BIMA - Examen (session 1)

11 janvier 2022 - 14h00

Le barème, sur 40, donné à titre indicatif, est susceptible d'être modifié.
Documents et calculatrice non autorisés. Téléphone portable éteint et rangé.
Durée de l'examen : 1 heure 30

The grading scale for each exercise is only an indication and can change.
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Exam duration: 1.5 hour

Exercice 1   Questions de cours (10 points)

1. On réalise une analyse en composantes principales d'un ensemble de données.
   — Que représente la valeur propre associée à une composante et quelle interprétation physique peut-on lui donner ?
   — Définir le rapport d'information expliquée (inertie) lorsqu'on ne retient que K composantes principales. Comment choisir la valeur de K dans un problème de reconnaissance de formes (par exemple des visages) ?
   A principal component analysis is performed on a dataset.
   — *What does the eigenvalue associated with a principal component represent? What could be a physical interpretation?*
   — *Define the ratio of explained information (inertia) when K principal components are kept. How to choose the value of K in a pattern recognition problem (e.g. faces)?*

2. Quelles sont les différences entre l'étirement d'histogramme et l'égalisation d'histogramme ? Dans quels cas utiliser l'un plutôt que l'autre ?
   What are the differences between histogram stretching and histogran equalization? In which case is it best to use one or the other?

3. Peut-on localiser spatialement les structures (contours par exemple) à partir du spectre d'une image ?
   Is it possible to spatially localize structures (e.g. contours) from the spectrum of an image?

4. Expliquer ce qu'est le phénomène d'aliasing, et comment on peut le corriger.
   Explain what is the aliasing phenomenon, and how it can be corrected.

5. Quel critère doit-on maximiser dans une analyse linéaire indépendante ? En pratique, comment l'obtient-on ?
   Which criterion should be maximized in a linear independent analysis? How is this done in practice?
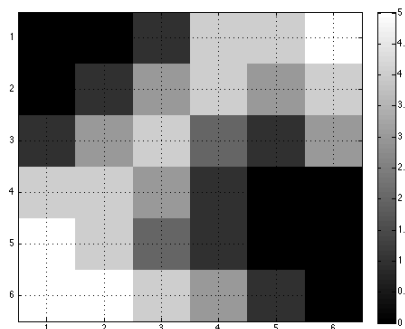
Answer of exercise 1

See the lecture slides.

Exercice 2   Traitement d'images (10 points)

Soit l'image I de taille 6×6 représentée ci dessous :
Let I be the 6×6 image represented below:

$$I = \begin{pmatrix} 0 & 0 & 1 & 4 & 4 & 5 \\ 0 & 1 & 3 & 4 & 3 & 4 \\ 1 & 3 & 4 & 2 & 1 & 3 \\ 4 & 4 & 3 & 1 & 0 & 0 \\ 5 & 4 & 2 & 1 & 0 & 0 \\ 5 & 5 & 4 & 3 & 1 & 0 \end{pmatrix}$$

1. Rappeler la définition d'un histogramme, d'un histogramme normalisé et d'un histogramme cumulé normalisé d'une image.
   Give the definition of histogram, normalized histogram, and normalized cumulative histogram of an image.

2. Calculer et représenter :
   — l'histogramme de I,
   — l'histogramme cumulé de I.
   Calculate and represent:
   — the histogram of I,
   — the cumulative histogram of I.

3. Calculer la moyenne μ et la variance σ^2 de I.
   Indication : on pourra exprimer la moyenne en fonction de l'histogramme et la variance en fonction de la moyenne et de la moyenne des carrés.
   Calculate the mean μ and variance σ^2 of I.
   Indication: the mean can be expressed from the histogram, and the variance from the mean and the mean of squares of the image values.

4. On va centrer et réduire l'image I, de telle sorte que les niveaux de gris de la nouvelle image I' vont s'écrire : $I' = \frac{I-\mu}{\sigma}$.
   — Quelle sont la moyenne et la variance de I' ? En utilisant une valeur à 10^{-1} près pour σ et μ, donner les valeurs minimales et maximales de I'.
   — La transformation pour passer de I à I' est-elle linéaire ? Justifier.
   The centered and reduced image is written as $I' = \frac{I-\mu}{\sigma}$.
   — What are the mean and variance of I'? Using approximate values of σ and μ (with a precision of 10^{-1}), what are the minimal and maximal values of the new image I'?
   — Is the transformation that maps I to I' linear? Justify your answer.

5. On va maintenant appliquer un filtrage médian 3×3 à l'image I.
   — Rappeler le principe du filtrage médian.
     N.B. : on effectuera une opération de zero-padding préalablement au filtrage.
   — Calculer le résultat de l'application d'un filtre médian 3×3 à l'image I.
   — Le filtrage médian est-il linéaire ? Justifier.
   Let us now compute a 3×3 median filter on I.
   — Recall the principle of the median filter.
     NB: apply a zero-padding before the filter.
   — Compute the result of the 3×3 median filter on I.
   — Is the median filter linear? Justify.

Answer of exercise 2

1. — The histogram describes the distribution of the gray levels in the image and is defined as : $H(k) = \mathrm{Card}\{0 \le i \le N-1, 0 \le j \le M-1 : I(i,j) = k\} = n_k$ where $N \times M$ defines the size of the image (i.e. number of pixels), and $I(i,j)$ is the gray value at point (i,j).
   — The normalized histogram is a discrete approximation of the probability density of the random variable "gray levels in an image". It is defined as $H_n(k) = \frac{H(k)}{N \times M}$.
   — The normalized cumulative histogram is a discrete approximation of the cumulative distribution of the random variable "gray levels in an image". It is defined as $H_c(k) = \sum_{i \le k} H_n(i)$.

2. Histogram : $H(0) = 8, H(1) = 7, H(2) = 2, H(3) = 6, H(4) = 9, H(5) = 4$.
   Cumulative histogram : $H_c(0) = 8, H_c(1) = 15, H_c(2) = 17, H_c(3) = 23, H(c)(4) = 32, H_c(5) = 36$
   (note that the last value is always equal to $N \times M$).

3. $\mu = \frac{\sum_k H(k) \times k}{\sum_k H(k)} = \frac{7+4+18+36+20}{36} \sim 2.4$
   $\sigma^2 = \frac{1}{N \times M} \sum_k H(k) \times k^2 - \mu^2 \sim 3.12$, and $\sigma \sim 1.8$.

4. $\mu' = 0, \sigma' = 1$.
   Since the transformation is monotonous, the minimal and maximal values of I' are directly derived
   from the ones of I : $I'_{\min} = \frac{0-\mu}{\sigma} \sim -1.3, I'_{\max} = \frac{5-\mu}{\sigma} \sim 1.4$.

   If I is replaced by λI, then the mean and variance become $\lambda \mu$ and $\lambda^2 \sigma^2$, hence I' becomes λI.

   If I is replaced by $I_1 + I2$, then the mean becomes $\mu_1 + \mu_2$, however the variance is not the sum
   $\sigma_1^2 + \sigma_2^2$.

5. The median filter is obtained by computing at each point x the median value of the gray levels
   of the 9 points $x_1...x_9$ in the 3×3 neighborhood of x. In the example, by using 0 padding, the
   median of I is equal to :

$$\begin{pmatrix} 0 & 0 & 1 & 3 & 4 & 0 \\ 0 & 1 & 3 & 3 & 4 & 3 \\ 1 & 3 & 3 & 3 & 2 & 0 \\ 3 & 4 & 3 & 1 & 1 & 0 \\ 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 4 & 2 & 1 & 0 & 0 \end{pmatrix}$$

   To have less border effect, another option is to compute the median over the only points that
   actually exist in the support of the image. In the case of a 3×3 neighborhood, the choice of one or
   the other option only impacts the first and last lines, and the first and last columns of the image.
   The median filter is not linear since the median value of $I_1(x_1) + I_2(x_1)...I_1(x_9) + I_2(x_9)$ is in
   general not equal to the median value of $I_1(x_1)...I_1(x_9)$ plus the median value of $I_2(x_1)...I_2(x_9)$.

Exercice 3   Echantillonnage (10 points)

Dans cet exercice, on considère le signal $x(t) = \cos(2\pi f_0 t + \alpha)$ défini sur $\mathbb{R}$.
In this exercise, we consider the signal $x(t) = \cos(2\pi f_0 t + \alpha)$ defined on $\mathbb{R}$.

1. Dessiner x sur l'intervalle $[0, T_0]$, avec $\alpha = \frac{\pi}{2}$ et $T_0 = \frac{1}{f_0}$.
   Draw x on the interval $[0, T_0]$, with $\alpha = \frac{\pi}{2}$ and $T_0 = \frac{1}{f_0}$.

2. Toujours avec $\alpha = \frac{\pi}{2}$, calculer et dessiner le spectre de x. Dans la suite, on considère une valeur
   quelconque pour α.
   Compute and draw the spectrum of x for $\alpha = \frac{\pi}{2}$.
   In the following questions, α can take any value.

3. Le signal x est échantillonné avec un pas d'échantillonnage T_s pour former le signal x_s. Exprimer
   x_s en fonction de x.
   *The signal x is sampled with a sampling step T_s, resulting in a signal x_s. Write x_s as a function
   of x.*

4. Calculer le spectre de x_s.
   Compute the spectrum of x_s.

5. Expliquer comment on peut reconstruire x à partir de x_s.
   Explain how x can be reconstructed from x_s.

6. Donner la valeur limite de T_s pour que le signal x puisse être reconstruit sans erreur à partir de
   x_s et montrer que cette valeur ne dépend pas de α.
   *Give the limit value of T_s such that x can be reconstructed without error from x_s, and prove that
   this value does not depend on α.*

Answer of exercise 3

1. $x(t) = \cos(2\pi f_0 t) \cos \alpha - \sin(2\pi f_0 t) \sin \alpha$. For $\alpha = \pi/2$, $\cos \alpha = 0$ and $\sin \alpha = 1$. Hence $x(t) = -\sin(2\pi f_0 t)$, and it is easy to draw its graph.

2.

$$\cos(2\pi f_0 t + \alpha) = \frac{e^{i(2\pi f_0 t + \alpha)} + e^{-i(2\pi f_0 t + \alpha)}}{2}$$

$$= \frac{e^{i\alpha}e^{i2\pi f_0 t} + e^{-i\alpha}e^{-i2\pi f0t}}{2}$$

$$X(f) = \frac{e^{i\alpha}\delta(f - f_0) + e^{-i\alpha}\delta(f + f_0)}{2}$$

which leads, for $\alpha = \pi/2$ to $X(f) = \frac{i}{2}(\delta(f - f_0) - \delta(f + f_0))$, i.e. two Dirac distributions on the imaginary axis, at frequencies f_0 and $-f_0$.

3. The sampling is expressed as the multiplication of x by a Dirac comb of parameter T_s : $x_s(t) = x(t)\sqcup\!\sqcup_{T_s}(t)$.

4.

$$X_s(f) = X \star \mathrm{FT}(\sqcup\!\sqcup_{T_s})$$

$$= \frac{1}{T_s}X \star \sqcup\!\sqcup_{1./T_s}$$

$$= \frac{1}{T_s}\sum_{k \in \mathbb{Z}} X \star \delta(f - k/T_s)$$

$$= \frac{1}{T_s}\sum_{k \in \mathbb{Z}} X(f - k/T_s)$$

5. Select one period of X_s, by multiplying X_s by a rectangular function of width $\frac{1}{T_s}$, and then apply the inverse Fourier transform.

6. The condition for a perfect reconstruction is that the periods of X_s do not overlap. The value of T_s should then satisfy $\frac{1}{T_0} \leq \frac{1}{2T_s}$, i.e. $T_s \leq \frac{T_0}{2}$. The limit value is $\frac{T_0}{2}$ (and does not depend on α).

Exercice 4   Filtres orientables (10 points)

Les filtres orientables sont des copies d'un même filtre après différentes rotations. On peut synthétiser ce filtre dans n'importe quelle orientation à partir d'une combinaison linéaire appropriée d'une base de filtres. Cela permet d'analyser la réponse du filtre non seulement selon un ensemble discret d'orientations, mais sur une plage continue d'orientations.

Steerable filters are all identical copies of the same filter, rotated to different orientations. One can synthesize this filter rotated to any orientation by taking an appropriate linear combination of the basis filters. This allows one to analyze oriented filter response not just at a discrete set of orientations, but over the continuous range of possible orientations.

Soit $F^\theta(x, y)$ le noyau d'un filtre obtenu par rotation d'angle θ de $F(x, y)$, c'est-à-dire $F^\theta(x, y) = F(x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta)$. On ignore les questions d'échantillonnage, et on considère que les points (x, y) sont définis dans l'espace continu. $F(x, y)$ est un filtre orientable nécessitant M filtres de base si on peut écrire

$$F^\theta(x, y) = \sum_{i=1}^{M} k_i(\theta)F_i(x, y)$$

où les F_i sont les filtres de base, qui ne dépendent que de (x, y) et pas de θ, et k_i sont des fonctions, appelées fonctions d'interpolation, qui ne dépendent que de θ et pas de (x, y).

Let $F^\theta(x, y)$ be a filter kernel obtained from the rotation of angle θ of $F(x, y)$. So $F^\theta(x, y) = F(x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta)$. We ignore sampling issues and work in a continuous (x, y) space. $F(x, y)$ is a steerable filter requiring M basis filters if we can write

$$F^\theta(x, y) = \sum_{i=1}^{M} k_i(\theta)F_i(x, y)$$

where F_i are the basis filters, that depend only on (x, y) and not on θ, and k_i are called the interpolation functions, that depend only on θ and not on (x, y).

Le filtre orientable le plus simple est la dérivée directionnelle d'une fonction à symétrie radiale $G(r)$, où $r = \sqrt{x^2 + y^2}$. Soit $G(r) = e^{-(x^2+y^2)}$ une fonction gaussienne.

The simplest possible steerable filter is the directional derivative of a radially symmetric function $G(r)$, where $r = \sqrt{x^2 + y^2}$. Let $G(r) = e^{-(x^2+y^2)}$ be a Gaussian function.

1. Montrer que $G(x, y)$ est invariant par rotation d'angle θ.
   Prove that $G(x, y)$ is invariant by a rotation of angle θ.

2. Montrer que $F = \frac{\partial G}{\partial x}$ est un filtre orientable avec $M = 2$. Donner la base des filtres et les fonctions d'interpolation.
   Prove that $F = \frac{\partial G}{\partial x}$ is a steerable filter with $M = 2$. Give the set of basis filters and interpolation functions.

3. Montrer que la dérivé seconde d'une gaussienne, $H = \frac{\partial^2 G}{\partial x^2}$ est aussi un filtre orientable. Donner la base des filtres et les fonctions d'interpolation, en utilisant le plus petit nombre M de termes possibles.
   Prove that the second derivative of a Gaussian $H = \frac{\partial^2 G}{\partial x^2}$ is also a steerable filter. Give a set of basis filters and interpolation functions with the smallest possible number M.

4. Expliquer comme échantillonner le filtre H, comment obtenir un filtre de support borné et proposer des bornes pertinentes pour ce support.
   Explain how to sample the filter H, how to obtain a filter with a bounded support and give relevant boundaries of this support.

5. (Question bonus) Écrire un code Python qui prend en argument un angle θ et qui retourne les filtres F^θ et H^θ.
   (Bonus question) Write a Python code that takes as argument an orientation θ and returns filters F^θ and H^θ.

6. Donner une application du filtre F^θ en analyse d'images.
   Give an application of filter F^θ for image processing.

Answer of exercise 4

1.

$$
\begin{aligned}
G(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) &= e^{-(x^2\cos^2\theta + y^2\sin^2\theta + 2xy\cos\theta\sin\theta + x^2\sin^2\theta - 2xy\cos\theta\sin\theta + y^2\cos^2\theta)} \\
&= e^{-x^2(\cos^2\theta+\sin^2\theta) - y^2(\sin^2\theta+\cos^2\theta)} = e^{-(x^2+y^2)} \\
&= G(x, y)
\end{aligned}
$$

2.

$$
\begin{aligned}
F(x, y) &= \frac{\partial G}{\partial x} \\
&= -2xG(x, y) \\
F^\theta(x, y) &= F(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) \\
&= -2(x\cos\theta + y\sin\theta)G^\theta(x, y) \\
&= -2\cos\theta\, xG(x, y) - 2\sin\theta\, yG(x, y)
\end{aligned}
$$

since $G^\theta = G$. This proves that F is steerable and we have :

$$
\begin{aligned}
k_1(\theta) &= -2\cos\theta \\
k_2(\theta) &= -2\sin\theta \\
F_1(x, y) &= xG(x, y) \\
F_2(x, y) &= yG(x, y)
\end{aligned}
$$

3.

$$
\begin{aligned}
H(x,y) &= \frac{\partial F}{\partial x} \\
&= -2G(x,y) + 4x^2 G(x,y) \\
&= (4x^2 - 2)G(x,y) \\
H^\theta(x,y) &= (4(x\cos\theta + y\sin\theta)^2 - 2)G\theta(x,y) \\
&= (4x^2\cos^2\theta + 8xy\cos\theta\sin\theta + 4y^2\sin\theta - 2)G(x,y)
\end{aligned}
$$

Then H is steerable with $M = 4$ (the smallest possible value) and we have :

$$
\begin{aligned}
k_1(\theta) &= -2 & F_1(x,y) &= G(x,y) \\
k_2(\theta) &= 4\cos^2\theta & F_2(x,y) &= x^2 G(x,y) \\
k_3(\theta) &= 8\cos\theta\sin\theta & F_4(x,y) &= xy G(x,y) \\
k_4(\theta) &= 4\sin^2\theta & F_4(x,y) &= y^2 G(x,y)
\end{aligned}
$$

4. We sample the Gaussian kernel G as seen in the course : $G(i,j)$ with $i = -N \cdots N$ and $j = -N \cdots N$. G is considered almost null for $|i|$ and $|j|$ greater than $\lceil 3\sigma \rceil$, and it is also the case for $iG(i,j)$, $jG(i,j)$... Here $2\sigma^2 = 1$, $\sigma = 1/\sqrt{2} \simeq 1/1.4 \simeq 0.7$. Then $N = 3$.

5.
```python
def steerable(theta):
    Ftheta = np.zeros((3,3))
    Htheta = np.zeros((3,3))
    C = math.cos(theta)
    S = math.sin(theta)
    C2 = C*C
    S2 = S*S
    for x in range(-1,2):
        for y in range(-1,2):
            x2=x*x
            y2=y*y
            G = math.exp(-(x2 + y2))
            Ftheta[x+1,y+1] = -2*G*(C*x+S*y)
            Htheta[x+1,y+1] = G*(-2 + 4*C2*x2 + 8*C*S*x*y + 4*S2*y2)
    return Ftheta, Htheta
```

A more efficient Numpy version :
```python
def steerable(theta):
    Y,X = np.meshgrid(np.arange(-1,2),np.arange(-1,2))
    G = np.exp(-X*X-Y*Y)
    CX = np.cos(theta)*X
    SY = np.sin(theta)*Y
    return -2*G*(CX+SY), G*(-2 + 4*CX*CX + 8*CX*SY + 4*SY*SY)
```

6. Local orientation maps, edge detection...

Annexes

Transformée de Fourier d'un signal x continu :
Fourier transform of a continuous signal x:

$$
f \mapsto \text{FT}(x)(f) = X(f) = \int_{\mathbb{R}} x(t)e^{-2i\pi ft}dt
$$

Propriétés de la transformée de Fourier continue – *Properties of the continuous Fourier transform*

	$x(t)$	$X(f)$
linéarité – *linearity*	$x(t) + \lambda y(t)$	$X(f) + \lambda Y(f)$
translation – *translation*	$x(t - t_0)$	$X(f)\, e^{-2i\pi f t_0}$
contraction – *contraction*	$x(\alpha t)$	$\frac{1}{|\alpha|} X(\frac{f}{\alpha})$
convolution – *convolution*	$x \star y(t)$	$X(f) \times Y(f)$
produit – *product*	$x(t) \times y(t)$	$X \star Y(f)$

Propriétés de la distribution de Dirac – *Properties of the Dirac distribution*

— $\int_{-\infty}^{+\infty} \delta(t) dt = 1$.
— $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
— $x \star \delta(t - t_0) = x(t - t_0)$
— scaling property : $|\alpha| \cdot \delta(\alpha t) = \delta(t)$
— $TF\left[\delta(t - t_0)\right] = e^{-2i\pi f t_0}$ et $TF\left[e^{2i\pi f_0 t}\right] = \delta(f - f_0)$

Transformée de Fourier de fonctions usuelles – *Some usual Fourier transforms*

Signal	FT
1	$\delta(f)$
$\delta(t)$	1
$e^{2i\pi f_0 t}$	$\delta(f - f_0)$
$\delta(t - t_0)$	$e^{-2i\pi f t_0}$
$\text{Ш}_T(t) = \sum\limits_{k=-\infty}^{+\infty} \delta(t - kT)$	$\frac{1}{T}\text{Ш}_{\frac{1}{T}}(f) = \frac{1}{T}\sum\limits_{k=-\infty}^{+\infty} \delta(f - \frac{k}{T})$