PHYS 4301
Homework 10

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Problem 1

As a refresher of your mastery of spherical coordinates (r, θ, ϕ), use their explicit relationship with cartesian (x, y, z) to demonstrate the following for the components of the operator

$$\hat{L} = r \times \hat{p} = -i\hbar r \times \nabla$$

of the orbital angular momentum:

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi} \tag{1}$$

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y = \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial \theta} + i \cot(\theta) \frac{\partial}{\partial \phi} \right). \tag{2}$$

As a result of these manipulations, you should be able to also show that, indeed,

$$-\frac{1}{\hbar^2} \hat{L}^2 = \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \phi^2}, \tag{3}$$

which is the combination that appears in the expression for the Laplace operator in spherical coordinates. Here the scalar square operator can be represented as

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \hat{L}_+ \hat{L}_- + \hat{L}_z^2 - \hbar \hat{L}_z.$$

$$\begin{aligned}
\hat{L} &= r \times p = r\hat{r}x - i\hbar\nabla \\
&= -i\hbar \left[r\hat{r}x \left(i\frac{\partial}{\partial x} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin(\theta)} \frac{\partial}{\partial \phi} \right) \right] \\
&= -i\hbar \left[r(\hat{r} \times \hat{r}) \frac{\partial}{\partial r} + \left(\hat{r} \times \hat{\theta} \right) \frac{\partial}{\partial \theta} + \frac{\hat{r} \times \phi}{\sin(\theta)} \frac{\partial}{\partial \phi} \right] \\
&= -i\hbar \left[0 + \phi \frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin(\theta)} \frac{\partial}{\partial \phi} \right] \\
&= -i\hbar \left(\hat{\phi} \frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin(\theta)} \frac{\partial}{\partial \phi} \right) \\
\hat{\theta} &= \cos(\theta) \cos(\phi) \hat{x} + \cos(\theta) \sin(\phi) \hat{y} - \sin(\theta) \hat{z} \\
\hat{\phi} &= -\sin(\theta) \hat{x} + \cos\left(\hat{\phi}\right) \hat{y}
\end{aligned}$$

$$\hat{x}L_x + \hat{y}L_y + \hat{z}L_z = -i\hbar \left[\left(-\sin(\phi) \hat{x} + \cos(\phi) \hat{y} \right) \frac{\partial}{\partial \theta} - \frac{1}{\sin(\theta)} \left(\cos(\theta) \cos(\phi) \hat{x} + \cos(\theta) \sin(\phi) \hat{y} - \sin(\theta) \hat{z} \right) \frac{\partial}{\partial \phi} \right]$$

$$
\begin{aligned}
L_x &= -i\hbar\left[-\sin(\phi)\frac{\partial}{\partial\theta} - \cos(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\right]\\
L_y &= -i\hbar\left[\cos(\phi)\frac{\partial}{\partial\theta} - \sin(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\right]\\
L_z &= -i\hbar\frac{\partial}{\partial\phi}
\end{aligned}
$$

$$
\begin{aligned}
L_+ &= L_x + iL_y\\
&= -i\hbar\left[\left(-\sin(\phi)\frac{\partial}{\partial\theta} - \cos(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\right) + i\left(\cos(\phi)\frac{\partial}{\partial\theta} - i\sin(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\right)\right]\\
&= -i\hbar\left[-\big(\sin(\phi) - i\cos(\phi)\big)\frac{\partial}{\partial\theta} - \cot(\theta)\big(\cos(\theta) + i\sin(\phi)\big)\frac{\partial}{\partial\phi}\right]\\
&= \hbar\left[\big(i\sin(\phi) + \cos(\phi)\big)\frac{\partial}{\partial\theta} + i\cot(\theta)\,e^{i\phi}\frac{\partial}{\partial\phi}\right]\\
&= \hbar\left(e^{i\phi}\frac{\partial}{\partial\theta} + i\cot(\theta)\,e^{i\phi}\frac{\partial}{\partial\phi}\right)\\
&= \boxed{\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot(\theta)\frac{\partial}{\partial\phi}\right)}\\
L_- &= L_x - iL_y\\
&= -i\hbar\left[\left(-\sin(\phi)\frac{\partial}{\partial\theta} - \cos(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\right) - i\left(\cos(\phi)\frac{\partial}{\partial\theta} - i\sin(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\right)\right]\\
&= i\hbar\left[\big(\sin(\phi) - i\cos(\phi)\big)\frac{\partial}{\partial\theta} - \cot(\theta)\big(\cos(\theta) + i\sin(\phi)\big)\frac{\partial}{\partial\phi}\right]\\
&= \hbar\left[\big(i\sin(\phi) + \cos(\phi)\big)\frac{\partial}{\partial\theta} - i\cot(\theta)\,e^{i\phi}\frac{\partial}{\partial\phi}\right]\\
&= \hbar\left(e^{i\phi}\frac{\partial}{\partial\theta} - i\cot(\theta)\,e^{i\phi}\frac{\partial}{\partial\phi}\right)\\
&= \boxed{\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} - i\cot(\theta)\frac{\partial}{\partial\phi}\right)}
\end{aligned}
$$

Problem 2

As we discussed in class, the eigenvectors $|jm\rangle$ corresponding to the generic angular momentum operator $\hat{J}$:

$$\hat{J}^2|jm\rangle = j(j+1)\hbar^2|jm\rangle, \quad \hat{J}_z|jm\rangle = m\hbar|jm\rangle, \tag{4}$$

comprise a $(2j+1)$-fold multiplet for a given j value (I used integer $m = -j, -j+1 \ldots j-1, j$ instead of j_z). This orthonormal basis (for fixed value of j) thereby defines a $(2j+1)$-dimensional space, within which the operator of the angular momentum actually "operates". The operator is then represented - within that space - by the matrices of corresponding dimensions.

As an explicit illustration of that matrix picture, you are asked to verify that the following 2×2 matrices indeed satisfy all the properties of the components of $\hat{J}$ for the case of $j = \frac{1}{2}$ (doublet, $2j+1 = 2$):

$$\hat{J}_k = \frac{\hbar}{2}\sigma_k. \tag{5}$$

Here σ's are the *famed Pauli matrices* for spin-$\frac{1}{2}$ particles:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

(a)

Show that components (5) obey commutation relations

$$\left[\hat{J}_k, \hat{J}_l\right] = i\hbar\epsilon_{klm}\hat{J}_m, \tag{7}$$

where ϵ_{klm} are the standard components (0 or ± 1) of the antisymmetric 3-rank tensor you also encounter in the definition of the vector cross-product.

$$\begin{aligned}
\left[\hat{J}_k, \hat{J}_l\right] &= \hat{J}_k\hat{J}_l - \hat{J}_l\hat{J}_k \\
\text{Let } (k,l) &= (x,y) \\
\left[\hat{J}_x, \hat{J}_y\right] &= \frac{\hbar^2}{u}\left(\sigma_x\sigma_y - \sigma_y\sigma_x\right) \\
\sigma_x\sigma_y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
\sigma_y\sigma_x &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
\sigma_x\sigma_y - \sigma_y\sigma_x &= \begin{pmatrix} i-(-i) & 0 \\ 0 & -i-i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\left[\hat{J}_x, \hat{J}_y\right] &= \frac{\hbar^2}{y}\left(2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\
&= \frac{\hbar}{y}\left(2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\frac{\hbar}{y} \\
&= \frac{\hbar}{2}(2i)\hat{J}_z \\
&= i\hbar\hat{J}_z
\end{aligned}$$

(b)

How does (the matrix of the) operator $\hat{J}^2$ look in this representation? Also, indicate the (column) eigenvectors corresponding to different m values in (4).

$$\begin{aligned}
\hat{J}^2 &= \begin{pmatrix} \langle\frac{1}{2},\frac{1}{2}|\hat{J}^2|\frac{1}{2},\frac{1}{2}\rangle & \langle\frac{1}{2},\frac{1}{2}|\hat{J}^2|\frac{1}{2},-\frac{1}{2}\rangle \\ \langle\frac{1}{2},-\frac{1}{2}|\hat{J}^2|\frac{1}{2},\frac{1}{2}\rangle & \langle\frac{1}{2},-\frac{1}{2}|\hat{J}^2|\frac{1}{2},-\frac{1}{2}\rangle \end{pmatrix} \\
&= \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}$$

With $J = \frac{1}{2}$, $m = \frac{1}{2}, -\frac{1}{2}$. This gives the eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.