PHYS 4301
Midterm

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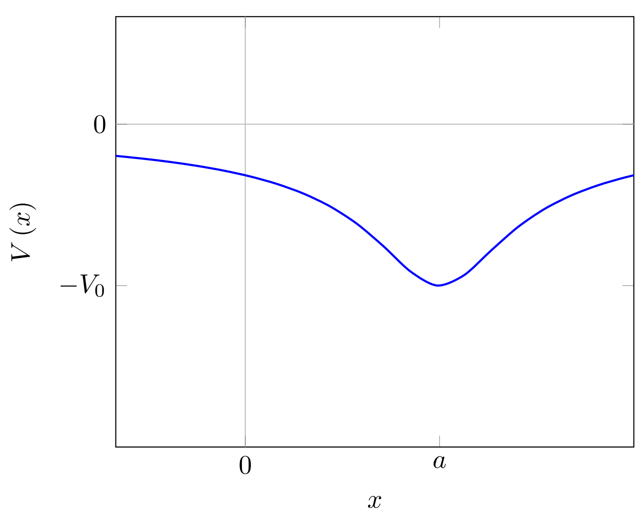
1. A particle of mass m lives on the x-axis, where it is exposed to the 1D potential (1). Here, V_0, a and b are given positive constants of appropriate dimensions. You are asked to analyze the ground state of this system in the "heavy-mass approximation" (no exact solution). In this context, the heavy-mass approximation means that you can assume mass m being as big as you want – but *not* infinite. Show your (approximate!) results as explicit functions of given system parameters.

$$V(x) = -\frac{V_0 b}{\sqrt{(x-a)^2 + b^2}} \tag{1}$$

a) Sketch the spatial behavior of this potential to guide you in your solution.

Spatial Behavior of Potential



b) What is the ground state energy E_0?

The ground state energy E_0 will be the minimum value of the potential function. To find the minimum, find the point on $V(x)$ where $\frac{d(V(x))}{dx} = 0$:

$$
\begin{aligned}
\frac{d\left(V(x)\right)}{dx} &= 0, \text{ s.t. } x = x_0 \\
0 &= \frac{V_0 b\left(x-a\right)}{\left(\left(x-a\right)^2 + b^2\right)^{\frac{3}{2}}} \\
0 &= V_0 b\left(x-a\right) \rightarrow 0 = x - a \\
x_0 &= a
\end{aligned}
$$

Then,

$$
\begin{aligned}
E_0 &= V\left(x_0\right) = V\left(a\right) \\
&= -\frac{V_0 b}{\sqrt{\left(a-a\right)^2 + b^2}} \\
&= -\frac{V_0 b}{\sqrt{b^2}} \\
&= -\frac{V_0 b}{b} \\
&= \boxed{-V_0}
\end{aligned}
$$

c) What is the expectation value $\hat{x} = \langle x \rangle$ for the particle position in the ground state?

As mass m increases, it is more likely that the particle will remain close to the position $x = a$, the lowest point on the potential function. Therefore, we can approximate the wave function as follows:

$$
\begin{aligned}
V\left(x\right) &\approx V\left(a\right) + \frac{V''\left(a\right)}{2}x^2 \\
&= -V_0 - \frac{x^2}{2}\left(\frac{V_0 b\left(2x^2 - 4ax - b^2 + 2a^2\right)}{\left(\left(x-a\right)^2 + b^2\right)^{\frac{5}{2}}}\right) \\
E\psi &= -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \left(V_0 + \frac{x^2}{2}\left(\frac{V_0 b\left(2x^2 - 4ax - b^2 + 2a^2\right)}{\left(\left(x-a\right)^2 + b^2\right)^{\frac{5}{2}}}\right)\right)\psi
\end{aligned}
$$

d) What is the variance $\sigma^2 = \langle (x - \hat{x})^2 \rangle$ for the particle position in the ground state?

$$\sigma^2 \propto \frac{1}{m}$$

e) Based on the obtained value of σ, could you formulate the condition of applicability of your approximate results (that is, how big mass m should be to make these results valid)?

Variance should decrease as mass increases so as $m \rightarrow \infty$, the results become more valid.

2. Let $\psi_n(x)\,(n = 0, 1, 2, \dots)$ be standard real-valued normalized stationary states of the 1D harmonic oscillator characterized by the spatial position (displacement) x, mass m, and frequency ω. You may want to use the algebraic approach for the following. At time $t = 0$, the oscillator is prepared in normalized non-stationary state (2) where A is the normalization constant and $i^2 = -1$.

$$\Psi(t=0) = A\,(2\psi_0 + i\psi_1 - \psi_2) \tag{2}$$

a) Find the normalization constant A.

Using the algebraic approach:

$$\begin{aligned}
|\Psi(0)\rangle &= A\left(2|0\rangle + i|1\rangle - |2\rangle\right) \\
1 = \langle\Psi(0)|\Psi(0)\rangle &= A^2\left(4\langle 0|0\rangle + \langle 1|1\rangle + \langle 2|2\rangle\right) \\
&= A^2(4+1+1) = 6A \\
A &= \boxed{\frac{1}{\sqrt{6}}} \\
|\Psi(t)\rangle &= \frac{1}{\sqrt{6}}\left(2|0\rangle + i|1\rangle - |2\rangle\right)
\end{aligned}$$

b) As this state $\Psi(t)$ evolves with time t, calculate the expectation value of the position:

$$\langle x\rangle(t) \tag{3}$$

$$\begin{aligned}
|\Psi(t)\rangle &= e^{-iHt/\hbar}|\Psi(0)\rangle \\
&= \frac{1}{\sqrt{6}}\left(e^{-iE_0t/\hbar}|0\rangle - ie^{-iE_1t/\hbar}|1\rangle + 2e^{-iE_3t/\hbar}|3\rangle\right) \\
\langle x\rangle(t) &= \langle\Psi(t)|\hat{x}|\Psi(t)\rangle
\end{aligned}$$

We can define $\hat{x}$ in terms of ladder operators:

$$\begin{aligned}
\hat{a} &= \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x}) \\
\hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x}) \\
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)
\end{aligned}$$

So,

$$\langle x\rangle(t) = \langle\Psi(t)|\hat{x}|\Psi(t)\rangle$$

$$= \frac{1}{6}\sqrt{\frac{\hbar}{2m\omega}}\left(2e^{iE_0t/\hbar}\langle 0| - ie^{iE_1t/\hbar}\langle 1| - e^{iE_3t/\hbar}\langle 3|\right)\left(\hat{a}+\hat{a}^\dagger\right)\left(2e^{-iE_0t/\hbar}|0\rangle + ie^{-iE_1t/\hbar}|1\rangle - e^{-iE_3t/\hbar}|3\rangle\right)$$

$$= \frac{1}{6}\sqrt{\frac{\hbar}{2m\omega}}\left(-ie^{i(E_1-E_0)t/\hbar} + ie^{i(E_0-E_1)t/\hbar}\right) = \frac{i}{6}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{i(E_0-E_1)t/\hbar} - e^{-i(E_0-E_1)t/\hbar}\right)$$

$$= -\frac{1}{3}\sqrt{\frac{\hbar}{2m\omega}}sin\left((E_0-E_1)\frac{t}{\hbar}\right) = -\frac{1}{3}\sqrt{\frac{\hbar}{2m\omega}}sin\left((-\hbar\omega)\frac{t}{\hbar}\right) = -\frac{1}{3}\sqrt{\frac{\hbar}{2m\omega}}sin(-\omega t)$$

$$\langle x\rangle(t) = \boxed{\frac{1}{3}\sqrt{\frac{\hbar}{2m\omega}}sin(\omega t)}$$

c) Repeat the calculation for the expectation value of the square of the position:

$$\langle x^2\rangle(t) \tag{4}$$

We can define $\hat{x}^2$ in terms of ladder operators:

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left(\hat{a}^2 + \hat{a}^{\dagger^2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right)$$

$$\left[\hat{a},\hat{a}^\dagger\right] = 1 \rightarrow \hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left(\hat{a}^2 + \hat{a}^{\dagger^2} + \left(1+\hat{a}^\dagger\hat{a}\right) + \hat{a}^\dagger\hat{a}\right)$$

$$= \frac{\hbar}{2m\omega}\left(\hat{a}^2 + \hat{a}^{\dagger^2} + 2\hat{a}^\dagger\hat{a} + 1\right)$$

$$\langle\hat{x}^2\rangle = \frac{1}{6}\frac{\hbar}{2m\omega}\left(\left(1+(1+2)+4(1+6)\right) - ie^{i(E_1-E_3)t/\hbar}\langle 1|\hat{a}^2+\hat{a}^{\dagger^2}|3\rangle + ie^{i(E_3-E_1)t/\hbar}\langle 3|\hat{a}^2+\hat{a}^{\dagger^2}|1\rangle\right)$$

$$= \frac{\hbar}{12m\omega}\left(14 - 2i\sqrt{6}\left(e^{i(E_1-E_3)t/\hbar} + e^{i(E_3-E_1)t/\hbar}\right)\right)$$

$$= \boxed{\frac{\hbar}{12m\omega}\left(14 - 4\sqrt{6}sin(2\omega t)\right)}$$

d) Compare the results for (3) and (4) and comment on the (origin of the) essential differences of their time dependence.

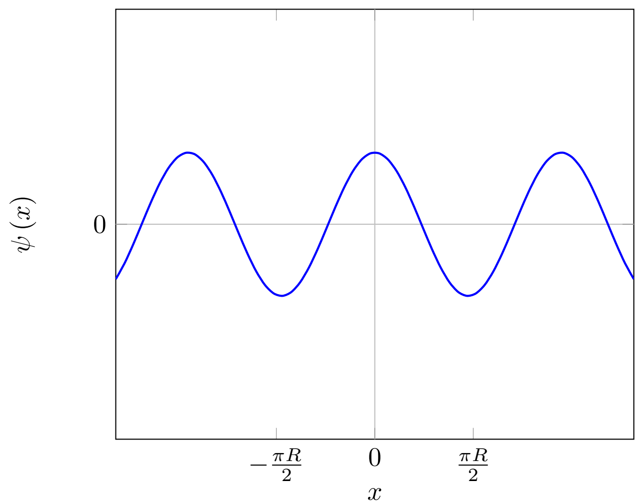
Both $\langle\hat{x}\rangle$ and $\langle\hat{x}^2\rangle$ are time-dependent. Although they share a wavelength, their primary difference would be amplitude, with the ratio of their amplitudes increasing as ω decreases, and as m decreases.

3. A particle of mass m is restricted to move along a circular ring of radius R. We use coordinate $-\pi R \leq x \leq \pi R$ to specify its position on the ring, where, of course, the lower and upper limits represent the same physical point. The particle is exposed to an attractive delta-function potential (5) where $\gamma > 0$ characterizes the strength of the potential. You are tasked with finding the *ground state* of the particle in this system.

$$V(x) = -\frac{\hbar^2}{m}\gamma\delta(x) \tag{5}$$

a) Sketch the expected qualitative behavior of the ground state wave function $\psi(x)$ on the "unrolled" ring $(-\pi R \leq x \leq \pi R)$ to guide you in your solution.

Qualitative Behavior of $\psi(x)$



b) Clearly outline and specify the principles that you will exploit to quantitatively build that $\psi(x)$.

Because the wave function is stated to behave such that $-\pi R$ and πR are the same point, the wave function ψ can be treated as continuous at these points.

c) Write down a (transcendental) equation that would allow you to find the energy and parameters of the ground state.

Let $x = R\Theta$. From the Schrodinger equation:

$$
\begin{aligned}
-\frac{\hbar^2}{2mR^2}\frac{\partial^2\psi_1}{\partial\Theta^2} &= E\psi_1 \\
\psi_1 &= Asin\left(\alpha\Theta\right) + Bcos\left(\alpha\Theta\right),\ \alpha^2 = \frac{2mER^2}{\hbar^2} \\
-\frac{\hbar^2}{2mR^2}\frac{\partial^2\psi_2}{\partial\Theta^2} + V\psi_2 &= E\psi_2 \\
\psi_2 &= Csin\left(\beta\Theta\right) + Dcos\left(\beta\Theta\right),\ \beta^2 = \frac{2mR^2\left(E-V\right)}{\hbar^2} \rightarrow \psi_1\left(-\frac{\pi}{2}\right) = \psi_2\left(-\frac{\pi}{2}\right) \\
Asin\left(\frac{\pi\alpha}{2}\right) + Bcos\left(\frac{\pi\alpha}{2}\right) &= Csin\left(\frac{\pi\beta}{2}2\right) + Dcos\left(\frac{\pi\beta}{2}\right) \\
Asin\left(\frac{\pi\alpha}{2}\right) &= Csin\left(\frac{\pi\beta}{2}\right),\ Bcos\left(\frac{\pi\alpha}{2}\right) = Dcos\left(\frac{\pi\beta}{2}\right) \\
\frac{A}{B}tan\left(\frac{\pi\alpha}{2}\right) &= \frac{C}{D}tan\left(\frac{\pi\beta}{2}\right) \rightarrow \frac{\alpha\pi}{2} = \frac{\beta\pi}{2} + n\pi,\ n\in\mathbb{Z} \\
\alpha - 2n &= \beta \\
\alpha^2 - 4n\alpha + 4n^2 &= \beta^2 \\
4n\alpha &= 4n^2 + \frac{2mR^2V}{\hbar^2} \\
\frac{R}{\hbar}\sqrt{2mE} &= n + \frac{mR^2\left(-\frac{\hbar^2}{m}\gamma\delta\left(x\right)\right)}{2n\hbar^2} \\
\frac{1}{\hbar}\sqrt{2mE} &= n + \frac{R\gamma\delta\left(x\right)}{2n}
\end{aligned}
$$

d) Illustrate how you would use the graphical approach to look for solutions of this equation.

To find E, we would restructure the above equation:

$$
\begin{aligned}
\left(\frac{1}{\hbar}\sqrt{2mE}\right)^2 &= \left(n + \frac{R\gamma\delta\left(x\right)}{2n}\right)^2 \\
\frac{2mE}{\hbar^2} &= \frac{\left(2n^2 + R\gamma\delta\left(x\right)\right)^2}{4n^2} \\
E &= \frac{\hbar^2\left(2n^2 + R\gamma\delta\left(x\right)\right)^2}{8mn^2}
\end{aligned}
$$

Plugging in any integer value for n provides a solution. At this point, the equation is defined in terms of R, γ, m, and $\delta\left(x\right)$. A search along the curve for an intersection with the x axis will provide solutions for E_n, such as E_0.

e) Presumably, as $R \rightarrow \infty$, your results should transition into our textbook case of a particle living on a straight line with the delta-function potential well. Do you see that you can indeed reproduce the textbook case in the corresponding limit?

Yes. Because the term containing R in the previous function also contains $\delta\left(x\right)$, as R grows it is still affected by the delta function potential, reproducing the behavior of the particle along a straight line.