PHYS 4302
Homework 8

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Problem 1

Two identical spinless bosons, mass m each, live in the free 3D space and are bound owing to their parabolic interaction potential energy

$$V = \frac{K\left|r_1 - r_2\right|^2}{2}, \tag{1}$$

where r_1 and r_2 are the particle position vectors and K the stiffness constant.

(a) What is the ground-state energy of this system?

Using the equation for the energy of a 3D harmonic oscillator $E = \left(n + \frac{3}{2}\right)\hbar\omega$, we have

$$\boxed{E = \frac{3}{2}\hbar\omega}$$

(b) What is the energy of the first (lowest-energy) excited *vibrational* level? How many-fold degenerate is this excited level?

Using the equation for the energy of a 3D harmonic oscillator $E = \left(n + \frac{3}{2}\right)\hbar\omega$, we have

$$\boxed{E = \frac{5}{2}\hbar\omega}$$

Because we're in 3D space, $n = n_x + n_y + n_z$, so one of these components must be 1 while both of the others are 0. This yields 3 combinations, so this excited level is $\boxed{\text{3-fold degenerate.}}$

(c) Does it really matter for the vibrational energy spectrum of this system that we are dealing with identical bosons and why?

Yes, it matters. In the case of non-identical bosons the degeneracy depends on the spin of the particles and their energies (can't be reduced to $n = n_x + n_y + n_z$).

Problem 2

Two electrons (mass m each) are confined to move on a circle of radius R. The electrons repel each other via a short-range force so that the potential energy of their interaction is well-represented by a δ-function:

$$U\left(x_1, x_2\right) = C\delta\left(x_1 - x_2\right), \;\; C > 0, \tag{2}$$

where x_1 and x_2 are one-dimensional electron coordinates *along* the circle. The repulsion is weak enough (coefficient C is small enough) to be treated as the lowest-order perturbation to the non-interacting two-particle states.

(a) Write down the energy spectrum of this two-electron system if the electrons were non-interacting.

(b) Determine corrections to the non-interacting energy spectrum that follow from the interaction (2) for *both* cases of the total system spin being equal to 0 (singlet case) and 1 (triplet case).

(c) Comment on the origin of the difference, if any, in these corrections for the singlet and triplet states

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Problem 3

A one-dimensional infinite square potential well of width a contains 14 identical *non-interacting* particles of mass m and spin $s = \frac{5}{2}$ each.

(a) What is the ground state energy $\mathscr{E}_0$ of this 14-particle system?

Each energy state can contain up to $n = 2s + 1 = 5 + 1 = 6$ particles, and the energy of each state is given by

$$\begin{aligned} E_n &= \frac{\pi^2 \hbar^2 n^2}{2ma^2} \\ &= \frac{\pi^2 \hbar^2}{2ma^2}\left(6 + 6 \cdot 2^2 + 6 \cdot 3^2\right) \\ &= \boxed{\frac{12\pi^2\hbar^2}{ma^2}} \end{aligned}$$

(b) What is the force F exerted on the walls of the potential well in the ground state?

$$\begin{aligned} F &= -\frac{d\mathscr{E}_0}{da} \\ &= -\frac{d}{da}\left(\frac{12\pi^2\hbar^2}{ma^2}\right) \\ &= \boxed{\frac{24\pi^2\hbar^2}{ma^3}} \end{aligned}$$

(c) If the well contained only 7 particles, would its ground state energy be $\frac{\mathscr{E}_0}{2}$ or not?

$$\begin{aligned} E_7 &= \frac{\pi^2 \hbar^2 n^2}{2ma^2} \\ &= \frac{\pi^2 \hbar^2}{2ma^2}\left(6 + 1 \cdot 2^2\right) \\ &= \frac{5\pi^2\hbar^2}{ma^2} \neq E_{14} \end{aligned}$$