Exo 1 $\quad A = \begin{bmatrix} 8 & -2 \\ -2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ -2 \end{bmatrix} \qquad f(x) = \frac{1}{2} x^T A x - b^T x$

$k = 0$

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to r_0 = b = \boxed{\begin{pmatrix} 8 \\ -2 \end{pmatrix}} \to d_0 = r_0 = \boxed{\begin{pmatrix} 8 \\ -2 \end{pmatrix}} \to s_0 = \|r_0\|^2 = 8^2 + (-2)^2 = 64 + 4 = \boxed{68}$$

0,25 $\quad$ 0,25 $\quad$ 0,25

$s_0 > 0$ donc

$$\rho_0 = \frac{s_0}{d_0^T A d_0} = \frac{68}{(8\ -2)\begin{pmatrix} 8 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ -2 \end{pmatrix}} = \frac{68}{(8\ -2)\begin{pmatrix} 68 \\ -20 \end{pmatrix}} = \frac{68}{8 \cdot 68 + (-2)(-20)} = \frac{17}{2 \times 68 + 10} = \boxed{\frac{17}{146}}$$ 0,25

$$x_1 = x_0 + \rho_0 d_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{17}{146}\begin{pmatrix} 8 \\ -2 \end{pmatrix} = \boxed{\begin{pmatrix} 68/73 \\ -17/73 \end{pmatrix}}$$ 0,25

$$r_1 = b - A x_1 = \begin{pmatrix} 8 \\ -2 \end{pmatrix} - \begin{pmatrix} 8 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 68/73 \\ -17/73 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix} - \begin{pmatrix} 578/73 \\ -170/73 \end{pmatrix} = \boxed{\begin{pmatrix} 6/73 \\ 24/73 \end{pmatrix}}$$ 0,25

$$\mu_1 = \|r_1\|^2 = \frac{6^2 + 24^2}{73^2} = \frac{6^2 + 6^2 \times 4^2}{73^2} = \frac{6^2 \times 17}{73^2} = \frac{36 \times 17}{73^2} = \boxed{\frac{9 \times 68}{73^2}}$$ 0,25

$$d_1 = r_1 + \frac{\mu_1}{s_0} d_0 = \begin{pmatrix} 6/73 \\ 24/73 \end{pmatrix} + \frac{9}{73^2}\begin{pmatrix} 8 \\ -2 \end{pmatrix} = \frac{1}{73^2}\begin{pmatrix} 6 \times 73 + 9 \times 8 \\ 24 \times 73 - 9 \times 2 \end{pmatrix} = \boxed{\begin{pmatrix} 510/73^2 \\ 1734/73^2 \end{pmatrix}}$$ 0,25

$$s_1 = \mu_1 = \boxed{\frac{9 \times 68}{73^2}}$$ 0,25

$s_1 > 0$ donc

$$\rho_1 = \frac{s_1}{d_1^T A d_1} = \frac{9 \times 68 / 73^2}{\left(\frac{510}{73^2}\ \frac{1734}{73^2}\right)\begin{pmatrix} 8 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 510/73^2 \\ 1734/73^2 \end{pmatrix}} = \frac{9 \times 68}{(612\ 2448)\begin{pmatrix} 510/73^2 \\ 1734/73^2 \end{pmatrix}}$$

$$= \frac{9 \times 68 \times 73^2}{612 \times 510 + 2448 \times 1734} = \frac{9 \times 17 \times 73^2}{153 \times 510 + 612 \times 1734} = \frac{9 \times 73^2}{153 \times 30 + 612 \times 102}$$

$$= \frac{3 \times 73^2}{153 \times 10 + 612 \times 34} = \frac{73^2}{51 \times 10 + 204 \times 34} = \frac{73^2}{51 \times (10 + 4 \times 34)} = \boxed{\frac{73^2}{51 \times 146}}$$ 0,25

$$x_2 = x_1 + \rho_1 d_1 = \begin{pmatrix} 68/73 \\ -17/73 \end{pmatrix} + \frac{73^2}{51 \times 146}\begin{pmatrix} 510/73^2 \\ 1734/73^2 \end{pmatrix} = \begin{pmatrix} 68/73 \\ -17/73 \end{pmatrix} + \begin{pmatrix} 10/146 \\ 34/146 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$ 0,25

$$r_2 = b - A x_2 = \begin{pmatrix} 8 \\ -2 \end{pmatrix} - \begin{pmatrix} 8 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$ 0,25

$$\mu_2 = \|r_2\|^2 = \boxed{0}, \quad d_2 = r_2 + \frac{\mu_2}{s_1} d_1 = \boxed{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}, \quad s_2 = \mu_2 = \boxed{0}$$

0,25

solution optimale : $\boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ 0,25

valeur optimale : $\frac{1}{2}(1\ 0)\begin{pmatrix} 8 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - (8\ -2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2}(8\ -2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{-4}$

Exo 2

① $\max\; -2(x_{11}-x_{12}) + (x_{21}-x_{22}) + x_3'$ 0,25 0,25 0,25 0,25

$$
\text{s.c}
\begin{cases}
2(x_{11}-x_{12}) - (x_{21}-x_{22}) \leq 2 & 0{,}25\\
-2(x_{11}-x_{12}) + (x_{21}-x_{22}) \leq -2 & 0{,}25\\
-(x_{21}-x_{22}) - 2x_3' \leq -1 & 0{,}25\\
x_{11}, x_{12}, x_{21}, x_{22}, x_3' \geq 0
\end{cases}
$$

② $\min\; -3x_1' - (x_{21}-x_{22})$ 0,25 0,25 0,25

$$
\text{s.c}
\begin{cases}
-x_1' - (x_{21}-x_{22}) - x_3 = 2 & 0{,}25\\
-x_1' + 2(x_{21}-x_{22}) + x_4 = 20 & 0{,}25\\
x_1', x_{21}, x_{22}, x_3, x_4 \geq 0
\end{cases}
$$

Exo 3

① $n=5,\ p=3,\ q=5 \rightarrow \binom{q}{n-p} = \binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5\times 4}{1\times 2} = \boxed{10}$ 0,25

② $A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 0 & 0 & 1 \end{pmatrix}$, $f^T = (3,-2,0,0,0)$

$A^{B_j}_{j\in\{1,2,3\}} = \begin{pmatrix} -1 & 1 & * \\ 1 & -1 & * \\ 2 & -2 & * \end{pmatrix} \rightarrow \boxed{\det(A^{B_j}) = 0 \text{ donc } A^{B_j} \text{ n'est pas inversible}}$ 0,25

du coup B_j n'est pas réalisable

③ $A^{B_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$ et $b = \begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix}$

inversible, or $x^{B_4} = b \ngeq 0$ donc B_4 $\boxed{\text{n'est pas réalisable}}$ 0,25

$A^{B_5} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $\det A^{B_5} = -1 \neq 0$ donc inversible

$$x^{B_5} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} > 0$$

donc B_5 est $\boxed{\text{réalisable}}$ $\boxed{\text{non-dégénérée}}$ 0,25 0,25

$$c^{H_5} = f_{H_5}^T - f_{B_5}^T (A^{B_5})^{-1} A^{H_5} = (-2,0) - (3,0,0)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ -2 & 0 \end{pmatrix}$$

$$= (-2,0) - (0,3,0)\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ -2 & 0 \end{pmatrix} = (-2,0) - (-3,3) = (1,-3)$$

$\boxed{c^{H_5} \neq 0}$ donc B_5 ne satisfait pas la CSO 0,25

$A^{B_6} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $\det(A^{B_6}) = -1 \neq 0$ donc inversible

$$x^{B_6} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} > 0$$

donc B_6 est $\boxed{\text{réalisable}}$ $\boxed{\text{non-dégénérée}}$ 0,25 0,25

$$c^{H_6} = (-2,0) - (3,0,0)\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ -2 & 0 \end{pmatrix} = (-2,0) - (-3,0,0)\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ -2 & 0 \end{pmatrix}$$

$$= (-2,0) - (-3,-3) = (1,3) \geq 0$$

$\boxed{c^{H_6} \geq 0}$ donc B^6 satisfait la CSO

0,25

$$\max \; x_1 + 4x_2 + 3x_3$$
$$\text{s.c} \begin{cases} x_1 + x_2 + x_3 = 3 \\ -x_1 \qquad + x_3 = 1 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$
0,25

$f^T = (1,4,3)$, $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$(P_a) \; \min \; y_1 + y_2$$
$$\text{s.c} \begin{cases} x_1 + x_2 + x_3 + y_1 = 3 \\ -x_1 \qquad + x_3 + y_2 = 1 \\ x_i, y_j \geq 0 \end{cases}$$

$\tilde{f}^T = (0,0,0,1,1)$

$\tilde{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \end{pmatrix}$, $\tilde{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

0,25

$B_0 = \{4,5\}$	x_1	x_2	x_3	y_1	y_2	
y_1	1	1	1	1	0	3
y_2	-1	0	1	0	1	1 ←
	0	-1	-2 ↑	0	0	4

0,25

$B_1 = \{4,3\}$	x_1	x_2	x_3	y_1	y_2	
y_1	2	1	0	1	-1	2 ←
x_3	-1	0	1	0	1	1
	-2 ↑	-1	0	0	2	2

$'x_3' = 'y_2'/1$ $\qquad$ $'Z' = 'Z' - 2 \times 1$

$'y_1' = 'y_1' - 'x_3'$

$'c' = 'c' - (-2) \times 'x_3'$

0,25

$B_2 = \{1,3\}$	x_1	x_2	x_3	y_1	y_2	
x_1	1	1/2	0	1/2	-1/2	1
x_3	0	1/2	1	1/2	1/2	2
	0	0	0	1	1	0

$'x_1' = 'y_1'/2$ $\qquad$ $'Z' = 'Z' - 2 \times 1$

$'x_3' = 'x_3' + 'x_1'$

$'c' = 'c' - (-2) \times 'x_1'$

0,25

(P)

$B_0 = \{1,3\}$	x_1	x_2	x_3	
x_1	1	1/2	0	1 ←
x_3	0	1/2	1	2
	0	2 ↑	0	7

$'Z' = f_{B_0}^T x_{B_0} = (1,3)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 + 6 = 7$

$c_{H_0} = f_{H_0}^T - f_{B_0}^T (A^{B_0})^{-1} A^{H_0}$

$= 4 - (1,3)\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= 4 - (1,3)\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = 4 - 2 = 2 \not\leq 0$

0,25

$B_1 = \{2,3\}$	x_1	x_2	x_3	
x_2	2	1	0	$\boxed{2}$
x_3	-1	0	1	$\boxed{1}$
	-4	0	0	$\boxed{11}$

$'x_2' = 2\,'x_1'$ $\qquad$ $'Z' = 'Z' + 2 \times 2$

$'x_3' = 'x_3' - \frac{1}{2}\,'x_2'$

$'c' = 'c' - 2\,'x_2'$

solution optimale : $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, valeur optimale : 11

0,25 $\qquad$ 0,25

Exo 5 ① (D) $\min\ 4y_1 + 2y_2$ 0,25

$$\text{s.c.} \begin{cases} y_1 + y_2 \boxed{\geq} 1 & 0{,}25 \\ -3y_1 - 4y_2 \boxed{\leq} 2 & 0{,}25 \\ -4y_1 + 3y_2 = 5 & 0{,}25 \\ y_1 \boxed{\geq} 0 & 0{,}25 \\ y_2 \boxed{\in} \mathbb{R} & 0{,}25 \end{cases}$$

② $A = \{x \in \mathbb{R}^3 : \begin{cases} x_1 - 3x_2 - 4x_3 \leq 4 \\ x_1 - 4x_2 + 3x_3 = 2 \end{cases}, x_1 \geq 0, x_2 \leq 0, x_3 \in \mathbb{R}\}$

A est non-vide car $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \in A$, donc (P) est réalisable 0,25

$A^* = \{y \in \mathbb{R}^2, \begin{cases} y_1 + y_2 \geq 1 \\ -3y_1 - 4y_2 \leq 2 \\ -4y_1 + 3y_2 = 5 \end{cases}, y_1 \geq 0, y_2 \in \mathbb{R}\}$

0,25

A^* est non-vide car $\begin{pmatrix} 1/4 \\ 2 \end{pmatrix} \in A^*$, donc (D) est réalisable

D'après Thm 19 (P) possède des optimums

0,25

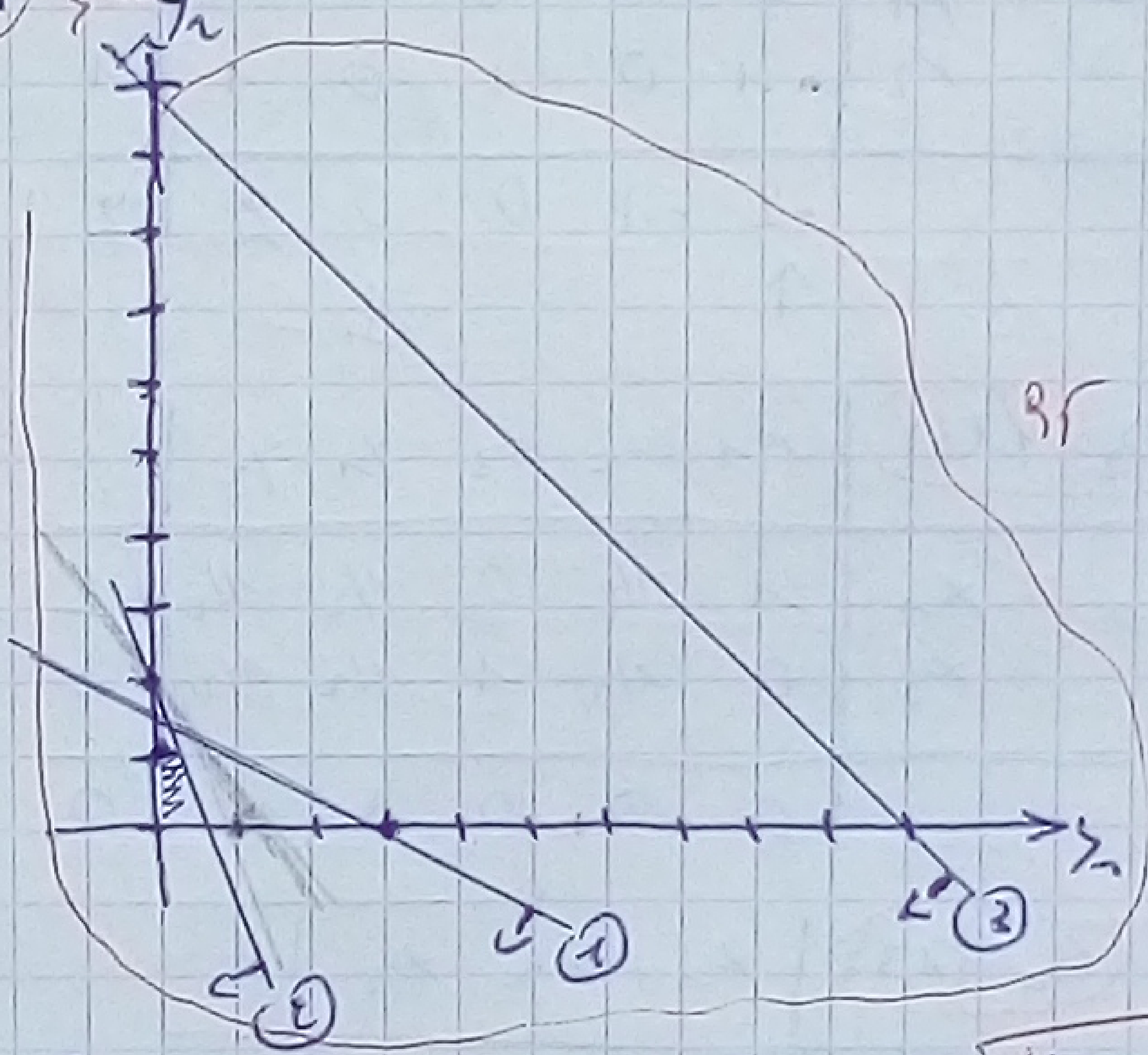
Exo 6 ① x_i = quantité [poids] 0,25 0,25

$\min\ (3x_1 + 2x_2 + 10x_3)$

$$(P)\quad \text{s.c.} \begin{cases} x_1 + 5x_2 + x_3 \geq 11 \\ 2x_1 + x_2 + x_3 \geq 4 \\ x_i \geq 0 \end{cases}$$ 0,25 0,25

② (P^*) $$\begin{array}{l} \max\ 11y_1 + 4y_2 \\ \text{s.c.} \begin{cases} y_1 + 2y_2 \leq 3 \\ 5y_1 + y_2 \leq 2 \\ y_1 + y_2 \leq 10 \\ y_j \geq 0 \end{cases} \end{array}$$

0,5



solution optimale : $\begin{cases} y_1 + 2y_2 = 3 \\ 5y_1 + y_2 = 2 \end{cases} \Leftrightarrow \boxed{\begin{cases} y_1 = 1/9 \\ y_2 = 13/9 \end{cases}}$ 0,5

valeur optimale : $\frac{11}{9} + \frac{52}{9} = \frac{63}{9} = \frac{21}{3} = \boxed{7}$ 0,25

③ $\begin{cases} x_3 = 0 \\ x_1 + 5x_2 = 11 \\ 2x_1 + x_2 = 4 \end{cases}$ 0,25 $\Leftrightarrow \boxed{x = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}$ 0,25 d'après Thm 21

④ D'après question ② et Thm 20, valeur optimale : $\boxed{7}$ 0,5

vérification : $3 \times 1 + 2 \times 2 = 7$