PHYS 4301
Homework 3

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1. Time evolution of states in the infinite square potential well

$$\Psi(x,0) = \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right) \quad (1)$$

$$\Psi_n = \sqrt{\frac{2}{a}} sin\left(\frac{n\pi x}{a}\right) \quad (2)$$

$$E_n = n^2 \frac{\pi^2 \hbar^2}{2ma^2} \quad (3)$$

$$\hat{H}\Psi_n = E_n \Psi_n \quad (4)$$

a) Let us prepare our system at time $t = 0$ in a non-stationary state described by (1). Study the time evolution of the probability density $\left|\Psi(x,t)\right|^2$

$$\begin{aligned}
\Psi(x,t) &= e^{-\frac{i\hat{H}t}{\hbar}}\Psi(x,0) \\
&= \frac{1}{\sqrt{2}}\left(\psi_1(x)\, e^{-\frac{i\hat{H}t}{\hbar}} + \psi_2(x)\, e^{-\frac{i\hat{H}t}{\hbar}}\right) \\
&= \frac{1}{\sqrt{2}}\left(\psi_1(x)\, e^{-\frac{iE_1t}{\hbar}} + \psi_2(x)\, e^{-\frac{iE_2t}{\hbar}}\right) \\
\Psi^*(x,t) &= \frac{1}{\sqrt{2}}\left(\psi_1^*(x)\, e^{\frac{iE_1t}{\hbar}} + \psi_2^*(x)\, e^{\frac{iE_2t}{\hbar}}\right) \\
\left|\Psi(x,t)\right|^2 &= \Psi^*(x,t)\,\Psi(x,t) \\
&= \left[\frac{1}{\sqrt{2}}\left(\psi_1^*(x)\, e^{\frac{iE_1t}{\hbar}} + \psi_2^*(x)\, e^{\frac{iE_2t}{\hbar}}\right)\right]\left[\frac{1}{\sqrt{2}}\left(\psi_1(x)\, e^{-\frac{iE_1t}{\hbar}} + \psi_2(x)\, e^{-\frac{iE_2t}{\hbar}}\right)\right] \\
&= \frac{1}{2}\left(\Psi_1^*(x)\,\Psi_1(x) + \Psi_2^*(x)\,\Psi_2(x) + \Psi_1(x)\,\Psi_2^*(x)\, e^{\frac{i(E_1-E_2)t}{\hbar}} + \Psi_1^*(x)\,\Psi_2(x)\, e^{\frac{-i(E_1-E_2)t}{\hbar}}\right) \\
\Psi_1(x) &= \sqrt{\frac{2}{a}} sin\left(\frac{\pi x}{a}\right) \\
\Psi_2(x) &= \sqrt{\frac{2}{a}} sin\left(\frac{2\pi x}{a}\right)
\end{aligned}$$

Both Ψ_1 and Ψ_2 are real

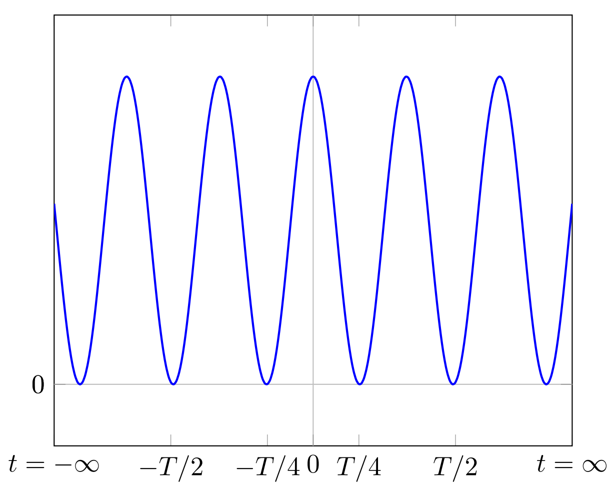
$$\left|\Psi(x,t)\right|^2 = \boxed{\frac{1}{2}\left(\left|\Psi_1(x)\right|^2 + \left|\Psi_2(x)\right|^2 + \Psi_1(x)\,\Psi_2^*(x)\, e^{\frac{i(E_1-E_2)t}{\hbar}} + \Psi_1^*(x)\,\Psi_2(x)\, e^{\frac{-i(E_1-E_2)t}{\hbar}}\right)}$$

The probability density **is** periodic:

$$\left|\Psi\left(x,t\right)\right|^2 = \frac{1}{2}\left(\Psi_1+\Psi_2\right)^2$$
$$t = \frac{\pi\hbar}{E_1-E_2} \to \Psi\left(x, \frac{\pi\hbar}{E_1-E_2}\right) = \frac{1}{2}\left(\Psi_1-\Psi_2\right)^2$$

$$\begin{aligned}\Psi\left(x,t\right) &= \frac{1}{\sqrt{2}}\left(\Psi_1\left(x\right)e^{-\frac{iE_1t}{\hbar}} + \Psi_2\left(x\right)e^{-\frac{iE_2t}{\hbar}}\right)\\ &= \frac{1}{\sqrt{2}}\left(\Psi_1\left(x\right) + \Psi_2\left(x\right)e^{-\frac{i\left(E_1-E_2\right)t}{\hbar}}\right) \text{ because } \Psi\left(x,t\right) = \Psi\left(x,t\right)e^{iE_1t/\hbar}\\ &= \frac{1}{\sqrt{2}}\left(\Psi_1\left(x\right)+\Psi_2\left(x\right)\right) = \Psi\left(x,0\right)\\ \text{So } T &= \boxed{\frac{2n\pi\hbar}{E_1-E_2}}\end{aligned}$$

$\Psi\left(x,t\right)$



b) Let us prepare our system in some arbitrary non-stationary state described by (5). Is $\Psi(x,t)$ periodic such that $\Psi(x,T) = \Psi(x,0)$ for some time T?

$$\Psi(x,0) = \sum_n c_n \psi_n(x) \tag{5}$$

$$\begin{aligned}
\Psi(x,0) &= \frac{1}{\sqrt{2}}\left(\Psi_n(x) + \Psi_m(x)\right) \\
\Psi(x,t) &= \frac{1}{\sqrt{2}}\left(\Psi_n(x)\, e^{-\frac{iE_n t}{\hbar}} + \Psi_m(x)\, e^{-\frac{iE_m t}{\hbar}}\right) \\
\Psi^*(x,t) &= \frac{1}{\sqrt{2}}\left(\Psi_n^*(x)\, e^{\frac{iE_n t}{\hbar}} + \Psi_m^*(x)\, e^{\frac{iE_m t}{\hbar}}\right), \Psi_n^* = \Psi_n, \Psi_m^* = \Psi_m \\
\Psi^*(x,t) &= \frac{1}{\sqrt{2}}\left(\Psi_n(x)\, e^{\frac{iE_n t}{\hbar}} + \Psi_m(x)\, e^{\frac{iE_m t}{\hbar}}\right) \\
\left|\Psi(x,t)\right|^2 &= \frac{1}{2}\left(\left|\Psi_1(x)\right|^2 + \left|\Psi_2(x)\right|^2 + \Psi_1(x)\,\Psi_2^*(x)\, e^{\frac{i(E_1-E_2)t}{\hbar}} + \Psi_1^*(x)\,\Psi_2(x)\, e^{\frac{-i(E_1-E_2)t}{\hbar}}\right) \\
&= \boxed{\frac{1}{\sqrt{2}}\left(\Psi_n(x) + \Psi_m(x)\right)} \\
T &= \frac{2\pi k\hbar}{E_n - E_m}, \ k \in \mathbb{Z}^+ \text{ from a)}
\end{aligned}$$

c) Comparing classical revival behavior to quantum revival behavior, at what energy E would these times be equal?

Classical revival time is the period of time it takes a classical particle to return to a position in a box of length l with velocity v, so $T_c = \frac{2l}{v}$. Therefore, the total kinetic energy is $E_c = \frac{mv^2}{2}$. Substituting, we get $T_c = \frac{2l}{v} = \frac{2l}{\sqrt{\frac{2E}{m}}} = 2l\sqrt{\frac{m}{2E}}$. With its energy given by the quantum mechanical energy of its state, we have $E = \frac{n^2\hbar^2}{8ml^2}$, so $T_c = 2l\sqrt{\frac{m}{2E}} = 2l\sqrt{\frac{m}{2\frac{n^2\hbar^2}{8ml^2}}} = \frac{4l^2 m}{n\hbar}$.

Consider the non-stationary state $\Psi(x,t) = c_n\psi_n(x)\, e^{-i_n t/\hbar} + c_m\psi_m(x)\, e^{-iE_m t/\hbar}$ with a quantum revival time of $T_q = \frac{\hbar}{E_n - E_m}$. $E_n - E_m = \frac{\hbar^2}{8ml^2}\left(n^2 - m^2\right)$. For T_q to equal T_c, $\frac{4l^2 m}{n\hbar} = \frac{8l^2 m}{\hbar(n^2 - m^2)} \rightarrow n^2 - m^2 = 2n$. So, if n is sufficiently large, $T_c = T_q$.

2. Degenerate states

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi \tag{6}$$
$$\psi(x\to\pm\infty)\to 0 \tag{7}$$

a) Prove that there are no degenerate bound states in an infinite ($-\infty<x<\infty$) one-dimensional space. That is, if $\psi_1(x)$ and $\psi_2(x)$ are two bound-state solutions of (6) for the same energy E, it will necessarily follow that $\psi_2=C\psi_1$, where C is just a constant. Bound-state solutions should of course vanish at $x\to\pm\infty$ (7), the mathematical statement essential for the proof.

Because $\psi_1(x)$, $\psi_2(x)$ are solutions of (6), we can write

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2}+V(x)\psi_1=E\psi_1$$
$$C\left(-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2}+V(x)\psi_1\right)=C(E\psi_1)\text{ similarly,}$$
$$-\frac{\hbar^2}{2m}\frac{d^2(\psi_2-C\psi_1)}{dx^2}+V(x)(\psi_2-C\psi_1)=E(\psi_2-C\psi_1)$$

Therefore, $\psi_2-C\psi_1$ is a solution with the same eigenvalue E. However, the question states that ψ_1 and ψ_2 are already solutions with an eigenvalue of E. Therefore, $\psi_2-C\psi_1=0\to\boxed{\psi_2=C\psi_1}$

b) Consider a ring path that our particle is confined to travel on with radius R. Find all stationary states and energy values for the particle on this ring in the absence of an external potential $\big(V(x)=0\big)$.

Let θ represent the coordinate of our particle with range $[0,2\pi]$.

$$-\frac{\hbar^2}{2m}\frac{1}{R^2}\frac{d^2}{2\theta^2}\psi(\theta)=E\psi(\theta)\text{, or}$$
$$\frac{d^2\psi(\theta)}{d\theta^2}+\frac{2mER^2}{\hbar}\psi(\theta)=0$$

Let $k^2=\frac{2mER^2}{\hbar^2}$, $\frac{d^2\psi(\theta)}{d\theta^2}+k^2\psi(\theta)=0$. The general solution for this form of differential equation is $\psi(\theta)=Ae^{ik\theta},-\infty<k<\infty$. Our boundary conditions are $\psi(\theta)=\psi(\theta+2\pi),\psi'(\theta)=\psi'(\theta+2\pi)$. The wave function is single-valued, so $Ae^{ik\theta}=Ae^{ik(\theta+2n\pi)},n\in\mathbb{Z}$. So, $e^{2\pi ik}=1,k\in\mathbb{Z}$.

For energy, $E=\frac{\hbar^2k^2}{2mR^2}=\boxed{\frac{n^2\hbar^2}{2mR^2}}$.