PHYS 4302

Homework 7

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Problem 1

You know very well that the classical trajectory of a charged particle (charge q, mass m) in the uniform magnetic field B (directed, say, along the z-axis) is helical: the particle moves with constant velocity along the field lines and executes a circular motion in the perpendicular (xy) plane with the cyclotron frequency

$$\omega_c = \left|\frac{qB}{m}\right|. \tag{1}$$

In this guided problem you will show that the energy spectrum of the orbital motion of a quantum particle in this magnetic field is given by

$$E = \frac{\hbar^2 k_z^2}{2m} + \left(n + \frac{1}{2}\right)\hbar\omega_c. \tag{2}$$

The first term in (2) represents, just as in the classical case, a free motion along the field with the continuous z-projection of the momentum $\hbar k_z$. The second term with discrete values of $n \in \mathbb{N}$ describes energy levels corresponding to the quantized motion in the xy-plane. These are famed Landau levels. The quantization of the orbital motion in the perpendicular plane is responsible for the diamagnetic part of the response of the electron gas. (We are not interested now in the interaction of the magnetic field with the spin-originated magnetic moment.)

As discussed in class, the Hamiltonian (orbital motion only) of a particle exposed to the magnetic field

$$B = \nabla \times A \tag{3}$$

is

$$\hat{H} = \frac{1}{2m}\left(\hat{p} - qA\right)^2, \tag{4}$$

where the canonical momentum operator $\hat{p} = -i\hbar\nabla$. There are many ways to choose the vector-potential A for the same field, here we choose the gauge in which the components of the vector potential

$$A = (0, Bx, 0)\,. \tag{5}$$

(a) Verify that the vector potential (5) indeed yields the uniform magnetic field B along the z-axis. Write down the stationary Schrödinger equation

$$\hat{H}\psi(x,y,z) = E\psi(x,y,z) \tag{6}$$

explicitly in Cartesian coordinates using this vector potential.

$$\begin{aligned}
\hat{H} &= \frac{1}{2m}(\hat{p} - qA)^2 \\
&= \frac{1}{2m}(\hat{p} - qA)(\hat{p} - qA) \\
&= \frac{\hat{p}^2 + q^2A^2}{2m} - \frac{q}{2m}(A\hat{p} + \hat{p}A)
\end{aligned}$$

$$\begin{aligned}
[A, \hat{p}] &= -i\hbar(\nabla \cdot A) \\
&= 0 \rightarrow A\hat{p} = p\hat{A}
\end{aligned}$$

$$\begin{aligned}
\hat{H} &= \frac{\hat{p^2}}{2m} + \frac{q^2A^2}{2m} - \frac{qA\hat{p}}{m} \\
&= \frac{\hat{p^2}}{2m} + \frac{q^2A^2}{2m} + \frac{i\hbar Bx}{m}\frac{\partial}{\partial y} \\
&= -\frac{\hat{h^2}\nabla^2}{2m} + \frac{q^2B^2}{2m}x^2 + \frac{i\hbar Bx}{m}\frac{\partial}{\partial y}
\end{aligned}$$

$$\hat{H}\psi = E\psi$$

$$\boxed{-\frac{\hat{h^2}\nabla^2}{2m}\psi(x,y,z) + \frac{q^2B^2}{2m}x^2\psi(x,y,z) + \frac{i\hbar Bx}{m}\frac{\partial}{\partial y}\psi(x,y,z) = E\psi(x,y,z)}$$

(b) Show that solutions of the derived Eq. (6) can be sought in the form

$$\psi(x,y,z) = e^{i(k_y y + k_z z)}\chi(x) \tag{7}$$

so that the eiegenvalue equation for the unknown function $\chi(x)$ has the form of a one-dimensional harmonic oscillator problem with the Hamiltonian

$$\hat{H}' = \frac{\hat{p_x}^2}{2m} + \frac{m\omega_c^2}{2}(x - x_0)^2. \tag{8}$$

What is the relationship of the center x_0 to k_y?

Substituting (7) into (6), we get

$$\left(-\frac{\hat{h^2}\nabla^2}{2m} + \frac{q^2B^2}{2m}x^2 + \frac{i\hbar Bx}{m}\frac{\partial}{\partial y} - E\right)e^{i(k_y y + k_z z)}\chi(x) = 0$$

Given the partial derivatives with the respect to (7), x_0 must be proportional to k_y.

(c) Given the eigenvalues of (8), do you confirm the resulting overall energy spectrum (2)? Now that the corresponding (unnormalized) stationary states can be labeled as

$$\psi_{nk_yk_z}(x,y,z) = e^{i(k_y y + k_z z)}\chi_{nk_y}(x), \tag{9}$$

we understand that energy (2) does not depend on k_y so there is a lot of degeneracy here. Write down explicitly what functions $\chi_{nk_y}(x)$ are, without normalization factors.

Problem 2

As we discussed in class, the strongest responses due to the interaction of charged particles with electro(magnetic) fields correspond to electric-dipole allowed transitions, for which the matrix elements of the position operator

$$\langle a'|r|a\rangle \neq 0 \tag{10}$$

between the unperturbed states $|a\rangle$ and $|a'\rangle$ do not vanish.

Suppose we are dealing with effectively one-electron spherically symmetric objects such as the Hydrogen atom, where the unperturbed eigenstates are characterized by their quantum numbers n, l and m:

$$|a\rangle \equiv |nlm\rangle.$$

(a) Using your textbook(s), e.g. Griffiths' *Introduction to Quantum Mechanics*, or on your own (!), study and derive constraints on the changes in those quantum numbers (such as, e.g., $l \to l'$) that may occur in electric-dipole allowed transitions $|a\rangle \to |a'\rangle$ in Eq. (10). Try not to just copy from a textbook but describe your own digest of derivations and explanations.

$$\begin{aligned} n' &= \{n-1, n, n+1\} \\ l' &= \{l-1, l+1\} \\ m' &= \{m-1, m, m+1\} \end{aligned}$$

These rules are provided by (10) such that $\langle n'l'm'|r|nlm\rangle \neq 0$.

(b) In the Hydrogen atom, if the electron is initially in the state $|320\rangle$, describe all possible sequences of the electric-dipole allowed transitions that lead to its return to the ground state $|100\rangle$ accompanied by the emission of the corresponding photons.

$$\begin{gathered} |320\rangle \to |210\rangle \to |100\rangle \\ |320\rangle \to |211\rangle \to |100\rangle \\ |320\rangle \to |220\rangle \to |120\rangle \to |100\rangle \\ |320\rangle \to |221\rangle \to |121\rangle \to |100\rangle \end{gathered}$$