PHYS 4301
Final Exam

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Problem 1

A particle of *unknown* mass (further denoted M) is bound in a spherically symmetric potential well

$$V(r) = V_0 \left(1 - \frac{1}{\cosh^4\left(\frac{r}{a}\right)}\right) \text{ with given parameters } V_0 > 0, a > 0. \tag{1}$$

The particle is known to be in some stationary state with energy

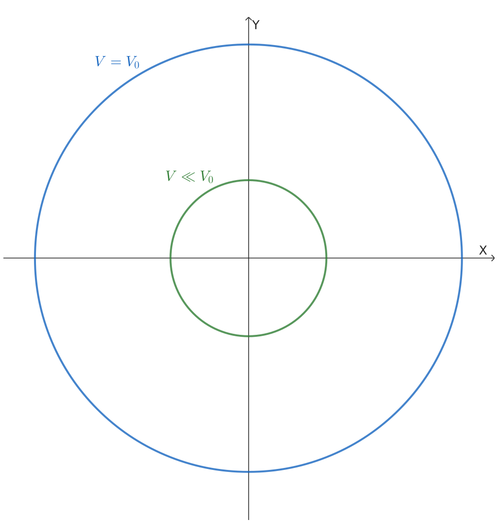
$$E \ll V_0 \tag{2}$$

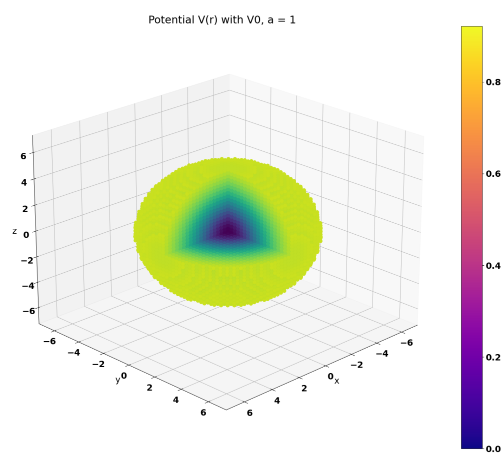
and having magnetic quantum number

$$m = -3. \tag{3}$$

We are *not* looking for exact solutions of the Schrödinger equation here. Instead, we want to have an *approximate* treatment based on condition (2).

(a) Sketch the overall behavior of the potential and indicate the spatial region of interest for specific limit (2) as you will be setting up your approach to the problem.





In the first plot, a projection of the potential onto the XY plane is shown. The outer blue ring shows the radial distance at which the potential reaches the value V_0. The inner green ring shows the radial distance at which the potential is much less than V_0, this is the area of interest. The potential is symmetrical, so this generalizes to the YZ and XZ planes as well.

In the second plot, I arbitrarily set $V_0 = 1$ for visualization, so the spatial region of interest for the specific limit (2) is the dark region in the center of the potential. Yellow areas have a potential close to V_0 so are not in the region of interest.

(b) With the additional information that given energy E is the *lowest* possible energy consistent with value (3) of m in potential (1), find the yet unknown mass M.

Given that E is the lowest possible energy consistent with m, we know that the principal quantum number n takes on the value $n = 4$, and the angular momentum quantum number ℓ takes on the value $\ell = 3$.

The $|4, 3, -3\rangle$ orbital for a Hydrogen atom describes a ring of bubble-shaped probability densities in a ring around the nucleus, similar to an inverse of our particle's likely probability density given the nature of the potential well. The Hydrogen atom's potential $U(r) = -\frac{q_e^2}{4\pi\epsilon_0 r}$ has a similiar shape to that of the provided potential, so we can analogize the two.

By scaling hypothetical values of V_0 and a to match the shape of the Hydrogen atom's potential distribution, we can estimate the mass of our unknown particle in terms of m_e, the mass of an electron. Both k and a determine the shape of their respective potentials. The ratio between k and e_m is approximately $1 \cdot 10^{40}$, so the ratio between a and M should be similar. Therefore, we can say that $\boxed{M \simeq a \cdot 10^{40}}$. This also states that M is inversely proportional to $V(r)$.

Problem 2

A particle of mass M lives in a three-dimensional spherically symmetric potential *shell* with impenetrable walls:

$$V(r) = \begin{cases} +\infty, & r \leq a \\ 0, & a < r < b \\ +\infty, & r \geq b \end{cases} \tag{4}$$

where $a < b$ are the radii of the inner and outer spherical walls, respectively. Let us label stationary states $|n_r lm\rangle$ of the particle with the *radial* quantum number $n_r \geq 1$, azimuthal quantum number l and magnetic quantum number m. In this notation, the ground state for the particle is $|100\rangle$.

In this problem, the particle is prepared in a non-stationary state, such that at time $t = 0$

$$|\Psi(t = 0)\rangle = \frac{1}{\sqrt{2}}|100\rangle + \frac{1}{\sqrt{2}}|200\rangle. \tag{5}$$

(a) Write down explicitly the properly normalized real-space wave function $\Psi(r, \theta, \psi; t)$ as state (5) evolves in time t.

$$\begin{aligned} |\Psi(t = 0)\rangle &= |\Psi(r, \theta, \psi; t = 0)\rangle \\ |\Psi(r, \theta, \psi; t)\rangle &= \frac{1}{\sqrt{2}}|100\rangle e^{-iE_1 \frac{t}{\hbar}} + \frac{1}{\sqrt{2}}|200\rangle e^{-iE_2 \frac{t}{\hbar}} \end{aligned}$$

(b) What is the expectation value $\overline{E} = \langle \hat{H} \rangle$ of particle energy in this state. Does this expectation value $\overline{E}$ depend on time?

$$\begin{aligned} \overline{E} = \langle \hat{H} \rangle &= p_1 E_1 + p_2 E_2 \\ &= \left(\frac{1}{\sqrt{2}}\right)^2 E_1 + \left(\frac{1}{\sqrt{2}}\right)^2 E_2 \\ &= \frac{E_1}{2} + \frac{E_2}{2} = \boxed{\frac{E_1 + E_2}{2}} \end{aligned}$$

The expectation value should depend on time, as the probability distribution is time-dependent. The expectation value is directly related to the probabilities associated with $|100\rangle$ and $|200\rangle$, so as these probabilities change, the expectation value will as well.

(c) Find pressure P_a that the particle in this state exerts on the inner wall of the potential shell. Does this pressure depend on time?

We will use the thermodynamic identity $P = -\frac{dU}{dV}$, where U is the energy of the system and V is the volume. P is the pressure within between the inner and outer spherical walls, and scrapping gravity and temperature gradients we can assume that $P = P_a = P_b$. We can substitute the previously defined $\overline{E}$ for U, giving $P_a = -\frac{d\overline{E}}{dV}$.

We previously stated that the expectation value of the particle's energy depends on time, and P_a depends on the expectation value of energy, so pressure P_a does depend on time.

(d) Find the expectation value of the particle distance r from the center as the system evolves in time. Does this expectation value $\overline{r}$ depend on time?

$$\overline{r} = \langle \Psi(t) | r | \Psi(t) \rangle \tag{6}$$

$$\begin{aligned}
\Psi(r,\theta,\psi;t) &= \frac{1}{\sqrt{2}}\Phi_1(r)\, e^{-iE_1\frac{t}{\hbar}} + \frac{1}{\sqrt{2}}\Phi_2(r)\, e^{-iE_2\frac{t}{\hbar}} \\
\Psi^*(r,\theta,\psi;t)\,\Psi(r,\theta,\psi;t) &= \left[\frac{1}{\sqrt{2}}\Phi_1^*(r)\, e^{iE_1\frac{t}{\hbar}} + \frac{1}{\sqrt{2}}\Phi_2^*(r)\, e^{iE_2\frac{t}{\hbar}}\right]\cdot\left[\frac{1}{\sqrt{2}}\Phi_1(r)\, e^{-iE_1\frac{t}{\hbar}} + \frac{1}{\sqrt{2}}\Phi_2(r)\, e^{-iE_2\frac{t}{\hbar}}\right] \\
&= \frac{\left|\Phi_1(r)\right|^2 + \left|\Phi_2(r)\right|^2}{2} + \frac{1}{2}\Phi_1^*(r)\,\Phi_2(r)\, e^{-\frac{i(E_2-E_1)t}{\hbar}} + \frac{1}{2}\Phi_2^*(r)\,\Phi_1(r)\, e^{\frac{i(E_2-E_1)}{\hbar}} \\
&= \frac{1}{2}\left[\left|\Phi_1(r)\right|^2 + \left|\Phi_2(r)\right|^2 + \Phi_1\Phi_2\left(e^{\frac{i(E_2-E_1)t}{\hbar}} + e^{-\frac{i(E_2-E_1)t}{\hbar}}\right)\right] \\
&= \frac{1}{2}\left[\left|\Phi_1(r)\right|^2 + \left|\Phi_2(r)\right|^2 + 2\Phi_1(r)\,\Phi_2(r)\cos\left((E_2-E_1)\frac{t}{\hbar}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\overline{r} &= \int_a^b dr\cdot r\left|\Psi(t)\right|^2\left(4\pi r^2\right) = 4\pi\int_a^b dr\cdot r^3\left|\Psi(t)\right|^2 \\
&= 4\pi\int_a^b dr\cdot r^3\frac{1}{2}\left[\left|\Phi_1(r)\right|^2 + \left|\Phi_2(r)\right|^2 + 2\Phi_1(r)\,\Phi_2(r)\cos\left((E_2-E_1)\frac{t}{\hbar}\right)\right] \\
&= 4\pi\left[\frac{1}{2}\int_a^b dr\cdot r^3\left|\Phi_1(r)\right|^2 + \frac{1}{2}\int_a^b dr\cdot r^3\left|\Phi_2\right|^2 + 2\cos\left((E_2-E_1)\frac{t}{\hbar}\right)\int_a^b dr\cdot r^3\Phi_1\Phi_2\right]
\end{aligned}$$

Applying the spherical Laplacian to Φ_1 and Φ_2:

$$\begin{aligned}
&= 4\pi\left[\frac{1}{2}\int_a^b dr\cdot r^3\frac{\sin^2\left(\frac{\pi r}{b-a}\right)}{2\pi r^2(b-a)} + \frac{1}{2}\int_a^b dr\cdot r^3\frac{\sin^2\left(\frac{2\pi r}{b-a}\right)}{2\pi r^2(b-a)} + 2\cos\left((E_2-E_1)\frac{t}{\hbar}\right)\int_a^b dr\cdot r^3\frac{\sin\left(\frac{\pi r}{b-a}\right)\sin\left(\frac{2\pi r}{b-a}\right)}{2\pi r^2(b-a)}\right] \\
&= \frac{4\pi}{4\pi(b-a)}\left[\int_a^b dr\cdot r\sin^2\left(\frac{\pi r}{b-a}\right) + \int_a^b dr\cdot r\sin^2\left(\frac{2\pi r}{b-a}\right) + 2\cos\left((E_2-E_1)\frac{t}{\hbar}\right)\int_a^b dr\cdot 2\sin\left(\frac{\pi r}{b-a}\right)\sin\left(\frac{2\pi r}{b-a}\right)\right] \\
&= \frac{1}{b-a}\left[\int_a^b dr\cdot r\sin^2\left(\frac{\pi r}{b-a}\right) + \int_a^b dr\cdot r\sin^2\left(\frac{2\pi r}{b-a}\right) + 4\cos\left((E_2-E_1)\frac{t}{\hbar}\right)\int_a^b dr\cdot\sin\left(\frac{\pi r}{b-a}\right)\sin\left(\frac{2\pi r}{b-a}\right)\right]
\end{aligned}$$

$$\int_a^b dr \cdot rsin^2\left(\frac{\pi r}{b-a}\right) = \frac{1}{2}\int_a^b dr \cdot r\left[1-\cos\left(\frac{2\pi r}{b-a}\right)\right]$$
$$= \frac{1}{2}\left[\int_a^b dr \cdot r - \int_a^b dr \cdot r\cos\left(\frac{2\pi r}{b-a}\right)\right]$$
$$= \frac{1}{2}\left[\frac{r^2}{2}\Big|_a^b - \left(r\frac{\sin\left(\frac{2\pi r}{b-a}\right)}{\frac{2\pi}{b-a}}\Big|_a^b + \int_a^b dr \cdot \frac{\sin\left(\frac{2\pi r}{b-a}\right)}{\frac{2\pi}{b-a}}\right)\right]$$
$$= \frac{1}{2}\left[\frac{b^2-a^2}{2} - \frac{b-a}{2\pi}\left(b\sin\left(\frac{2\pi b}{b-a}\right) - a\sin\left(\frac{2\pi a}{b-a}\right)\right) - \left(\frac{b-a}{2\pi}\right)^2\cos\left(\frac{2\pi r}{b-a}\right)\Big|_a^b\right]$$
$$= \frac{b^2-a^2}{4} - \frac{b-a}{4\pi}\left(b\sin\left(\frac{2\pi b}{b-a}\right) - a\sin\left(\frac{2\pi a}{b-a}\right)\right) - \frac{(b-a)^2}{32\pi^2}\left(\cos\left(\frac{2\pi b}{b-a}\right) - \cos\left(\frac{2\pi a}{b-a}\right)\right)$$

Similarly, $\int_a^b dr \cdot r\sin^2\left(\frac{2\pi r}{b-a}\right) = \frac{b^2-a^2}{2} - \frac{b-a}{2\pi}\left(b\sin\left(\frac{2\pi b}{b-a}\right) - a\sin\left(\frac{2\pi a}{b-a}\right)\right) - \frac{(b-a)^2}{16\pi^2}\left(\cos\left(\frac{2\pi b}{b-a}\right) - \cos\left(\frac{2\pi a}{b-a}\right)\right)$

$$\int_a^b dr \cdot \sin\left(\frac{\pi r}{b-a}\right)\sin\left(\frac{2\pi r}{b-a}\right) = \frac{b-a}{\pi}\int_a^b du \cdot \sin(u)\sin(2u),\ u = \frac{\pi r}{b-a}$$
$$= 2\frac{b-a}{\pi}\int_a^b du \cdot \cos(u)\sin^2(u) = 2\frac{b-a}{\pi}\int_a^b dv \cdot v^2,\ v = \sin(u)$$
$$= 2\frac{b-a}{\pi}\frac{v^3}{3}\Big|_a^b = 2\frac{b-a}{\pi}\frac{2\sin^3(u)}{3}\Big|_a^b$$
$$= 2\frac{b-a}{3\pi}\sin^3(u) = \frac{2(b-a)\sin^3\left(\frac{\pi r}{b-a}\right)}{3\pi}\Big|_a^b$$
$$= -\frac{1}{6\pi}(b-a)\left(\sin\left(\frac{3\pi b}{b-a}\right) - 3\sin\left(\frac{\pi b}{b-a}\right) - \sin\left(\frac{3\pi a}{b-a}\right) + 3\sin\left(\frac{\pi a}{b-a}\right)\right)$$

Plugging these integrals back into $\bar{r}$:

$$\bar{r} = \frac{1}{b-a}\left[\int_a^b dr \cdot r\sin^2\left(\frac{\pi r}{b-a}\right) + \int_a^b dr \cdot r\sin^2\left(\frac{2\pi r}{b-a}\right) + 4\cos\left((E_2-E_1)\frac{t}{\hbar}\right)\int_a^b dr \cdot \sin\left(\frac{\pi r}{b-a}\right)\sin\left(\frac{2\pi r}{b-a}\right)\right]$$
$$= \frac{1}{b-a}\left[\left\{\frac{b^2-a^2}{4} - \frac{b-a}{4\pi}\left(b\sin\left(\frac{2\pi b}{b-a}\right) - a\sin\left(\frac{2\pi a}{b-a}\right)\right) - \frac{(b-a)^2}{32\pi^2}\left(\cos\left(\frac{2\pi b}{b-a}\right) - \cos\left(\frac{2\pi a}{b-a}\right)\right)\right\} + \right.$$
$$\left\{\frac{b^2-a^2}{2} - \frac{b-a}{2\pi}\left(b\sin\left(\frac{2\pi b}{b-a}\right) - a\sin\left(\frac{2\pi a}{b-a}\right)\right) - \frac{(b-a)^2}{16\pi^2}\left(\cos\left(\frac{2\pi b}{b-a}\right) - \cos\left(\frac{2\pi a}{b-a}\right)\right)\right\} +$$
$$\left. 4\cos\left((E_2-E_1)\frac{t}{\hbar}\right)\left\{-\frac{1}{6\pi}(b-a)\left(\sin\left(\frac{3\pi b}{b-a}\right) - 3\sin\left(\frac{\pi b}{b-a}\right) - \sin\left(\frac{3\pi a}{b-a}\right) + 3\sin\left(\frac{\pi a}{b-a}\right)\right)\right\}\right]$$

$$\bar{r} = \frac{1}{b-a}\left[-\left\{\frac{b^2-a^2}{4} - \frac{b-a}{4\pi}\left(b\sin\left(\frac{2\pi b}{b-a}\right) - a\sin\left(\frac{2\pi a}{b-a}\right)\right) - \frac{(b-a)^2}{32\pi^2}\left(\cos\left(\frac{2\pi b}{b-a}\right) - \cos\left(\frac{2\pi a}{b-a}\right)\right)\right\} + \right.$$
$$\left. 4\cos\left((E_2-E_1)\frac{t}{\hbar}\right)\left\{-\frac{1}{6\pi}(b-a)\left(\sin\left(\frac{3\pi b}{b-a}\right) - 3\sin\left(\frac{\pi b}{b-a}\right) - \sin\left(\frac{3\pi a}{b-a}\right) + 3\sin\left(\frac{\pi a}{b-a}\right)\right)\right\}\right]$$

Problem 3

Let $|j, m\rangle$ be normalized eigenvectors corresponding to the generalized angular momentum j and its projection m on the z-axis:

$$\hat{J}^2|j, m\rangle = j\,(j+1)\,\hbar^2|j, m\rangle, \quad \hat{J}_z|j, m\rangle = m\hbar|j, m\rangle. \tag{7}$$

As stated in (7), the z-component of the angular momentum in state $|j, m\rangle$ has a definite value of $m\hbar$. As we know, the x- and y-components, however, would not have definite values in that state and their measurements would thus be characterized by some uncertainties. As usual, we quantify the uncertainty in the measurement of physical observable A via

$$\sigma_A^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2, \tag{8}$$

where the expectation values are taken over the state of interest.

In this problem, you are given the state

$$|3, -2\rangle \tag{9}$$

from the $j = 3$ septet.

Find the uncertainties σ_{J_x} and σ_{J_y} in the measurement of the x- and y-projections in state (9). These uncertainties are respectively determined with $\hat{A} = \hat{J}_x$ or $\hat{A} = \hat{J}_y$ in (8).

$$\begin{aligned}
\hat{J}_x &= \frac{\hat{J}_+ + i\hat{J}_-}{2} \\
\hat{J}_y &= \frac{\hat{J}_+ - i\hat{J}_-}{2i} \\
\hat{J}_\pm|j, m\rangle &= \hbar\sqrt{(j \pm m)\,(j \pm m + 1)}|j, m \pm 1\rangle \\
\langle \hat{J}_x^2 \rangle = \langle \hat{J}_y^2 \rangle &= \langle \hat{J}^2 \rangle - \langle \hat{J}_z \rangle^2 \\
&= j\,(j+1)\,\hbar^2 - m^2\hbar^2 \\
&= 3\,(3+1)\,\hbar^2 - (-2)^2\,\hbar^2 \\
&= (12-4)\,\hbar^2 = 8\hbar^2 \\
\sigma_{j_x} &= \sqrt{\langle J_x^2 \rangle - \langle \hat{J}_x \rangle^2} \\
&= \sqrt{\langle J_x^2 \rangle - 0} = \sqrt{8\hbar^2} = \sigma_{j_y} \\
\sigma_{j_x} = \sigma_{j_y} &= \boxed{2\hbar\sqrt{2}}
\end{aligned}$$