PHYS 4302
Homework 6

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The purpoose of these somewhat artificial examples is for you to practice some calculations that may be involved in applications of Fermi's golden rule:

$$W = \frac{2\pi}{\hbar} \int df \left|\langle f|F|i\rangle\right|^2 \delta\left(E_f - E_i - \hbar\omega\right) \tag{1}$$

for the rate of transitions into a continuum spectrum as caused by periodic perturbations

$$V(t) = Fe^{-i\omega t} + F^\dagger e^{i\omega t}. \tag{2}$$

Equation (1) is written for transitions to higher energies $E_f > E_i$ and assumes that continuum states are normalized as $\langle f|f'\rangle = \delta\left(f - f'\right)$.

Given that the initial discrete state $|i\rangle$ has energy $E_i = -\epsilon < 0$, you are asked to calculate the transition rate (1) as a function of the perturbation frequency ω, $W(\omega)$, for the cases specified below. On your way you are supposed to take care of the appropriate normalization factors, etc.

(a) Transitions into a "one-dimensional continuum."
The initial state's wave function behaves as $\psi_i(x) \propto e^{-\alpha|x|}$ while the final states $|f\rangle$ are eigenstates of the Hamiltonian $\hat{H}_1 = -\left(\frac{\hbar^2}{2m}\right)\frac{\partial^2}{\partial x^2}$. The perturbation interaction operator is given as

$$F = -iU\frac{\partial}{\partial x}. \tag{3}$$

$$\begin{aligned}
W &= \frac{2\pi}{\hbar} \int df \left|\langle f|F|i\rangle\right|^2 \delta\left(E_f - E_i - \hbar\omega\right) \\
\langle f|F|i\rangle &= -iU \int dx f^\dagger \frac{\partial}{\partial x}\psi_i(x) \\
&= iU\alpha\left(\int_{-\infty}^{\infty} dx f^\dagger e^{-\alpha x} - dx f^\dagger e^{\alpha x}\right) \\
&= \boxed{2iU\alpha\int_0^\infty dx f^\dagger(x) e^{-\alpha x}} \\
E_f &= \frac{\hbar^2 k^2}{2m} \\
W &= \frac{2\pi}{\hbar} \int df \left|\langle f|F|i\rangle\right|^2 \delta\left(\frac{\hbar^2 k^2}{2m} + \epsilon - \hbar\omega\right) \\
&= \boxed{\frac{4\pi U^2\alpha^2}{\hbar^3}\sqrt{2m\left(\hbar\omega - \epsilon\right)}}
\end{aligned}$$

(b) Transitions into a "two-dimensional continuum."

The initial state's wave function behaves as $\psi_i(x,y) \propto e^{-\alpha\left(x^2+y^2\right)}$ while the final states $|f\rangle$ are eigenstates of the Hamiltonian $\hat{H}_2 = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. The perturbation interaction operator is given as

$$F = U. \tag{4}$$

$$
\begin{aligned}
W &= \frac{2\pi}{\hbar}\int df \left|\langle f|F|i\rangle\right|^2 \delta\left(E_f - E_i - \hbar\omega\right) \\
\langle f|F|i\rangle &= U\int\int dx\ dy\ \psi^\dagger e^{-\alpha\left(x^2+y^2\right)}\psi \\
&= U\int\int dx\ dy\ \psi e^{-\alpha\left(x^2+y^2\right)} \\
&= U\int_0^{2\pi}\int_0^{\infty} d\theta\ dr\ r\psi^\dagger e^{-\alpha r^2} \\
&= U\int_0^{2\pi}\int_0^{\infty} d\theta\ dr\ r\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(r-\rho)^2}{2}}e^{-\alpha r^2} \\
&\text{(where the final state is a Gaussian distribution with radius } \rho \text{ and spread } \sigma \\
&= U\frac{\sigma^2}{\sqrt{2\pi}}\int_0^{2\pi}\int_0^{\infty} d\theta\ du\ \frac{1}{\sigma}e^{-\frac{(u-\rho\sqrt{\alpha})^2}{2\sigma^2}}e^{-u^2},\ u = r\sqrt{\alpha} \\
&= \boxed{U\sigma\rho\sqrt{\frac{\pi}{2}}e^{-\alpha\rho^2}} \\
W &= \frac{2\pi U^2}{\hbar}\int dx\ dy\ \psi\psi_i^2\delta\left(E_f - E_i - \hbar\omega\right)
\end{aligned}
$$

Parameter U appearing in (3) and (4) as well as parameter α of the spatial extent of the initial wave function are just given constants of appropriate dimensions.