Fundamentals of Image Processing

▶ Lecture 3: Fourier transform ◀

Master of Computer Science
Sorbonne University

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Context and goals

- Fourier transform (FT): fundamental tool for image processing
- Fourier transform and its applications are discussed in the four next lectures:
  - Today: definition and application
  - Lecture 4: sampling and discrete Fourier transform (DFT)
  - Lecture 5: spatial filtering $\Rightarrow$ filtering in Frequency domain
  - Lecture 6: edge detection and spatial filtering
- Understand the Fourier representation space
  - several representations of a same function: of time (or space) $\Rightarrow$ of frequency
- Definition of mathematical tools (convolution, Dirac, usual FT) crucial for this lecture
- Definition and properties of the 2D Fourier transform
  - Calculus in Fourier spaces, visualize and interpret a spectrum

Outline

Context: analysis of 1D signals in the Frequency domain

- Fourier Series

Fourier transform of signals (1D)

Fourier transform of images (2D)

2D Fourier transform: application to image processing

Decomposition of periodic functions into Fourier series

- Let x be a real (or complex) T-periodic function. We have for $t \in \mathbb{R}$:

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{+\infty}\left(a_k \ \cos\left(\frac{2k\pi t}{T}\right) + b_k \sin\left(\frac{2k\pi t}{T}\right)\right) \quad (1)$$

- Coefficients a_k et b_k are given by:

$$\begin{aligned} a_k &= \frac{2}{T}\ \int_0^T x(t)\cos\left(\frac{2k\pi t}{T}\right) dt \\ b_k &= \frac{2}{T}\ \int_0^T x(t)\sin\left(\frac{2k\pi t}{T}\right) dt \end{aligned} \quad (2)$$

Fourier series: decomposition of periodic functions

Writing x with amplitude and phase

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{+\infty}\left(a_k\ \cos\left(\frac{2k\pi t}{T}\right) + b_k \sin\left(\frac{2k\pi t}{T}\right)\right)$$

- Amplitude: $r_k = \sqrt{a_k^2 + b_k^2}$
  Phase: θ_k, such that $\cos\theta_k = \frac{a_k}{r_k}$ and $\sin\theta_k = \frac{b_k}{r_k}$
- Equation (1) becomes:

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{+\infty} r_k\ \cos\left(\frac{2k\pi t}{T} - \theta_k\right) \tag{3}$$

(remember that $\cos a \cos b + \sin a \sin b = \cos(a-b)$)

Fourier series decomposition of periodic functions

Writing x with the complex Exponential function

Some useful recalls:

- Euler formula:

$$\exp(iz) = \cos(z) + i\ \sin(z)$$

- de Moivre formula:

$$\exp(ikz) = \ (\cos(z) + i\sin(z))^k = \ \cos(kz) + i\ \sin(kz)$$

$\Rightarrow$

$$\begin{aligned} \cos(kz) &= \frac{\exp(ikz) + \exp(-ikz)}{2} \\ \sin(kz) &= \frac{\exp(ikz) - \exp(-ikz)}{2i} \end{aligned}$$

Fourier series decomposition of periodic functions

Writing x with the complex Exponential function (cont'd)

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{+\infty}\left(a_k\ \cos\left(\frac{2k\pi t}{T}\right) + b_k\ \sin\left(\frac{2k\pi t}{T}\right)\right)$$

- Using Euler and de Moivre formulas, we derive:

$$\begin{aligned}
x(t) &= \frac{1}{2}a_0 + \sum_{k=1}^{+\infty}\frac{a_k}{2}\left[\exp\left(\frac{i2\pi kt}{T}\right) + \exp\left(\frac{-i2\pi kt}{T}\right)\right] + \\
&\quad \sum_{k=1}^{+\infty}\frac{b_k}{2i}\left[\exp\left(\frac{i2\pi kt}{T}\right) - \exp\left(\frac{-i2\pi kt}{T}\right)\right] \\
x(t) &= \frac{1}{2}a_0 + \sum_{k=1}^{+\infty}\exp\left(\frac{i2\pi kt}{T}\right)\left[\frac{a_k}{2} - i\frac{b_k}{2}\right] + \\
&\quad \sum_{k=1}^{+\infty}\exp\left(\frac{-i2\pi kt}{T}\right)\left[\frac{a_k}{2} + i\frac{b_k}{2}\right]
\end{aligned}$$

Fourier series decomposition of periodic functions

Writing x with the complex Exponential function (cont'd)

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{+\infty} \exp\left(\frac{i2\pi kt}{T}\right)\left[\frac{a_k}{2} - i\frac{b_k}{2}\right] + \sum_{k=1}^{+\infty} \exp\left(\frac{-i2\pi kt}{T}\right)\left[\frac{a_k}{2} + i\frac{b_k}{2}\right]$$

- change of summation index $k' \leftarrow -k$ ($2^{nd} sum$), $a_{-k} = a_k$, $b_{-k} = -b_k$:

$$\begin{aligned} x(t) &= \frac{1}{2}a_0 + \sum_{k=1}^{+\infty} \exp\left(\frac{i2\pi kt}{T}\right)\left[\frac{a_k}{2} - i\frac{b_k}{2}\right] + \\ & \sum_{k=-\infty}^{-1} \exp\left(\frac{i2\pi kt}{T}\right)\left[\frac{a_k}{2} - i\frac{b_k}{2}\right] \end{aligned}$$

Fourier series decomposition of periodic functions

Writing x with the complex Exponential function

$x(t) = \frac{1}{2} a_0 + \sum_{k=1}^{+\infty} \exp\left(\frac{i2\pi kt}{T}\right)\left[\frac{a_k}{2} - i\frac{b_k}{2}\right] + \sum_{k=-\infty}^{-1} \exp\left(\frac{i2\pi kt}{T}\right)\left[\frac{a_k}{2} - i\frac{b_k}{2}\right]$

Let us define: $c_k = \frac{a_k}{2} - i\frac{b_k}{2}$ and $c_0 = \frac{a_0}{2}$, we have:

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \, \exp\left(\frac{2ik\pi t}{T}\right) \tag{4}$$

with:

$$c_k = \frac{1}{T} \int_0^T x(t) \, \exp\left(\frac{-2i\pi kt}{T}\right) dt \tag{5}$$

- x writes as a linear combination of elementary functions Φ_k:
  - $\Phi_k(t) = \exp\left(\frac{2ik\pi t}{T}\right)$
  - $\frac{k}{T}$ is a “pure” frequency ($k \in \mathbb{Z}$, may be negative)

Vector spaces, function spaces and scalar product

- Let us consider the space of T-periodic functions. Admitted: it is a vector space of infinite dimension (basis size is infinite)
- Projection of a function f on a function g: defined by the scalar (or dot) product, $\langle ., . \rangle$, as:

$$\langle f, g \rangle = \frac{1}{T} \int_0^T f(t)\bar{g}(t)dt \quad \text{with} \quad f, g : \mathbb{R} \mapsto \mathbb{C} \tag{6}$$

  $\bar{g}(t)$ is the complex conjugate of $g(t)$
- Recall (complex conjugate): $z = x + iy \Rightarrow \bar{z} = x - iy$

Fourier series: interpretation

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \underbrace{\exp\left(\frac{2ik\pi t}{T}\right)}_{\Phi_k(t)} = \sum_{k=-\infty}^{+\infty} c_k \, \Phi_k(t)$$

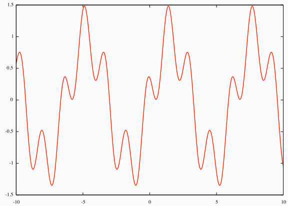
- Interpretation of the decomposition into Fourier series:

$$\begin{aligned} c_k &= \frac{1}{T} \int_0^T x(t) \, \exp\left(\frac{-2i\pi kt}{T}\right) dt \\ &= \langle x, \Phi_k \rangle \end{aligned}$$

$\Rightarrow c_k$ is the projection of x on the complex function $\exp(\frac{2i\pi kt}{T})$

- Exercise: show that $\left(\exp(\frac{2i\pi kt}{T})\right)_{k\in\mathbb{Z}}$ is an orthonormal basis

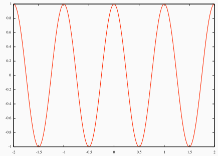
Let x be a signal $\left(x(t) = \sum_{j=1}^{4} \cos(2\pi k_j \frac{t}{T})\right)$



$$x(t) = \sum_{k} c_k \ \exp(\frac{2ik\pi t}{T})$$

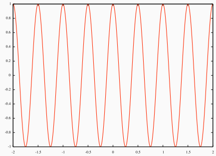
$$c_k = \left\langle x, \exp(\frac{2i\pi kt}{T})\right\rangle$$

$$e^{\frac{\pm 2i\pi k_1 t}{T}}$$



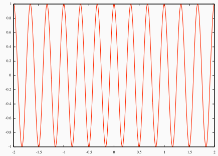
$$c_{k_1} = c_{-k_1} = \frac{1}{2} a_{k_1} = \frac{1}{2}$$

$$e^{\frac{\pm 2i\pi k_2 t}{T}}$$



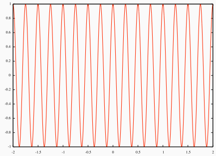
$$c_{k_2} = \frac{1}{2}$$

$$e^{\frac{\pm 2i\pi k_3 t}{T}}$$



$$c_{k_3} = \frac{1}{2}$$

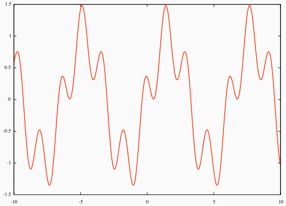
$$e^{\frac{\pm 2i\pi k_4 t}{T}}$$



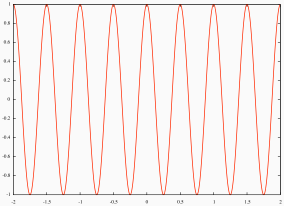
$$c_{k_4} = \frac{1}{2}$$

Fourier series: geometric interpretation

signal $x(t)$



basis function $\Phi_k(t) = \exp(\frac{2ik\pi t}{T})$



$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \; \exp\left(\frac{2ik\pi t}{T}\right) \Longleftrightarrow c_k = \langle x, \Phi_k \rangle$$

$\Longrightarrow$ the scalar product c_k measures the similarity beetween the signal x and an element Φ_k of the basis

- c_k is the **amount** of pure frequency $\exp(\frac{2i\pi kt}{T})$ contained in the signal x

Fourier series: conclusion

Changes the way a function is described

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \ \exp\left(\frac{2ik\pi t}{T}\right) \Longleftrightarrow c_k = \langle x, \Phi_k \rangle$$

- All periodic functions may be represented in the Fourier basis. A periodic function is then described as an infinite (and countable) set of complex numbers: $c_k, k \in \mathbb{Z}$
- A non null c_k coefficient means: $\exp\left(\frac{2i\pi kt}{T}\right)$ element is contained in signal x with "quantity" c_k
- Example: $x(t) = \cos(2\pi \frac{t}{T}) : c_k = \begin{cases} 1/2 & \text{if } k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$
  - projection of x on $\left\{\exp(\frac{\pm 2i\pi t}{T})\right\}$ gives $\frac{1}{2}$
  - projection of x on $\left\{\exp(\frac{2i\pi kt}{T})\right\}$ gives $0 \ \forall k \neq \pm 1$

Outline

Continuous Fourier transform of a signal (1D)

Fourier series $\rightarrow$ Fourier transform

- Any periodic signal can be represented by its Fourier series
- Non periodic signal? limit case of a periodic signal when $T \rightarrow \infty$
- If x is non periodic, the Fourier series does not hold: the countable basis $\left\{\exp(\frac{2i\pi kt}{T})\right\}$, $k \in \mathbb{Z}$ must be changed to a continuous basis!
- **Definition**: Fourier transform X of a signal x

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-i2\pi ft}dt \quad \text{with} \quad f \in \mathbb{R} \tag{7}$$

- Definition: $X(f)$ is the frequency response of x
- The Fourier transform is a generalization of Fourier series to non-periodic functions

Continuous Fourier transform of a signal

Interpretation

$$\begin{aligned} X(f) &= \int_{-\infty}^{+\infty} x(t) e^{-i2\pi ft} dt \quad f \in \mathbb{R} \\ &= \langle x, \Phi_f \rangle \end{aligned}$$

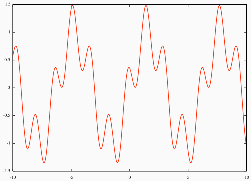
with $\Phi_f(t) = \exp(i2\pi ft)$ and $\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t)\bar{g}(t)dt$

- x is **projected** on a uncountable set[1] of continuous functions $(\exp(i2\pi ft))_{f \in \mathbb{R}}$: f is a real parameter, not a discrete index
- $X(f)$ gives information about the frequency f of signal x:
  - Is the frequency f present in signal x? To which amount?
  - The variable f is real: a continuum of frequencies versus a discrete set of frequencies for Fourier series

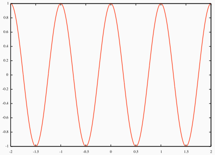
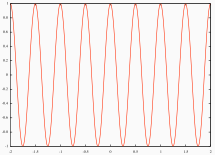
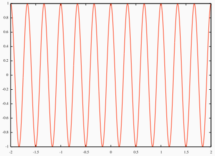
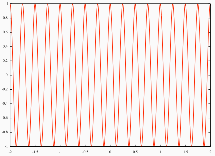
[1] a set whose elements cannot be counted, such as $\mathbb{R}$

Fourier transform: geometrical interpretation

- A signal $x(t)$

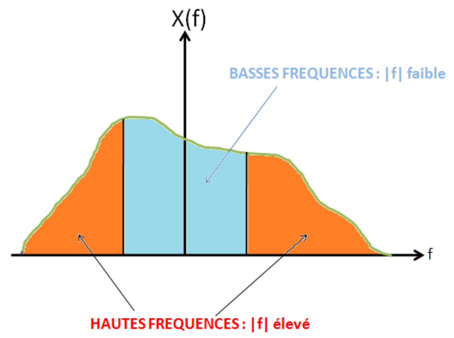


$$X(f) = \langle x, \exp(2i\pi ft)\rangle$$

$e^{\pm 2i\pi f_1 t}$  $X(f_1) = \langle x, \exp(\pm 2i\pi f_1 t)\rangle$	$e^{\pm 2i\pi f_2 t}$  $X(f_2) = \langle x, \exp(\pm 2i\pi f_2 t)\rangle$
$e^{\pm 2i\pi f_3 t}$  $X(f_3) = \langle x, \exp(\pm 2i\pi f_3 t)\rangle$	$e^{\pm 2i\pi f_4 t}$  $X(f_4) = \langle x, \exp(\pm 2i\pi f_4 t)\rangle$

Interpretation: high and low frequencies of a signal

- High frequencies of $x(t)$: significant values of $X(f)$ for high values of $|f|$ means quick variations of $x(t)$
- Low frequencies of $x(t)$: significant values of $X(f)$ for low values of $|f|$ means slow variations of $x(t)$



Fourier transform of a signal

Remarks

- $X(f)$ is a complex number
  - The real part of X is $X_R(f) = \int_{-\infty}^{+\infty} x(t) \cos(2\pi f t)\, dt$, an even function
  - The imaginary part of X is $X_I(f) = -\int_{-\infty}^{+\infty} x(t) \sin(2\pi f t)\, dt$, an odd function
- Real and imaginary parts are not interpretable, prefer:
  1. module or amplitude spectrum: $|X(f)| = \sqrt{X_R(f)^2 + X_I(f)^2}$
     $\hookrightarrow$ the norm of $X(f)$ seen as a vector in the complex plane. $|X(f)|$ represents the amount of pure frequency f present in signal x
  2. phase: $\Theta(f) = \arctan\left(\frac{X_I(f)}{X_R(f)}\right)$
     $\hookrightarrow$ the angle of $X(f)$ seen as a vector in the complex plane with the real axis. The phase represents the shift between x and $\exp(2i\pi f t)$

Remarks (cont'd)

- A "special" frequency, $f = 0$: $X(0)$ is the mean of the signal!

$$
\begin{aligned}
X(0) &= \int_{-\infty}^{+\infty} x(t) \exp(-i2\pi 0 t) dt \\
&= \int_{-\infty}^{+\infty} x(t) dt
\end{aligned}
$$

Inverse Fourier transform

Signal reconstruction

- The signal x can be reconstructed from its Fourier transform X:

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{i2\pi ft} df$$

- Projection on basis $\left(\bar{\Phi}_f\right)_{f \in \mathbb{R}}$, with $\bar{\Phi}_t(f) = \exp\left(-i2\pi ft\right)$:

$$x(t) \;\; = \;\; \left\langle X, \bar{\Phi}_t \right\rangle$$

- We can therefore reconstruct the signal by "summing" the projections...
  - Direct consequence of a projection on an orthogonal basis

Convolution product

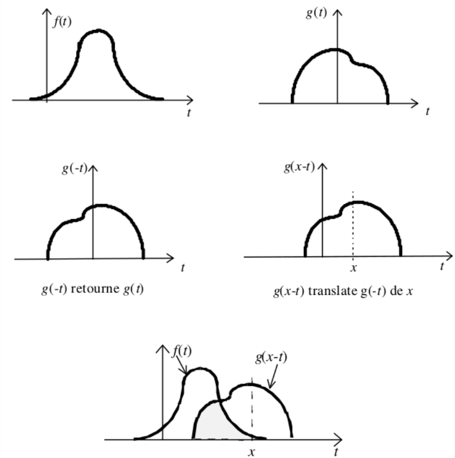
- Operator $\star$ denotes the convolution product taking two signals x and y as operands and returning a signal z:

$$\begin{aligned} z(t) &= x \star y(t) = \int_{-\infty}^{+\infty} x(\tau) y(t-\tau) d\tau \\ z &= x \star y \end{aligned}$$

- Steps:
  1. Signal reversal: $y(\tau) \rightarrow y(-\tau)$
  2. Signal translation by t : $\rightarrow y(t-\tau)$
  3. Product with x: $x(\tau)y(t-\tau)$
  4. Summation of $x(\tau)y(t-\tau)$

Convolution product: geometric interpretation

1. Signal reversal: $g(t) \rightarrow g(-t)$
2. Signal translation by t: $\rightarrow g(x-t)$
3. Product with f: $f(t)g(x-t)$
4. Summation of $f(t)g(x-t) \rightarrow f \star g(x)$



Convolution product: important properties

Convolution operator is

- commutative:

$$x \star y = y \star x$$

- distributive:

$$x \star (y + z) = x \star y + x \star z$$

- and associative:

$$(x \star y) \star z = x \star (y \star z)$$

Important properties of 1D Fourier transform

- Linearity: $\mathrm{FT}[t \mapsto ax(t) + by(t)](f) = aX(f) + bY(f)$
- Time scaling: $\mathrm{FT}\left[t \mapsto x(\alpha t)\right](f) = \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)$
- Time shifting : $\mathrm{FT}\left[t \mapsto x(t - t_0)\right](f) = X(f)e^{-i2\pi t_0 f}$
- Frequency shifting: $\mathrm{FT}\left[t \mapsto x(t)e^{-i2\pi f_0 t}\right](f) = X(f - f_0)$
- Convolution theorem (proof in tutorial works):
  - $\mathrm{FT}\left[t \mapsto x \star y(t)\right](f) = X(f) \times Y(f)$
  - $\mathrm{FT}\left[t \mapsto x(t) \times y(t)\right](f) = X \star Y(f)$

  $\Rightarrow$ Very important property showing the equivalence between spatial filtering and filtering in the frequency domain (Lecture 5)
- $x(t) \in \mathbb{R} \Rightarrow$ Hermitian symmetry: $X(f) = \bar{X}(-f)$, implying:
  - $|X(-f)| = |X(f)|$: module of FT (spectrum amplitude) is an even function
  - Phase and imaginery part of FT are odd functions

  $\Rightarrow$ Signals (and images) are real functions, their amplitude spectrum is always even

Fourier transform of some usual signals

Fourier transform of a rectangular function

- Rectangular function ("gate"):

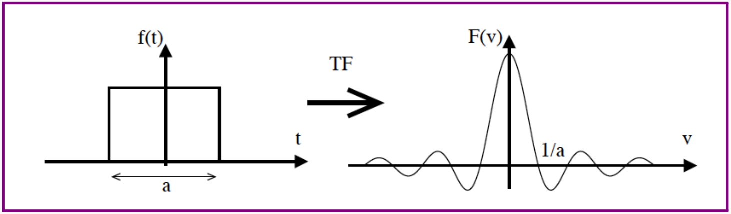
$$\mathrm{Rect}(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

- Fourier transform of a rectangular signal: sinc (sinus cardinal) function (see tutorial works)

$$\mathrm{FT}\left[\mathrm{Rect}\left(\frac{t}{a}\right)\right](f) = \int_{-\frac{a}{2}}^{+\frac{a}{2}} e^{-i2\pi ft} dt = a\frac{\sin(\pi fa)}{\pi fa} = a\,\mathrm{sinc}(\pi fa)$$

Fourier transform of a rectangular function

$$\mathsf{FT}\left[\mathrm{Rect}\left(\frac{t}{a}\right)\right](f) = a\,\mathrm{sinc}(\pi f a)$$



- Very useful for signal digitization – signal windowing and sampling - (Lecture 4)

Fourier transform of a Gaussian function

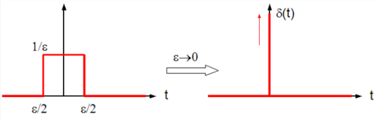
- Fourier transform of a Gaussian function: also a Gaussian function
$$\mathrm{FT}\left[e^{-b^2 t^2}\right](f) = \frac{\sqrt{\pi}}{|b|} e^{-\frac{\pi^2 f^2}{b^2}}$$
- The standard deviation in the frequency domain is inversely proportional to the standard deviation in the time domain: a direct consequence of the "time scaling" property

A very important function: Dirac delta function $\delta(t)$

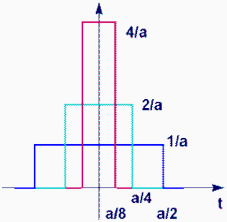
- Informal (and not rigorous) definition:

$$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ +\infty & \text{otherwise} \end{cases}$$

  with $\int_{-\infty}^{+\infty} \delta(t)dt = 1$.
  More rigorous: $\delta(t)$ is null everywhere except for $t = 0$, has integral equal to 1



- Can be defined as a limit of a sequence of Rectangular functions: $\delta(t) = \lim_{a \to 0} \left[\frac{1}{a}\mathrm{Rect}\left(\frac{t}{a}\right)\right]$



Fourier transform of some usual signals

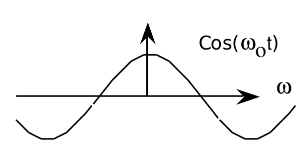
Dirac function $\delta(t)$ or Impulse

- $\delta(t)$ is not an ordinary function but a **distribution**
  - Generalization of the notion of function
- $\delta(t)$ has a central role in signal and image processing
  - neutral element for convolution product, useful for FT calculus
  - used to model the sampling of signals (1D) and images (2D): Lecture 4
- Essential properties:
  - $\int_{-\infty}^{+\infty} \delta(t) dt = 1$
  - $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
  - $x \star \delta(t - t_0) = x(t - t_0)$: translation by t_0 of signal x
  - Fourier transform:
    - $\text{FT}\left[e^{2i\pi f_0 t}\right](f) = \delta(f - f_0)$: pure frequency $f_0 \rightarrow$ Delta function
    - $\text{FT}\left[\delta(t - t_0)\right](f) = e^{-2i\pi f t_0}$
  - *Scaling property* : $|\alpha|\delta(\alpha t) = \delta(t)$

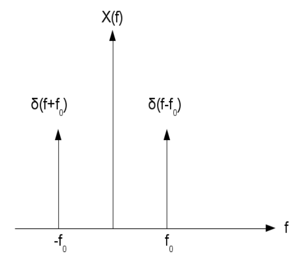
Fourier transform of some usual signals

cos function: $x(t) = \cos(2\pi f_o t)$

$$\text{FT}\left[\cos(2\pi f_0 t)\right](f) = \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \tag{9}$$



$w_0 = 2\pi f_0$



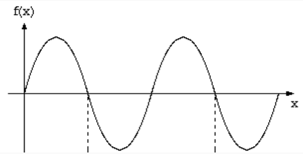
Two pure frequencies $\pm f_0$

- See tutorial works, FT of cos is a real function.

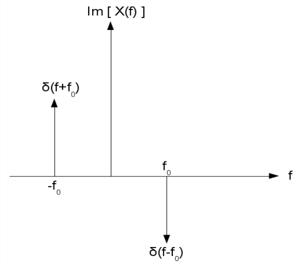
Fourier transform of some usual signals

sin function: $x(t) = \sin(2\pi f_o t)$

$$\mathrm{FT}\left[\sin(2\pi f_0 t)\right](f) = \frac{i}{2}\left[\delta(f+f_0) - \delta(f-f_0)\right] \qquad (10)$$



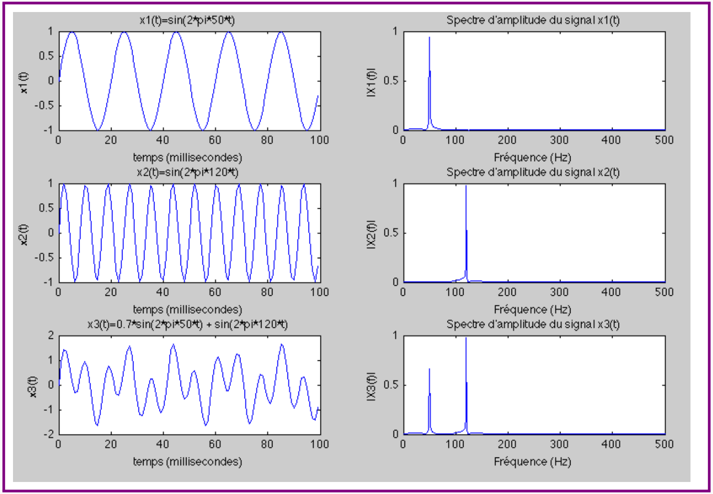
$x(t) = \sin(2\pi f_0 t)$



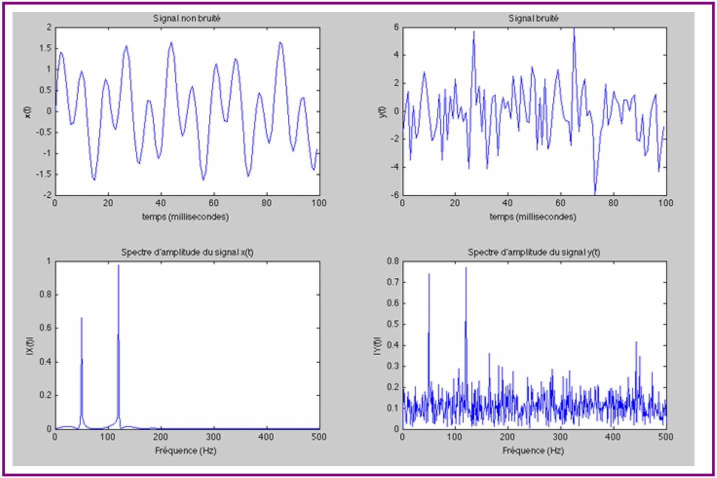
Two pure frequencies $\pm f_0$

- See tutorial works: FT of sin is an imaginary function

Fourier transform of some usual signals



An example with noise



Outline

Continuous Fourier transform of a 2D signal

Definition

- Given a 2D and non periodic signal $(t, u) \mapsto x(t, u)$, its Fourier transform is a function $X : \mathbb{R}^2 \to \mathbb{C}$ defined as:

$$X(f, g) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(t, u) e^{-i2\pi(ft+gu)} dtdu$$

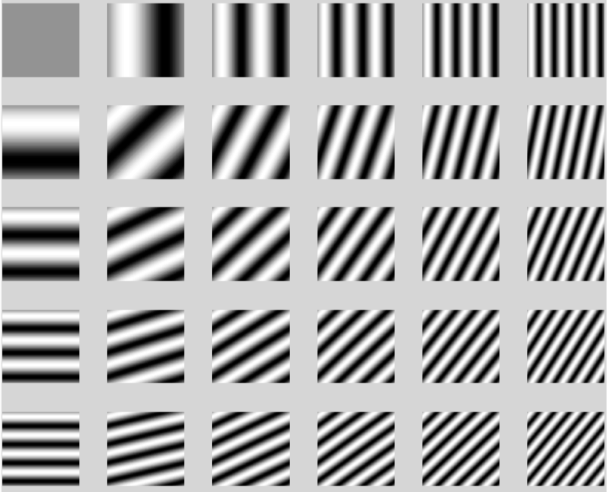
- Interpretation: X is the projection of x on a basis of functions $\{\exp(2i\pi(ft + gu))\}_{(f,g)\in\mathbb{R}^2}$, each function may be viewed as an elementary image:

$$X(f, g) = \langle x, \Phi_{f,g} \rangle$$

with $\Phi_{f,g}(t, u) = e^{i2\pi(ft+gu)}$

Fourier transform of a 2D signal

$\Phi_{f,g}(t,u) = e^{-i2\pi(ft+gu)}$: **elementary images**



Fourier transform of a 2D signal

Some definitions

- Real part:
  $X_R(f,g) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t,u)\cos\left(2\pi(ft+gu)\right) dtdu$
  an even function
- Imaginary part:
  $X_I(f,g) = -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t,u)\sin\left(2\pi(ft+gu)\right) dtdu$
  an odd function
- Module, or amplitude spectrum:
  $|X(f,g)| = \sqrt{X_R(f,g)^2 + X_I(f,g)^2}$
- Phase: $\Theta(f,g) = \arctan\left(\frac{X_I(f,g)}{X_R(f,g)}\right)$
- The null frequency, $f = g = 0$:
  $X(0,0) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t,u) dtdu$

Inverse Fourier transform

- Signal x can be reconstructed from its Fourier representation X:

$$x(t,u) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(f,g) e^{i2\pi(ft+gu)} dfdg$$

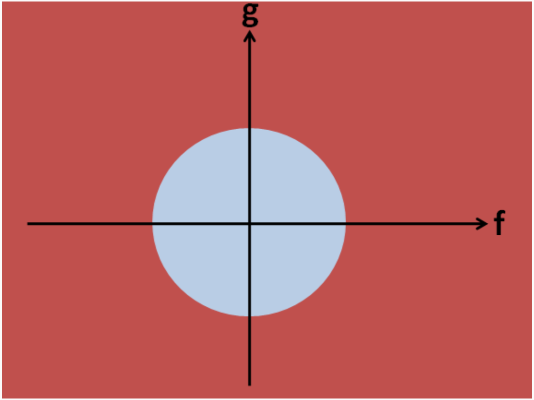
- Direct consequence of a projection on an orthonormal basis:

$$x(t,u) = \left\langle X(t,u), \bar{\Phi}_{f,g}(t,u) \right\rangle \tag{11}$$

with $\bar{\Phi}_{f,g}(t,u) = e^{-i2\pi(ft+gu)}$

Fourier transform of a 2D signal

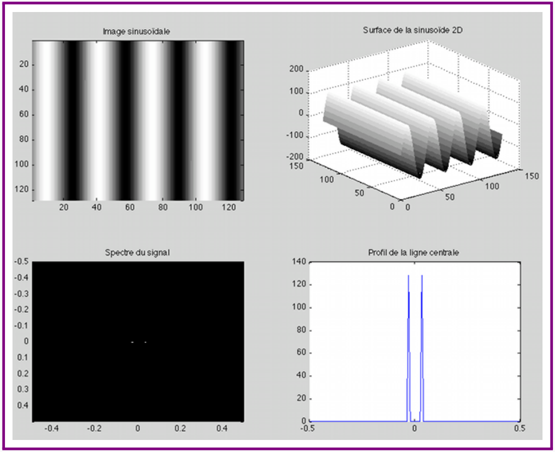
Frequency components



- Low spatial frequency: $f^2 + g^2$ small
- High spatial frequency: $f^2 + g^2$ large

Fourier transform of a 2D signal: first example

$x(t,u) = \cos(2\pi f_0 t)$
$X(f,g) = \frac{\delta(f-f_0)+\delta(f+f_0))}{2}$
(see tutorial works)

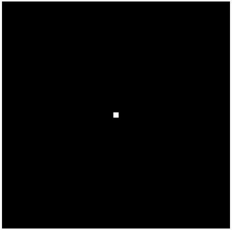


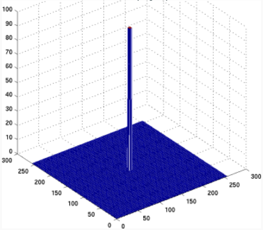
$\mathrm{FT}\left[A\cos\left[2\pi f_o\left(x\cos(\theta)+y\sin(\theta)\right)\right]\right] =??$ (see tutorial works)

Fourier transform of a 2D signal: second example

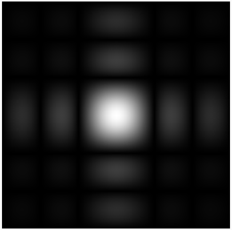
2D rectangular function: $x(t,u) = \mathrm{Rect}\left(\frac{t}{T}\right) \times \mathrm{Rect}\left(\frac{u}{T}\right)$

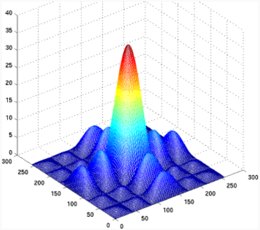
$x(t,u)$





$X(f,g)$





$X(f,g)$ sinc 2D (see tutorial work)

Outline

Fourier transform of a 2D signal

From 1D to 2D

- 2D Fourier transform: a natural extension of the 1D case
  - Many common properties
  - Some 2D specificities
- 2D Fourier transform: $\sim$ 2 successive 1D Fourier transforms (see tutorial works)
  - first transform: $Z(f, u) = \mathrm{FT}[t \mapsto x(t, u)]$
  - second transform: $X(f, g) = \mathrm{FT}\left[u \mapsto Z(f, u)\right]$
- if x separable, $x(t, u) = z(t) \times k(u)$, we have:

$$X(f, g) = Z(f) \times K(g)$$

  with $Z = TF\left[z\right]$ and $K = TF\left[k\right]$
  - Simple product of two mono-dimensional Fourier transforms

1D and 2D FT: common properties

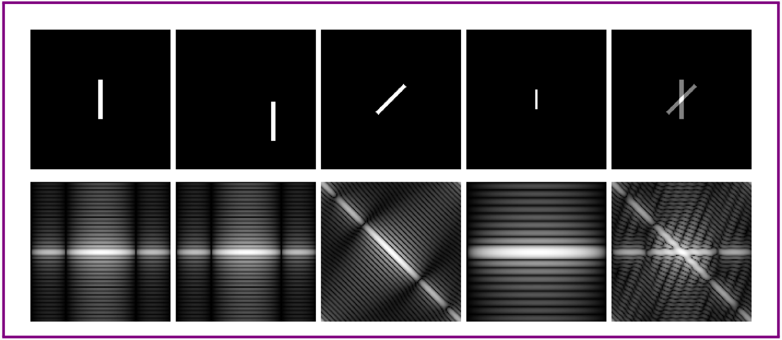
	Continuous 1D FT		Continuous 2D FT	
	x(t)	X(f)	x (t,u)	X(f,g)
1	$x(t) + \lambda y(t)$	$X(f) + \lambda Y(f)$	$x(t,u) + \lambda y(t,u)$	$X(f,g) + \lambda Y(f,g)$
2	$x(t - t_0)$	$X(f)\ e^{-2i\pi f t_0}$	$x(t - t_0, u - u_0)$	$X(f,g)\ e^{-2i\pi(f t_0 + g u_0)}$
3	$x(\alpha t)$	$\frac{1}{\lvert\alpha\rvert} X(\frac{f}{\alpha})$	$x(\alpha t, \beta u)$	$\frac{1}{\lvert\alpha\rvert\lvert\beta\rvert} X(\frac{f}{\alpha}, \frac{g}{\beta})$
4	$x \star y(t)$	$X(f) \times Y(f)$	$x \star y(t,u)$	$X(f,g) \times Y(f,g)$
5	$x(t) \times y(t)$	$X \star Y(f)$	$x(t,u) \times y(t,u)$	$X \star Y(f,g)$

Table 1: Properties of 1D and 2D Fourier transforms

1. linearity
2. translation
3. domain contraction
4. convolution
5. product

1D and 2D FT: common properties

Illustrations

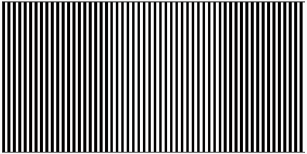


2D Fourier transform: specificities

Notion of "spatial frequency"

- Image $I : \mathbb{R}^2 \rightarrow \mathbb{R}$: function of two variables (spatial position)
- Fourier transform $\text{FT}(I) : \mathbb{R}^2 \rightarrow \mathbb{C}$: function of two variables (spatial frequency)
- Spatial frequency $\rightarrow$ couple of frequencies:
  - horizontal frequency: how fast signal varies along x-axis
  - vertical frequency: how fast signal varies along y-axis

Low horizontal frequency
Null vertical frequency



High horizontal frequency
Null vertical frequency

2D Fourier transform: specificities

Notion of "spatial frequency"

- Natural images: non-stationary signals
- Different spatial frequencies (high/low) in different regions of the image

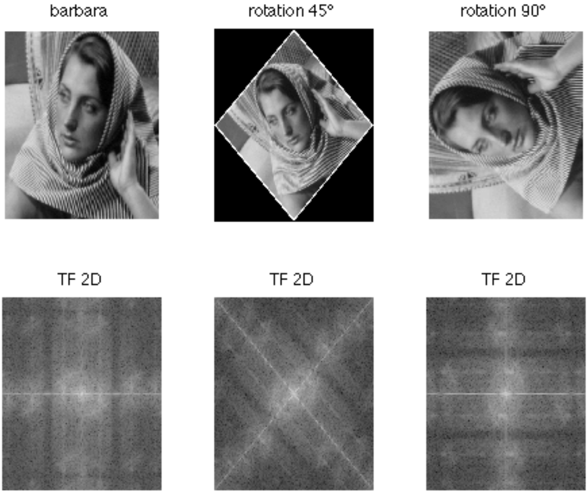
Rotation

- A rotation of angle θ in the spatial domain induces the same rotation in the frequency domain:

$$
\begin{aligned}
y(t,u) &= x(t\cos\theta + u\sin\theta, -t\sin\theta + u\cos\theta) \\
Y(f,g) &= X(f\cos\theta + g\sin\theta, -f\sin\theta + g\cos\theta)
\end{aligned}
$$

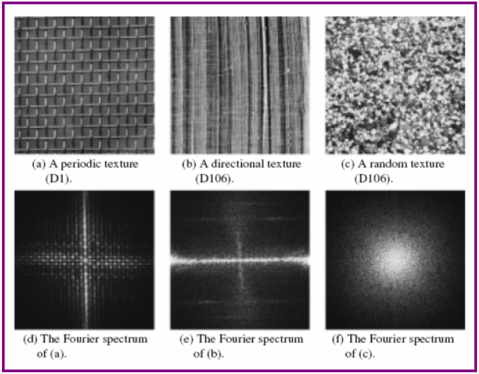
- Important: the frequency response $X(f,g)$ gives an information on the orientation of image structures

2D Fourier transform: example with rotation



2D Fourier transform: orientations in images

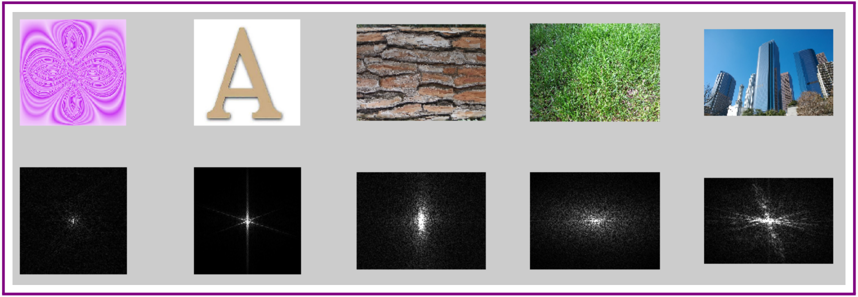
- Important: the frequency response $X(f, g)$ gives an information on the orientation of structures in the image
- Examples with textured images



- Main structure orientations are highlighted in the amplitude spectrum

2D Fourier transform: orientations in images

- Examples with natural images

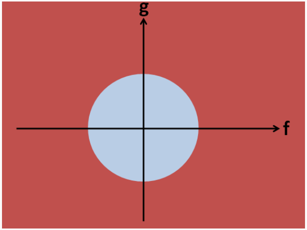


- Main structure orientations are highlighted in the amplitude spectrum

2D Fourier transform: visualizing the spectrum

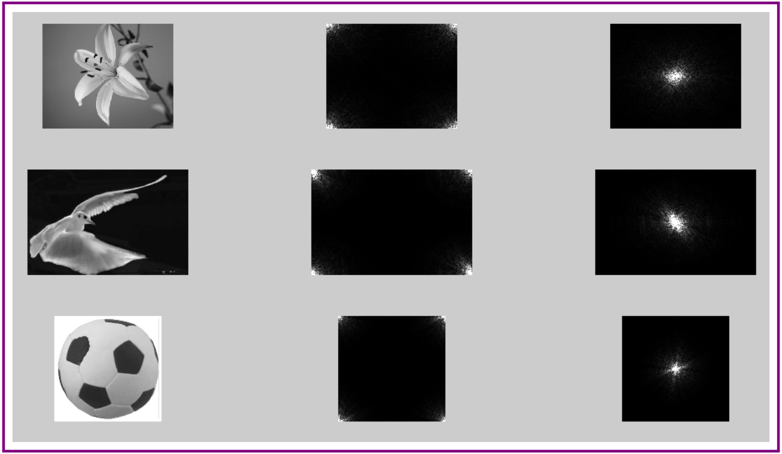
Centering the spectrum:

- Function `fft2()` from `numpy.fft` to compute a 2D Fourier transform
  - `fft2()` is a discrete Fourier transform (DFT) $\rightarrow$ see Lecture 4
- `fft2()`: spectrum origin at left-top corner of the image
- To shift the origin of the frequencies at the center of the spectrum (more natural visualization), use the function `fftshift()` from `numpy.fft`



- Spectrum values are not changed, only a rearrangement
- Highlights the symmetry property of real signals spectrum ($|F(u,v)| = |F(-u,-v)|$)

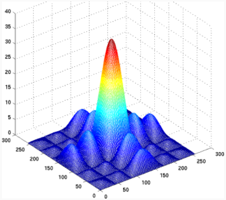
Centering the spectrum: illustration

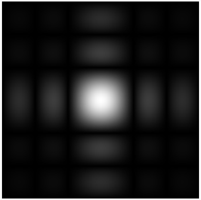


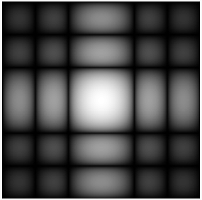
2D Fourier transform: visualizing the spectrum

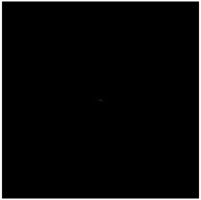
Natural images: high/low frequencies

- Energy of low frequencies $\gg$ Energy of high frequencies
  $\Rightarrow$ apply a log transform $1 + \log(|X(f,g)|)$

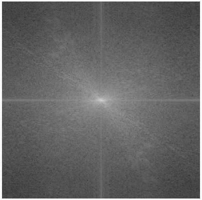








$|X(f,g)|$



$1 + \log(|X(f,g)|)$

Outline

Fourier transform: application to image processing

- Fourier transform: representation in the frequency domain
- Time representation $\rightarrow$ Frequency representation:
  1. Time (or spatial for 2D case) representation: projection on a basis of Dirac functions $(\Psi_t)_{t\in\mathbb{R}}, \Psi_t(u) = \delta(u-t)$:
     $$x(t) = x \star \delta(t) = \int_{\mathbb{R}} x(u)\delta(t-u)du = \langle x, \Psi_t \rangle \tag{12}$$
     $\Rightarrow x$ is described in time
  2. Frequency representation: projection on a sinusoidal basis $\left(\bar{\Phi}_t\right)_{t\in\mathbb{R}}, \bar{\Phi}_t(f) = e^{-2i\pi ft}$:
     $$x(t) = \int_{\mathbb{R}} X(f)e^{2i\pi ft}df = \left\langle X, \bar{\Phi}_t \right\rangle \tag{13}$$
     $\Rightarrow x$ is described in frequency
- Use the frequency representation if the useful information is better described than with the time representation
- "useful" depends on the context

2D Fourier transform: application to image processing

Noise removal

- Problem formulation: isolate the signal from the noise
- Hypothesis: signal and noise have not the same frequency components, signal $\Leftrightarrow$ low freq, noise $\Leftrightarrow$ high freq
- Method: high frequencies $\leftarrow 0$

Original image

Noisy image

Result image

Compression

- Formulation: represent the signal with fewer data
- Hypothesis: high frequencies have low energy, and contain irrelevant information
- Method: keep only the low frequencies

Original

Compressed image (/2)

$\Rightarrow$ principle of JPEG compression, see practical work 4

Numerous applications of frequency analysis

- Noise removal, compression, image restoration: low level processing
- Useful for higher level processing
- Linear filtering (Lecture 5): convolution in spatial domain $\leftrightarrow$ multiplication in frequency domain: dual operations
- Filtering: preprocessing step for many applications :
  - Edge detection (Lecture 6), detection and description of points of interest, *e.g.* Harris, SIFT (Lectures 7-8), texture characterization...

Frequency representation: the pro

- well adapted to represent stationary and smooth signals

Frequency representation: the cons

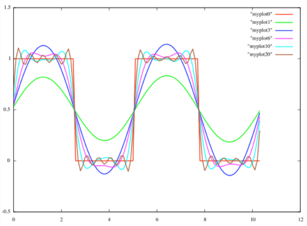
Non stationary signals

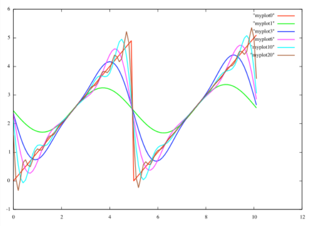
- Fourier transform describes the image structures only in terms of frequencies
    - Spectrum amplitude is invariant by translation and cannot localize the image structures in time/space
- Images are non stationary signals: the frequency components are not the same everywhere the image domain (see example on Slide 49)
- Solution: use representations describing the image *both* in time and space: wavelets for instance

Frequency representation: the cons

Non derivable signals

- A non derivable signal needs an infinity of frequencies to be described
  - Gibbs phenomenon





- Not well adapted to describe image contours

⇒ JPEG not suited to images having many contours, (for example text images)