PHYS 4301
Homework 4

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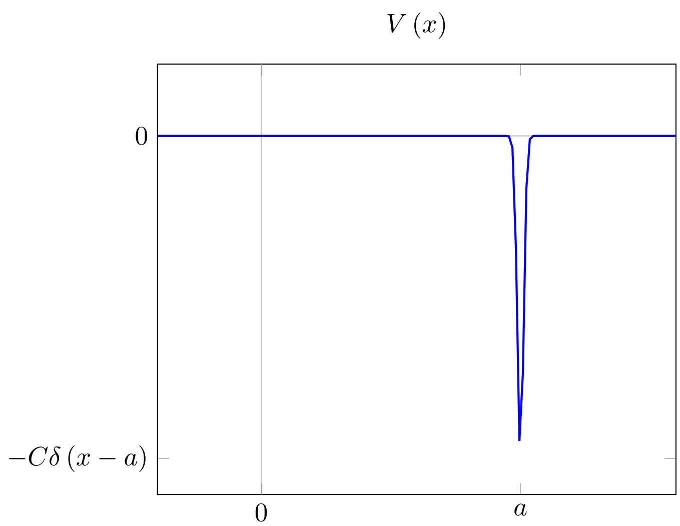
1. Given the following potential energy, explore the formation of a bound state for a particle of mass m by solving the time-independent Schrödinger equation

$$V(x) = \begin{cases} +\infty, & x < 0, \\ -C\delta(x-a), & x > 0 \end{cases} \tag{1}$$

where a is the position of the potential well and C is its strength.

a) Find the (transcendental) equation that would allow you to determine the energy of the bound state. Indicate how you would be solving this equation graphically.



$$\frac{d^2\Psi(x)}{dx^2} + \left(\frac{2mc}{\hbar^2}\delta(x-a) + \frac{2mE}{\hbar^2}\right)\Psi(x) = 0$$

$$k^2 = \frac{2mE}{\hbar^2} \rightarrow$$

$$\frac{d^2\Psi(x)}{dx^2} + \left(\frac{2mc}{\hbar^2}\delta(x-a) + k^2\right)\Psi(x) = 0 \rightarrow$$

$$\Psi(x) = \begin{cases} \Psi_1(x) = Ae^{kx} + Be^{-kx}, & 0 < x < a, \\ \Psi_2(x) = De^{-kx}, & x > a, \end{cases} \tag{2}$$

The wave function has to fulfill the following boundary conditions:

$$
\begin{aligned}
\Psi_0\left(0\right) = 0 \rightarrow \\
A + B = 0 \rightarrow B = -A \\
\Psi_1\left(a\right) = \Psi_2\left(a\right) \rightarrow \\
Ae^{ka} - Ae^{-ka} = De^{-ka}
\end{aligned}
$$

Integrating the time-independent Schrödinger equation, we get

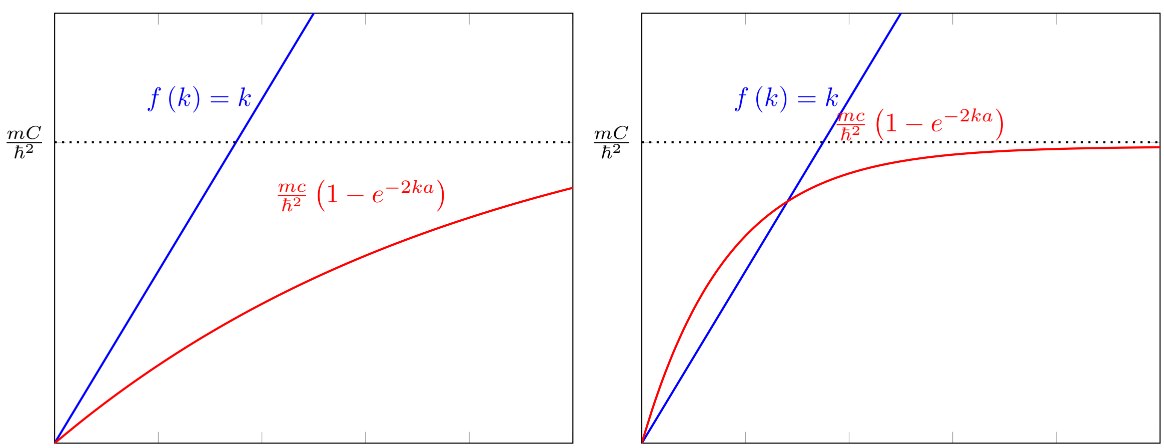
$$
\begin{aligned}
\lim_{\epsilon \to a}\left(\Psi_2'\left(a + \epsilon - \Psi_1'\left(a - \epsilon\right)\right)\right) + \frac{2mC}{\hbar^2}\Psi_2\left(a\right) = 0 \\
-kDe^{-ka} - kAe^{ka} - kAe^{-ka} + \frac{2mC}{\hbar^2}De^{-ka} = 0 \\
-kAe^{ka} + Ake^{-ka} - kAe^{ka} - kAe^{-ka} + \frac{2mC}{\hbar^2}\left(Ae^{ka} - Ae^{-ka}\right) = 0 \\
-kAe^{ka} - kAe^{ka} + \frac{2mC}{\hbar^2}\left(Ae^{ka} - Ae^{-ka}\right) = 0
\end{aligned}
$$

Solving the above equation for k, we can find the energy of the bound state:

$$
\begin{aligned}
k^2 = -\frac{2mE}{\hbar^2} \\
E = -\frac{k^2\hbar^2}{2m}
\end{aligned}
$$

Graphical Solution

$a < \frac{\hbar^2}{2mC}$ (no bound state) $\qquad$ $a > \frac{\hbar^2}{2mC}$ (one bound state)



b) In the absence of the neighboring wall the bound state is formed for any value of strength C. But now, do we have a restriction on the value of C for a bound state to exist? If so, what is the minimal possible value of C in terms of other given parameters of the problem?

If the neighboring wall does not exist, we can model the system with an attractive dirac delta potential. Therefore, a approaches ∞, so $\exists$ a bound value for every value of C. Given the state $V = -C\delta(x-a)$, we know that C must be positive: $\boxed{C > 0}$.

c) Can you bring simple physical arguments to rationalize the results in item (b)?

2. Study the bound state(s) of a particle of mass m exposed to two identical delta function potential wells displaced in space with respect to each other, described below

$$V(x) = -C\left(\delta(x-a) + \delta(x+a)\right) \tag{3}$$

a) How many bound states will be formed due to the described potential?

There should be three bound states corresponding to the three distinct regions:

$$\begin{cases} \Psi_1, & x < -a, \\ \Psi_2, & -a \leq x \leq a, \\ \Psi_3, & x > a \end{cases}$$

This does not change if $a \to \infty$, as $a < \infty$. However, if $a \to 0$, bound states appear at $x < 0$ and $x > 0$.

b) Find equations that would allow you to determine the energies of the bound states

$$\begin{aligned} &\frac{d^2\Psi}{dx^2} + \frac{2mC}{\hbar^2}\left(\delta(x-a) + \delta(x+a)\right)\Psi(x) + \frac{2mE}{\hbar^2}\Psi(x) = 0, \\ &x \neq \pm a \implies \frac{d^2\Psi}{dx^2} + \frac{2mE}{\hbar^2}\Psi(x) = 0 \\ &\qquad\qquad\quad \frac{d^2\Psi}{dx^2} + k^2\Psi(x) = 0, \text{ where } k^2 = \frac{2mE}{\hbar^2} \\ &\qquad\quad \Psi(x) = \begin{cases} Ae^{-ikx}, & x > a, \\ \frac{B}{2}\left(e^{ikx} \pm e^{-ikx}\right), & -a < x < a, \\ \pm Ae^{ikx}, & x < -a, \end{cases} \end{aligned}$$

c) Suppose the particle is in its ground state. Does it produce any force between the potential wells?

Yes, the particle will produce force between the potential walls if it is in its ground state.