PHYS4301
Homework 1

Charles Averill
charles@utdallas.edu

September 2022

1.

Let a be a real, positive parameter of length. Let A be an unknown normalization constant.

$$\text{For the Gaussian packet, } \sigma_x\sigma_k >= \frac{1}{2}, \text{ or } \sigma_x\sigma_p = \frac{\hbar}{2} \tag{1}$$

$$Ae^{-\frac{|x|}{a}} \tag{2}$$

$$\frac{A}{cosh\left(\frac{x}{a}\right)} \tag{3}$$

a) Find the normalization constant A

(2)

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |Ae^{-|x|/a}|^2 dx \\
&= A^2 \int_{-\infty}^{\infty} e^{-2|x|/a} dx \\
&= 2A^2 \int_0^{\infty} e^{-2\frac{x}{a}} dx, u = -\frac{2x}{a}, \frac{du}{dx} = -\frac{2}{a}, dx = -\frac{a}{2} \\
&= 2A^2 \int_0^{\infty} e^u \left(-\frac{a}{2}\right) du = -aA^2 \int_0^{\infty} e^u du \\
&= -aA^2 e^{-2x/a}\Big|_0^{\infty} \\
&= \lim_{x\to\infty} \left(-aA^2e^{-x}\right) - \left(-aA^2e^0\right) = aA^2 \\
A^2 &= \frac{1}{a} \to \boxed{A = \frac{1}{\sqrt{a}}}
\end{aligned}$$

(3)

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\frac{A}{cosh\left(\frac{x}{a}\right)}|^2 \\
&= A^2 \int_{-\infty}^{\infty} cosh^{-2}\left(\frac{x}{a}\right), u = \frac{x}{a}, dx = a \cdot du \\
&= aA^2 \int_{-\infty}^{\infty} cosh^{-2}(u)\, du \\
&= aA^2 \int_{-\infty}^{\infty} sech^2(u)\, du \\
&= aA^2 tanh(u) = aA^2 tanh\left(\frac{x}{a}\right)\Big|_{-\infty}^{\infty} \\
&= \left(aA^2\right) - \left(-aA^2\right) = 1 \\
A^2 &= \frac{1}{2a} \to \boxed{A = \frac{1}{\sqrt{2a}}}
\end{aligned}$$

b) Find standard deviation σ_x for the position uncertainty

$$\sigma_x = \sqrt{\langle \left(x - \langle x \rangle\right)^2 \rangle} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \tag{4}$$

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} \Psi^\star(x)\, f(x)\, \Psi(x)\, dx \tag{5}$$

(2)

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \Psi^\star(x)\, x \Psi(x)\, dx \text{ from (5)} \\
&= \int_{-\infty}^{\infty} x |Ae^{-|x|/a}|^2 dx = \int_{-\infty}^{\infty} x A^2 e^{-2|x|/a} dx \\
&= \int_{-\infty}^{\infty} x \left(\frac{1}{\sqrt{a}}\right)^2 e^{-2|x|/a} dx = \int_{-\infty}^{\infty} x \frac{1}{a} e^{-2|x|/a} dx \\
\text{Let } g(x) &= x\left(\frac{1}{a} e^{-2|x|/a}\right) \\
\rightarrow -g(-x) &= -\left(-x\left(\frac{1}{a}e^{-2|-x|/a}\right)\right) = x\left(\frac{1}{a}e^{-2|x|/a}\right) \\
g(x) &= -g(-x) \text{ so } g(x) \text{ is symmetric about the origin} \\
\text{Therefore, } \langle x \rangle &= \int_{-\infty}^{\infty} g(x)\, dx = 0
\end{aligned}$$

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \Psi^\star(x)\, x^2 \Psi(x)\, dx \text{ from (5)} \\
&= \int_{-\infty}^{\infty} x^2 |Ae^{-|x|/a}|^2 dx = \int_{-\infty}^{\infty} x^2 A^2 e^{-2|x|/a} dx \\
&= \int_{-\infty}^{\infty} x^2 \left(\frac{1}{\sqrt{a}}\right)^2 e^{-2|x|/a} dx = \int_{-\infty}^{\infty} x^2 \frac{1}{a} e^{-2|x|/a} dx \\
&= \frac{2}{a}\int_0^{\infty} x^2 e^{-2x/a} dx \text{ , } u = -\frac{x}{a}, dx = -a \cdot du \\
&= \frac{2}{a}\int_0^{\infty} a^2 u^2 e^{-2u} (-a)\, du = -2a^2 \int_0^{\infty} u^2 e^{2u} du \\
&= -2a^2 \left[\frac{u^2 e^{2u}}{2} - \int_0^{\infty} u e^{2u} du\right]\Bigg|_0^{\infty} = -2a^2\left[\frac{u^2e^{2u}}{2} - \left(\frac{ue^{2u}}{2} - \int_0^{\infty} \frac{e^2 u}{2} du\right)\right]\Bigg|_0^{\infty} \text{ , via integration by parts} \\
&= -2a^2\left[\frac{u^2e^{2u}}{2} - \left(\frac{ue^{2u}}{2} - \frac{e^{2u}}{4}\right)\right]\Bigg|_0^{\infty} = -2a^2\left[\frac{e^{2u}}{2}\left(u^2 - u + \frac{1}{2}\right)\right]\Bigg|_0^{\infty} \\
&= -a^2\left[e^{-2x/a}\left(\frac{x^2}{a^2} + \frac{x}{a} + \frac{1}{2}\right)\right]\Bigg|_0^{\infty} \\
\langle x^2 \rangle &= -a^2\left[-\frac{1}{2}\right] = \frac{a^2}{2}
\end{aligned}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{a^2}{2} - 0} = \boxed{\frac{a}{\sqrt{2}}} \text{ from (4)}$$

(3)

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty}\Psi^{\star}(x)\,x\Psi(x)\,dx \text{ from (5)}\\ &= \int_{-\infty}^{\infty} x\left|\frac{A}{cosh\left(\frac{x}{a}\right)}\right|^2 dx\\ &= \int_{-\infty}^{\infty} x\left|\frac{1}{cosh\left(\frac{x}{a}\right)\sqrt{2a}}\right|^2 dx\\ &= \int_{-\infty}^{\infty} x\frac{sech^2\left(\frac{x}{a}\right)}{2a}dx\end{aligned}$$

$$\text{Let } h(x) = x\frac{sech^2\left(\frac{x}{a}\right)}{2a}$$

$$\rightarrow -h(-x) = -\left(-x\frac{sech^2\left(\frac{-x}{a}\right)}{2a}\right) = x\frac{sech^2\left(\frac{x}{a}\right)}{2a}$$

$$h(x) = -h(-x) \ \text{ so } h(x) \ \text{ is symmetric about the origin}$$

$$\text{Therefore, } \langle x\rangle = \int_{-\infty}^{\infty} h(x)\,dx = 0$$

$$\begin{aligned}\langle x^2\rangle &= \int_{-\infty}^{\infty}\Psi^{\star}(x)\,x^2\Psi(x)\,dx \text{ from (5)}\\ &= \int_{-\infty}^{\infty} x^2\left|\frac{A}{cosh\left(\frac{x}{a}\right)}\right|^2 dx\\ &= \int_{-\infty}^{\infty} x^2\left|\frac{1}{cosh\left(\frac{x}{a}\right)\sqrt{2a}}\right|^2 dx\\ &= \frac{1}{2a}\int_{-\infty}^{\infty}\frac{x^2}{cosh^2\left(\frac{x}{a}\right)}dx,\ u=\frac{x}{a}, dx = a\cdot du \rightarrow \langle x^2\rangle = \frac{a}{2}\int_{-\infty}^{\infty}\frac{u^2}{cosh^2(u)}du\\ &= a\int_{0}^{\infty}\frac{u^2}{cosh^2(u)}du = \frac{a\pi^2}{12}\end{aligned}$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a\pi^2}{12} - 0} = \boxed{\frac{\pi}{2}\sqrt{\frac{a}{3}}} \text{ from (4)}$$

c) Find wave function $f(k)$ in the wave-vector space

$$f(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\cdot\Psi(x)\,e^{--ikx} \tag{6}$$

(2)

$$\begin{aligned}f(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\cdot\Psi(x)\,e^{-ikx} \text{ from (6)}\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Ae^{-\frac{|x|}{a}}e^{-ikx}dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{a}}e^{-\frac{|x|}{a}}e^{-ikx}dx\\ &= \frac{1}{\sqrt{2\pi a}}\left[\int_{-\infty}^{0} e^{\frac{x}{a}}e^{-ikx}dx + \int_{0}^{\infty} e^{-\frac{x}{a}}e^{-ikx}dx\right] = \frac{1}{\sqrt{2\pi a}}\left[\int_{-\infty}^{0} e^{x\left(\frac{1}{a}-ik\right)}dx + \int_{0}^{\infty} e^{-x\left(\frac{1}{a}+ik\right)}dx\right]\\ &= \frac{1}{\sqrt{2\pi a}}\left[\frac{1}{\frac{1}{a}-ik} + \frac{1}{\frac{1}{a}+ik}\right] = \boxed{\frac{\sqrt{a}}{\pi\left(a^2k^2+1\right)}}\end{aligned}$$

(3)

$$
\begin{aligned}
f(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \cdot \Psi(x)\, e^{-ikx} \text{ from } (6) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{A}{cosh\left(\frac{x}{a}\right)} e^{-ikx} dx \\
&= \frac{1}{\sqrt{4a\pi}} \int_{-\infty}^{\infty} sech\left(\frac{x}{a}\right) e^{-ikx} dx \\
&= \frac{1}{\sqrt{4a\pi}} \int_{-\infty}^{\infty} 2\left(e^{x/a} + e^{-x/a}\right)^{-1} \left(e^{ikx}\right)^{-1} dx \\
&= \mathscr{F}^{-1}\left[\frac{1}{\sqrt{4a\pi}} 2\left(e^{x/a} + e^{-x/a}\right)^{-1}\right] \\
f(k) &= \boxed{\frac{\sqrt{a\pi}}{2} sech\left(\frac{ak\pi}{2}\right)}
\end{aligned}
$$

d) Find standard deviation σ_k for the wave-vector uncertainty

$$
\langle g(k) \rangle = \int_{-\infty}^{\infty} f^{\star}(k)\, g(k)\, f(k)\, dx \tag{7}
$$

(2)

$$
\begin{aligned}
\langle k \rangle &= \int_{-\infty}^{\infty} f^{\star}(k)\, k f(k)\, dk \text{ from } (7) \\
&= \int_{-\infty}^{\infty} k \left| \frac{\sqrt{a}}{\pi\left(a^2k^2+1\right)} \right|^2 dk = \int_{-\infty}^{\infty} \frac{ak}{\pi^2\left(a^2k^2+1\right)^2} dk \\
\text{Let } b(k) &= \frac{ak}{\pi^2\left(a^2k^2+1\right)^2} \\
\to -b(-k) &= -\left(\frac{a(-k)}{\pi^2\left(a^2(-k)^2+1\right)^2}\right) = \frac{ak}{\pi^2\left(a^2k^2+1\right)^2} \\
b(k) &= -b(-k) \text{ so } b(k) \text{ is symmetric about the origin} \\
\text{Therefore, } \langle k \rangle &= \int_{-\infty}^{\infty} b(k)\, dk = 0
\end{aligned}
$$

$$
\begin{aligned}
\langle k^2 \rangle &= \int_{-\infty}^{\infty} f^{\star}(k)\, k^2 f(k)\, dk \text{ from } (7) \\
&= \int_{-\infty}^{\infty} k^2 \left| \frac{\sqrt{a}}{\pi\left(a^2k^2+1\right)} \right|^2 dk = \int_{-\infty}^{\infty} \frac{ak^2}{\pi^2\left(a^2k^2+1\right)^2} dk \\
&= \frac{a}{\pi^2} \int_{-\infty}^{\infty} \frac{k^2}{\left(a^2k^2+1\right)^2} dk = \frac{1}{a\pi^2} \int_{-\infty}^{\infty} \frac{1}{a^2k^2+1} - \frac{1}{\left(a^2k^2+1\right)^2} dk = \frac{1}{a\pi^2} \left[\frac{arctan(ak)}{a} - \frac{k}{a^2k^2+1}\right] \Bigg|_{-\infty}^{\infty} \\
&= \frac{1}{2\pi a^2}
\end{aligned}
$$

$$
\sigma_k = \sqrt{\langle k^2 \rangle - \langle k \rangle^2} = \sqrt{\frac{1}{2\pi a^2} - 0} = \boxed{\frac{1}{a\sqrt{2\pi}}} \text{ from } (4)
$$

(3)

$$\langle k\rangle = \int_{-\infty}^{\infty} f^{\star}(k)\, k f(k)\, dk \text{ from } (7)$$

$$= \int_{-\infty}^{\infty} k \left|\frac{\sqrt{a\pi}}{2} sech\left(\frac{ak\pi}{2}\right)\right|^2 dk = \frac{a\pi}{4}\int_{-\infty}^{\infty} k \cdot sech^2\left(\frac{ak\pi}{2}\right) dk$$

$$\text{Let } j(k) = k \cdot sech^2\left(\frac{ak\pi}{2}\right)$$

$$\rightarrow -j(-k) = -\left((-k)\cdot sech^2\left(\frac{a(-k)\pi}{2}\right)\right) = k \cdot sech^2\left(\frac{ak\pi}{2}\right)$$

$$j(k) = -j(-k) \text{ so } j(k) \text{ is symmetric about the origin}$$

$$\text{Therefore, } \langle k\rangle = \int_{-\infty}^{\infty} j(k)\, dk = 0$$

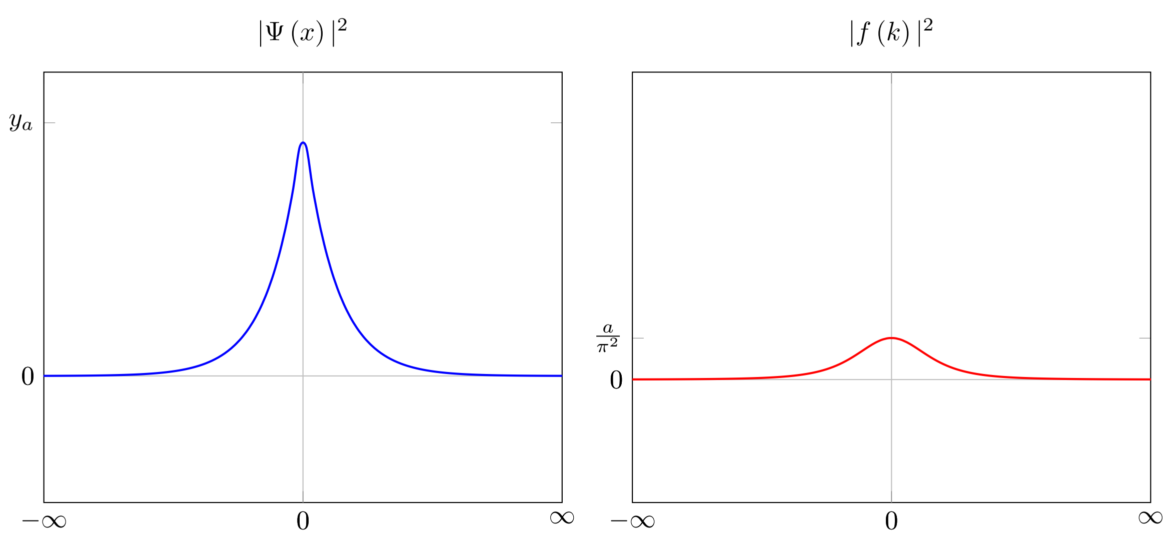
$$\langle k^2\rangle = \int_{-\infty}^{\infty} f^{\star}(k)\, k^2 f(k)\, dk \text{ from } (7)$$

$$= \int_{-\infty}^{\infty} k^2 \left|\frac{\sqrt{a\pi}}{2} sech\left(\frac{ak\pi}{2}\right)\right|^2 dk = \frac{a\pi}{4}\int_{-\infty}^{\infty} k^2 \cdot sech^2\left(\frac{ak\pi}{2}\right) dk = \frac{4}{3\pi a^3}$$

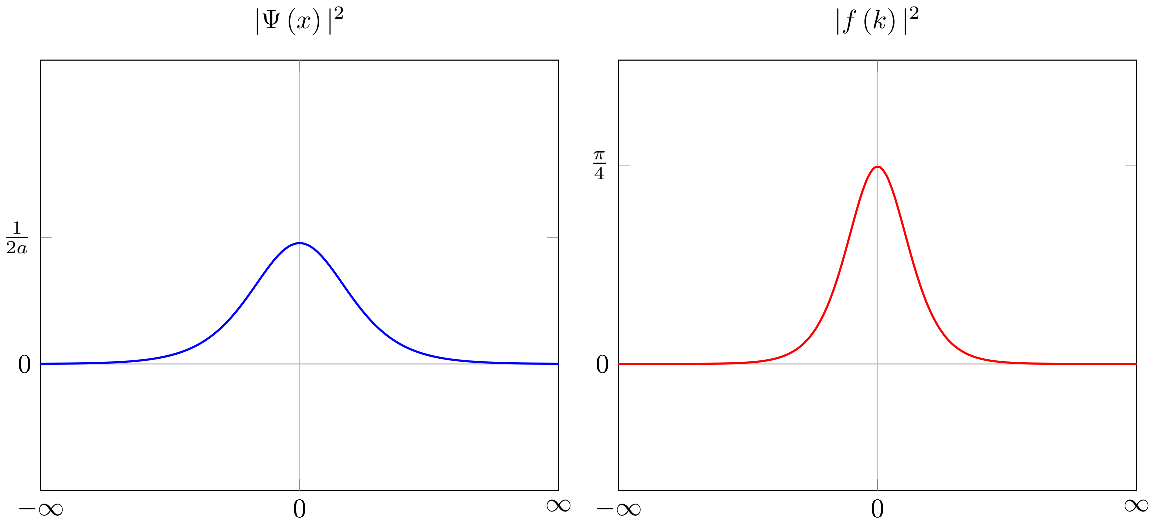
$$\sigma_k = \sqrt{\langle k^2\rangle - \langle k\rangle^2} = \sqrt{\frac{4}{3\pi a^3} - 0} = \boxed{\frac{2}{\sqrt{3\pi a^3}}} \text{ from } (4)$$

e) Sketch the behavior of $\left|\Psi(x)\right|^2$ and $\left|f(k)\right|^2$

(2)



(3)



f) Compare the resulting $\sigma_x \sigma_k$ to (1) and comment

(2)

$$\begin{aligned}\sigma_x \sigma_k &= \frac{a}{\sqrt{2}} \cdot \frac{1}{a\sqrt{2\pi}} \\ &= \frac{1}{2\sqrt{\pi}}\end{aligned}$$

(3)

$$\begin{aligned}\sigma_x \sigma_k &= \frac{\pi}{2}\sqrt{\frac{a}{3}} \cdot \frac{2}{\sqrt{3\pi a^3}} \\ &= \pi\sqrt{\frac{1}{9\pi a^2}} = \frac{\pi}{3a}\sqrt{\frac{1}{\pi}}\end{aligned}$$

2.

Wave packets can also undergo spacial spreading:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \cdot f(k)\, e^{i(kx-\omega(k)t)} \tag{8}$$

Given the standing Gaussian wave packet $f(k)$:

$$\Psi(x,0) = \left(2\pi a^2\right)^{-1/4} e^{-x^2/4a^2} \tag{9}$$

where a is the parameter of the initial spatial extent, explore this spacial spreading. Dispersion of waves is assumed to be parabolic, where m is the particle's mass:

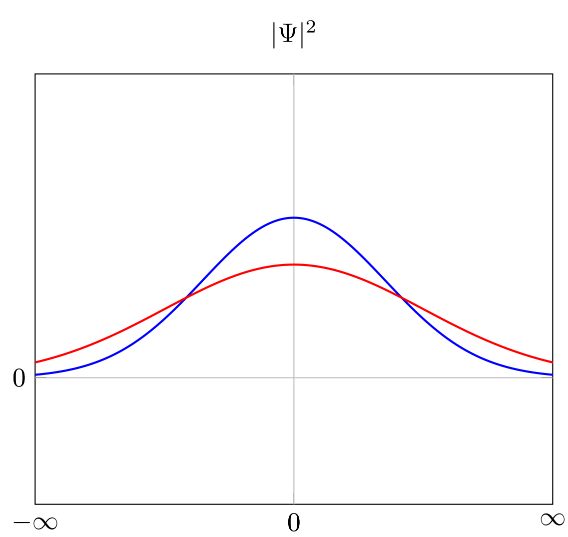
$$\omega(k) = \frac{\hbar k^2}{2m} \tag{10}$$

a) Calculate explicitly the evolution of the wave function (8) and the probability density $|\Psi(x,t)|^2$ with time

$$
\begin{aligned}
f(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)\,e^{-ikx}dx \text{ from } (6)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(2\pi a^2\right)^{-1/4}e^{-x^2/4a^2}e^{-ikx}dx\\
&= \frac{\left(2\pi a^2\right)^{-1/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left(x^2/4a^2+ikx\right)}dx\\
&= \frac{\left(2\pi a^2\right)^{-1/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left[\left(\frac{x}{2a}\right)^2+2ika\left(\frac{x}{2a}\right)+(ika)^2-(ika)^2\right]}dx\\
&= \frac{\left(2\pi a^2\right)^{-1/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left(\frac{x}{2a}+ika\right)^2}e^{(ika)^2}dx\\
&= \frac{e^{-k^2a^2}\left(2\pi a^2\right)^{-1/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left(\frac{x}{2a}+ika\right)^2}dx, t=\left(\frac{x}{2a}+ika\right), dx=2a\cdot dt\\
&= \frac{e^{-k^2a^2}\left(2\pi a^2\right)^{-1/4}}{\sqrt{2\pi}}2\sqrt{\pi}a \to \boxed{f(k)=\sqrt{a}\left(2\frac{1}{\pi}\right)^{1/4}e^{-k^2a^2}}
\end{aligned}
$$

$$
\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(k)\,e^{-i(kx-\omega t)}dk\\
&= \left(2\frac{a^2}{\pi}\right)^{-1/4}\int_{-\infty}^{\infty}e^{-k^2a^2}e^{i\left(kx-\frac{k^2\hbar t}{2m}\right)}dk\\
&= \left(2\frac{a^2}{\pi}\right)^{-1/4}\int_{-\infty}^{\infty}e^{-k^2a^2+ikx-\frac{i\hbar k^2 t}{2m}}dk, \alpha=\left(a^2+\frac{i\hbar t}{2m}\right), \beta=x\\
&= \left(2\frac{a^2}{\pi}\right)^{-1/4}\int_{-\infty}^{\infty}e^{-k^2\alpha+i\beta k}dk=\sqrt{\frac{\pi}{\alpha}}e^{-\beta^2/4\alpha}=\boxed{\sqrt{\frac{\pi}{\left(a^2+\frac{i\hbar t}{2m}\right)}}e^{\frac{-x^2}{4\left(a^2+\frac{i\hbar t}{2m}\right)}}}
\end{aligned}
$$

b) Does the probability density retain its Gaussian shape? Sketch its spatial behavior at $t=0$ and at some "remote" time t on the same plot



$|\Psi(x,0)|^2$, $|\Psi(x,t)|^2$

c) Calculate the time-dependence of the following expectation values for the particle position and momentum: $\langle x\rangle, \langle x^2\rangle, \langle p\rangle,$ and$\langle p^2\rangle$. If $\langle x^2\rangle$ spreads over time, find at what time t it becomes twice bigger than the initial value of a^2

d) Based on above, calculate if there is any time-dependence in the uncertainty product $\sigma_x\sigma_p$ as compared to (1) and comment.