)) - \alpha_{3,t}(r+1) \log(c_t^{\star}). \end{aligned}$$

This last step of the Gibbs sampler combined with the previous steps give a completely new solution to the model.

Sampling from the Feasible Space in 5 Easy Steps

Suppose an initial solution $r = 0$ to the household problem is given (e.g., by solving a mixed-integer program). Then,

1. Given r , get $(\lambda_t^i(r+1))_{t \in \mathcal{T}}$ as outlined in Step 1.
2. Given 1, get $(W_t(r+1), l_t^{\star i}(r+1), h_t^{\star i}(r+1))_{t \in \mathcal{T}}$ as outlined in Step 2.
3. Given 1-2, get $(U_t^i(r+1), \mathcal{P}_t^i(r+1), P_t^i(r+1))_{t \in \mathcal{T}}$ as outlined in Step 3.
4. Given 1-3, get $(c_t^{\star}(r+1))_{t \in \mathcal{T}}$ as outlined in Step 4.
5. Given 1-4, get $(\alpha_t(r+1), \epsilon_t(r+1))_{t \in \mathcal{T}}$ as outlined in Step 5.

6. Set $r = r + 1$ and repeat 1-5 until $r = R > 0$.

C.1 Miscellaneous

This subsection provides additional details about the sampling procedure.

Target Distribution

We ensure that the sampling procedure yields the desired least favorable distribution on measurement error by using a Metropolis-Hastings algorithm. Once a complete new solution is obtained from the Gibbs sampler, update the Markov chain with the appropriate acceptance ratio. Note that the acceptance ratio depends on the target distribution. In our application, the target distribution is proportional to a normal distribution:

$$d\tilde{\eta}(\cdot|x_i) \propto \exp\left(-\|g_i^m(x_i, e_i)\|^2\right).$$

As pointed out by [Schennach \(2014\)](#), under mild regularity conditions the mean and variance of the distribution are inconsequential for the validity of Proposition 5.

Length of the Monte Carlo Markov Chain

The Gibbs sampler generates a Markov Chain that suffers from autocorrelation. For this reason, it is good practice to only keep a subset of the R solutions, a technique known as thinning. In our application, we keep 5% of all solutions. Also, the theory of stochastic processes tells us that convergence to the stationary distribution may take some time —its existence follows by construction of the Metropolis-Hastings algorithm. Accordingly, it is good practice to leave out the first few solutions. In our application, we leave out the first 100000 solutions. We then draw another 100000 solutions from the feasible space.

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