BIMA - Exam

January 2019

The grading scale for each exercise is only an indication and can change. No document authorized. Exam duration: 2 hours

Exercise 1 Questions on the lectures (10 points)

1. In the Fourier domain, is it possible to spatially localize the structures? Is it possible to measure their size? Justify.
2. Compare edge detectors using first order and second order derivatives, in terms of robustness to noise and edge localization.
3. Is the Harris operator invariant under translation? rotation? scale? Justify.
4. Is the Harris detector invariant under an affine change of intensity $I' = aI + b$ (where I is the image and a, b two constant coefficients)? Explain your answer and discuss what should be changed in the detector criterion.
5. Explain the minimization criterion on the variance projected on a vectorial line defined by a vector v in PCA method. How can such a system be solved in practice?
6. Explain the criterion that is maximized in the LDA method and how this optimization problem is solved. What is the fundamental difference in comparison to PCA?

Answer of exercise 1

1. In the Fourier domain, it is not possible to spatially localize (i.e. in the image domain) the structures, since the spectrum is invariant under translation. Size can be measured. It corresponds to the periods of the sin functions.
2. First order: more robust to noise, less accurate localization (except if the local extrema in the gradient direction are selected). Second order: less robust to noise, better localization.
3. The Harris detector is invariant under rotation (it is based on the eigenvalues of a symmetric definite positive matrix, which are not changed if a rotation matrix is applied). It is also invariant under translation since it relies on a high-pass linear filter. It is not invariant under scale. Multi-scale extensions are possible.
4. If the Harris detector is applied to image $I' = aI+b$, b is not involved in the derivatives, and therefore not in the computation of the matrix M. The new criterion writes $R' = det(M') - kTr(M')^2 = a^4R$. To obtain a detector that is invariant under such transformation, it is sufficient to change the threshold on R according to this proportionality relation.

5. The criterion writes $v^T \Sigma v$ where Σ is the variance/covariance matrix. In practice, this amounts to compute the eigenvalues of Σ and to sort them by decreasing order.

6. The criterion writes $v^T Bv/v^T \Sigma v$ where v is a vector, B the inter-class variance (between the centroid of different classes) and Σ the variance/covariance matrix. In practice, this amounts to compute the eigenvalues of $\Sigma^{-1}B$ and to sort then by decreasing order. In contrast to PCA (which performs dimension reduction in a non-supervised way), the notion of class is here involved in the computation of B.

Exercise 2 Fourier transform (4 points)

The continuous Fourier transform (FT) X of a continuous signal x is defined as: $X(f) = \int_{\mathbb{R}} x(t)e^{-2i\pi ft}dt$. The inverse Fourier transform (IFT) is defined as: $x(t) = \int_{\mathbb{R}} X(f)e^{2i\pi ft}df$.

1. The "gate" function is defined as: $\text{Rect}(f) = \begin{cases} 1 & \text{is } |f| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$.

   Compute its inverse Fourier transform

2. Does this gate function define an ideal filter? What are its drawbacks?

3. Show that the spectrum of the continuous Fourier transform of an image is invariant under translation $\vec{t} = (t_1, t_2)^T$. *Indication*: compute the Fourier transform of $(x, y) \mapsto I(x + t_1, y + t_2)$ and apply a change of variable.

Answer of exercise 2

1.

$$\begin{aligned} IFT(\text{Rect})(t) &= \int_{\mathbb{R}} \text{Rect}(f)e^{2i\pi ft}df \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2i\pi ft}df \\ &= \left[\frac{e^{2i\pi ft}}{2i\pi t}\right]_{f=-\frac{1}{2}}^{f=\frac{1}{2}} \\ &= \frac{e^{i\pi t} - e^{-i\pi t}}{2i\pi t} \\ &= \frac{\sin(\pi t)}{\pi t} = \text{sinc(t)} \end{aligned}$$

2. Rect defines an ideal filter since its support is bounded. The main drawback is that it induces aliasing when the spatial signal is reconstructed.

3.

$$
\begin{aligned}
FT((x,t) \mapsto I(x+t_1, y+t_2))(f,g) &= \iint_{\mathbb{R}^2} I(x+t_1, y+t_2) e^{2i\pi(fx+gy)} dxdy \\
\text{define} \quad x' = x+t_1, y' = y+t_2 : & \\
&= \iint_{\mathbb{R}^2} I(x', y') e^{2i\pi(f(x'-t_1)+g(y'-t_2))} dx'dy' \\
&= \iint_{\mathbb{R}^2} I(x', y') e^{2i\pi(fx'+gy')} e^{2i\pi(-ft_1-gt_2)} dx'dy' \\
&= e^{-2i\pi(ft_1+gt_2)} FT((x,y) \mapsto I(x,y))(f,g)
\end{aligned}
$$

The two spectra are equal up to a coefficent ($e^{-2i\pi(ft_1+gt_2)}$), but which has a modulus equal to 1.

Exercise 3 Signal reconstruction (10 points)

1. Compute the FT of the gate function.
2. Let x be a signal with bounded support. Prove that its FT is not with bounded support. (*Indication*: use the gate function).
3. Let $x(t) = \sin(2\pi f_0 t)$. Draw x on a complete period centered at 0. What is the value of this period?
4. Compute the FT of x and draw its spectrum.
5. In practice, signals are limited in time, i.e. they have a bounded support. Therefore the actual signal is $x_L(t) = x(t)\,\mathrm{Rect}(\frac{t}{L})$.
   Compute the FT of x_L. Is is possible to retrieve the FT of x from the one of x_L? Is is possible to reconstruct x?

Answer of exercise 3

1. See question 1 of Exercise 2.
2. If x has a bounded support, then there exists a large enough value of L such that $x(t) = x(t)Rect(\frac{t}{L})$. The TF is then $LX \star \mathrm{sinc}(f/L)$, whose support is not bounded.
3. The period is $T_0 = \frac{1}{f_0}$.
4.

$$
\begin{aligned}
X(f) &= FT\left(\frac{e^{2i\pi f_0 t} - e^{-2i\pi f_0 t}}{2i}\right) \\
&= \frac{1}{2i}\left(\delta(f-f_0) - \delta(f+f_0)\right)
\end{aligned}
$$

5. $X_L(f) = \frac{1}{2i}\left(\delta(f-f_0) - \delta(f+f_0)\right) \star \mathrm{sinc}(\frac{f}{L}) = \frac{1}{2i}(\mathrm{sinc}(\frac{f}{L} - f_0) - \mathrm{sinc}(\frac{f}{L} + f_0))$, i.e. two sinc functions centered at f_0 and $-f_0$.

   The FT of x can be retrieved if it is possible to apply a deconvolution to $X_L = X \star \mathrm{sinc}(f/L)$. Then x is reconstructed by computing the inverse FT.

   Note that this is possible because x is a continuous signal and its continuous FT is computed. If the signal is sampled to get a discrete signal, then this is no more possible.

Exercise 4 Spatial filtering (16 points)

Let us consider a family of Gabor filters, defined on the spatial domain, with impulse response defined as:

$$(x,y) \mapsto h_{\theta,\sigma,\gamma,f_0}(x,y) = \exp\left(-\frac{x'^2 + \gamma y'^2}{2\sigma^2}\right)\cos(2\pi f_0 x') \quad (1)$$

$$x' = x\cos\theta + y\sin\theta, \quad x \in \mathbb{R} \quad (2)$$

$$y' = -x\sin\theta + y\cos\theta, \quad y \in \mathbb{R} \quad (3)$$

Part I

1. Which spatial transformation do Equations 2 and 3 represent? What is the role of the parameter θ? What is its domain of definition?
2. Assume $\theta = 0$, $f_0 = 0$ and $\gamma = 1$. Write the mathematical expression of h. Which spatial filter is it? What is the role of the parameter σ? What is its domain of definition?
3. Assume $\theta = 0$, $f_0 = 0$, $\sigma = 1$, and $\gamma = 1$. What is the value of h at points (x, y) located at Euclidean distance σ of the space origin? (Provide a formal expression, without approximations). The obtained value is now denoted by z_0.
4. Derive the shape of the intersection of the graph of $h_{\theta,1,1,0}$, i.e. $(x, y, z = h_{\theta,1,1,0}(x, y))$, with the plane $z = z_0$ in $\mathbb{R}^3$. What happens when θ varies?
5. Assume now $\gamma = \frac{1}{2}$. What is the shape of the intersection of graph of $h_{\theta,1,\frac{1}{2},0}$ with the plane $z = z_0$?
   Indication: you may recognize the spatial transform $(x, y) \mapsto (x, \gamma y)$.
   What is the shape of $(x, y, z = h_{\theta,1,\frac{1}{2},0})$ in $\mathbb{R}^3$? What is the definition domain of γ?
6. Derive the influence of parameters θ and γ on the shape and orientation of the Gaussian function.
7. Assume $\sigma = 1$, $f_0 = 1$ and $\theta = 0$. Draw roughly the graph of function $x \mapsto h_{0,1,\gamma,1}(x, 0)$ on the interval $[-2, 2]$.

Part II

Assume $\theta = 0$.

1. Compute the FT of $(x, y) \mapsto \cos(2\pi f_0 x')$.

2. Compute the FT of $(x, y) \mapsto \exp\left(-\frac{x'^2+\gamma y'^2}{2\sigma^2}\right)$.

3. Derive the transfer function H of $h_{f_0,\sigma,\gamma,0}$ and draw roughly the profile of the spectrum $H(u, 0)$.

4. Now, if θ varies, what is the influence of its value on the spectrum of the transfer function H?

5. Is it a low-pass, high-pass or band-pass filter? Is it an ideal filter?

Answer of exercise 4

Part I:

1. Rotation of angle $\theta \in [0, 2\pi]$ and center the space origin.

2. $h_{0,\sigma,1,0}(x, y) = \exp(-\frac{x^2+y^2}{2\sigma^2})$. It is a 2D Gaussian function, with standard deviation $\sigma > 0$.

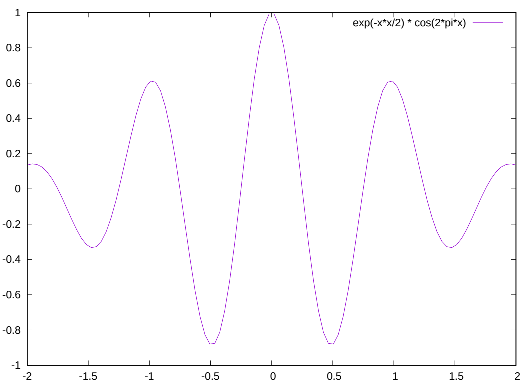
3. $h_{0,1,1,0}(x, y) = \exp(-\frac{x^2+y^2}{2})$. $z_0 = e^{-1/2}$

4. Circle of radius 1 and centered at origin. The rotation does not change the shape of the circle, nor of the Gaussian.

5. Contraction of γ along the y axis. The circle is transformed into an ellipse with smallest radius equal to 1/2. γ takes values in $\mathbb{R}^+$.

6. These two parameters control the shape (γ) and the main direction of the Gaussian (θ).

7. $h(x) = e^{-x^2/2}\cos(2\pi x)$. Graph:



Part II:

1. The FT of $(x', y) = \cos(2i\pi x')$ is $\frac{\delta(f-f0,g)+\delta(f+f_0,g)}{2}$. The spatial rotation acts in a similar way in the frequency domain. Hence the FT is still two Dirac distributions, oriented in direction θ. Here $\theta = 0$, hence $x' = x$ (and $y' = y$).

2. The FT of a Gaussian with standard deviation σ is a Gaussian with standard deviation proportional to $\frac{1}{\sigma}$.
   We have $e^{-\frac{x^2+\gamma y^2}{\sigma^2}} = e^{-\frac{x^2}{\sigma^2}} e^{-\frac{\gamma x^2}{\sigma^2}}$.
   The FT is approximately: $\exp(-\sigma^2 \frac{f^2+g^2/\gamma}{2})$

3. The transfer function is the continuous FT, here a Gaussian convolved with the two Dirac distributions.

4. If is a band-pass or low-pass filter, depending on the value of f_0. It is not an ideal filter (because of the Gaussian).