PHYS 4302

Homework 5

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Problem 1

An electron is confined to move in one dimension (coordinate x) and resides in a standard potential well of width a with infinitely high walls, the potential energy being

$$U(x) = 0 \text{ for } |x| < \frac{a}{2} \text{ and } U(x) = \infty \text{ for } x \geq \frac{x}{2}. \tag{1}$$

The electron is in its ground state ($n = 1$) when it is subject to a Lorentzian pulse of the *weak* electric field (along x):

$$\mathcal{E}(t) = \frac{\mathcal{E}_0}{\left(\frac{t}{T}\right)^2 + 1}, \quad -\infty < t < \infty \tag{2}$$

where parameter T signifies the time "duration" of the pulse.

(a) Establish the operator to the system Hamiltonian due to the applied electric field.

$$U(x) = \begin{cases} 0 & |x| < \frac{a}{2} \Rightarrow -\frac{a}{2} < x < \frac{a}{2} \\ \infty & |x| > \frac{a}{2} \Rightarrow x > \frac{a}{2}, x < -\frac{a}{2} \end{cases}$$

$$\Psi(x) = \begin{cases} \phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), & n \text{ is even} \\ \psi_n(x) = \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right), & n \text{ is odd} \end{cases}$$

Let the force on the electron $f = -q_e \mathcal{E}(t)$

$$-\frac{dU(x)}{dx} = -\frac{q_e \mathcal{E}_0}{\left(\frac{t}{T}\right)^2 + 1}$$

$$\hat{H} = U(x) = \boxed{\frac{q_e \mathcal{E}_0 x}{\left(\frac{t}{T}\right)^2 + 1}}$$

(b) Calculate probabilities P_2 and P_3 for the electron to be found in respectively $n=2$ and $n=3$ excited state levels after the electric-field caused perturbation is over. We need an explicit answer in terms of the relevant system parameters.

$$\begin{aligned}
P_2 &= \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} e^{i\omega t}\langle\phi_1(n)|\hat{H}|\psi_1(n)\rangle dt\right|^2 \\
&= \frac{1}{\hbar^2}\left(q_e\mathcal{E}_0\right)^2\left|\int_{-\infty}^{\infty}\frac{e^{i\omega t}\langle\phi_1(n)|x|\psi_1(n)\rangle}{\left(\frac{t}{T}\right)^2+1}\right|^2 \\
&= \frac{\left(q_e\mathcal{E}_0\right)^2}{\hbar^2}\left|\langle\phi_2(n)|x|\psi_1(n)\rangle\right|^2\left|\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\left(\frac{t}{T}\right)^2+1}\right|
\end{aligned}$$

$$\begin{aligned}
\langle\phi_2(n)|x|\psi_1(n)\rangle &= \frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx \\
&= \frac{2}{a}\int_{0}^{\frac{a}{2}} x\left[\cos\left(\frac{\pi x}{a}\right)-\cos\left(\frac{3\pi x}{a}\right)\right]dx \\
&= \frac{2}{a}\left[\int_{0}^{\frac{a}{2}} x\cos\left(\frac{\pi x}{a}\right)dx-\int_{0}^{\frac{a}{2}} x\cos\left(\frac{3\pi x}{a}\right)dx\right] \\
&= \frac{2}{a}\left[\left(x\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\Big|_0^{\frac{a}{2}}-\frac{a}{\pi}\int_0^{\frac{a}{2}}\sin\left(\frac{\pi x}{a}dx\right)\right)-\left(x\frac{a}{3\pi}\sin\left(\frac{3\pi x}{a}\right)\Big|_0^{\frac{a}{2}}-\frac{a}{3\pi}\int_0^{\frac{a}{2}}\sin\left(\frac{3\pi x}{a}dx\right)\right)\right] \\
&= \frac{2}{\pi}\left[\frac{a}{2}\sin\left(\frac{\pi}{2}\right)+\frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)\Big|_0^{\frac{a}{2}}-\frac{a}{6}\sin\left(\frac{3\pi}{2}\right)-\frac{a}{9\pi}\cos\left(\frac{3\pi x}{a}\right)\Big|_0^{\frac{a}{2}}\right] \\
&= \frac{2a}{\pi}\left[\frac{2}{3}-\frac{8}{9\pi}\right]\approx 0.2443a
\end{aligned}$$

$$P_2 = \boxed{\frac{\left(q_e\mathcal{E}_0\right)^2}{\hbar^2}\left(0.05968249a^2\right)\left|\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\left(\frac{t}{T}\right)^2+1}\right|^2}$$

$$\begin{aligned}
P_3 &= \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} e^{i\omega t}\langle\psi_3(n)|\hat{H}|\psi_1(n)\rangle\right|^2 \\
&= \frac{\left(q_e\mathcal{E}_0\right)^2}{\hbar^2}\left|\langle\psi_3(n)|x|\psi_1(n)\rangle\right|^2\left|\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\left(\frac{t}{T}\right)^2+1}\right|^2
\end{aligned}$$

$$\begin{aligned}
\langle\psi_3(n)|x|\psi_1(n)\rangle &= \int_{-\frac{a}{2}}^{\frac{a}{2}}\psi_3^*(n)\,x\psi_1(n)\,dx \\
&= \frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos\left(\frac{3\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)dx \\
&= 0,\ \text{because integrand is odd} \\
\rightarrow P_3 &= \boxed{0}
\end{aligned}$$

Problem 2

A quantum particle of mass m is bound in the *ground state* of the one-dimensional parabolic potential well $K_0 \frac{x^2}{2}$ until time $t = 0$. Between the time moments of $t = 0$ and $t = T$ the stiffness of the spring is ramped up as

$$K(t) = K_0 + \frac{t}{T}(K_1 - K_0), \quad 0 \le t \le T, \tag{3}$$

and it stays equal to K_1 afterwards, at $t > T$. The overall stiffness of the change is small: $|K_1 - K_0| \ll K_0$.

(a) Does any work need to be done to exercise this ramp-up? Can this work be dependent on the duration T of the ramp-up and why?

$$\begin{aligned}
W &= \frac{K(t)x^2}{2} - \frac{K_0 x^2}{2} \\
&= \frac{K_0 x^2}{2} + \frac{1}{2}\frac{t}{T}(K_1 - K_0)x^2 - \frac{K_0 x^2}{2} \\
&= \boxed{\frac{t}{2T}(K_1 - K_0)x^2}
\end{aligned}$$

Yes, this work is dependent on the duration T of the ramp-up, because the longer it takes to ramp-up, the higher the spring constant becomes, resulting in more work needing to be done.

(b) If the answer to the last question is positive, evaluated the needed work in the lowest-order of the perturbation theory.

$$\begin{aligned}
W &= E(t) - E(0) \\
&= \frac{\hbar}{2}\sqrt{\frac{K(t)}{m}} - \frac{\hbar}{2}\sqrt{\frac{K_0}{m}} \\
&= \frac{\hbar}{2}\sqrt{\frac{K_0 + \frac{t}{T}(K_1 - K_0)}{m}} - \frac{\hbar}{2}\sqrt{\frac{K_0}{m}} \\
&= \frac{\hbar}{2}\sqrt{\frac{K_0}{m}}\sqrt{1 + \frac{t}{T}\frac{K_1 - K_0}{K_0}} - \frac{\hbar}{2}\sqrt{\frac{K_0}{m}} \\
&= \frac{\hbar}{2}\sqrt{\frac{K_0}{m}}\left(1 + \frac{t}{2T}\left(\frac{K_1 - K_0}{K_0}\right) - 1\right) \\
&= \boxed{\frac{\hbar t}{4T}\sqrt{\frac{K_0}{m}}\left(\frac{K_1 - K_0}{K_0}\right)}
\end{aligned}$$

(c) Analyze the T-dependent part of the work, if any, and find out how much of the variation in the amount of work is achieved between the adiabatic $(T \to \infty)$ and nearly instantaneous $(T \to 0)$ limits?

The variation in the amount of work between the adiabatic and nearly instantaneous limits should be insignificant compared to the total amount of work.

Problem 3

An electron is confined to move in one dimension (coordinate x within a parabolic potential well $K\frac{x^2}{2}$. The electron occupies the ground state (state $n = 0$).

At some point in time, a uniform electric field $\mathcal{E}$ is suddenly (nearly instantaneously) applied to this system in the positive x-direction and remains at this value afterwards. The strength of the field is *not* limited to perturbatively small values.

(a) What is the probability P_0 that the electron will stay in the ground state of the system after the field is applied?

$$\begin{aligned}
P_n &= \frac{1}{\hbar^2}\left|\int_0^\infty \langle n|q_e\mathcal{E}\,(x)\,|0\rangle e^{i\omega t}dt\right|^2 \\
&= \frac{(q_e\mathcal{E})^2}{\hbar^2}\left|\langle n|x|0\rangle\right|^2\left|\int_0^\infty e^{i\omega t}dt\right|^2 \\
&= -\frac{(q_e\mathcal{E})^2}{\hbar}\frac{1}{2m\omega}\langle n|1\rangle^2\left(\omega e^{i\omega t}\right)^2
\end{aligned}$$

$$P_0 = \boxed{0}$$

(b) Can the electron be found in states other than the ground state? If so, evaluate probabilities P_1 and P_2 to find the electron in states $n = 1$ and $n = 2$, respectively.

$$\begin{aligned}
P_1 &= \boxed{\frac{(q_e\mathcal{E})^2}{\hbar}\frac{1}{2m\omega}\omega^2 e^{2i\omega t}} \\
P_2 &= \boxed{0}
\end{aligned}$$

(c) Of course, in reality it takes some time τ for the field to be turned on. Can you comment on the physical conditions related to justification of the assumption of a "nearly instantaneous" application in this specific case?

By the uncertainty principle, we know that $\Delta t \simeq \frac{\hbar}{\Delta E}$. This is the smallest time interval corresponding with a given energy change ΔE. Therefore, if $\tau \propto \Delta t$, then the "nearly instantaneous" application is justified.