Fundamentals of Image Processing

Harris corner detector - Detailed solution

Master 1 Computer Science - IMAge & VCC

Sorbonne Université

Year 2020-2021

1. Gaussian window $w(x,y) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2+y^2}{2\sigma^2}}$, used as a weighting function in the mean squared intensity variation:

$$E^1_{u,v}(x_1,y_1) = \sum_{x,y} w(x-x_1,y-y_1) \times [I(x+u,y+v) - I(x,y)]^2$$

Note that in practice the sum is taken only over a limited domain (usually of size 3σ).

2. The Taylor series expansion of $I(x+u,y+v)$ at order 1 is:

$$I(x+u,y+v) = I(x,y) + u \cdot \frac{\partial I}{\partial x} + v \cdot \frac{\partial I}{\partial y} + \mathcal{O}(u^2,v^2) \approx I(x,y) + u \cdot I_x(x,y) + v \cdot I_y(x,y) \quad (1)$$

Note that this holds for small displacements u, v.
In the following derivations, we will write in short I_x for $I_x(x,y)$, and similarly for the other derivatives.

3. Matricial expression of $E^1_{u,v}(x_1,y_1)$:
Using the first order approximation, we get:

$$E^1_{u,v}(x_1,y_1) \approx \sum_{x,y} w(x-x_1,y-y_1) \times [u \cdot I_x + v \cdot I_y]^2 = \sum_{x,y} w(x-x_1,y-y_1) \left[u^2 I_x^2 + 2uvI_xI_y + v^2 I_y^2\right]$$

$$= u^2 \sum_{x,y} w(x-x_1,y-y_1) I_x^2 + v^2 \sum_{x,y} w(x-x_1,y-y_1) I_y^2 + 2uv \sum_{x,y} w(x-x_1,y-y_1) I_x I_y$$

$$= (u,v) \times \begin{bmatrix} \sum\limits_{x,y} w(x-x_1,y-y_1) \cdot I_x^2 & \sum\limits_{x,y} w(x-x_1,y-y_1) \cdot I_x\ I_y \\ \sum\limits_{x,y} w(x-x_1,y-y_1) \cdot I_x\ I_y & \sum\limits_{x,y} w(x-x_1,y-y_1) \cdot I_y^2 \end{bmatrix} \times \begin{pmatrix} u \\ v \end{pmatrix}$$

$$= (u,v) \times M(x_1,y_1) \times \begin{pmatrix} u \\ v \end{pmatrix}$$

with

$$M(x_1,y_1) = \begin{bmatrix} \sum\limits_{x,y} w(x-x_1,y-y_1) \cdot I_x^2 & \sum\limits_{x,y} w(x-x_1,y-y_1) \cdot I_x\ I_y \\ \sum\limits_{x,y} w(x-x_1,y-y_1) \cdot I_x\ I_y & \sum\limits_{x,y \in W} w(x-x_1,y-y_1) \cdot I_y^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sum\limits_{x,y} w(x_1-x,y_1-y) \cdot I_x^2 & \sum\limits_{x,y} w(x_1-x,y_1-y) \cdot I_x\ I_y \\ \sum\limits_{x,y} w(x_1-x,y_1-y) \cdot I_x\ I_y & \sum\limits_{x,y} w(x_1-x,y_1-y) \cdot I_y^2 \end{bmatrix} = \begin{bmatrix} w \star I_x^2 & w \star I_x\ I_y \\ w \star I_x\ I_y & w \star \cdot I_y^2 \end{bmatrix} = \begin{bmatrix} A & C \\ C & B \end{bmatrix}$$

Note that the coefficients of the matrix M, denoted by A, B, C for short, are values, that depend on each pixel position (x_1, y_1).

The matrix M, called auto-correlation matrix, represents the local structure of function $E^1(x_1,y_1)$ in a neighborhood of pixel (x_1,y_1).

4. Reminder on linear algebra:

   A real symmetric matrix $M = \begin{bmatrix} A & C \\ C & B \end{bmatrix}$ has positive eigenvalues. An eigenvector (u, v) associated with an eigenvalue λ is defined as:

$$\begin{bmatrix} A & C \\ C & B \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

   Solving for non zero vectors leads to

$$\lambda^2 - (A+B)\lambda + AB - C^2 = 0$$

   $A + B = Tr(M)$ is the trace of the matrix, and $AB - C^2 = Det(M)$ is its determinant.

   The solutions in λ of the this second degree equation are the eigenvalues:

$$\lambda_1 = \frac{Tr(M) + \sqrt{Tr(M)^2 - 4Det(M)}}{2}$$

$$\lambda_2 = \frac{Tr(M) - \sqrt{Tr(M)^2 - 4Det(M)}}{2}$$

   Hence the sum and product of eigenvalues are then:

$$\lambda_1 + \lambda_2 = Tr(M)$$

$$\lambda_1 \lambda_2 = Det(M)$$

5. Harris & Stephens proposed to compute the eigenvalues λ_1 and λ_2 of E^1 to identify different types of local behavior of the intensity function (Figure 1(a)). To reduce the computational complexity, these authors proposed the following criterion $R(x, y)$ (Figure 1(b)):

$$R(x, y) = Det(M) - k \cdot [Tr(M)]^2 \tag{2}$$

   where k is a small positive value.

   Let us show the equivalence between Figures 1(a) and 1(b):

   (a) If λ_1 and λ_2 are approximately equal, we note $\lambda_1 \sim \lambda_2 = \lambda$, and get

$$R(x, y) \sim \lambda^2 - k(4\lambda^2) = \lambda^2(1 - 4k)$$

   Since in practice k is chosen as a small value, i.e. $k << 1$, we get $R(x, y) \sim \lambda^2$. Then:

   i. If $\lambda \to 0$, this means that the derivative of I are close to 0, i.e. the region is flat (homogenous gray levels), and $R \to 0$.

   ii. If $\lambda > 0$ then $R > 0$, and we have locally a corner.

   (b) If $\lambda_1 >> \lambda_2$ (or the reverse) then

$$R(x, y) \sim \lambda_1\lambda_2 - k\lambda_1^2 = \lambda_1^2(\frac{\lambda_2}{\lambda_1} - k) \sim -k\lambda_1^2$$

   If the pixel is an edge, it means that the variations of I are in one direction only, i.e. $\lambda_1 >> \lambda_2$, and we get $R < 0$.

6. The Harris detector is invariant by rotation: Indeed the trace and determinant are intrinsic properties, that do not depend of the coordinate frame. To prove this, let us write the rotation in a matricial way as $P\begin{pmatrix} u \\ v \end{pmatrix}$ where P is a rotation matrix. The the previous computations leads to

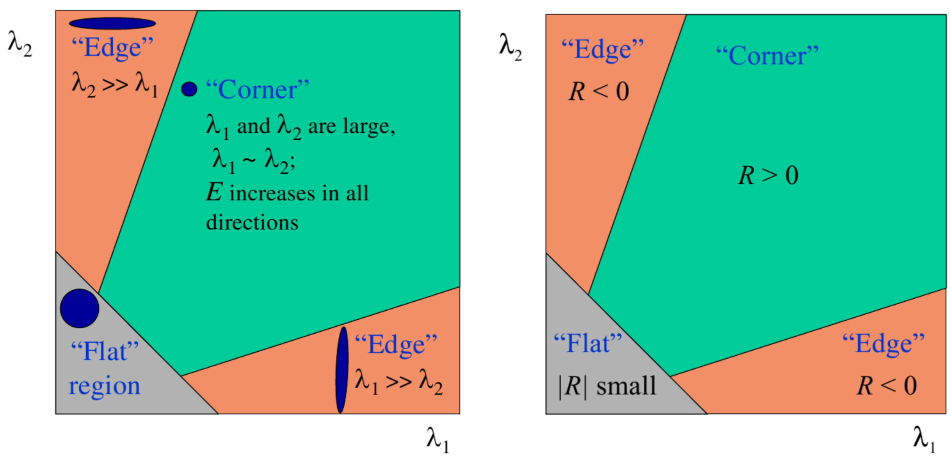
$$(u, v) \times P^t \times M(x_1, y_1) \times P \times \begin{pmatrix} u \\ v \end{pmatrix}$$

i.e. M is replaced by P^tMP. The determinant is equal to the product of the determinants of each matrix, and $Det(P) = 1$ is P is a rotation matrix. Hence $Det(P^tMP) = Det(P)$. A straighforward calculation with $P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, where θ is the rotation angle, leads to $Tr(P^tMP) = A + B = Tr(M)$.

7. The local maxima of $R(x, y)$ are invariant with respect to affine intensity variations, i.e. that can be written as $I'(x, y) = a \cdot I(x, y) + b$: Indeed each derivative writes $I'_x = aI_x$, etc. This means that every coefficient in M is multiplied by the constant a^2, and the new matrix writes $M' = a^2M$. Then we get $R' = Det(M') - kTr(M')^2 = a^4Det(M) - k(a^2Tr(M))^2 = a^4R$. Then R' is locally maximal at the same points as R.

8. The detector is not scale invariant since it depends on w.

9. A simple method to detect corners at different scales, based on the Harris detector, would consist in choosing different sizes of w (i.e. different values of σ).



(a) Eigenvalues (b) Harris criterion

Figure 1: Harris detector.