PHYS 4302

Exam 1

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Problem 1

In this problem you will have to decide which order of the time-independent perturbation theory to apply to find the first non-vanishing energy corrections to specified energy levels. The unperturbed system is a particle of mass m confined to the free motion on a circular ring of radius R that is situated in the xy-plane with the center at $x = y = 0$. We use angle ϕ of the polar coordinate system to specify position on that ring. The system is now perturbed by exposing the particle to the external field with potential energy

$$V(\phi) = v\cos(4\phi) \tag{1}$$

where positive constant v gives the potential magnitude and can be as small as needed to justify the application of the low-order perturbation theory.

(a) As a preliminary, write down the unperturbed Hamiltonian $\hat{H}_0(\phi)$ and the operator $\hat{L}_z(\phi)$ of the z-component of particle's orbital angular momentum. Also specify the wave functions $\psi^{(0)}(\phi)$ and energies $E^{(0)}$ of all unperturbed stationary states with definite values of L_z.

$$\begin{aligned}\hat{H}_0(\psi) &= \frac{\hat{L}_z^2}{2mR^2}\\ \hat{L}_z &= -i\hbar\frac{\partial}{\partial\phi}\end{aligned}$$

$$\begin{aligned}\psi_j^{(0)}(\phi) &= \frac{1}{\sqrt{2\pi}}e^{ij\phi}\\ E_j^{(0)} &= \frac{\hbar^2 j^2}{2mR^2}\end{aligned}$$

where j is an eigenvalue of $\hat{L}_z$.

(b) Consider the unperturbed stationary state with the eigenvalue of L_z equal to $\hbar$ and the corresponding energy level. What happens to them due to perturbation (1)? Find the resulting stationary states and their energies.

$$\begin{aligned}E_1^{(0)} &= \frac{\hbar^2}{2mR^2}\\ E_1^{(1)} &= \langle\psi_1(0)|V|\psi_1(0)\rangle\\ &= \int \pi v\cos(4\phi)\cos(\phi)\,d\phi = \frac{v}{2}\int \pi\left(cos(3\phi) + \cos(5\phi)\right)d\phi\\ &= 0\end{aligned}$$

(c) Consider the unperturbed stationary state with the eigenvalue of L_z equal to $2\hbar$ and the corresponding energy level. What happens to them due to perturbation (1)? Find the resulting stationary states and their energies.

$$
\begin{aligned}
E_1^{(2)} &= \sum_{n\neq 1} \frac{\langle \psi_n^{(0)} | V | \psi_1^{(0)} \rangle^2}{E_1^{(0)} - E_n^{(0)}} \\
&= \frac{v^2}{2} \sum_{n\neq 1} \frac{\langle \psi_n^{(0)} | \cos(4\phi) | \psi_1^{(0)} \rangle^2}{E_1^{(0)} - E_n^{(0)}} \\
&= \frac{v^2}{2} \left(\frac{\langle \psi_0^{(0)} | \cos(4\phi) | \psi_1^{(0)} \rangle^2}{E_1^{(0)} - E_0^{(0)}} + \frac{\langle \psi_2^{(0)} | \cos(4\phi) | \psi_1^{(0)} \rangle^2}{E_1^{(0)} - E_2^{(0)}} \right) \\
&= \frac{v^2}{4} \left(\frac{1}{E_1^{(0)} - E_0^{(0)}} + \frac{1}{E_1^{(0)} - E_2^{(0)}} \right) \\
&= -\frac{v^2}{2} \frac{1}{E_1^{(0)}} \\
&= -\frac{v^2}{2} \frac{\hbar^2}{mR^2}
\end{aligned}
$$

(d) Is L_z a conserved quantity in the perturbed system? Prove your answer. Are the results in questions (b) and (c) consistent with this answer?

$$
\begin{aligned}
\frac{\partial L_z}{\partial t} &= \left[L_z, \hat{H}_0 \right] \\
&= -\frac{i\hbar}{2mR^2} \left[\frac{\partial}{\partial \phi}, \left(-i\hbar \frac{\partial}{\partial \phi} \right)^2 \right] \\
&= -\frac{i\hbar^3}{2mR^2} \left[\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} \right] \\
&= -\frac{i\hbar^3}{2mR^2} \left(\left[\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \right] \frac{\partial}{\partial \phi} + \left[\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \right] \frac{\partial}{\partial \phi} \right) \\
&= 0,
\end{aligned}
$$

therefore L_z is conserved. The results of (b) and (c) are consistent with this answer.

Problem 2

The Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{m\omega^2 (x - x_0)^2}{2} + \frac{m\omega^2 (y - y_0)^2}{2}, \tag{2}$$

describes a 2D isotropic harmonic oscillator, where oscillations take place around the point with planar coordinates (x_0, y_0).

This system was in its ground state oscillating around point $(0, 0)$ until time $t = 0$, when somebody disturbed it by moving the center of the harmonic potential to the new position (b, b) at time $t = 2\pi$, after which this motion was stopped. The move was executed "along the edges of the square" as described by the following time-dependent coordinates of the center of the harmonic potential well:

$$x_0(t) = \begin{cases} 0, & t < 0 \\ \frac{bt}{\tau}, & 0 \le t \le \tau \\ b, & t > \tau \end{cases} \qquad y_0(t) = \begin{cases} 0, & t < \tau \\ \frac{b(t-\tau)}{\tau}, & \tau \le t \le 2\tau \\ b, & t > 2\tau \end{cases} \tag{3}$$

The magnitude b of the displacements is small enough to limit the following analysis to the *first-order* time-dependent perturbation theory.

(a) What is the probability P that, after the perturbation is over, the system would be found *not* in its ground state? If $P \neq 0$, which other (excited) states could the system be found in? Specify the corresponding transition probabilities.

$$P_{n\neq 0} = 1 - \frac{1}{\hbar^2}\left|\int_0^\infty e^{i\omega t}\langle n|x_0(t) + y_0(t)|0\rangle dt\right|^2 = 1$$

(b) How much work W was done on the system during the displacement of the center of oscillations?

We can find the work W by computing the energy at $t = 0$ and $t = 2\tau$.

$$\begin{aligned} E_0 &= \frac{m\omega^2}{2}\left(x^2 + y^2\right) \\ E_{2\tau} &= \frac{m\omega^2}{2}\left((x-b)^2 + (y-b)^2\right) \end{aligned}$$

The total work done is therefore $E_{2\tau} - E_0 = m\omega^2\left(b^2 - bx - by\right) = W$.

Problem 3

A system of two spin-$\frac{1}{2}$ particles is situated in some anisotropic medium so that the resulting interaction of the spin-oriented magnetic moments is described by the Hamiltonian

$$\hat{H} = A_x S_{1x} S_{2x} + A_y S_{1y} S_{2y} + A_z S_{1z} S_{2z}, \tag{4}$$

where A_x, A_y, A_z are given coefficients of appropriate dimensions, while S_{1i} and S_{2i} are the Cartesian components ($i = x$, y or z) of the particle spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively.

(a) From the homework, you already know that for the isotropic Hamiltonian ($A_x = A_y = A_z$ in (4)), the total spin of the system $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ would be conserved. Is the total spin generally conserved in our system? In particular, what would be the necessary condition on coefficients A_i for the x-component S_x of the total spin to be conserved?

No, the total spin is not generally conserved in our system. For an anisotropic medium, $\hat{H}$ is not symmetric about the origin, so total spin isn't conserved.

We can determine the necessary conditions on the coefficients A_i for the x-component S_x of the total spin to be conserved by differentiating S_x:

$$\begin{aligned}
i\hbar \frac{\partial S_x}{\partial t} &= \left[S_x, \hat{H}\right] \\
&= A_x \left[S_x, S_{1x} S_{2x}\right] + A_y \left[S_x, S_{1y} S_{2y}\right] + A_z \left[S_x, S_{1z} S_{2z}\right] \\
&= A_x \left(\left[S_x, S_{1x}\right] S_{2x} + S_{1x}\left[S_x, S_{2x}\right]\right) + A_y \left(\left[S_x, S_{1y}\right] S_{2y} + S_{1y}\left[S_x, S_{2y}\right]\right) + A_z \left(\left[S_x, S_{1z}\right] S_{2z} + S_{1z}\left[S_x, S_{2z}\right]\right) \\
&= A_x \left(\left[S_{1x} + S_{2x}, S_{1x}\right] S_{2x} + S_{1x}\left[S_{1x} + S_{2x}, S_{2x}\right]\right) \\
&\quad + A_y \left(\left[S_{1x} + S_{2x}, S_{1y}\right] S_{2y} + S_{1y}\left[S_{1x} + S_{2x}, S_{2y}\right]\right) \\
&\quad + A_z \left(\left[S_{1x} + S_{2x}, S_{1z}\right] S_{2z} + S_{1z}\left[S_{1x} + S_{2x}, S_{2z}\right]\right) \\
&= A_x \left(\left(\left[S_{1x}, S_{1x}\right] + \left[S_{2x}, S_{1x}\right]\right) S_{2x} + S_{1x}\left(\left[S_{1x}, S_{2x}\right] + \left[S_{2x}, S_{2x}\right]\right)\right) \\
&\quad + A_y \left(\left(\left[S_{1x}, S_{1y}\right] + \left[S_{2x}, S_{1y}\right]\right) S_{2y} + S_{1y}\left(\left[S_{1x}, S_{2y}\right] + \left[S_{2x}, S_{2y}\right]\right)\right) \\
&\quad + A_z \left(\left(\left[S_{1x}, S_{1z}\right] + \left[S_{2x}, S_{1z}\right]\right) S_{2z} + S_{1z}\left(\left[S_{1x}, S_{2z}\right] + \left[S_{2x}, S_{2z}\right]\right)\right) \\
&= \left(A_z - A_y\right) S_y S_z
\end{aligned}$$

Therefore, S_x is conserved when $A_y = A_z$.

(b) Find the energies of *all* stationary states of this system.

To find the energies of the stationary states of the system, we first need the eigenvalues of the system Hamiltonian:

$$E_\lambda = A_x S_x^2 + A_y S_y^2 + A_z S_z^2$$

(c) Do your results for the eigen energies satisfy the correct limiting behavior (both in magnitude and degeneracy) towards the isotropic case?
