BIMA - Exam

January 2020

The grading scale for each exercise is only an indication and can change. No document authorized. Exam duration: 2 hours

Exercise 1 Questions on the lectures (10 points)

1. Consider an image whose values are encoded on 1 unsigned octet, and are all smaller than 50. What will be the visual appearance of the image? How could its contrast be significantly enhanced? Same question if the image values are all greater than 200?
2. Define the following terms: impulse response, transfer function.
3. Among the following filters, which ones are linear (justify!): (a) histogram stretching, (b) histogram equalization, (c) spatial convolution, (d) frequential filtering, (e) Harris filter.
4. Let x be a continuous signal on $\mathbb{R}$. Which are the continuous filters which are linear and invariant under translation? Same question if the signal is digitized on $\mathbb{Z}$.
5. Hyperspectral images are acquired using tens or even hundreds of wavelengths. This is the case for some satellite imaging systems, for example. Propose a method to visualize this type of images using colors. How could the loss of information be quantified?
6. What is the objective of the SIFT detector? Describe its principle.
7. Assume we want to segment an image into honogeneous regions. What is the main difference between k-means method and split-and-merge method?

Answer of exercise 1

1. Using an exponential function applied on the values in the first case, and a logarithm in the second case.
2. Impulse response: response of a filter applied to a Dirac distribution. Its Fourier transform is the transfer function.
3. (a) Stretching is an affine operation on the gray levels, hence not linear. (b) Not linear since $H(I + J) \neq H(I) + H(J)$. (c) Linear by definition. (d) Linear since the FT is linear. (e) Not linear (contains quadratic terms).
4. Continuous and discrete convolutions.
5. Apply PCA, select the three first components and use them as RGB channels. The sum of the 3 largest eigenvalues measures the amount of information the 3 first components contain.

6. SIFT is a filter that summarizes and image into a vector of 128 values, which are the histograms of gradient orientations. This representation is sensitive to scale and rotation, and allows comparing images.
7. k-means does not guarantee to obtain connected regions in a same class, by contrast to split and merge.

Exercise 2 Computation of the Fourier Transform (10 points)

The continuous Fourier transform (FT) of a continuous signal x is defined as $f \mapsto X(f) = \int_{\mathbb{R}} x(t)e^{-2i\pi ft}dt$. The FT of the function $t \mapsto e^{-2i\pi f_0 t}$ is the function $f \mapsto \delta(f - f_0)$, where δ is the Dirac distribution (equal to 0 everywhere except at point 0, and with integral equal to 1).

1. The "gate" function is defined as: $\text{Rect}(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$
   Compute the FT of this function.
2. Prove the "scale property" of the FT, i.e.
$$FT(t \mapsto x(\alpha t)) = X\left(\frac{f}{\alpha}\right)\frac{1}{|\alpha|}$$
   where X is the FT of signal x.
3. Derive the FT of the function $\text{Rect}_L : t \mapsto \text{Rect}\left(\frac{t}{L}\right)$.
4. Prove the translation property of the FT, i.e.:
$$TF(t \mapsto x(t - t_0))(f) = e^{-2i\pi f t_0}X(f)$$
   where X is the FT of signal x.
5. Let $x_1(t) = \text{Rect}(\frac{t}{2T_0})\,(\cos(2\pi t f_0) + \cos(4\pi t f_0))$ where $T_0 = \frac{1}{f_0}$. Compute the FT of x_1.
6. Draw the spectrum amplitude of x_1 as well as the graph of x_1. (No computer allowed...)
7. Let x_2 be the signal defined as:
$$x_2(t) = \begin{cases} \cos(2\pi f_0 t) & \text{if } t \in [-T_0, 0] \\ \cos(4\pi f_0 t) & \text{if } t \in [0, T_0] \\ 0 & \text{otherwise} \end{cases}$$
   Write $x_2(t)$ using the gate function.
8. Derive its FT.
9. Draw the graph of x_2 as well as its spectrum amplitude.
10. What do you conclude on the FT and its ability to localize in time the modes of a non-stationary signal?

Answer of exercise 2

1. $TF(\text{Rect})(f) = \int_{\mathbb{R}} \text{Rect}(t)e^{-2i\pi ft}dt = \int_{-1/2}^{1/2} e^{-2i\pi ft}dt = [\frac{e^{-2i\pi ft}}{-2i\pi f}]_{t=-1/2}^{t=1/2} = \frac{e^{-i\pi f}-e^{i\pi f}}{-2i\pi f} = \frac{\sin(\pi f)}{\pi f} = \text{sinc}(\pi f)$

2. $FT(t \mapsto x(\alpha t)) = \int_{-\infty}^{+\infty} x(\alpha t)e^{-2i\pi ft}dt$
   Apply a change of variable: $t' = \alpha t$. Then $dt' = \alpha dt$. If α is positive, then t' ranges from $-\infty$ to $+\infty$ when t ranges from $-\infty$ to $+\infty$. If α is negative, then t' ranges from $+\infty$ to $-\infty$. Then the integral becomes:

   - For α positive: $\int_{-\infty}^{+\infty} x(t')e^{-2i\pi f\frac{t'}{\alpha}}\frac{1}{\alpha}dt' = \frac{1}{\alpha}X(\frac{f}{\alpha})$.
   - For α negative: $\int_{+\infty}^{-\infty} x(t')e^{-2i\pi f\frac{t'}{\alpha}}\frac{1}{\alpha}dt' = -\frac{1}{\alpha}X(\frac{f}{\alpha})$.

   Grouping the two cases together leads to $FT(t \mapsto x(\alpha t)) = \frac{1}{|\alpha|}X(\frac{f}{\alpha})$.

3. $\alpha = \frac{1}{L} \Rightarrow X(f) = L\,\text{sinc}(L\pi f)$

4. Apply the change of variable $t' = t - t_0$ (hence $dt' = dt$):
   $$\int_{\mathbb{R}} x(t-t_0)e^{-2i\pi ft}dt = \int_{\mathbb{R}} x(t')e^{-2i\pi f(t'+t_0)}dt' = e^{-2i\pi ft_0}\int_{\mathbb{R}} f(t')e^{-2i\pi ft'}dt' = e^{-2i\pi ft_0}X(f)$$

5. $FT(\cos(4\pi f_0 t)) = \frac{\delta(f-2f_0)+\delta(f+f_0)}{2}$. From the Fourier transforms of the cosine function and of the gate function, applying the convolution theorem leads to:
   $$X_1(f) = 2T_0\,\text{sinc}(2\pi T_0 f) \star \left(\frac{\delta(f-f_0)+\delta(f+f_0)}{2} + \frac{\delta(f-2f_0)+\delta(f+2f_0)}{2}\right)$$

6. To draw the graph x_1 just superimpose the graphs of the two sine functions (one with twice the frequency of the other, with same origin) and to add them. The graph of X_1 is the spectrum of the 4 sinc functions, scaled and centered at $\pm 2f_0$ and $\pm f_0$.

7. $x_2(t) = \text{Rect}(\frac{t-T_0/2}{T_0})\cos(2\pi f_0 t) + \text{Rect}(\frac{t+T_0/2}{T_0})\cos(4\pi f_0 t)$

8. Use the translation formula: $FT(\text{Rect}(\frac{t}{T_0} \pm \frac{1}{2})(f) = e^{\pm i\pi f}FT(\text{Rect}(\frac{t}{T_0}))$, and derive X_2:
   $$X_2(f) = e^{-i\pi f}T_0\,\text{sinc}(\pi T_0 f)\star\left(\frac{\delta(f-f_0)+\delta(f+f_0)}{2}\right)+e^{i\pi f}T_0\,\text{sinc}(\pi T_0 f)\star\left(\frac{\delta(f-2f_0)+\delta(f+2f_0)}{2}\right)$$

   The FT of the translated gate functions contain factors of modulus 1, which have no impact on the spectrum. We still get a sinc function, convolved with 4 translated Dirac distributions.

   The overlapping in the sinc are more rapid since the support of the gate function is smaller.

9. The spectrum is almost the same as the one of x_1. The graph of x_2 is easy to draw.

10. Since the two spectra are almost identical, it is not possible to use the FT to localize in time the two modes of signal x_2. However their frequencies are well localized.

Exercise 3 Discrete Fourier transform (4 points)

The discrete Fourier transform (DFT) of a signal x of length N is defined as:

$$X(n) = \sum_{k=-N/2}^{N/2} x(k)e^{-2i\pi\frac{nk}{N}}$$

1. Prove that the filter whose impulse response is

$$f_1(n) = \begin{cases} -2 & \text{if } n = 0 \\ 1 & \text{if } n = \pm 1 \end{cases}$$

   is a high-pass filter. What is the cut-off frequency of this filter?

2. Prove that the filter $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is a low-pass filter.

Answer of exercise 3

1. We have $N = 3$ and for $x = f_1$:

$$\begin{aligned} X(n) &= \sum_{k=-1}^{1} x(k)e^{-2i\pi\frac{nk}{3}} \\ &= e^{2i\pi\frac{n}{3}} - 2 + e^{-2i\pi\frac{n}{3}} \\ &= 2\cos(\frac{2\pi n}{3}) - 2 \\ X(0) &= 2 - 2 = 0 \\ X(\pm 1) &= 2\cos(\frac{\pm 2\pi}{3}) - 2 = 2(-\frac{1}{2}) - 2 = -3 \end{aligned}$$

   We have $|X(\pm 1) > |X(0)|$ hence it is a high-pass filter. The cut-off frequency is $f_c = \pm 1$.

2.

$$\begin{aligned} X(n) &= \sum_{k=-1}^{1} x(k)e^{-2i\pi\frac{nk}{3}} \\ &= e^{2i\pi\frac{n}{3}} + 1 + e^{-2i\pi\frac{n}{3}} \\ &= 2\cos(\frac{2\pi n}{3}) + 1 \\ X(0) &= 2 + 1 = 3 \\ X(\pm 1) &= 2\cos(\frac{\pm 2\pi}{3}) + 1 = 2(-\frac{1}{2}) + 1 = 0 \end{aligned}$$

   We have $|X(\pm 1) < |X(0)|$, hence it is a low-pass filter.

Exercise 4 Spatial filtering (4 points)

1. Is the filter $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ separable? If yes, write the separated filters. What does A represent?

2. Same questions for $B = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix}$.

3. Let $X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T$ be a discrete 1D signal of length n (X is noted as a vector). Let A be a symmetrical matrix with three bands such that:

   - the diagonal elements are all equal to a_2
   - the elements of the upper diagonal are all equal to a_3
   - the elements of the lower diagonal are all equal to a_1

   Prove that AX can be written as a discrete convolution, with a kernel to be determined.

4. Generalize this result for a symmetrical matrix with $p < n$ diagonals with constant values $a_1, \cdots, a_p$.

Answer of exercise 4

1. A is separable: $A = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$. It represents an approximation of the derivative in horizontal and vertical directions: $A \simeq \frac{\partial^2}{\partial x \partial y}$

2. B is not separable since it cannot be written as the product of a line vector with a column vector. B is an approximation of the Laplacian.

3. The computation of the i^{th} line of AX yields: $a_1 x_{i-1} + a_2 x_i + a_3 x_{i+1}$. It is the convolution of X with the kernel $\begin{pmatrix} a_3 & a_2 & a_1 \end{pmatrix}$. Points at the boundary are handled by zero-padding.

4. The computation of the i^{th} line of AX yields: $a_1 x_{i-p/2} + \cdots + a_{p/2} x_i + \cdots + a_p x_{i+p/2}$, which is the convolution with the kernel $\begin{pmatrix} a_p & a_{p-1} & \cdots & a_1 \end{pmatrix}$.

Exercise 5 Hessian detector (12 points)

The Hessian matrix $H(x, y)$ at each pixel (x, y) of an image I is defined as:

$$H = \begin{bmatrix} \frac{\partial^2 I}{\partial x^2} & \frac{\partial^2 I}{\partial x \partial y} \\ \frac{\partial^2 I}{\partial x \partial y} & \frac{\partial^2 I}{\partial y^2} \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{bmatrix}$$

The eigenvectors of $H(x, y)$, with the associated eigenvalues, define the principal directions of variation of I at (x, y). Let us remind that:

- for a 2×2 symmetrical matrix $M = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$, the determinant is $\det(M) = ab - c^2$.

- the determinant is invariant under change of orthonormal coordinate frame. It follows that the determinant is the product of the two eigenvalues of M.

1. Hessian detector
   (a) Compute the trace of $H(x, y)$. Which detector do you recognize?
   (b) What are the expected eigenvalues of $H(x, y)$ if (x, y) is a corner point? if (x, y) is in a homogeneous region? if (x, y) is an edge point?
   (c) The Hessian detector is based on the criterion $C_h(x, y) = |\det(H)|$.
      - Explain how to use $C_h(x, y)$ to detect corners.
      - Does $C_h(x, y)$ allow distinguishing between corners, edges, and homogeneous regions, as Harris detector does?
   (d) Detection algorithm. Provide the python code:
      - of a function `computeH(I)` which computes the Hessian criterion $C_h(x, y)$ at each pixel (x, y) of animage I. Assume we already have a function `convolution(I,M)` which computes the convolution of image I with the kernel M.
      - of a function `HessianDetection(I)` which detects the key points in I from the Hessian criterion. In particular, explain the post-processing steps.
2. Invariance of the Hessian detector
   (a) Is the Hessian detector invariant under rotation? Justify.
   (b) Consider an affine change of intensity: $I'(x, y) = aI(x, y) + b, a \in \mathbb{R}^*, b \in \mathbb{R}$.
      i. Let $H'(x, y)$ be the Hessian matrix of $I'(x, y)$. Write $H'(x, y)$ as a function of $H(x, y)$.
      ii. Is the Hessian criterion $C_h(x, y)$ invariant under such affine changes of intensity? Justify.
      iii. Are the local maxima of $C_h(x, y)$ invariant under such affine changes of intensity? If this is the case, propose a method to adapt the detection threshold from I to I'. Otherwise, justify.

Answer of exercise 5

1. Hessian detector:
   (a) $Trace(H(x, y)) = I_{xx} + I_{yy}$, which is the Laplacian.
   (b) Corner: two large eigenvalues; edge: one large eigenvalue and one small; homogeneous regions: two small eigenvalues (0 in a constant region).
   (c) $C_h(x, y) = |det(M)| = |\lambda_1| \cdot |\lambda_2|$.
      - Then applying a threshold on $C_h(x, y)$ indicates variation of the intensity in two orthogonal directions. The absolute value allows ignoring the sign of the eigenvalues.

- Is it less easy than with the Harris dector. For instance, if $|\lambda_1| = 10^{-4}$ and $|\lambda_2| = 1$, the determinant will be the same as for $|\lambda_1| = |\lambda_2| = 10^{-2}$. Note that Harris criterion also involves $-kTrace(H)$ which would lead to large negative values in the first case, and the criterion would be more discriminant.

(d)
- Function `computeH(I)`: to compute the second derivative, a possibility is to apply two times a convolution using Sobel kernels S_x and S_y:

$$I_{xx} = \texttt{convolution(convolution(}I, S_x), S_x)$$

$$I_{xy} = \texttt{convolution(convolution(}I, S_x), S_y)$$

$$I_{yy} = \texttt{convolution(convolution(}I, S_y), S_y)$$

- Function `HessianDetection(I)`: computes C_h from H, and then computes the local maxima of C_h which are then thresholded.

2. Invariance of the Hessian detector:

   (a) The determinant is invariant under change of orthonormal frame, which corresponds to a rotation, hence the determinant is invariant under rotation.

   (b) $I'(x, y) = aI(x, y) + b$
   - $H'(x, y) = aH(x, y)$
   - Not invariant since $C'_h(x, y) = a^2 C_h(x, y)$
   - The local maxima are the same. However the threshold has to be changed. If image I_1 has a range dynamic d_1 and an image I_2 has a range dynamic d_2, the threshold δ that applies to I_1 has to be changed to $\delta(\frac{d_2}{d_1})^2$ for I_2.