BIMA - Exam

January 2018

The grading scale for each exercise is only an indication and can change. No document authorized. Exam duration: 2 hours

Exercise 1 Questions on the lectures (10 points)

1. Given a pixel of an image and its autocorrelation matrix M computed in its neighborhood, what is the Harris criterion for corner detection? Is this detector invariant by rotation? Justify.
2. What is the dimension of the SIFT descriptor? Enumerate transformations for which SIFT is invariant.
3. Given a set of data characterized by their variance-covariance matrix Σ and a unit vector v, what is the criterion defined for the minimization of the variance projected on the straight line span by v? How to solve this equation in a practical way?
4. Explain the aliasing artefact and give the conditions under which it appears.
5. What is the principle that allows generalizing the 1D Fourier transform properties to higher dimensions? Show the link between the 1D Fourier transform and the 2D Fourier transform.
6. We want to classify the pixels of an image into ten classes. To this end, we make use of k-means method. In this case, what is the value of k? How many connected regions can be obtained?
7. What is the difference between histogram equalization and histogram stretching? Are these transformations reversible?

Answer of exercise 1

Grading: 2-2-2-1-1-1-1

1. The Harris criterion is defined as: $H = det(M) - k.trace(M)^2$. H is invariant under rotations (which is not the case for Moravec detector) since applying a rotation matrix does not change the eigenvalues of H. The multi-scale extension of the criterion is scale invariant.
2. The SIFT descriptor has dimension 128. It is invariant under rotations (if the analysis window is rotated with respect to the main orientation), as well as under global intensity variations (thanks to the L2 normalization) and under saturation phenemena (hysteresis thresholding).

3. The total projected variance is equal to $v^T \Sigma v$, and its minimization under constraint amounts to compute the eigenvalues and eigenvectors of Σ, and to select the largest eigenvalue and the corresponding subspace.

4. Aliasing occurs when Shannon condition is not satisfied, i.e. the maximum frequency is larger than twice the one of sampling. To avoid this phenomenon, it is possible to apply a Gaussian filter on the image, so as to suppress the high frequencies and satisfy Shannon condition.

5. Fourier transform is separable. This means that properties that are satisfied in one direction are satisfied in all other directions. We have $TF(I)(f,g) = TF(y \mapsto K(f,y))(g)$ and $K(f,y) = TF(x \mapsto I(x,y))(f)$.

6. $k = 10$. If the image is classified based only on the gray levels as feature, then the number of connected components cannot be known (i.e. a class can be composed of several connected components).

7. Stretching changes the bins distribution in the histogram, but not their size. Equalization changes the bins distribution in the histogram, as well as their size.

Exercise 2 Fourier transform (10 points)

We recall the definition of

- the "gate" function: $\mathrm{Rect}(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$,
- the Fourier transform of a signal x: $X(f) = \int_{\mathbb{R}} x(t) e^{-2i\pi f t} dt$,
- and the translation property of TF: $TF(t \mapsto x(t - t_0))(f) = X(f) e^{-2i\pi f t_0}$.

1. Calculate the Fourier transform of Rect.
2. Let Sign be the function defined as: $\mathrm{Sign}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{otherwise} \end{cases}$

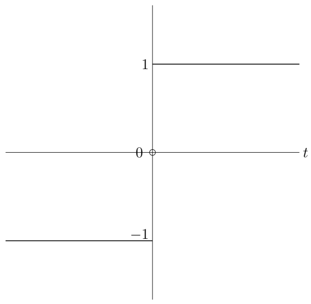
   Draw the graph of this function.
3. We define $U(t) = \mathrm{Sign}(t + \frac{1}{2})$ and $V(t) = \mathrm{Sign}(t - \frac{1}{2})$. Draw their graph and find a simple relation between Rect, U et V.
4. Derive the Fourier transform of Sign.
5. The Heavyside function is defined as: $H(t) = \frac{1}{2}(1 + \mathrm{Sign}(t))$. Derive its Fourier transform.

Answer of exercise 2

Grading: 2-1-2-3-2

1. $TF(\mathrm{Rect})(f) = \int_{\mathbb{R}} \mathrm{Rect}(t) e^{-2i\pi f t} dt = \int_{-1/2}^{1/2} e^{-2i\pi f t} dt = [\frac{e^{-2i\pi f t}}{-2i\pi f}]_{t=-1/2}^{t=1/2} = \frac{e^{-i\pi f} - e^{i\pi f}}{-2i\pi f} = \frac{\sin(\pi f)}{\pi f} = \mathrm{sinc}(\pi f)$

2. Graph of the sign function:



3. Similar drawing up to a translation.
   $\mathrm{Rect}(t) = \frac{1}{2}(U(t) - V(t))$

4. $TF(\mathrm{Rect}) = \frac{1}{2}(TF(S(t+1/2) - S(t-1/2))) = \frac{1}{2}(TF(S)e^{2i\pi f/2} - TF(S)e^{-2i\pi f/2}) = TF(S)(i\sin(\pi f))$ or $TF(\mathrm{Rect})(f) = \mathrm{sinc}(\pi f) = TF(S)(i\sin(\pi f))$, hence $TF(S) = \frac{1}{i\pi f}$

5. $H(t) = \frac{1}{2}(1 + S(t))$, hence $TF(H) = \frac{1}{2}(TF(1) + TF(S)) = \frac{1}{2}(\delta(f) + \frac{1}{i\pi f})$

Exercise 3 Sampling (10 points)

We recall that the Dirac comb is defined as: $\text{Ш}_T(t) = \sum_{k\in\mathbb{Z}} \delta(t - kT)$, and its Fourier transform is: $\frac{1}{T}\text{Ш}_{\frac{1}{T}}(f)$.

1. Let $x(t)$ be a continuous signal that we sample with a period T_e to obtain a signal $x_e(t) = \text{Ш}_{T_e}(t)x(t)$. Write the relation between $X(f)$ and $X_e(f)$ that are the Fourier transforms of signals $x(t)$ and $x_e(t)$, respectively.

2. The spectrum of $x(t)$ is supposed bounded, and its highest frequency is denoted by f_m. We define $X_L(f) = X_e(f)\,\mathrm{Rect}(\frac{f}{L})$. What is the condition on L to retrieve exactly X from X_L?

3. In the following and to simplify, we set $L = 1$. Let Tri be the function defined as: $\mathrm{Tri}(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$

   and its Fourier transform is: $\mathrm{sinc}^2(2\pi f)$. Draw the graph of Tri.

4. X_e is truncated with Tri: we set $X_T(f) = X_e(f)\,\mathrm{Tri}(f)$. Give the condition to retrieve exactly X from X_T.

5. Give the interpolating formula that reconstructs X from X_T.

6. Is this interpolation better than Shannon's one? Justify.

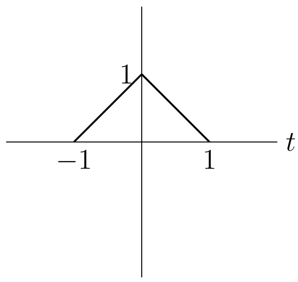
Answer of exercise 3

Grading: 2-2-1-1-2-2-

1. From $x_e(t) = Ш_{T_e}(t)x(t)$, we derive $X_e(f) = \frac{1}{T_e}\sum_{k\in\mathbb{Z}} X(f - \frac{k}{T_e})$. It is the periodized Fourier transform.

2. In order to retrieve exactly X, we must have $f_m < 2f_e = 2/T_e$ (Shannon's theorem). Since X_e is obtained with a gate function of width L, we have to take $L = T_e$.

3. Graph of the Tri function:



4. Shannon condition.

5. $X(f) = X_T(f) = \text{Tri}(f)X_e(f)$. Using inverse TF: $x(t) = \text{sinc}^2(\pi t) \star x_e(t) = \sum_k x_e(kT_e)\,\text{sinc}^2(\pi(t - kT_e))$

6. The square sinc decreases faster than sinc, and the band-pass effects are then stronger, resulting in less aliasing.

Exercise 4 Spatial filtering (10 points)

Let A be an image. Consider the following algorithm:

- step 1: apply a mean filter of size 3×3 on A to obtain an image B.
- step 2: subtract B from A to obtain C.
- step 3: multiply C by a scalar coefficient $k > 0$ to obtain D.
- step 4: add D to A to obtain the final image E.

1. Associate images B, C, D and E to images 1, 2, 3 and 4 respectively displayed in Figure 1. In this example, k has been set to 3. To help you, we display some statistics of these images:

	min	mean	max
image 1	-0.059	0.277	1.0
image 2	0.000	0.276	0.996
image 3	-0.129	0.000	0.294
image 4	-0.043	0.000	0.098
original image A	0.000	0.276	0.996

2. Consider the small region extracted from the original image A:

$$\begin{pmatrix} 0 & 1 & 8 & 32 & 79 & 136 & 177 & 185 \\ 0 & 1 & 8 & 32 & 79 & 135 & 176 & 184 \\ 0 & 1 & 8 & 31 & 78 & 134 & 175 & 183 \end{pmatrix}$$

   Apply the all steps of the filter on the second line of this region, excluding the first and last column (then, there are 6 values to calculate at each step) with parameter k set to 3.

3. Explain and justify the behavior of this filter. Is it a linear filter? If yes, is it a high, low or band pass filter? Give the kernel.

4. Explain the role of parameter k.

5. Write a Python code of the filter.

Answer of exercise 4

Grading: 2-3-2-2-1

1. Correspondence:
   - B: image 2
   - C: image 4
   - D: image 3
   - E: image 1

2. Filtering steps:
   - B: 3 14 39 82 130 165
   - C: -2 -6 -7 -3 5 11
   - D: -6 -18 -21 -9 15 33
   - E: -5 -10 11 70 150 209

3. This filter enhance the contours. It is a linear high-pass filter. The convolution kernel can be computed as:

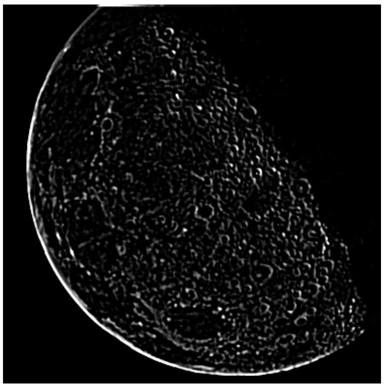
$$\begin{aligned}
E &= A + k(A - A \star \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}) \\
&= A \star \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + k(A \star \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}) \\
&= A \star (\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{k}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}) \\
&= A \star \begin{pmatrix} -\frac{k}{9} & -\frac{k}{9} & -\frac{k}{9} \\ -\frac{k}{9} & 1 + \frac{8k}{9} & -\frac{k}{9} \\ -\frac{k}{9} & -\frac{k}{9} & -\frac{k}{9} \end{pmatrix}
\end{aligned}$$

4. If $k = 0$ then the filter is equal to the identity. When k increases, the contours are more and more enhanced.

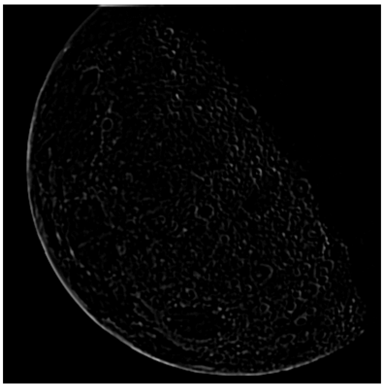
(a) original image original A

(b) image 1

(c) image 2



(d) image 3



(e) image 4

Figure 1: Images.