PHYS 4302
Homework 8

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Problem 1

Determine semi-classical energy levels E_n for a particle of mass m in a one-dimensional system

(a) with the quartic potential energy

$$U(x) = \lambda x^4,\ \lambda > 0, \tag{1}$$

By substituting $U(x)$ into the Schrodinger equation:

$$\frac{d^2}{dx^2}\psi(x) - \frac{2m}{\hbar^2}\left(\lambda x^4 - E\right)\psi(x) = 0$$

Now we perform the WKB approximation under the assumption that the wave function varies slowly compared to the potential:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{3}{4}\hbar\omega\left(\frac{\lambda}{\hbar^2}\right)^2(2n+1)$$

where $n \in \mathbb{N}$ is the principal quantum number.

(b) with the linear potential energy

$$U(x) = \lambda\left|x\right|,\ \lambda > 0. \tag{2}$$

By substituting $U(x)$ into the Schrodinger equation:

$$\frac{d^2}{dx^2}\psi(x) - \frac{2m}{\hbar^2}\left(\lambda\left|x\right| - E\right)\psi(x) = 0$$

Now we perform the WKB approximation under the assumption that the wave function varies slowly compared to the potential:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{3}{4}\hbar\omega\left(\frac{\lambda}{\hbar^2}\right)^2\left(2n^2 + 2n + \frac{1}{3}\right)$$

where $n \in \mathbb{N}$ is the principal quantum number.

(c) Compare your results for E_n as functions of the quantum number n in potentials (1) and (2) with the familiar dependence of the harmonic oscillator and comment on the energy spacing between the neighboring levels in different potentials.

Problem 2

Consider a particle of mass M in a one-dimensional system (coordinate x) with attractive potential energy

$$V(x) = -\frac{A}{(x^2 + a^2)^s}, \tag{3}$$

where $A > 0, a > 0$, and $s > 0$ are positive system parameters.

Use the semi-classical (Bohr-Sommerfeld) quantization rule to find out for which values of exponent s this potential would support an infinite (as opposed to finite!) number of bound states. (You do NOT need to find those bound states to answer the question!)

Problem 3

(a) Determine, as a function of the particle energy $E > 0$, the semi-classical probability of tunneling through the potential barrier defined by the potential energy

$$U(x) = \begin{cases} -U_0, & x < x_0, \\ \frac{A}{x}, & x > x_0, \end{cases} \tag{4}$$

where U_0, x_0, and A are positive parameters.

$$\begin{aligned} T &\simeq e^{-2\gamma} \\ \gamma &= \frac{1}{\hbar}\int_{x_0}^{x_1} \left|p(x)\right| dx,\ x_1 = \frac{A}{E} \\ &= \frac{1}{\hbar}\int_{x_0}^{x_1} \sqrt{\left(\frac{A}{x} - E\right) 2mdx} \\ &= \frac{\sqrt{2mE}}{\hbar}\int_{x_0}^{x_1} \sqrt{\frac{A}{Ex} - 1dx} \end{aligned}$$

(b) This model problem is at the heart of the theory of the alpha-decay of certain radioactive nuclei. Use your textbook(s), e.g. Griffiths' *Introduction to Quantum Mechanics*, to learn more about this theory, to detail your understanding of the relevance and limitations of the specified model problem, including quantification and appropriate ranges of parameters involved. Provide a brief description of the essentials as you perceive them.

Alpha decay is a process in which an alpha particle within the nucleus of certain radioactive atoms is electrically repelled by the remaining particles in the nucleus until its energy exceeds the potential defined by the strong nuclear force. The aforementioned potential is governed by the radius of the nucleus, represented by a square well up to one nucleus radius and a Coulombic tail after one nucleus radius, where a greater difference from the center of the radius correlates with a lower potential barrier. Alpha particles escape the potential barrier via quantum tunneling, with the distance to which they can tunnel to being given by

$$R = \frac{Ze^2}{2\pi\epsilon_0}.$$