Fundamentals of Image Processing

▶ Lecture 4: Advanced Fourier transform - Digitization ◀

Master of Computer Science
Sorbonne University

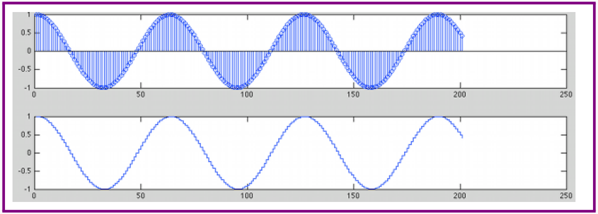
September 2022

Outline

From continuous to discrete signals

Principle

- A continuous signal (analog signal) must be digitized in order to be recorded and processed by a computer.
- Three steps:
  - Signal **windowing**: limit the signal support (domain) in time
  - Signal **sampling**: a discrete collection of equally spaced samples, the time between two samples is the sampling period
  - Signal **quantization**: conversion of the measured continuous intensity value into a discrete quantity

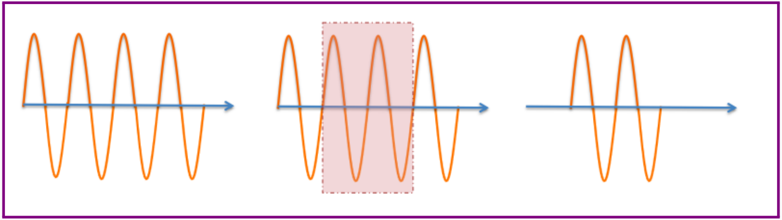


Outline

Windowing: 1D signal

Principe

- The signal must have a compact support (i.e. defined on a bounded time domain)
  $\hookrightarrow$ obtained by applying a signal windowing:

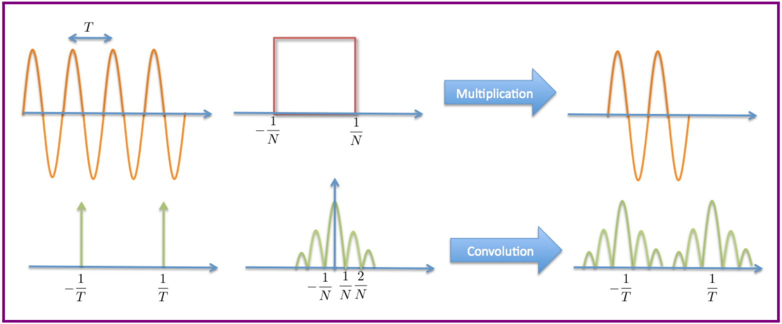


$$x_L(t) = x(t)\,\mathrm{Rect}\left(\frac{t}{L}\right) \qquad \mathrm{Rect}(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$

Principle

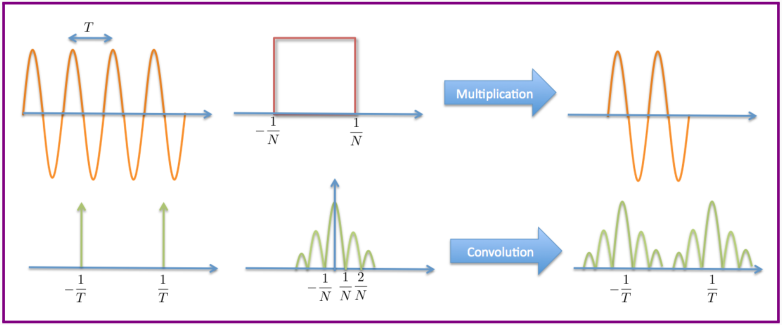
- Multiplication by a rectangular function in the time domain

  $\Leftrightarrow$ Convolution with a sinc function in the frequency domain



Windowing: 1D signal

Principle



Question: What is the window size that ensures a negligible degradation between continuous and windowed signals?
$\hookrightarrow$ tutorial work

Outline

Windowing: 2D signal (image)

Principle

- Same principle as in 1D
- The 2D signal must have a compact support (a bounded domain)
- Multiplication by a 2D rectangular function in the spatial domain

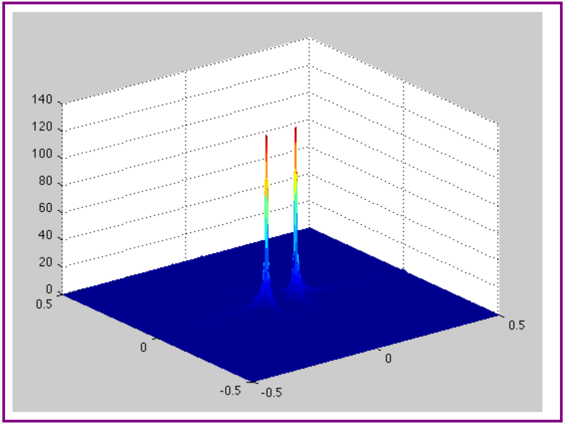
  $\Leftrightarrow$ Convolution with a 2D sinc function in the frequency domain

Example: 2D sine wave (sinusoid)

Spatial domain	Frequency domain
$x(t,u) = \cos(2\pi f_0 t)$	$X(f,g) = \frac{\delta(f-f_0)+\delta(f+f_0))}{2}$
$x_{N,M}(t,u) = x(t,u)\,\mathrm{Rect}\left(\frac{t}{N}\right)\mathrm{Rect}\left(\frac{u}{M}\right)$	$X_{N,M}(f,g) = ?$

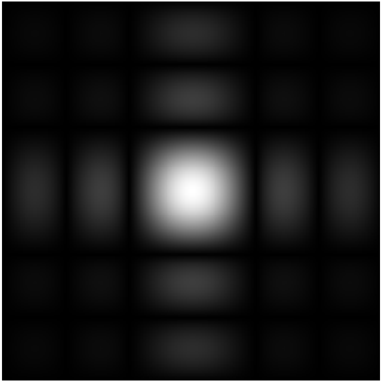
Windowing: image

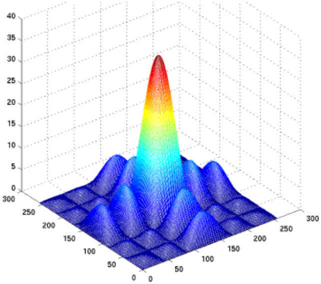
Illustration: Fourier transform of a 2D sine wave: $x_{N,M}(t,u) = \cos(2\pi f_0 t)\,\mathrm{Rect}\left(\frac{t}{N}\right)\mathrm{Rect}\left(\frac{u}{M}\right)$



Windowing: image

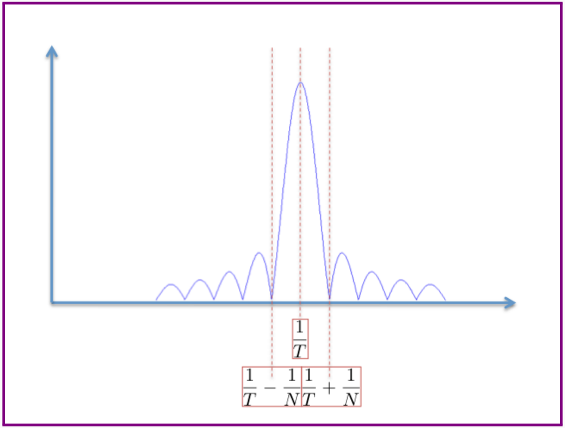
Close view of a peak





Windowing: image

Image of size $N \times N$, profile of the median line



with $T = \frac{1}{f_0}$

Outline

Sampling in time domain (1D), in spatial domain (2D)

Principle

- Goal: transform an analog signal into a numeric and discrete signal
- The signal is sampled: only punctual measures are retained (on a regular grid), the remaining of the signal is lost!
- The frequency of sampling (number of samples by time/space unit) must be carefully chosen:
    1. high enough to allow for a signal reconstruction
    2. not too high to respect storage space constraints

Outline

1D sampling: mathematical modeling

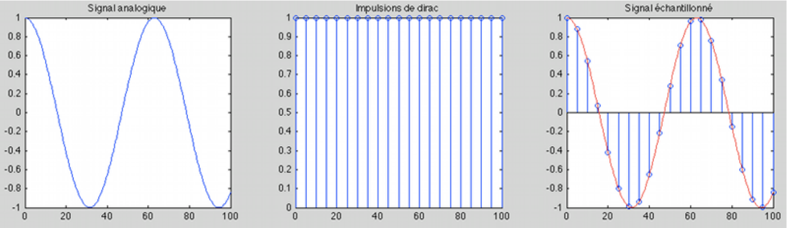
What happens in the time domain

- Signal sampling is modeled by a multiplication with a series of Dirac functions with period Δt
- Let $x_a(t)$ be the analog signal, $x_s(t)$ the signal sampled by a Dirac comb (or impulse train) of period Δt, $\text{Ш}_{\Delta t}$, is:

$$
\begin{aligned}
x_s(t) = x_a(t)\text{Ш}_{\Delta t}(t) &= x_a(t) \sum_{n=-\infty}^{+\infty} \delta(t - n\Delta t) \\
&= \sum_{n=-\infty}^{+\infty} x_a(n\Delta t)\delta(t - n\Delta t)
\end{aligned}
$$

1D sampling: mathematical modeling

What happens in the time domain



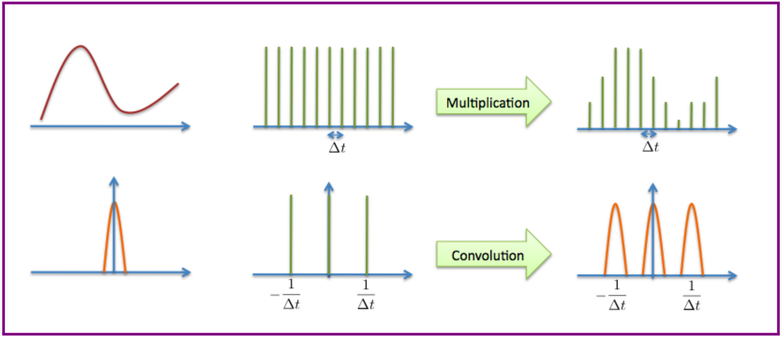
What happens in the frequency domain

- The Fourier transform of $\text{Ш}(t)$ is also a Dirac comb (admitted):
  $\mathrm{FT}(\text{Ш}_{\Delta t})(f) = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty}\delta(f - \frac{n}{\Delta t}) = \frac{1}{\Delta t}\text{Ш}_{\frac{1}{\Delta t}}(f)$
- Spectrum of the sampled signal:

$$X_s(f) = \frac{1}{\Delta t}X_a \star \text{Ш}_{\frac{1}{\Delta t}}(f) = \frac{1}{\Delta t}\sum_{k=-\infty}^{+\infty} X_a\left(f - \frac{k}{\Delta t}\right)$$

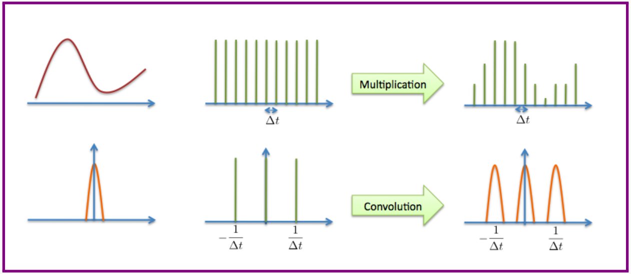
- **Interpretation**: $X_s(f)$ is a periodization of $X_a(f)$
  - Period of $\Delta f = \frac{1}{\Delta t}$ (inverse of the period sampling Δt in the temporal domain)

Sampling, temporal and frequency deterioration: conclusion



1D sampling

Loss of information



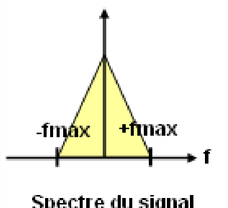
Fundamental questions:

- Is there necessarily a loss of information during sampling?
- If not, how to sample without loss?

1D sampling

Loss of information

- Question: is there necessarily a loss of information during sampling?
- Answer: yes and no!
  - No, if we consider signals having a bounded support in the frequency domain: **band-limited signal**
  - Yes in the general case
- "Band-limited" signal: signal $x(t)$ whose frequency components, $X(f)$, are null for $|f| > f_{max}$



Example of band-limited signals

Exercises

- $x(t) = \text{Rect}(t)$: band-limited signal? If yes, $f_{max} =$?
- $x(t) = \text{sinc}(t)$: band-limited signal? If yes, $f_{max} =$?
- $x(t) = \sin(2\pi f_0 t)$: band-limited signal? If yes, $f_{max} =$?
- $x(t) = \cos(2\pi f_0 t)$: band-limited signal? If yes, $f_{max} =$?
- $x(t) = \delta(t - t_0)$: band-limited signal? If yes, $f_{max} =$?

Band-limited signals

Which sampling frequency?

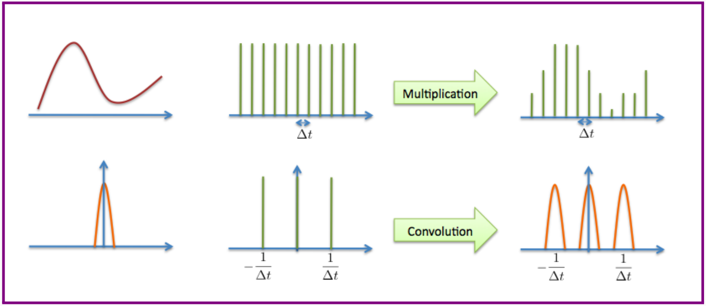
- $x_a(t)$ a band-limited signal $\rightarrow f_{max}$
- $x_a(t)$ is sampled at frequency $f_s = 1/T_s \Rightarrow x_s(t)$
- **Theorem (Nyquist-Shannon)**: no loss of information between $x_a(t)$ and $x_s(t)$ if:

$$f_s \geq 2f_{max} \Leftrightarrow T_s \leq \frac{1}{2}T_{max} \tag{1}$$

- Sampling frequency f_s: twice higher than the maximal frequency of the signal

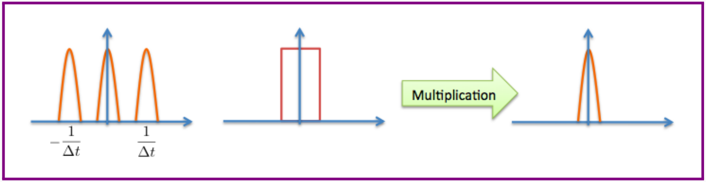
Correct sampling: interpretation in the frequency domain

- $f_s \geq 2f_{max}$: no overlapping of each period of the spectrum



Correct sampling: interpretation in the frequency domain

- Frequency domain: how to retrieve $X_a(f)$ from $X_s(f)$?

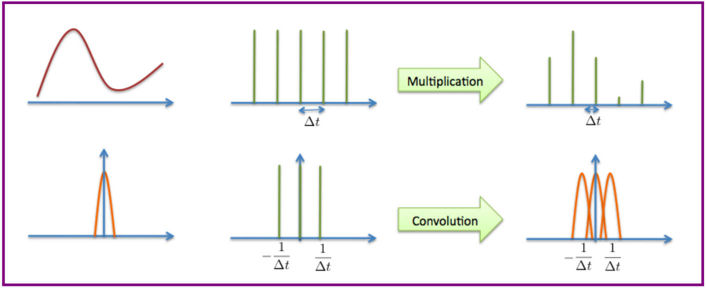


- Retrieve the central period of the spectrum: multiplication by a rectangular function
- Compute the inverse Fourier transform of this spectrum
- Temporal domain: convolution by a function sinc, Shannon's interpolation formula (see below)

1D sampling: Shannon theorem

Incorrect sampling: interpretation in the frequency domain

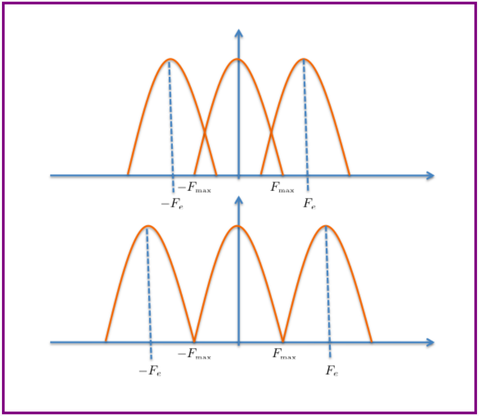
- $f_s < 2f_{max}$: periodization $\rightarrow$ **spectral folding** or **aliasing**



- High frequencies interfere with periodization: each period intersects with adjacent periods
- No possible frequency processing to isolate the initial spectrum, other periods always involved
- We cannot reconstruct the original signal in the time domain...

Incorrect sampling and limit case

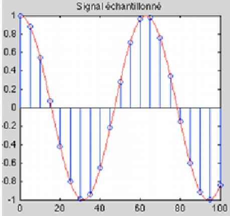
- $f_s < 2f_{max}$: spectrum periodization implies lobes overlapping
- $f_s = 2f_{max}$: limit case



Which sampling frequency?

- **Shannon theorem condition**: $f_s \geq 2f_{max} \Leftrightarrow T_s \leq \frac{1}{2}T_{max}$

If Shannon's condition is verified:



- No loss of information between $x_a(t)$ and $x_s(t)$
- $x_a(t)$ is the unique continuous function with $f_{max} \leq f_s/2$ interpolating the samples $x(kT_s)$
- It is possible to reconstruct exactly $x_a(t)$ from the samples $x_s(kT_s)$, $k \in \mathbb{Z}$

- Reconstruction: Shannon interpolation formula (tutorial works)

$$x_r(t) = \sum_{k=-\infty}^{+\infty} x_s(kT_s)\frac{sin(\pi f_s(t-kT_s))}{\pi f_s(t-kT_s)} = \sum_{k=-\infty}^{+\infty} x_s(kT_s)\,\mathrm{sinc}(\pi f_s(t-kT_s)) \quad (2)$$

Aliasing effect

- The continuous signal cannot be reconstructed from the samples
- Shannon interpolation formula (Eq. (2)) retrieves an incorrect signal: reconstructed signal $x_r \neq$ original signal x_a
- Degradations x_r / x_a hard to predict, strongly depend on the frequency content of the signal
  - Some interesting cases are studied (basic functions) below
  - Degradation visually important for images (see 2D sampling)

Case of a sine wave function

- Consider the sine wave function:

$$x_a(t) = A\cos(2\pi f_0 t) \tag{3}$$

- Is x_a a band-limited signal? If yes, $f_{max} =$?

1D sampling: Shannon theorem

Case of a sine wave function

- Consider the sine wave function:

$$x_a(t) = A\cos(2\pi f_0 t) \tag{3}$$

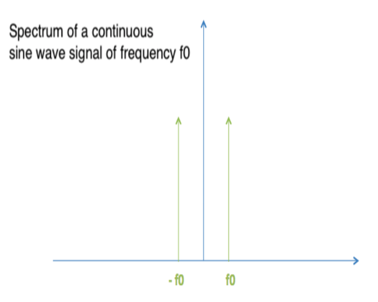
- Is x_a a band-limited signal? If yes, $f_{max} = ?$
  - Yes! $X_a(f) = \mathrm{FT}\left[\cos(2\pi f_o t)\right] = \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$

  $\Rightarrow$ $f_{max} = f_0$
  - Shannon theorem: x_a can be sampled without information loss if $f_s \geq 2f_0$: at least two samples per period!

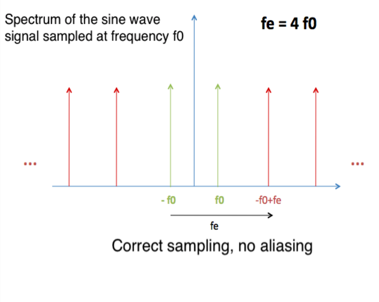
Case of sine wave function: $x_a(t) = A\cos(2\pi f_0 t)$

Correct sampling: $f_s = 4f_0$

Sampling: visualization in the frequency domain



$X_a(f)$

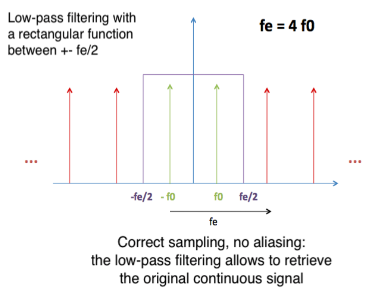


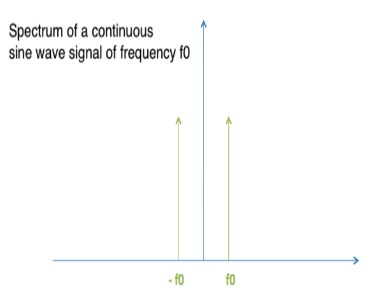
$X_s(f)$

Case of sine wave function: $x_a(t) = A\cos(2\pi f_0 t)$

Correct sampling: $f_s = 4f_0$

Reconstruction: visualization in the frequency domain





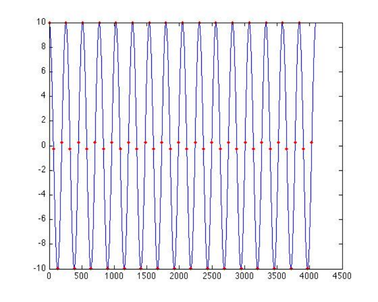
$$X_r(f) = X_s(f)\,\mathrm{Rect}\left(\frac{f}{f_s}\right)$$

$$X_r(f) = X_a(f)$$

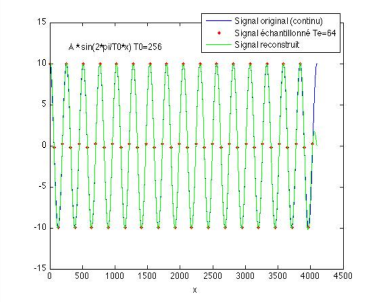
Case of sine wave function: $x_a(t) = A\cos(2\pi f_0 t)$

Correct sampling: $f_s = 4f_0$

Sampling & reconstruction: visualization in the temporal domain



$x_a(t)$ & $x_s(t)$



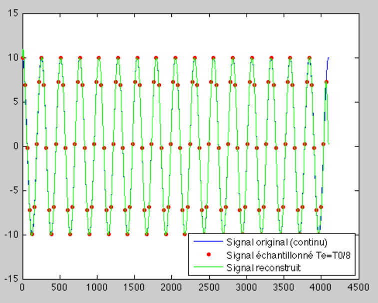
$x_r(t)$

- The reconstructed signal is identical to the original signal

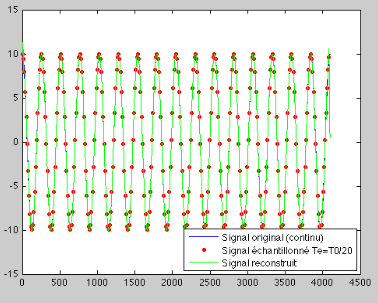
Case of sine wave function: $x_a(t) = A\cos(2\pi f_0 t)$

Correct sampling: $f_s \geq 2f_0$

Lossless reconstruction $\forall f_s \geq 2f_0$



$f_s = 8f_0$

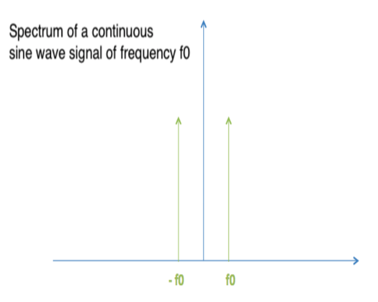


$f_s = 20f_0$

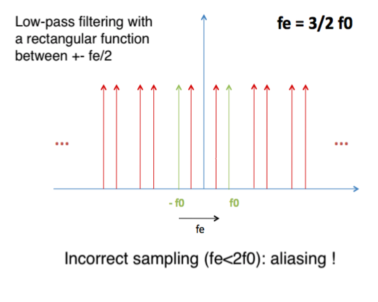
Case of sine wave function: $x_a(t) = A\cos(2\pi f_0 t)$

Incorrect sampling: $f_s = \frac{3}{2} f_0$

Sampling: visualization in the frequency domain



$X_a(f)$

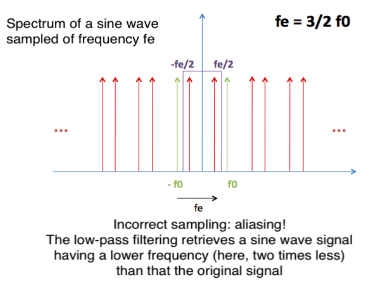


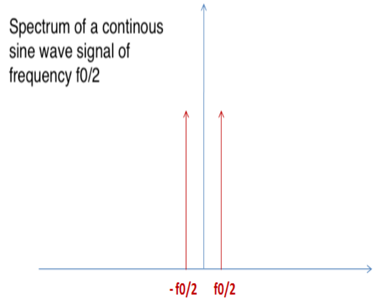
$X_s(f)$

Case of sine wave function: $x_a(t) = A\cos(2\pi f_0 t)$

Incorrect sampling: $f_s = \frac{3}{2} f_0$

Reconstruction: visualization in the frequency domain

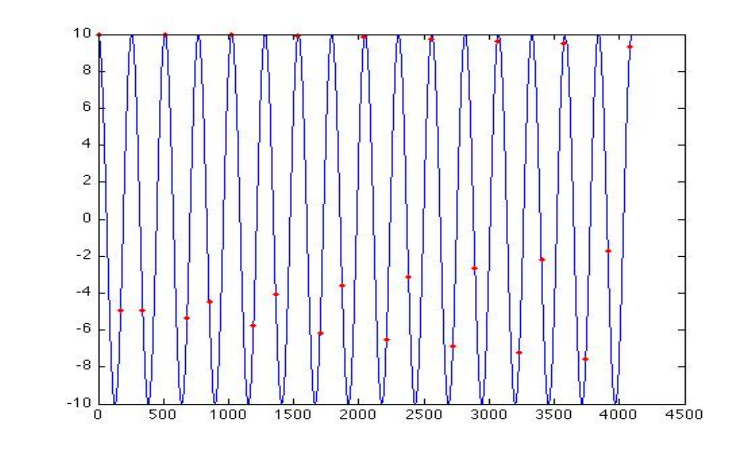




$$X_r(f) = X_s(f)\,\mathrm{Rect}\left(\frac{f}{f_s}\right) \qquad X_r(f) \neq X_a(f)$$

Reconstructed spectrum: a wave sine of frequency $f_0' = \frac{f_0}{2}$ **!**

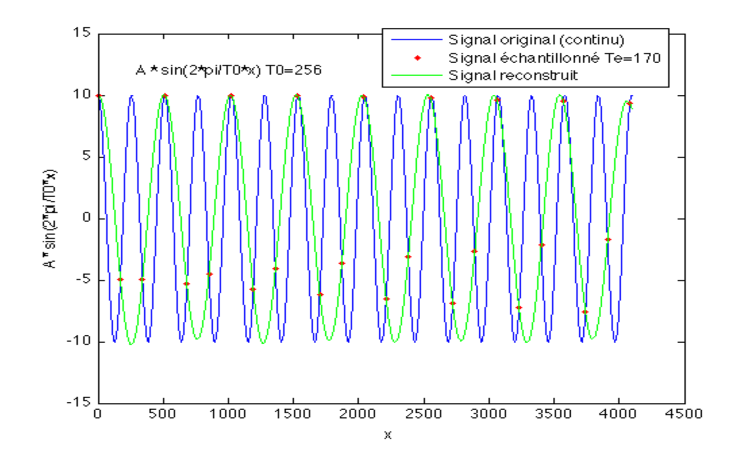
Incorrect sampling: $f_s = \frac{3}{2} f_0$



$x_a(t)$ & $x_s(t)$

Case of sine wave function: $x_a(t) = A\cos(2\pi f_0 t)$

Incorrect sampling: $f_s = \frac{3}{2} f_0$

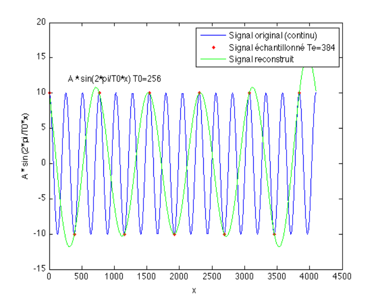


$x_r(t)$: a wave sine of frequency $f_0' = \frac{f_0}{2}$!

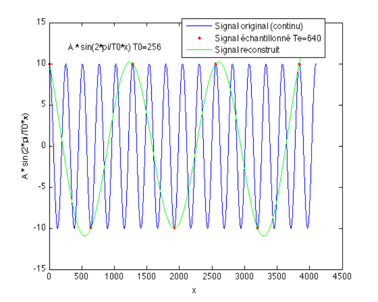
Case of sine wave function: $x_a(t) = A\cos(2\pi f_0 t)$

Incorrect sampling: $f_s < 2f_0$

False reconstructions due to aliasing for various f_s values

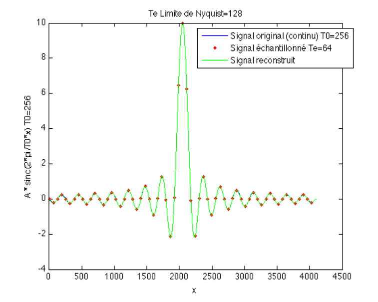


$f_s = \frac{2}{3} f_0$

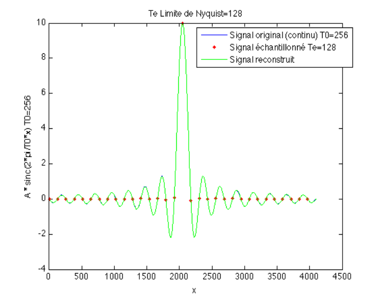


$f_s = 0.4 f_0$

Band-limited signal
Correct sampling: $f_s \geq 2f_0$



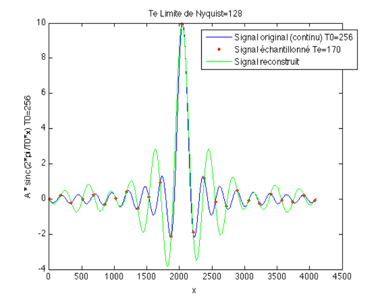
$f_s = 4f_0$



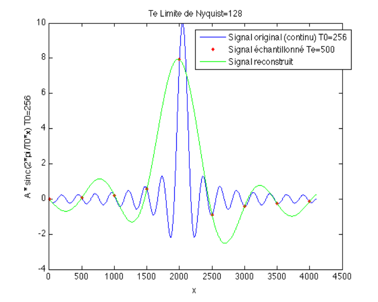
$f_s = 2f_0$ (limit case)

Another example: $x_a(t) = A\,\mathrm{sinc}\,(2\pi f_0 t)$

Incorrect sampling: $f_s < 2f_0$



$f_s = \frac{2}{3} f_0$



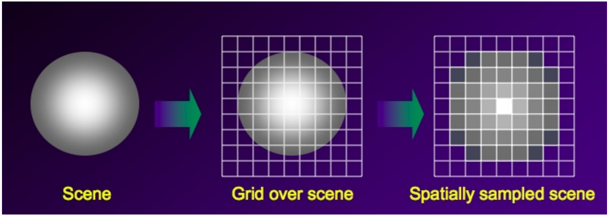
$f_s = 0.4 f_0$

Outline

Sampling 2D signals

Difference/similarity between 1D and 2D

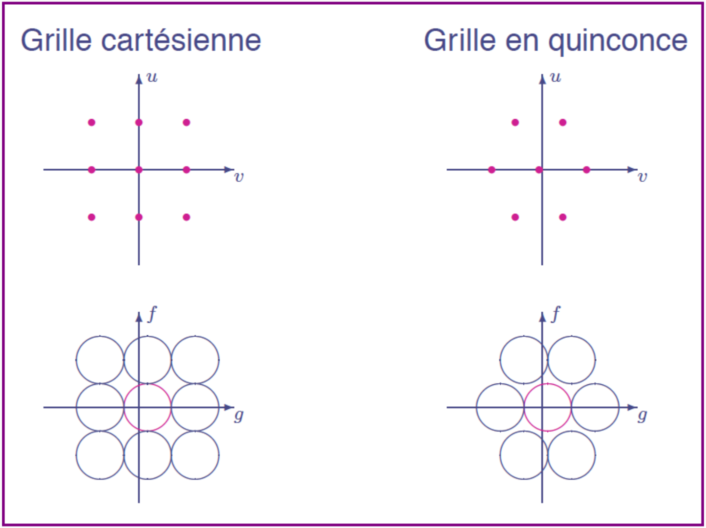
- Shannon's condition: as in 1D, signal must be band-limited
- Condition more restrictive for images than 1D signals
  - see discussion at the end of this lecture
- For images as band-limited signals:
  $\rightarrow$ 2D sampling theorem: a generalization of 1D case



Sampling grid

- Most of the time: Cartesian grid, pixels are equally spaced, at distance Δt et Δu from each other for horizontal and vertical directions
- Generally: $\Delta t = \Delta u$, but depends on sensor characteristics
- Hexagonal or staggered grids are also available:
  - on cathodic TV and old computer screens, hexagonal grids have an optimal spatial resolution for the same number of samples
  - some computer vision applications are more efficient on such grids (e.g. topology, invariance by rotation)

Example of sampling grids

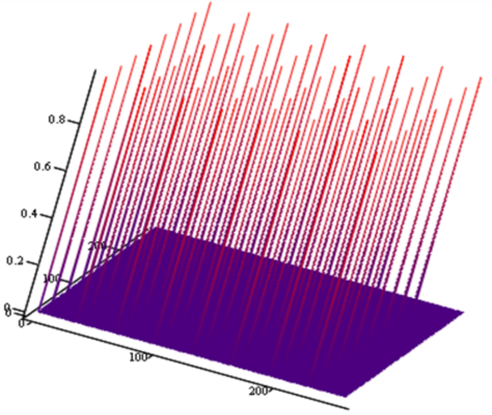


Mathematical modeling

- Let $x_a(t, u)$ be a continuous 2D signal
- Consider a Cartesian grid, (Δ_t, Δ_u), and an ideal model of sampling (Dirac function)
- The sampled signal $x_s(t, u)$ is formally defined as a product between $x_a(t, u)$ and a series of 2D Dirac functions, a 2D impulse train s, bi-periodic function:

$$
\begin{aligned}
x_s(t, u) &= x_a(t, u) s(t, u) \\
s(t, u) &= \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \delta(t - k\Delta t, u - l\Delta u) \\
\delta(t - k\Delta t, u - l\Delta u) &= \delta(t - k\Delta t)\delta(u - l\Delta u)
\end{aligned}
$$

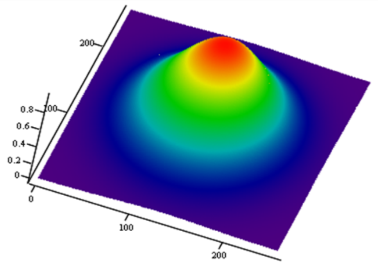
2D sampling: 2D impulse train



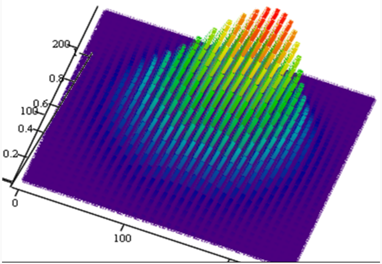
The sampling function in 2D:

$$s_{\Delta t,\Delta u}(t,u) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \delta(t - k\Delta t, u - l\Delta u)$$

2D sampling with impulse train



continuous signal $x_a(t, u)$



sampled signal
$x_s(t, u) = x_a(t, u)s(t, u)$

Fourier transform of a 2D sampled signal

- $x_s(t, u) = x_a(t, u)s(t, u)$
- $X_s(f, g) = X_a \star S(f, g)$
- As in 1-D, we have:

$$S(f, g) = \frac{1}{\Delta t \Delta u} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta \left(f - \frac{m}{\Delta t}, g - \frac{n}{\Delta u} \right)$$

$\Rightarrow$ the Fourier transform of a 2D impulse train is also a 2D impulse train (of inverse periods)

- Hence:

$$X_s(f, g) = \frac{1}{\Delta t \Delta u} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} X_a \left(f - \frac{m}{\Delta t}, g - \frac{n}{\Delta u} \right)$$

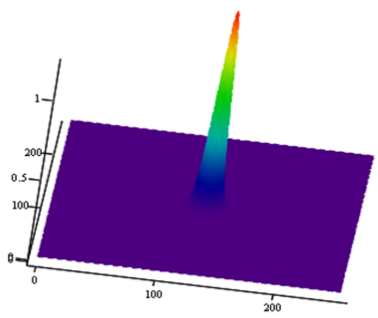
2D sampling

Fourier transform of a 2D sampled signal

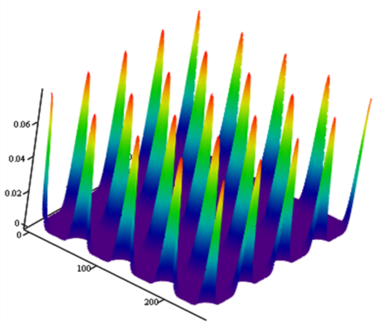
$$X_s(f,g) = \frac{1}{\Delta t \Delta u} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} X_a\left(f - \frac{m}{\Delta t}, g - \frac{n}{\Delta u}\right)$$

- Generalization from 1D to 2D: $X_s(f,g)$ obtained by periodization of $X_a(f,g)$ in the both canonical directions according to the inverse sampling periods $\frac{1}{\Delta t}$ et $\frac{1}{\Delta u}$
- The 2D sampling produces a repetition of the spectrum in two directions

2D sampling with impulse train



continuous signal $X_a(f, g)$



sampled signal $X_s(f, g)$

$$\begin{aligned} X_s(f,g) &= X_a \star S(f,g) \\ &= \frac{1}{\Delta t \Delta u} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} X_a\left(f - \frac{m}{\Delta t}, g - \frac{n}{\Delta u}\right) \end{aligned}$$

2D sampling

Shannon theorem (in 2D)

- Let $x_a(t, u)$ be a band-limited signal with f_{max}, g_{max} the highest frequencies
- Let $x_s(t, u)$ be the sampled signal with frequencies $f_s = \frac{1}{\Delta t}$ and $g_s = \frac{1}{\Delta u}$ (multiplication by $s_{\Delta t, \Delta u}$)
- **Theorem (Nyquist-Shannon)**: no loss of information between x_a and x_s if :

$$f_s \geq 2f_{max} \text{ and } g_s \geq 2g_{max} \tag{4}$$

2D sampling: reconstruction

Reconstruction of x_a
If conditions of Eq. (4) hold:

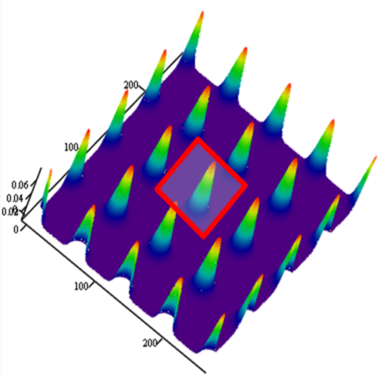
- No loss of information between $x_a(t, u)$ et $x_s(t, u)$
- It is possible to retrieve x_a from $x_s(k\Delta t, k\Delta u)$, $k \in \mathbb{Z}$
- Reconstruction: Shannon interpolation formula:

$$x_r(t, u) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} x_s\left(k\Delta t, k\Delta u\right) \mathrm{sinc}\left(\frac{\pi(t - k\Delta t)}{\Delta t}\right) \mathrm{sinc}\left(\frac{\pi(u - l\Delta u)}{\Delta u}\right) \tag{5}$$

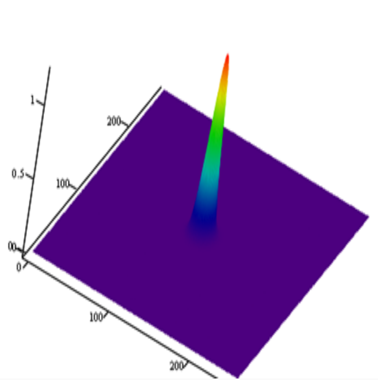
- Method: low-pass filtering of $X_a(f, g)$:

$$X_r(f, g) = X_a(f, g)\,\mathrm{Rect}(\Delta t f)\,\mathrm{Rect}(\Delta u g)$$

2D sampling: reconstruction



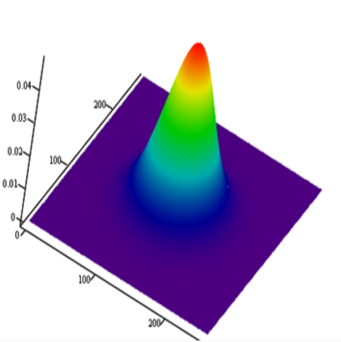
$X_a(f, g)\,\mathrm{Rect}(\Delta t f)\,\mathrm{Rect}(\Delta u g)$



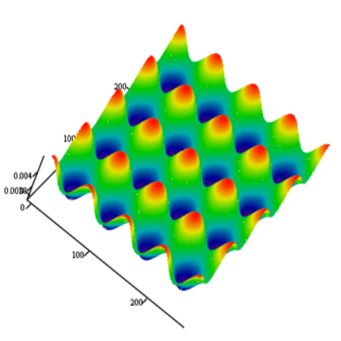
$X_r(f, g)$

Aliasing

- As in 1D: the spectrum periods overlap if Shannon's conditions are not respected



$X_a(f, g)$

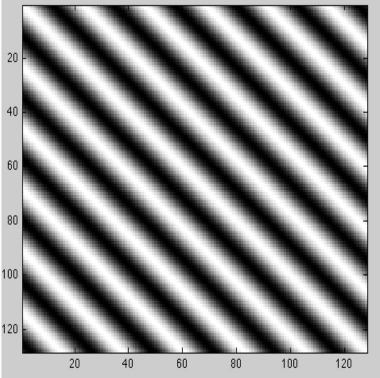


$X_e(f, g)$

Aliasing in 2D: example with sine wave function

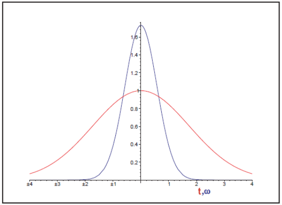
$$x(t, u) = A\cos\left[2\pi f_o\left(t\cos(\theta) + u\sin(\theta)\right)\right]$$

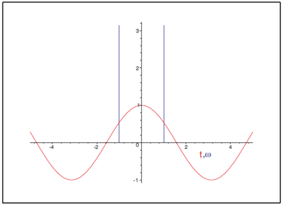
- TF$[x(t, u)]$ = ? is it band-limited? maximal frequencies? Shannon frequency limit?
- Aliasing: what happens if Shannon's conditions are not respected?
- Answer: see practical works!



Band-limited signal: realistic hypothesis?

- In theory: no!
  - Heisenberg's uncertainty principle ⇒ a signal cannot be located with infinite precision both in time and in frequency
  - Windowed signal: temporal support is bounded ⇒ frequency domain is NOT bounded





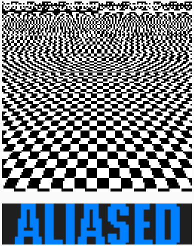
  - There is no integrable function having a compact support both in time and in frequency
  - Band-limited signal ⇒ unbounded temporal domain

2D sampling and aliasing: Discussion

Band-limited signal: realistic hypothesis?

- In practice: yes for smooth and stationary signals, *e.g.* biological signals (ECG)
  - High frequencies energy $\ll$ low frequency energy
  - Aliasing occurs but is negligible

- For natural images:
  - Generally: high frequencies energy $\ll$ low frequency energy
  - But locally not: for instance edges are high frequencies (non smooth structures)
  - Images structures are often non stationary, implying high frequencies (see tutorial works)



Aliasing: Example on natural images

Original image (512×512)

Sub-sampled image by 2 (256×256)

Aliasing: Example on natural images

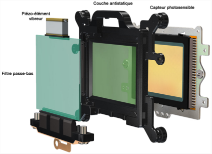
over-sampled image
by 2 (256×256)

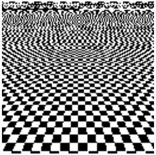
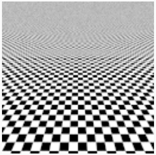
over-sampled image
by 4 (128×128)

- What are the effects of aliasing? How are they interpreted?
- See practical work

How to avoid aliasing: anti-aliasing filter

- Remove frequencies > Nyquist frequency ⇒ band-limited signal
  1. Analog filtering, performed electronically before the digitization process
     - Audio signal: the human ear does not hear beyond 22 kHz ⇒ low pass filtering with 22 kHz as cutoff frequency then sampling at 44 kHz (actual standard)
     - Images: there are analog anti-aliasing filters in some cameras, using the same principle




  2. Digital filtering:
     - apply a digital low pass filter at $f_s/2$

Under-sampling and anti-aliasing filter

- Given a digital image correctly sampled $\rightarrow$ under-sampling (useful for numerous applications)
- How to insure that under-sampling does not induce aliasing? $\Rightarrow$ Apply a digital low-pass filter before under-sampling

raw under-sampling

anti-aliasing + under-sampling ($\downarrow 2$)

In practice

- Fax: 2 preset resolutions
- Printers: finite number of user-selectable resolutions
- TV: 25 or 30 frames per second, 625 or 525 lines per picture height
- Digital TV:

HDTV	TV
$1920 \times 1152 \times 50$	$720 \times 576 \times 25$

Outline

Digitization of images

Recall

$$f(x,y) \longrightarrow \boxed{\text{sampling}} \longrightarrow \boxed{\text{Quantization}} \longrightarrow f(i,j)$$

continuous image — digital image

- The **spatial resolution** is determined after **sampling**
  - Determines the smallest perceptible detail in the image
  - What is the best sampling rate?
- The **gray scale resolution** is obtained after quantization.
  - Determines the smallest discernible gray level change in the image
  - Is there an optimal quantifier?

Quantization

- Goal: reduce the number of bits needed to encode intensity values $\rightsquigarrow$ counter part is a lost of signal precision
- Associate an integer to a real number $\rightsquigarrow$ implicit compression
- Two types of quantifiers:
  - Scalar quantizer: each sample is quantized
  - Vector quantizer: a sequence of samples is quantized

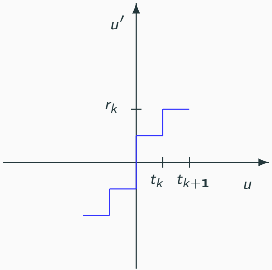
Scalar quantification

Definition

- Let u be a random variable with probability density $p_u(u)$
- Quantization is a function $U \longrightarrow Q, u \longmapsto u'$, $U \subset \mathbb{R}, Q \subset \mathbb{Z}$
- U is an interval, Q is a discrete and finite set
- U subdivided into smaller intervals: $U = \bigcup_k [t_k, t_{k+1}[$
- a value $r_k \in Q$ is associated to each sub-interval
- choices of transition levels t_k may by guided by the density function p_u and an optimality criterion
- The density function p_u is generally unknown, and approximated by the image histogram

Scalar quantization

Principle



$u' \in \{r_1, r_2, \ldots, r_L\}$ is the quantized signal

$t_k \leq u < t_{k+1}$ and $u \to r_k$

$t_k : k = 1, 2, \ldots, L+1$ transition level

r_k : k^{th} quantized level

- Example: uniform quantizer with $u \in [0, 10.0]$
  - We want $u' \in \{0, 1, \ldots, 255\}$ (1-byte dynamics)
  - $t_1 = 0,\ t_{256} = 10.0$
  - With an uniform subdivision, we have $t_k = (k-1) \times \frac{10}{255}$, $k = 2, \ldots, 256$

Outline

Digitization

Discrete Fourier transform

DFT is a sampling of Fourier transform of the sampled signal

1. Fourier transform of the sampled signal:

$$
\begin{aligned}
X_s(f) &= \int_{-\infty}^{+\infty} \left(\sum_{n=-\infty}^{+\infty} x(t)\delta(t - nT_s) \right) e^{-i2\pi ft} dt \\
&= \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left(x(t)\delta(t - nT_s) e^{-i2\pi ft} \right) dt \\
&= \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left(x(nT_s) e^{-i2\pi fnT_s} \delta(t - nT_s) \right) dt \\
&= \sum_{n=-\infty}^{+\infty} x(nT_s) e^{-i2\pi fnT_s} \int_{-\infty}^{+\infty} \delta(t - nT_s) dt \\
&= \sum_{n=-\infty}^{+\infty} x(nT_s) e^{-i2\pi fnT_s}
\end{aligned}
$$

DFT is a sampling of Fourier transform of the sampled signal

2. The signal x must be windowed, we choose a rectangular function of length $N \times T_s, N \in \mathbb{N}$ number of samples

$$X_{tr}(f) = \sum_{n=0}^{N-1} x(nT_s)e^{-i2\pi fnT_s}$$

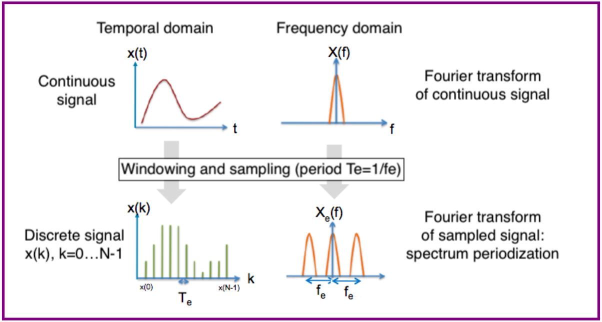
3. Discrete Fourier transform:
   X_{tr} is sampled with period $\frac{1}{N\times T_s}$: $X(k) = X_{tr}\left(\frac{k}{N\times T_s}\right)$, $k = 0, \ldots, N-1$

$$X(k) = \sum_{n=0}^{N-1} x_n e^{-i\frac{2\pi kn}{N}}$$

   with $x_n = x(nT_s)$, the nth sample

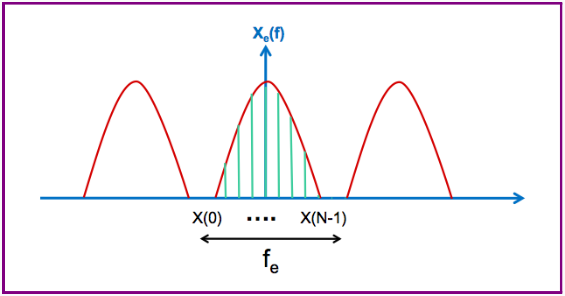
DFT is a sampling of Fourier transform of the sampled signal

- Between 0 and f_s: N values (the same number of signal samples x_k)



Centered spectrum

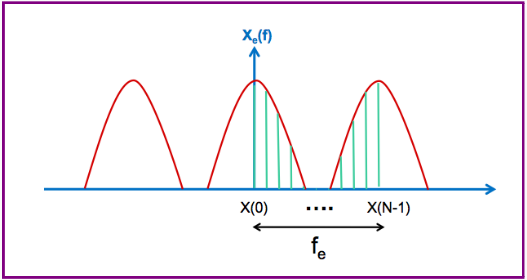
- Samples are spaced by $\frac{f_s}{N}$ between $-\frac{f_s}{2}$ and $\frac{f_s}{2}$
- $X(0) \leftrightarrow -\frac{f_s}{2}$ and $X(N-1) \leftrightarrow +\frac{f_s}{2}$
- **Very important**: coefficients $X(k)$ are the frequency componements relatively to the sampling frequency f_s



1D Discrete Fourier transform

Non-centered spectrum: samples are spaced by $\frac{f_s}{N}$ between 0 and f_s

$X(0) \leftrightarrow 0$ and $X(N-1) \leftrightarrow f_s$



- **It is what Python calculates when function `fft()` is called!**
- Need to use `fftshift()` to have the low frequencies in the center of the spectrum

Définition / Conclusion

- Assume a digital signal x_n with N samples, then its discrete Fourier transform $X(k)$ is given by:

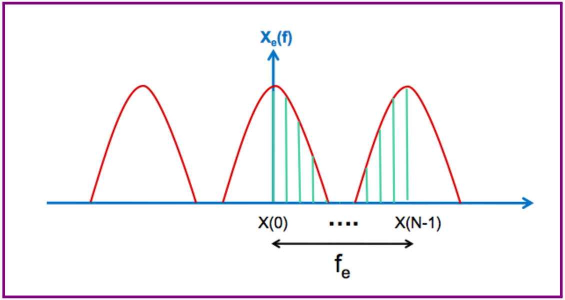
$$X(k) \;=\; \sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi kn}{N}} \quad \text{with} \quad k \in \left\{-\frac{N}{2}, \cdots, \frac{N}{2} - 1\right\}$$

- The digital signal x_n can be reconstructed from its frequency components $X(k)$ using the discrete inverse Fourier transform:

$$x_n = \sum_{k=0}^{N-1} X(k) e^{\frac{2\pi ikn}{N}}$$

1D Discrete Fourier transform

Notion of frequency resolution

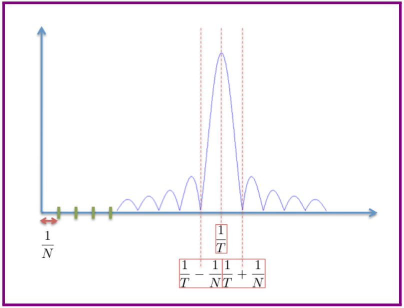


- The number of signal samples is the same as the number of spectrum samples (N)
- It is the frequency resolution

Frequency resolution: example

Sine wave function: remember that

- the Fourier transform of sine wave function with unbounded support, $x(t) = \cos(2\pi f_0 t)$, is a couple of Dirac functions centered on $\pm f_0$
- the Fourier transform of a sine wave function with bounded support, $x_N(t) = x(t)\,\mathrm{Rect}\left(\frac{t}{N}\right)$, is a couple of sinc functions centered on $\pm f_0$

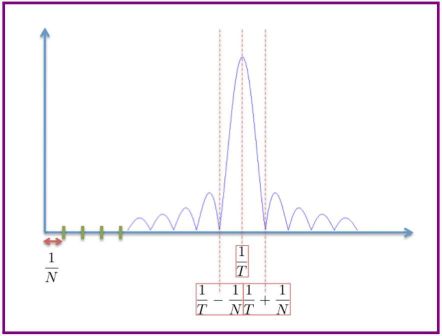


Frequency resolution: example

$$\mathbf{DFT}\left[\cos(2\pi f_0 t)\,\mathrm{Rect}\left(\frac{t}{N}\right)\right] = ?$$

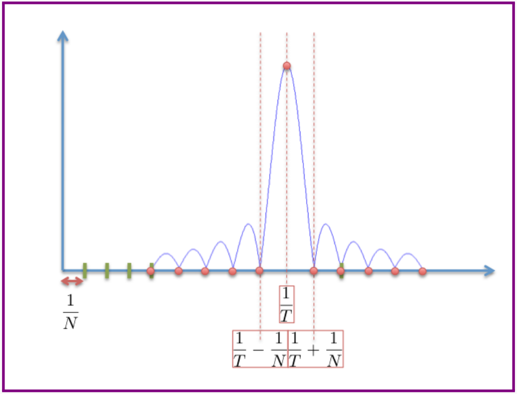
- How does its discrete spectrum look like?

  $\hookrightarrow$ it depends on the frequency resolution: see tutorial work



Frequency resolution: example (studied in tutorial work)

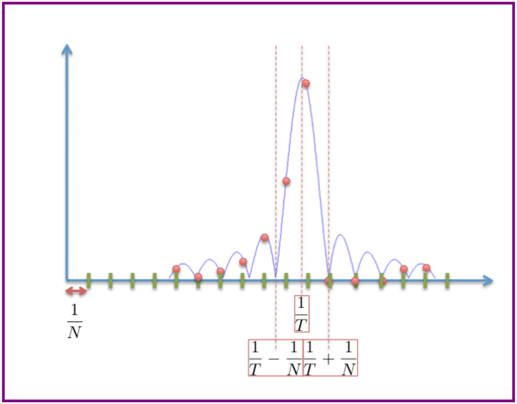
Frequency resolution inversely proportional to sine wave period: $T_0 = \frac{k}{f_s}$



- Except at the main lobe of sinc, only its zeros are sampled: nothing to see here!

Frequency resolution: example (studied in tutorial work)

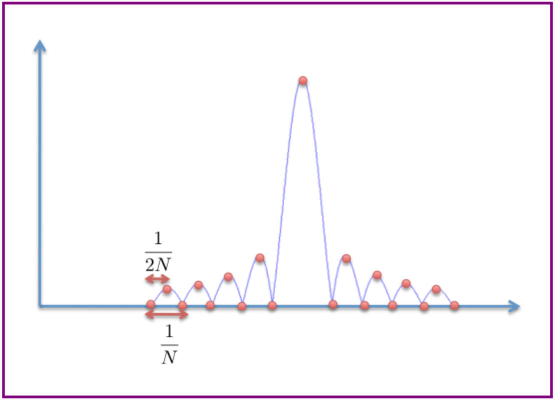
Any sine wave period



- Non null values of sinc are sampled

Still for $T_0 = \frac{k}{f_e}$, how to visualize more than sinc zeros?

- Zero-padding technique: signal is extended with zero values
- Artificially increasing the frequency resolution: here $\times$ 2



2D discrete Fourier transform

Mathematical formulation

- For an image x of size $N \times M$, the 2D discrete Fourier transform is:

$$X(u, v) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} x(n, m) e^{-2\pi i \left(\frac{un}{N} + \frac{vm}{M}\right)}$$

  with: $0 \leq n \leq N-1$ and $0 \leq m \leq M-1$

- Interpretation: as in 1D $\rightarrow$ sampling of the Fourier transform of the sampled signal with the same number of samples

2D discrete inverse Fourier transform

$$\boxed{\text{Frequency domain}} \xrightarrow{DFT^{-1}} \boxed{\text{Spatial domain}}$$

(u, v) $\qquad\qquad$ (n, m)

Mathematical formulation

- For an image of size $N \times M$, the discrete inverse Fourier transform is:

$$x(n, m) = \frac{1}{N \times M} \sum_{u=-\frac{N}{2}}^{\frac{N}{2}-1} \sum_{v=-\frac{M}{2}}^{\frac{M}{2}-1} X(u, v) e^{2\pi i \left(\frac{un}{N} + \frac{vm}{M}\right)}$$

with: $0 \leq n \leq N - 1$ and $0 \leq m \leq M - 1$

Visualization of the 2D spectrum

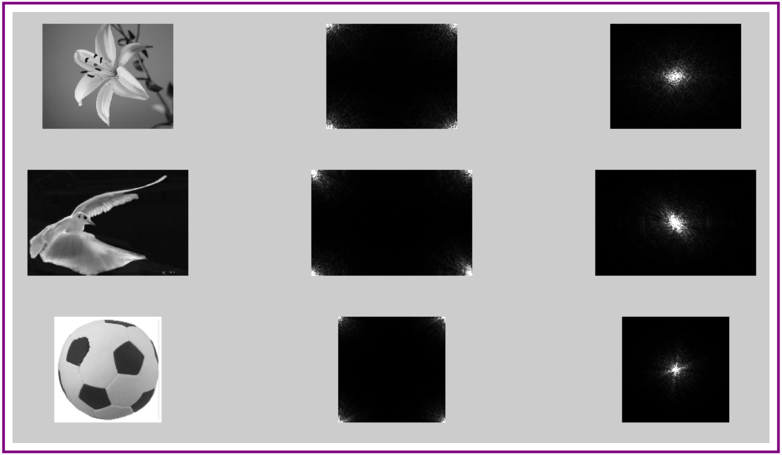
Centered spectrum

$$X(u, v) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} x(n, m) e^{-2\pi i \left(\frac{un}{N} + \frac{vm}{M}\right)}$$

- As in 1D, the Python function `fft2()` samples the frequency space on the domain $0 \leq n \leq N-1$ and $0 \leq m \leq M-1$
- To have low frequencies at the center of the spectrum, use `fftshift()`

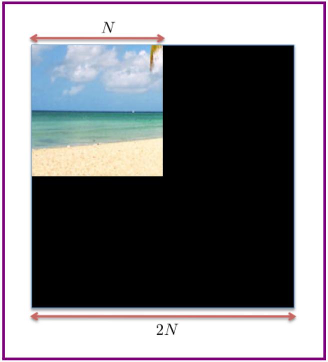
Visualization of the 2D spectrum

Centered spectrum



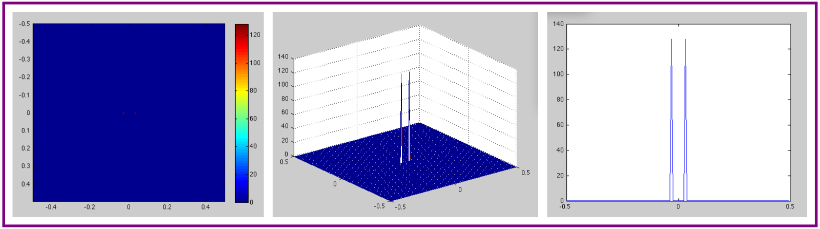
How to increase the frequency resolution?

- As in 1D: image is extended with zero values
- Zero-padding technique: here the resolution is multiplied by 2

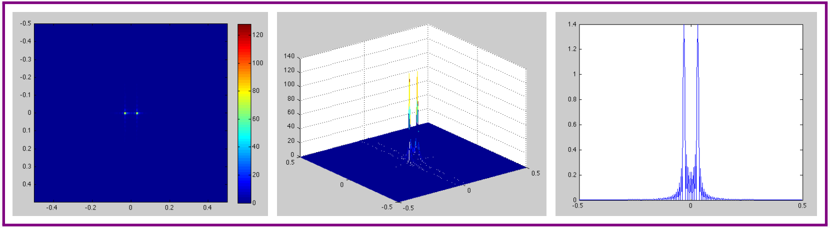


2D DFT: zero-padding

- Same number of samples

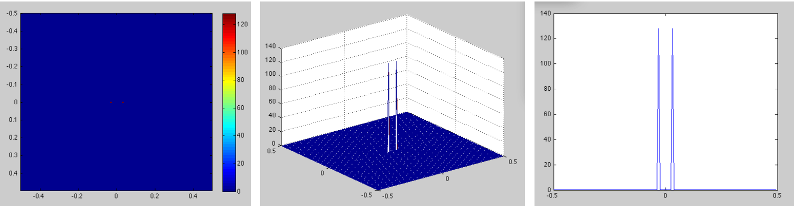


- Twice more samples

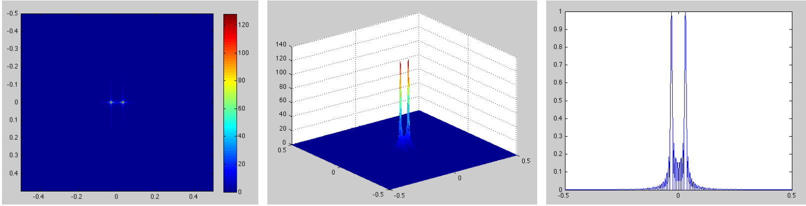


2D DFT: zero-padding

- Same number of samples



- Five times more samples



Digitization: conclusion

What you need to know

- The difference between:
  1. Fourier transform of a continuous signal

     $\Rightarrow$ a continuous function
  2. Fourier transform of a windowed signal

     $\Rightarrow$ a continuous function
  3. Fourier Transform of a discrete (sampled) signal

     $\Rightarrow$ again a continuous function
  4. discrete Fourier transform of a discrete signal

     $\Rightarrow$ a discrete function
- Understand the spectrum computed and visualized with Python
  1. `fftshift()` to have low frequencies in the center
  2. Notion of frequency resolution: use of zero-padding to enhance visualization