CheatSheet

Wednesday, January 4, 2023 3:27 PM

I HMM

$\Pi = P(s_1 = i)$

$A = P(s_t = j \mid s_{t-1} = i)$ $\quad B$ ($x_1 \dots x_c$ in columns)

$B = P(x_t \mid s_t = i)$

apprentissage = Baum-Welch | Si étiquetté : Comptage ez

Décodage : Viterby | Evaluation : Forward

1) • Forward : Calculer la proba de $X = (x_1, \dots, x_m) = x_1^T$

But : Trouver $P(x_1^T \mid \lambda) = \sum P(x_1^T, s_1^T \mid \lambda)$ ← Problème de combinatoire

• $\alpha_0(j) = \Pi_j \, B_j(x_1)$

• $\alpha_t(j) = P(x_1^t, s_t = j)$

$j \to$ chaque état

$= b_j(x_t) \sum_i^{\text{état}} a_{ij} \, \alpha_{t-1}(i)$

$$A = \begin{pmatrix} 0{,}99 & 0{,}01 \\ 0{,}01 & 0{,}99 \end{pmatrix}, \quad B = \begin{pmatrix} 0{,}8 & 0{,}1 \\ 0{,}2 & 0{,}9 \end{pmatrix}$$

t	x_t	$\alpha_t(A)$ $(j=1)$	$\alpha_t(B)$ $(j=2)$
1	1	$0{,}99 \times 0{,}8 = 0{,}792$	$0{,}01 \times 0{,}1 = 0{,}001$
2	0	$0{,}2(0{,}99 \times 0{,}792 + 0{,}01 \times 0{,}001) = 0{,}156818$	$0{,}9(0{,}01 \times 0{,}792 + 0{,}99 \times 0{,}001) = 8{,}019 \times 10^{-3}$
3	1	$0{,}8(0{,}99 \times 0{,}156818 + 0{,}01 \times 8{,}019 \times 10^{-3}) \approx 0{,}1237$	$0{,}1(0{,}01 \times 0{,}156818 + 0{,}99 \times 8{,}019 \times 10^{-3}) \approx 9{,}5 \times 10^{-4}$

$\Sigma \approx 0{,}1247$

2) • Viterby : Seq Observation ⟹ Seq Etat

Pb : $\hat{s}_1^T = \underset{s_1^T}{\text{argmax}} \prod_{t=1} P(s_t \mid s_{t-1}) \, P(x_t \mid s_t)$

$\delta_1(i) = \Pi_j \, B_i(x_1)$ $\quad | \quad \Psi_t(j) = \underset{i \in [1,N]}{\arg\max} \, \delta_{t-1} \, a_{ij}$

$\delta_t(i) = b_j(x_t) \left[\max_i \delta_{t-1} \, a_{ij} \right]$

$$\begin{pmatrix} \delta(1) & \delta(1) \\ \delta(2) & \delta(2) \end{pmatrix} \cdot A = \begin{pmatrix} \cdot & \cdot \\ \cdot & \vdots \end{pmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

max max argmax

x_t		max 1	max 2	$\delta_t(A)$	$\delta_t(B)$
0		$0{,}792$	$0{,}001$		
1	A	$0{,}792 \times 0{,}99 = \boxed{0{,}7840}$	$0{,}792 \times 0{,}01 = \boxed{0{,}00792}$	$0{,}7840 \times 0{,}2 = 0{,}1568$	$0{,}00792 \times 0{,}9 = 7{,}128 \times 10^{-3}$
	B	$0{,}001 \times 0{,}01 = 1 \cdot 10^{-5}$	$0{,}001 \times 0{,}99 = 0{,}00099$		
0	A	$0{,}1568 \times 0{,}99 = \boxed{0{,}1552}$	$0{,}1568 \times 0{,}01 = 1{,}568 \cdot 10^{-3}$	$0{,}1552 \times 0{,}8 = 0{,}12416$	$7{,}0567 \cdot 10^{-3} \times 0{,}1 = 7{,}05 \cdot 10^{-4}$
	B	$7{,}128 \times 0{,}01 \times 10^{-3} = /$	$7{,}128 \times 0{,}99 \times 10^{-3} = \boxed{7{,}0567 \cdot 10^{-3}}$	max	

max max

II Regression

1) Regression normal

$Y = \alpha + BX + \varepsilon \qquad \varepsilon \sim \mathcal{N}(0, \sigma)$

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$Y_i \sim \mathcal{N}(\alpha + Bx_i, \sigma)$

• Fonction de cout

$F(\alpha, B) = \sum (y_i - \alpha + Bx_i)^2 \rightsquigarrow \underset{\alpha, B}{\text{argmin}} \, F(\alpha, B)$

$$\nabla F = 0 \iff \begin{cases} \bar{y} = \alpha + B\bar{x} \\ \sum_{i=1}^m x_i (y_i - (\alpha + Bx_i)) = 0 \end{cases}$$

$$\iff \begin{cases} b = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \dfrac{\text{cov}(X, Y)}{V(X)} \\ a = \bar{y} - \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \bar{x} = \bar{y} - \dfrac{\text{cov}(X,Y)}{V(X)} \bar{x} \end{cases}$$

• Max Vraissemblance

$Y = f(X) \overset{\text{exemple}}{=} \alpha X^2 + BX + \gamma + \varepsilon$

$Y_i \sim \mathcal{N}(\alpha x_i^2 + Bx_i + \gamma, \sigma)$

$$P(y_i \mid x_i, \theta, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i - f(x))^2}$$

$$L = \prod_i \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - f(x_i))^2}$$

$$LL = \sum -\log(\sqrt{2\pi}\sigma) - \frac{1}{2\sigma^2}(y_i - f(x_i))^2$$

$\text{argmax} \, \ell = \text{argmax} - \sum -(y_i - f(x_i))^2$

$$\nabla = 0 \iff \begin{cases} \sum 2x_i^2 (y_i - \alpha x_i^2 - Bx_i - \gamma) = 0 \\ \sum 2x_i (y_i - \alpha x_i^2 - Bx_i - \gamma) = 0 \\ \sum 2 (y_i - \alpha x_i^2 - Bx_i - \gamma) = 0 \end{cases}$$

$\iff A \begin{pmatrix} \alpha \\ B \\ \gamma \end{pmatrix} = B$

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• R^2 & Variance

$$\sigma_y = \frac{1}{m} \sum (y_i - \bar{y})^2 = \frac{1}{m} \sum (\hat{y}_i + e_i - \bar{y})^2$$

⟹ Developpement + $\text{cov}(e, \hat{y}) = 0$

$= \frac{1}{m} \sum (\hat{y} - \bar{y})^2 + \frac{1}{m} \sum (e_i)^2 = $ Var expliquée + Var erreur résiduelle

$$R^2 = 1 - \frac{\text{Variance Residuel}}{\text{Variance Totale}} = 1 - \frac{\sum (y_i - \hat{y})^2}{\sum (y_i - \bar{y})^2}$$

$$= 1 - \frac{\sum (e_i)^2}{\sigma_y}$$

• Multidim

$Y_i = B_0 + B_1 x_{i1} + B_2 x_{i2} + \varepsilon_i$

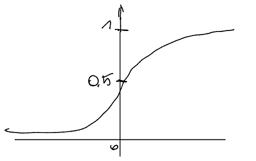
$\iff Y_i = [1 \;\; x_{i1} \;\; x_{i2}] \begin{pmatrix} B_0 \\ B_1 \\ B_2 \end{pmatrix} + \varepsilon_i = \mathbb{X}B + \varepsilon$

$\mathbb{X} = \begin{bmatrix} 1 & x_1^T \\ \vdots & \vdots \\ 1 & x_m^T \end{bmatrix} \quad \hat{B} = (X^T X)^{-1} X^T Y$

$\sim \mathcal{N}(B, \sigma^2 (X^T X)^{-1})$

2) Regression Logistique

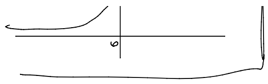
$$P(Y = 1 \mid X = \mathbf{x}) = f(\mathbf{x}) = \frac{1}{1 + e^{-(\mathbf{x}w + b)}}$$



$L = \prod P(Y = y_i \mid X = \mathbf{x}_i)$

$LL = \sum y_i \log(f(\mathbf{x}_i)) + (1 - y_i) \log(1 - f(\mathbf{x}_i))$

$\frac{\partial LL}{\partial w_j} = \sum x_{ij} (y_i - f(\mathbf{x}_i))$



$\frac{\partial LL}{\partial w_j} = \sum \ldots$

$\frac{\partial LL}{\partial b} = \sum y_i - f(x_i)$

$\Rightarrow$ Montée de gradient OBLIGATOIRE

$\boxed{\text{Algo Descente de gradient}}$

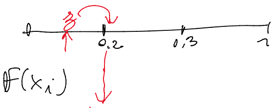
$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{pmatrix}$, $X_i \in \mathbb{R}^d$

```
def fcn (X, Y, max_iter=100, E=, γ=
    N, d = X.shape        # Dimention
    w = np.zeros(d,1)     # param
    for _ in range(max_iter):
        w_0 = w.copy()
        w = w + E x ∇f(X, w)
        if ((w - w_0)^2).sum() < gamma
            Break
    Return w
```

II Sampling

1) Somme cumulée fct discrète

Cum sum des probas



$z \sim \mathrm{Unif}([0,1[)$

$X_0 = 0$, $F(X_{i-1}) \leq z \leq F(X_i)$

Renvoyer X_i

On prend le x_i supp

2) Inversion de la fonction de rep

$F^{-1} :]0,1[\to \mathbb{R}$

$\alpha \to \mathrm{Inf}\{t \in \mathbb{R}, F_X(t) \geq \alpha\}$

- Sur le graph de $F(x)$, placer α sur l'ordonnée
- Faire varier α sur chaque intervalle tq $F_X \geq \alpha$
- Puis prendre l'inf de la borne inférieure de cette intervalle

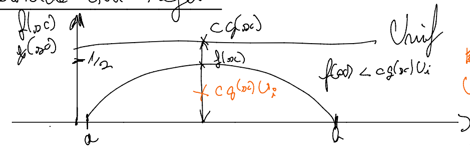
Exemple analytique : loi exp

$\forall \alpha \in]0,1[, F_X^{-1} = \mathrm{Inf}\{t \in \mathbb{R}, F_X(t) \geq \alpha\}$

$= \mathrm{Inf}\{t \in \mathbb{R}_*^+, 1 - e^{-\lambda t} \geq x\}$

$= \mathrm{Inf}\{t \in \mathbb{R}_*^+, t \geq \frac{\ln(1-\alpha)}{-\lambda}\} = -\frac{\ln(1-\alpha)}{\lambda}$

3) Méthode du rejet



tirage

U_i tirage $\sim \mathrm{Unif}[0,1]$

$X \sim f(x)$, $g(x) \sim \mathrm{Unif}[a, b]$

$\Rightarrow \frac{1}{b-a} \mathbb{1}_{]a,b[}(x)$

? ensemble pas fini

1) Tirer z selon $g(x)$
2) calculer $k\, g(z) = mq$
3) Tirer $u \sim \mathrm{Unif}[0, mq)$
4) accepter z ssi $u \leq f(z) = mq\, f(\ldots))$

$P(\text{acceptés}) = \int \frac{g(x)}{k \times g(z)}$

4) MC MC

Construction d'une chaine de Markov de loi stationaire $\pi(\cdot)$

- Metropolis Hasting

$\pi(x) P(y|x) \alpha(x, y) = \pi P(x|y) \alpha(y, x)$

$\alpha(x, y) = \frac{\pi(y) P(x|y)}{\pi(x) P(y|x)}$ $\quad \alpha(y, x) = 1$

Proba de realiser transition

- Generer x_{t+1} à partir de x_t
  1) tirer z selon $P(\cdot | x_t)$ (à choisir ?)
  2) $\alpha(x_t, z) = \min\{1, \alpha(x_t, z)\}$
  3) tirer $u \sim \mathrm{Unif}[0,1[$
  4) $x_{t+1} = z$ (si $u \leq \alpha(x_t, z)$ / x_t)

$P(z | x_t) = \begin{cases} q(\cdot) \leadsto \text{Ui random walk ?} \\ q(z) : \text{independent (r.s) ?} \end{cases}$

- Metropolis-Hastings par bloc

Generer $x_{t+1} = \{x_{t+1}^1 \ldots x_{t+1}^n\}$ de x_t

1) Choisir σ : permutation $1, \ldots, n$

$\forall i$ in σ :

a) $y = \{x_{t+1}^{\sigma(1)}, \ldots, x_{t+1}^{\sigma(i-1)}, x_t^{\sigma(i+1)}, \ldots, x_t^{\sigma(n)}\}$

b) tirer $z^{\sigma(i)} \sim P(\cdot | x_t^{\sigma(i)}, y)$

c) calculer $\alpha(x_t^{\sigma(i)}, z^{\sigma(i)} | y)$

d) Tirer $u \sim \mathrm{Unif}[0,1[$

e) $x_{t+1}^{\sigma(i)} = \begin{cases} z^{\sigma(i)} \text{ si } u \leq \alpha \\ x_t \text{ sinon} \end{cases}$

4) Echantillonage de Gibbs

$P(z^{\sigma(i)} | x_t^{\sigma(i)}, y) = \pi(z^{\sigma(i)} | y)$

$x_{t+1}^{\sigma(i)} = z^{\sigma(i)}$