PHYS 4302

Homework 4

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Problem 1

In this guided problem, you will explore some ideas from a general topic of "addition of angular momenta." Let $\mathbf{J}_1$ and $\mathbf{J}_2$ be two angular momentum operators (I am dropping hats on operators). They may refer to angular momenta of two different particles or to the orbital angular momentum and to the spin of the same particle; that is, they act on different "coordinates" in the description of our system. It means that these operators commute with each other:

$$[J_{1i}, J_{2k}] = 0, \quad \text{for all Cartesian indices } i, k. \tag{1}$$

These momenta "interact", e.g. due to the associated magnetic momenta, the relevant part of the system Hamiltonian is their scalar product ("energy is scalar"):

$$H = \beta \mathbf{J}_1 \cdot \mathbf{J}_2, \tag{2}$$

where β is an appropriate coefficient. Hamiltonian (2) could, e.g., describe an important phenomenon of the so-called spin-orbit interaction.

We also now define the total angular momentum of the system:

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2. \tag{3}$$

(a) Show that J is indeed the angular momentum operator, that is, it satisfies the same commutation relations as each of $\mathbf{J}_1$ and $\mathbf{J}_2$.

By the definition of momentum operators, we should have that

$$\left[L_x, L_y\right] = i\hbar L_z. \quad \text{(Griffiths 4.99)}$$

By (3), we should have $J_x = J_{1x} + J_{2x}$, $J_y = J_{1y} + J_{2y}$, and $J_z = J_{1z} + J_{2z}$. Therefore, we have

$$\begin{aligned}
\left[J_x, J_y\right] &= \left[J_{1x} + J_{2x}, J_{1y} + J_{2y}\right] \\
&= \left[J_{1x}, J_{1y}\right] + \left[J_{1x}, J_{2y}\right] + \left[J_{2x}, J_{1y}\right] + \left[J_{2x}, J_{2y}\right] \\
&= \left[J_{1x}, J_{1y}\right] + 0 + 0 + \left[J_{2x}, J_{2y}\right] \\
&= i\hbar J_{1z} + i\hbar J_{2z} = \boxed{i\hbar J_z}
\end{aligned}$$

(b) Show that $\mathbf{J}_1$ and $\mathbf{J}_2$ are no longer conserved due to (2) but $\mathbf{J}$ is a conserved quantity. (Doesn't it make sense, the conservation of the total momentum?)

$$\begin{aligned}
\frac{\partial J_1}{\partial t} &= \frac{1}{i\hbar}\left[J_1 \beta J_1 \cdot J_2\right] \\
&= \frac{\beta}{i\hbar}\left[\left[J_1, J_1\right] J_2 + J_1\left[J_1, J_2\right]\right] \\
&= \frac{\beta}{i\hbar}\left[0 J_2 + J_1\left[J_1, J_2\right]\right] \\
&= \boxed{\frac{\beta}{i\hbar}\left[J_1, J_2\right] J_1 \neq 0} \quad \text{so } J_1 \text{ is not conserved}
\end{aligned}$$

J_2 is similarly not conserved.

(c) Show that all three squares: $\mathbf{J}_1^2, \mathbf{J}_2^2$, and $\mathbf{J}^2$ are conserved. In addition, these three squares and operator $\mathbf{J}$, let us consider its J_z component as usual, all commute with each other. That is, these four operators represent compatible observables.

An important consequence of the statements above is that we can label the stationary eigenstates as $|jmj_1j_2\rangle$ by the respective eigenvalues of the conserved compatible observables:

$$\mathbf{J}^2|jmj_1j_2\rangle = \hbar^2 j\left(j+1\right)|jmj_1j_2\rangle, \tag{4}$$

$$J_z|jmj_1j_2\rangle = \hbar m|jmj_1j_2\rangle, \tag{5}$$

$$\mathbf{J}_1^2|jmj_1j_2\rangle = \hbar^2 j_1\left(j_1+1\right)|jmj_1j_2\rangle, \tag{6}$$

$$\mathbf{J}_2^2|jmj_1j_2\rangle = \hbar^2 j_2\left(j_2+1\right)|jmj_1j_2\rangle. \tag{7}$$

We will have a separate discussion of the allowed values j.

$$\begin{aligned}
\frac{\partial J_1^2}{\partial t} &= \frac{1}{i\hbar}\left[J_1^2, \beta J_1 \cdot J_2\right] \\
&= \frac{\beta}{i\hbar}\left[J_1^2, J_1 \cdot J_2\right] \\
&= \frac{\beta}{i\hbar}\left[\left[J_1^2, J_1\right] J_2 + J_1\left[J_1^2, J_2\right]\right] \\
&= \frac{\beta}{i\hbar}\left[0+0\right] \\
&= \boxed{0} \quad \text{so } J_1^2 \text{ is conserved}
\end{aligned}$$

J_2^2 is similarly conserved.

$$\begin{aligned}
\frac{\partial \mathbf{J}^2}{\partial t} &= \frac{1}{i\hbar}\left[\mathbf{J}^2, \beta J_1 \cdot J_2\right] \\
&= \frac{\beta}{i\hbar}\left[\left(J_1 + J_2\right)^2, J_1 \cdot J_2\right] \\
&= \frac{\beta}{i\hbar}\left[\left[\mathbf{J}^2, J_1\right] J_2 + J_1\left[\mathbf{J}^2, J_2\right]\right] = \frac{\beta}{i\hbar}\left[0 \cdot J_2 + J_1 \cdot 0\right] \\
&= \boxed{0} \quad \text{so } \mathbf{J}^2 \text{ is conserved}
\end{aligned}$$

(d) You should be able now to show that $|jmj_1j_2\rangle$ is indeed the stationary state of (2):

$$H|jmj_1j_2\rangle = E|jmj_1j_2\rangle, \tag{8}$$

with energy

$$E = \frac{\beta\hbar^2}{2}\left[j\left(j+1\right) - j_1\left(j_1+1\right) - j_2\left(j_2+1\right)\right]. \tag{9}$$

All you need to do for this is square (3) and then follow our already established relationships (4)-(7).

Squaring (3) we get

$$\begin{aligned}
\mathbf{J}\cdot\mathbf{J} &= \left(J_1 + J_2\right)\left(J_1 + J_2\right) \\
&= J_1^2 + J_1J_2 + J_2J_1 + J_2^2 \\
&= J_1^2 + 2J_1J_2 + J_2^2, \quad 2J_1J_2 = \mathbf{J}^2 - J_1^2 - J_2^2 \\
H &= \frac{\beta}{2}\left(\mathbf{J}^2 - J_1^2 - J_2^2\right) \\
E|jmj_1j_2\rangle &= H|jmj_1j_2\rangle \\
&= \frac{\beta}{2}\left(\mathbf{J}^2 - J_1^2 - J_2^2\right)|jmj_1j_2\rangle \\
&= \frac{\beta}{2}\left(\mathbf{J}^2|jmj_1j_2\rangle - J_1^2|jmj_1j_2\rangle - J_2^2|jmj_1j_2\rangle\right) \\
&= \boxed{\frac{\beta\hbar^2}{2}\left(j\left(j+1\right) - j_1\left(j_1+1\right) - j_2\left(j_2+1\right)\right)|jmj_1j_2\rangle} \text{(9)}
\end{aligned}$$

Problem 2

Two atoms carrying spin-$\frac{1}{2}$ each are localized in neighboring spatial locations, vector $\mathbf{a}$ indicates the position of one atom with respect to the other. The resulting interaction between the atoms turns out to be predominantly magnetic so that the system Hamiltonian (no hats on operators) is

$$H = \frac{\mu_0}{4\pi}\left[\frac{\mathbf{m}_1\cdot\mathbf{m}_2}{a^3} - 3\frac{\left(\mathbf{m}_1\cdot\mathbf{a}\right)\left(\mathbf{m}_2\cdot\mathbf{a}\right)}{a^5}\right], \quad a = |\mathbf{a}|, \tag{10}$$

a familiar dipole-dipole interaction from EM courses. Here $\mathbf{m}_1 = \gamma\mathbf{S}_1$ and $\mathbf{m}_2 = \gamma\mathbf{S}_2$ are spin-originated magnetic moment operators, with γ being the gyromagnetic ratio and $\mathbf{S}_1$ and $\mathbf{S}_2$ the respective spin operators. It should be clear that this system is *not* spherically symmetric as there is now a distinct direction provided by vector $\mathbf{a}$.

(a) For increased convenience, choose the z-axis along vector a and rewrite the Hamiltonian (10) entirely in terms of the familiar spin-$\frac{1}{2}$ operators, while lumping all other parameters into a convenient combination (that would eventually define the energy scale for the system).

$$\begin{aligned}
\mathbf{m}_1 &= \gamma S_1 z \\
\mathbf{m}_2 &= \gamma S_2 z
\end{aligned}$$

$$\begin{aligned}
H &= \frac{\mu_0}{4\pi}\left[\frac{\gamma S_1 z\cdot\gamma S_2 z}{a^3} - 3\frac{\left(\gamma S_1 z\cdot\mathbf{a}\right)\left(\gamma S_2 z\cdot\mathbf{a}\right)}{a^5}\right] \\
&= W\left(\frac{S_z z S_2 z}{a^3} - 3\frac{S_z z S_2 z a_z^2}{a^5}\right), \quad W = \frac{\mu_0\gamma^2}{4\pi}
\end{aligned}$$

(b) Whether straightforwardly or by using some symmetry considerations, find energies of all 4 stationary states of this system.

(c) Specify those stationary states, their degeneracies and appropriate quantum numbers to label them.

(d) In your interpretation of the results, compare them to the classical picture of interacting dipoles as dependent on their orientations.