PHYS 4301
Homework 6

Charles Averill
charles@utdallas.edu

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1. Let ψ_n be normalized stationary states of the oscillator $(n = 0, 1, 2, \dots)$. We established in class the relations in (1). Other operators $\hat{A}$ can now be represented as combinations of ladder operators to find the result of their actions on ψ_n as seen in (2). The RHS of (2) is an expansion of the result, which is some wave function, over the set of our eigenstates, b_m being the expansion coefficients. You will find there would be only very few non-zero b_m in our examples. In the following, use the representation of the position x and momentum $\hat{p}$ operators via ladder operators.

$$\hat{a}\psi_n = \sqrt{n}\psi_{n-1}, \qquad \hat{a}^\dagger\psi_n = \sqrt{n+1}\psi_{n+1} \tag{1}$$
$$\hat{A}\psi_n = \sum_m b_m\psi_m \tag{2}$$

a) Find $x\psi_n$ and $\hat{p}\psi_n$ in a form similar to equations (1) and (2)

$$\begin{aligned}
\hat{a} &= \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x}) \\
\hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x}) \\
\hat{a} + \hat{a}^\dagger &= 2\sqrt{\frac{m\omega}{2\hbar}}\hat{x} \\
\hat{a} - \hat{a}^\dagger &= i\sqrt{\frac{2}{\hbar m\omega}}\hat{p} \\
\hat{x} &= \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) \\
x\psi_n &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}\psi_n + \hat{a}^\dagger\psi_n\right) = \boxed{\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\psi_{n-1} + \sqrt{n+1}\psi_{n+1}\right)} \\
\hat{p} &= -\sqrt{-\frac{\hbar m\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right) \\
\hat{p}\psi_n &= -\sqrt{-\frac{\hbar m\omega}{2}}\left(\hat{a}\psi_n - \hat{a}^\dagger\psi_n\right) = \boxed{-\sqrt{-\frac{\hbar m\omega}{2}}\left(\sqrt{n}\psi_{n-1} - \sqrt{n+1}\psi_{n+1}\right)}
\end{aligned}$$

b) Use now *only* results of a) to find $x^2\psi_n$, $\hat{p}^2\psi_n$ and $x\hat{p}\psi_n$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right)$$

$$\hat{x}^2 = \hat{x}\hat{x} = \frac{\hbar}{2m\omega}\left(\hat{a}+\hat{a}^\dagger\right)\left(\hat{a}+\hat{a}^\dagger\right) = \frac{\hbar}{2m\omega}\left(\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\right)$$

$$= \frac{\hbar}{2m\omega}\left(\hat{a}^2+\left(\hat{a}^\dagger\right)^2+2\hat{a}^\dagger\hat{a}+1\right) \text{ because } \hat{a} \text{ and } \hat{a}^\dagger \text{ are orthogonal, so } \hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a}+1$$

$$x^2\psi_n = \frac{\hbar}{2m\omega}\left(\hat{a}^2\psi_n+\left(\hat{a}^\dagger\right)^2\psi_n+2\hat{a}^\dagger\hat{a}\psi_n+\psi_n\right) = \boxed{\frac{\hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\psi_{n+2}+\sqrt{n(n-1)}\psi_{n-2}+(2n+1)\psi_n\right)}$$

$$\hat{p} = -\sqrt{-\frac{\hbar m\omega}{2}}\left(\hat{a}-\hat{a}^\dagger\right)$$

$$\hat{p}^2 = \hat{p}\hat{p} = -\frac{\hbar m\omega}{2}\left(\hat{a}-\hat{a}^\dagger\right)\left(\hat{a}-\hat{a}^\dagger\right) = -\frac{\hbar m\omega}{2}\left(\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\right)$$

$$\hat{p}^2\psi_n = -\frac{\hbar m\omega}{2}\left(\hat{a}^2\psi_n+\left(\hat{a}^\dagger\right)^2\psi_n-2\hat{a}^\dagger\hat{a}\psi_n-\psi_n\right) = \boxed{-\frac{\hbar m\omega}{2}\left(\sqrt{(n+1)(n+2)}\psi_{n+2}+\sqrt{n(n-1)}\psi_{n-2}-(2n+1)\psi_n\right)}$$

$$\hat{x}\hat{p} = -\sqrt{\frac{\hbar}{2m\omega}}\sqrt{-\frac{\hbar m\omega}{2}}\left(\hat{a}+\hat{a}^\dagger\right)\left(\hat{a}-\hat{a}^\dagger\right) = -\sqrt{-\frac{\hbar^2}{4}}\left(\hat{a}+\hat{a}^\dagger\right)\left(\hat{a}-\hat{a}^\dagger\right)$$

$$= -\frac{i\hbar}{2}\left(\hat{a}^2-\left(\hat{a}^\dagger\right)^2+\hat{a}^\dagger\hat{a}-\hat{a}\hat{a}^\dagger\right) = -\frac{i\hbar}{2}\left(\hat{a}^2-\left(\hat{a}^\dagger\right)^2-1\right)$$

$$x\hat{p}\psi_n = -\frac{i\hbar}{2}\left(\hat{a}^2\psi_n-\left(\hat{a}^\dagger\right)^2\psi_n-\psi_n\right) = \boxed{-\frac{i\hbar}{2}\left(\sqrt{(n+1)(n+2)}\psi_{n+2}-\sqrt{n(n-1)}\psi_{n-2}-\psi_n\right)}$$

c) And, come on! Why don't you calculate - with the least expense - $x^3\psi_n$?

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right)$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left(\hat{a}^2+\left(\hat{a}^\dagger\right)^2+2\hat{a}^\dagger\hat{a}+1\right)$$

$$\hat{x}^3 = \hat{x}^2\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right)\frac{\hbar}{2m\omega}\left(\hat{a}^2+\left(\hat{a}^\dagger\right)^2+2\hat{a}^\dagger\hat{a}+1\right)$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(\left(\hat{a}^\dagger\right)^3+\hat{a}^3+\left(\hat{a}^\dagger\right)^2\hat{a}+\hat{a}^2\hat{a}^\dagger+2\hat{a}^\dagger\hat{a}\hat{a}^\dagger+2\hat{a}^\dagger\hat{a}^2+\hat{a}^\dagger+\hat{a}\right)$$

$$x^3\psi_n = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\Big(\sqrt{(n+1)(n+2)(n+3)}\psi_{n+3}+\sqrt{n(n-1)(n-2)}\psi_{n-3}+\sqrt{n^2(n+1)}\psi_{n+1}$$
$$+\sqrt{(n+1)^2 n}\psi_{n-1}+2(n+1)\sqrt{n+1}\psi_{n+1}+2(n-1)\sqrt{n}\psi_{n-1}+\sqrt{n}\psi_{n-1}+\sqrt{n+1}\psi_{n+1}\Big)$$

d) Use the derived in b) to check if indeed $\hat{H}\psi_n = \left(n + \frac{1}{2}\right)\hbar\omega\psi_n$ where $\hat{H}$ is, of course, the Hamiltonian for our oscillator. Can you also confirm that the expectation values of the kinetic and potential energies in any stationary state are equal to each other?

$$
\begin{aligned}
\hat{p}^2 &= -\frac{\hbar m\omega}{2}\left(\hat{a}^2 + \left(\hat{a}^\dagger\right)^2 - 2\hat{a}^\dagger\hat{a} - 1\right) \\
\hat{x}^2 &= \frac{\hbar}{2m\omega}\left(\hat{a}^2 + \left(\hat{a}^\dagger\right)^2 + 2\hat{a}^\dagger\hat{a} + 1\right) \\
\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2} \\
&= \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} \\
&= -\frac{1}{2m}\frac{\hbar m\omega}{2}\left(\hat{a}^2 + \left(\hat{a}^\dagger\right)^2 - 2\hat{a}^\dagger\hat{a} - 1\right) + \frac{m\omega^2}{2}\frac{\hbar}{2m\omega}\left(\hat{a}^2 + \left(\hat{a}^\dagger\right)^2 + 2\hat{a}^\dagger\hat{a} + 1\right) \\
\hat{H}\psi_n &= -\frac{\hbar\omega}{4}\left(\sqrt{(n+1)(n+2)}\psi_{n+2} + \sqrt{n(n-1)}\psi_{n-2} - (2n+1)\psi_n\right) \\
&+ \frac{\hbar\omega}{4}\left(\sqrt{(n+1)(n+2)}\psi_{n+2} + \sqrt{n(n-1)}\psi_{n-2} + (2n+1)\psi_n\right) \\
&= -\frac{\hbar\omega}{4}\left(\sqrt{(n+1)(n+2)}\psi_{n+2} + \sqrt{n(n-1)}\psi_{n-2} - (2n+1)\psi_n - \sqrt{(n+1)(n+2)}\psi_{n+2} - \sqrt{n(n-1)}\psi_{n-2} - (2n+1)\psi_n\right) \\
&= -\frac{\hbar\omega}{4}\left(-(2n+1)\psi_n - (2n+1)\psi_n\right) = -\frac{\hbar\omega}{4}\left(-2(2n+1)\right)\psi_n = \frac{\hbar\omega}{2}(2n+1)\psi_n = \boxed{\left(n + \frac{1}{2}\right)\hbar\omega\psi_n}
\end{aligned}
$$

$$
\begin{aligned}
T &= \frac{\hat{p}^2}{2m} \\
\langle T\rangle &= \int_{-\infty}^{\infty}\psi_n^*\frac{\hat{p}^2}{2m}\psi_n dx \\
&= \frac{1}{2m}\int_{-\infty}^{\infty}\psi_n^*\hat{p}^2\psi_n dx \\
&= \frac{1}{2m}\int_{-\infty}^{\infty}\psi_n^*\left[\frac{\hbar m\omega}{2}\left(-\hat{a}^2 - \left(\hat{a}^\dagger\right)^2 + 2\hat{a}^\dagger\hat{a} + 1\right)\right]\psi_n dx \\
&= \frac{\hbar\omega(2n+1)}{4}\int_{-\infty}^{\infty}\psi_n^*\psi_n dx = \boxed{\frac{\hbar\omega(2n+1)}{4}} \\
V(x) &= \frac{1}{2}m\omega^2\hat{x}^2 \\
\langle V(x)\rangle &= \frac{1}{2}m\omega^2\int_{-\infty}^{\infty}\psi_n^*\hat{x}^2\psi_n dx \\
&= \frac{1}{2}m\omega^2\int_{-\infty}^{\infty}\psi_n^*\left[\frac{\hbar}{2m\omega}\left(\hat{a}^2 + \left(\hat{a}^\dagger\right)^2 + 2\hat{a}^\dagger\hat{a} + 1\right)\right]\psi_n dx \\
&= \frac{\hbar\omega}{4}(2n+1)\int_{-\infty}^{\infty}\psi_n^*\psi_n dx = \boxed{\frac{\hbar\omega(2n+1)}{4}} \\
\langle T\rangle = \langle V(x)\rangle &= \boxed{\frac{\hbar\omega(2n+1)}{4}}
\end{aligned}
$$

2. Let ψ_n $(n = 0, 1, 2, \dots)$ be standard normalized stationary states of the 1D harmonic oscillator characterized by spatial displacement x, mass m, and frequency ω. You may want to use the algebraic approach in your calculations.

a) At time $t = 0$, the oscillator is prepared in (normalized) state (3). Find the time dependence of the expectation value of the kinetic energy operator (4) in this state $\Psi(t)$ at times $t > 0$

$$\Psi(t=0) = \frac{1}{\sqrt{2}}\psi_2 + \frac{i}{\sqrt{2}}\psi_4 \tag{3}$$

$$\langle \hat{T} \rangle (t), \qquad \hat{T} = \frac{\hat{p}^2}{2m} \tag{4}$$

$$\begin{aligned}
|\psi(t=0)\rangle &= \frac{1}{\sqrt{2}}|2\rangle + \frac{i}{\sqrt{2}}|4\rangle \\
|\psi(t)\rangle &= \frac{1}{\sqrt{2}}e^{-i\frac{5}{2}\omega t}|2\rangle + \frac{i}{\sqrt{2}}e^{-i\frac{9}{2}\omega t}|4\rangle \\
\langle \hat{T} \rangle (t) &= \langle \psi(t)|\frac{\hat{p}^2}{2m}|\psi(t)\rangle \\
&= -\frac{\hbar\omega}{4}\left(\left[\frac{1}{\sqrt{2}}e^{i\frac{5}{2}\omega t}\langle 2| - \frac{i}{\sqrt{2}}e^{i\frac{9}{2}\omega t}\langle 4|\right]\left(\hat{a}^\dagger - \hat{a}\right)^2\left[\frac{1}{\sqrt{2}}e^{-i\frac{5}{2}\omega t}|2\rangle + \frac{i}{\sqrt{2}}e^{-i\frac{9}{2}\omega t}|4\rangle\right]\right) \\
&= -\frac{\hbar\omega}{4}\left(\frac{1}{2}(-2-3) + \frac{i}{2}e^{-2i\omega t}\sqrt{12} - \frac{i}{2}e^{2i\omega t}\sqrt{12} + \frac{1}{2}(-4-5)\right) \\
&= -\frac{\hbar\omega}{4}\left(-\frac{5}{2} - \frac{9}{2} - \frac{i}{2}\sqrt{12}\frac{e^{2i\omega t} - e^{-2i\omega t}}{2i}2i\right) = -\frac{\hbar\omega}{4}\left(-7 + \sqrt{12}sin(2\omega t)\right) = \boxed{\frac{\hbar\omega}{4}\left(7 - 2\sqrt{3}sin(2\omega t)\right)}
\end{aligned}$$

b) What would be the result for time dependence (4) if the initial state was (5) instead of (3)?

$$\Psi(t=0) = \frac{1}{\sqrt{2}}\psi_2 + \frac{i}{\sqrt{2}}\psi_6 \tag{5}$$

$$\begin{aligned}
|\psi(t=0)\rangle &= \frac{1}{\sqrt{2}}|2\rangle + \frac{i}{\sqrt{2}}|6\rangle \\
|\psi(t)\rangle &= \frac{1}{\sqrt{2}}e^{-i\frac{5}{2}\omega t}|2\rangle + \frac{i}{\sqrt{2}}e^{-i\frac{13}{2}\omega t}|6\rangle \\
\langle \hat{T} \rangle (t) &= \langle \psi(t)|\frac{\hat{p}^2}{2m}|\psi(t)\rangle \\
&= -\frac{\hbar\omega}{4}\left(\left[\frac{1}{\sqrt{2}}e^{i\frac{5}{2}\omega t}\langle 2| - \frac{i}{\sqrt{2}}e^{i\frac{13}{2}\omega t}\langle 6|\right]\left(\hat{a}^\dagger - \hat{a}\right)^2\left[\frac{1}{\sqrt{2}}e^{-i\frac{5}{2}\omega t}|2\rangle + \frac{i}{\sqrt{2}}e^{-i\frac{13}{2}\omega t}|6\rangle\right]\right) \\
&= -\frac{\hbar\omega}{4}\left(\frac{1}{2}\langle 2|\left(\hat{a}^\dagger - \hat{a}\right)^2|2\rangle + \frac{1}{2}\langle 6|\left(\hat{a}^\dagger - \hat{a}\right)^2|6\rangle\right) \\
&= -\frac{\hbar\omega}{4}(-2-3-6-7) = \boxed{-\frac{9\hbar\omega}{4}}
\end{aligned}$$