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PANCAKE NETWORKS AND PANCAKE PROBLEMS

by

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DISSERTATION

Presented to the Faculty of

The University of Texas at Dallas

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy in Computer Science

THE UNIVERSITY OF TEXAS AT DALLAS

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PANCAKE NETWORKS AND PANCAKE PROBLEMS

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I dedicate this work to my very special daughters, Beth and Claire,

and to my loving family whose support has been unflagging.

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Pancake Networks and Pancake Problems

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Low dilation embeddings are used to compare the computational capabilities of star and pancake networks. Comparisons with the burnt pancake and cycle prefix networks are also discussed. Among the embeddings presented are a one-to-many dilation one embedding of the pancake network of dimension n into the star network of dimension 2n; a one-to-many dilation one embedding of the star network of dimension n into the pancake network of dimension $(n^3 - 4n^2 + 5n + 4)/2$; a one-to-many dilation two embedding of the star network of dimension n into the pancake network of dimension 2n-2; and a one-to-many dilation three embedding of the pancake network of dimension n into the star network of dimension 2n-1.

The cycle structure of the pancake network and the burnt pancake network are investigated. In particular, it is proved that the pancake network of dimension n

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has cycles of size i, for all i ($6 \leq i \leq n!$), and the burnt pancake network of dimension n has cycles of size i, for all i ($8 \leq i \leq 2^n n!$).

The Deterministic Pancake Problem asks for the maximum number of steps, for all permutations on n symbols, for the symbol 1 to be placed in the first position. This is accomplished through prefix reversals whose size is determined by the integer in position one. The Deterministic Pancake Problem is also known by the names "Reverse Card Shuffle" and "Topswaps". A quadratic lower bound is exhibited, disproving an earlier conjecture of a linear lower bound.

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Chapter 1

Introduction

A multitude of scientific problems can be modeled by permutations. For example, in computer science, message routing in interconnection networks and network design for efficient parallel computation are often recast as problems about permutations [1,5,8,13,14,19,20,27,28,29,31]. In phylogeny, attempts to reconstruct the "tree of life" have been facilitated by comparing gene sequences of separate species and determining the minimum number of mutation operations needed to transform one sequence to another. That is, genomes of separate species can often be viewed as permutations on a common set of genes, with differences accounted for by inversions and transpositions of various subsequences of the genes [2,3,17,18,21-23]. The number of such inversions and transpositions is viewed as a good approximation for the distance between species and thereby suggests a method to reconstruct their relative positions in the tree of life. Finally, the everyday activity of sorting can be viewed as rearranging an arbitrary permutation of input values into some pre-specified order [4,11,12,15,19,31].

Parallel computer architectures are of great interest to the research and industrial communities. Many interconnection topologies, such as the hypercube, the butterfly, and the mesh are commercially available. Networks are often judged by comparing their diameter and maximum node degree as a function of the total number of processors. The distance between a pair of nodes (*i.e.* processors) is defined to be the length of the shortest path joining these nodes. The diameter of a network is the maximum distance between any pair of processors in a network. We define the degree of a node to be number of connections, or edges, incident to

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it. The degree of a network is the largest number of connections to any node in the network.

A large variety of interconnection strategies have been defined in the literature. See, for example, [1,13,25,29]. Processors can be arranged in meshes, pyramids, hypercubes, and trees, and also in less familiar structures such as meshes-of-trees, butterflies, pancake networks, star networks, cycle prefix networks, shuffle exchange networks and cube connected cycles. Formal definitions for each of these networks can be found in [29].

An informal description of the properties of some of these networks provides an intuitive understanding of ways in which networks can be compared. A hypercube of dimension n has 2^n processors, diameter n and degree n. Both the degree and diameter are logarithmic in the number of processors. Each processor in the hypercube of dimension n is labeled with a unique n-bit binary string. Two processors are adjacent if their labels differ in exactly one bit. If they differ in the i^{th} bit, then they are i-dimensional neighbors.

The cube connected cycle network is a variant of the hypercube in which each hypercube node is replaced with a cycle of length n. In each such cycle, the nodes are labeled with integers in increasing order around the cycle. For a pair of cycles corresponding to i-dimensional neighbors in the underlying hypercube, the i^{th} node in one cycle is connected to the i^{th} node in the neighboring cycle. The cube connected cycle network of dimension n has $n2^n$ processors, diameter $2n + \lfloor n/2 \rfloor$ and degree 3. Observe that the structural modification to the hypercube created by introducing these cycles to obtain the cube connected cycle network makes the node degree constant (regardless of the number of processors), but makes the diameter is greater.

Another seemingly related network is the shuffle exchange network of dimension n. This network also has 2^n processors, each labeled with a unique

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n-bit binary string. Two nodes with labels $a\alpha$ and αa are joined by a shuffle edge where $a \in \{0,1\}$ and α is a binary string of length n-1. Two nodes with labels αa and αb are joined by an exchange edge where $a,b \in \{0,1\}$, $b \neq a$, and α is a binary string of length n-1. The shuffle exchange network has diameter 2n-1 and degree 3. Like the cube connected cycle network, this network has constant degree, but a larger diameter than the hypercube of the same size.

In this thesis, we are particularly interested in networks whose nodes are labeled by permutations on n symbols, and whose edges are described by transformations of the permutations. Informally, the pancake network of dimension n has n! processors, each labeled with a unique permutation on n symbols. Two processors are adjacent if the label of one processor can be obtained from the other by a prefix reversal of size i, $(2 \leq i \leq n)$. The network has degree n-1, and its diameter has been shown to be between $\frac{15}{14}n$ and $\frac{5}{3}n$ [15,19]. The star network of dimension n also has n! processors, each labeled with a unique permutation on n symbols. Two processors in the star network are adjacent if the label of one processor can be obtained from the other by a transposition that exchanges an arbitrary symbol with the symbol in the first position. The star network of dimension n has degree n-1, and its diameter is $\left\lfloor \frac{3(n-1)}{2} \right\rfloor$ [1]. Both the star and the pancake network have degree and diameter sublogarithmic in the number of processors, hence, by these criteria, they compare favorably with a hypercube of similar size.

With such a rich collection of interesting topologies, one naturally wishes to compare the computing power of processor networks based on these interconnection strategies. Many interesting and difficult questions arise. For example, can algorithms for one network run efficiently on another? If there are differences in efficiency, which structural properties account for the differences?

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Networks can be compared by describing how to simulate one by another, that is, by defining embeddings or mappings of these networks into each other. In any embedding of a network G into an network H, adjacent nodes in G may need to be mapped to nodes that are not adjacent in H. Informally, dilation (of an embedding) measures the distance in H between the images of nodes that are adjacent in G. The dilation is d if images of adjacent nodes are at distance at most d in H. By means of embeddings, simulations can be quantified with measurable parameters such as dilation and congestion. These and other related concepts are defined formally in the next section.

The subject of this thesis is two specific areas in the pantheon of permutation problems. The first area concerns interconnection networks defined as Cayley graphs on the symmetric group of order n, *i.e.*, the group of all permutations on n objects. In general, Cayley graphs are defined by a group $\mathcal{G}$ and a set Γ of generators for the group. A set of generators for a group is a subset of group elements, say $\Gamma=\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$, such that every group element x can be expressed as a product of elements of the set Γ. Cayley graphs have a node for each element in the group $\mathcal{G}$, and an edge between nodes x and y when there is a generator γ in Γ such that $y = \gamma \bullet x$. Formal definitions are given in the next section.

For example, the n-dimensional hypercube, Q_n, is a Cayley graph. An illustration of Q_3, the 3-dimensional hypercube, appears in Figure 1.1.

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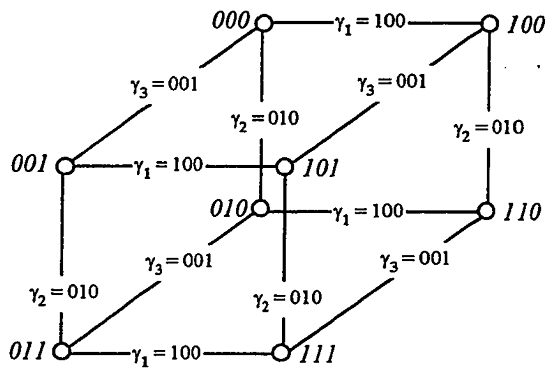

Figure 1.1. Q_3, the hypercube of dimension 3, is a Cayley graph.

Q_3 has a node for each binary string of length three, and an edge between two nodes x and y if x and y differ in exactly one bit. Let $\oplus_3$ denote bitwise modulo 2 addition of binary strings of length 3. The set of all binary strings of length 3, denoted by Z_2^3, forms a group with the operation $\oplus_3$. Let $(Z_2^3, \oplus_3)$ be this group, and let Γ be the set $\{\gamma_1 = 100, \gamma_2 = 010, \gamma_3 = 001\}$. Observe that every binary string of length 3 can be expressed as a bitwise modulo 2 sum of strings in Γ. That is, Γ is a set of generators for the group $(Z_2^3, \oplus_3)$. Hence, in Cayley graph terminology, Q_3 is the Cayley graph $G((Z_2^3, \oplus_3), \Gamma)$ where the nodes are labeled by the elements of the group $(Z_2^3, \oplus_3)$ and the each edge is labeled with a generator from Γ. For instance, the nodes x = 101 and y = 111 are adjacent because $y = \gamma_2 \oplus_3 x$, *i.e.*, $111 = 010 \oplus_3 101$.

We are specifically interested in Cayley graphs on the symmetric group of order n. The symmetric group has many sets of generators. One such set is the set of permutations corresponding to all possible prefix reversals. The resulting

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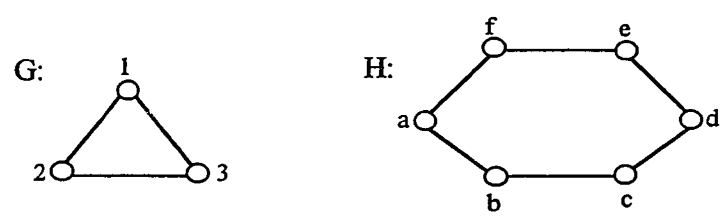
Cayley graph is the interconnection network known as the *pancake network,* which has been a frequent subject of investigation [1,8,11,12,15,19]. We study the computing power of pancake networks by comparing their ability to simulate other common networks of parallel processors. Our simulation studies are accomplished by describing embeddings (*i.e.*, mappings) of guest processors into host processors. We consider traditional one-to-one embeddings, as well as the more general one-to-many embeddings. In one-to-many embeddings, each guest processor is mapped to a disjoint subset of host processors.

An example of a straightforward one-to-many embedding is given in Figure 1.2. The triangle on the left is the guest network, G, and the hexagon is the host network, H. The embedding has dilation one. Note that there is no one-to-one dilation one embedding of G into H. In this and many other cases, dilation one is impossible to achieve with a one-to-one embedding. However, in the same instances, one-to-many dilation one embeddings are often possible.



guest node	image in host
1	{a,d}
2	{c,f}
3	{e,b}

Figure 1.2. A one-to-many dilation one embedding of G into H.

Let f be an embedding and let x be a processor in the guest network. A processor x is simulated by each processor in its image f(x) in the host network.

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That is, every host processor in f(x) does the computations assigned to x. The dilation of an embedding f is the maximum distance between each processor in f(x) to a nearest processor in f(y), where this maximum is taken over all neighboring processors x and y in the guest network. Hence, the dilation of an embedding is related to the amount of slowdown in a simulation due to differences in topology. That is, if the dilation is d, then the worst case slowdown factor is d, as routing messages may take up to d times as many steps. Of course, slowdown is also affected by edge congestion, hence, for a given embedding, routings with high edge congestion experience greater slowdown than routings with low edge congestion. In general, however, the lower the dilation of an embedding, the better the simulation of the guest network by the host.

We describe the ability of the pancake network to simulate the *star network* [1]. The n-dimensional star network is the Cayley graph defined by the symmetric group and the set of generators consisting of all transpositions that exchange an arbitrary symbol with the symbol in the first position. The study is conducted in a precise mathematical manner by describing embeddings of star networks into pancake networks and the dilation of these embeddings. That is, we consider mappings of star networks into pancake networks that minimize the maximum distance in the pancake network between images of adjacent star network processors.

We also consider interconnection networks defined as Cayley graphs on extensions of the symmetric group or with other sets of generators. For example, we consider the *burnt pancake network* [11,15,19] and the *cycle prefix network* [13]. The n-dimensional burnt pancake network is closely related to the n-dimensional pancake network. Its edges represent prefix reversals as do the edges in the pancake network, however, each symbol in the burnt pancake network has an associated sign that is changed when the symbol is part of a prefix

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reversal. The cycle prefix network is the Cayley graph defined by the symmetric group and the set of generators consisting of all permutations that make a cyclic shift of some prefix. We compare the relative computing power of parallel machines based on these alternative interconnection strategies with the pancake network. Again, the study is accomplished in a precise mathematical manner by examining embeddings of burnt pancake networks and cycle prefix networks into pancake networks, and vice-versa.

A second area of study is a specific problem about permutations with some history in the field of combinatorics [31]. The topic is the *Deterministic Pancake Problem* (also known as *Reverse Card Shuffle* and *Topswaps* [4]). This problem was described by D. Berman and M. S. Klamkin [4] as follows:

> *A deck of cards is numbered 1 to n in random order. Perform the following operations on the deck. Whatever the number on the top card is, count down that many in the deck and turn the whole block over on top of the remaining cards. Then, whatever the number of the (new) top card, count down that many cards in the deck and turn this whole block over on top of the remaining cards. Repeat the process. Show that the number 1 will eventually reach the top.*

(We will refer to this process as the *deterministic pancake process.*) A deck of n cards is modeled by a permutation on the set of n natural numbers {1,...,n}, and the process of turning over the top block of i cards is simulated by a prefix reversal of size i. For example, a deck of 7 cards arranged in the order 3, 6, 5, 2, 4, 7, 1 is modeled by the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 5 & 2 & 4 & 7 & 1 \end{pmatrix}$.

We write π as the string 3652471 for short. The number on the top card is 3, so according to the deterministic pancake process, we flip over the top three cards,

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bringing the card numbered 5 to the top. We simulate this move by applying a prefix reversal of size 3 to π, yielding the string (*i.e.*, permutation) 5632471. Now the number on the top card is 5, so we flip over the top 5 cards, bringing the card numbered 4 to the top. We simulate this move by applying a prefix reversal of size 5 to the previous permutation, yielding the permutation 4236571. Continuing in this fashion, the deterministic pancake process eventually terminates when the card labeled 1 is brought to the top. Our simulation of the deterministic pancake process, when applied to the string 3652471 is represented by the sequence $3652471 \mapsto 5632471 \mapsto 4236571 \mapsto 6324571 \mapsto 7543261 \mapsto 1623457$. Initially, it was not even clear that the process terminated for every permutation π. In 1974, Knuth showed that the process terminates for every permutation and gave an exponential upper bound on the maximum number of steps needed to terminate [4].

For a given permutation π on $\{1, ..., n\}$ representing the initial order of the cards, let *run*(π) denote the sequence of permutations obtained by the deterministic pancake process. Let $|run(\pi)|$ denote the length of this sequence of permutations, and define h(n) by:

$$h(n) = \max \{ |\mathrm{run}(\pi)| \mid \pi \text{ is a permutation of } \{1, ... ,n\}\}.$$

Berman and Klamkin describe two specific problems:

(1) Determine the asymptotic behavior of the function h(n), and,

(2) For any n and any permutation π on $\{1, ..., n\}$ that takes the maximum number of steps (so that $|\mathrm{run}(\pi)| = h(n)$), show that the sequence $\mathrm{run}(\pi)$ terminates with the identity permutation, *i.e.*, the process terminates with a sorted sequence of integers. (The permutations π on $\{1, ...,n\}$ such that $|\mathrm{run}(\pi)| = h(n)$ are the *long-winded permutations*. Thus, the problem asks to show that all long-winded permutations terminate in sorted order.)

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D. E. Knuth noted that the problem was shown to him in 1973 by J. H. Conway [4], who proposed it and named it *Topswaps*. Knuth included part of problem (1) on a take-home exam, asking that the students show that h(n) is bounded above by the $(n+1)^{st}$ Fibonacci number. (In particular, this shows that the process always stops and that there is an exponential upper bound on h(n)). Knuth described a proof of this upper bound in his solution to the students and conjectured that, in fact, the function h(n) has a linear upper bound. We show that Knuth's conjecture is false. In particular, we describe infinite families of permutations such that, for a given constant c>0, and for arbitrarily large n, these families contain permutations π_n on $\{1, ..., n\}$ such that $|run(\pi_n)| > cn^2$. That is, we show that h(n) is $\Omega(n^2)$.

Berman and Klamkin also computed exact values of h(n) for all $n \leq 9$. We describe exact values of h(n) for all $n \leq 15$ and give (surprisingly) large values for h(n), for $16 \leq n \leq 30$. That is, for all n, $16 \leq n \leq 30$, we describe permutations that take a surprisingly large number of steps, although this number may not be the maximum number of steps. Our work also shows that Berman and Klamkin's conjecture, stated in (2) above, is not true. That is, for n=12 and 15, the long-winded permutations do not end with the identity permutation. (In fact, for n=6, some long-winded permutations terminate in sorted order and others do not.)

The Deterministic Pancake Problem is closely related to the Pancake Problem [12] which asks for the number of prefix reversals required to sort permutations on n symbols, for each positive integer n. Let this number be represented by r(n), for each n. Gates and Papadimitriou [15] show that

$$\frac{17}{16}n \leq r(n) \leq \frac{5}{3}n.$$

Furthermore, if one considers permutations of signed integers with the property that a prefix reversal changes the integers' signs as well as their relative ordering,

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then one has the so-called Burnt Pancake Problem [15]. Let s(n) be the number of steps required to sort permutations of signed integers, for each positive integer n. Gates and Papadimitriou [15] and Cohen and Blum [11] show that

$$\frac{n}{2} \leq s(n) \leq 2n-3.$$

Cohen and Blum [11] conjecture that the particular sequence $-I_n$ of signed integers given by -1,-2,....,-n is hardest to sort, for all n. They describe a $\frac{23}{14}n$ upper bound for sorting $-I_n$ and describe exact values of s(n) for $n \leq 10$. Heydari and Sudborough [19] improve the upper bound for sorting $-I_n$ to $\frac{3(n+1)}{2}$, describe exact values of r(n) for $n \leq 13$, and improve the lower bound on r(n) to $\frac{15}{14}n$.

As the name suggests, the Deterministic Pancake Problem can be considered a deterministic form of the Pancake Problem. That is, in the Deterministic Pancake Problem the size of the prefix reversal is determined by the first integer in the permutation. It is interesting to note that the deterministic version of this sorting problem is $\Omega(n^2)$, as we show, whereas the nondeterministic version is O(n).

The Deterministic Pancake Problem also represents a simple (but, as we see, inefficient) routing strategy in the pancake network. Routing represents the process of choosing a path from an arbitrary processor with label α to another processor with label β. It is sufficient to view routing as the process of sending a message from an arbitrary permutation to the identity, as a routing from α to β is equivalent to a routing from $\alpha\beta^{-1}$ to the identity permutation. To see this, consider a pair of arbitrary processors in the pancake network, say those labeled with permutations α and β. To route a message from α to β means choosing a

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path, say $\alpha_0, \alpha_1, \alpha_2, ..., \alpha_p$ for some p from α to β, *i.e.*, $\alpha_0 = \alpha$ and $\alpha_p = \beta$. Each edge $\{\alpha_i, \alpha_{i+1}\}$ along this path corresponds to some prefix reversal and the composition of these p prefix reversals can be expressed as a permutation γ. Thus, one can express a path from α to β as a permutation γ such that $\gamma \circ \alpha = \beta$. Algebraically, this means that $\gamma = \beta \circ \alpha^{-1}$ and $\gamma^{-1} = \alpha \circ \beta^{-1}$. Conversely, this means that finding a path γ from α to β is equivalent to determining a way to express $\gamma = \beta \circ \alpha^{-1}$ as a product of prefix reversals. And it follows that such a product of prefix reversals must express a route from $\gamma^{-1} = \alpha \circ \beta^{-1}$ to the identity permutation e, because by group properties, $\gamma \circ \gamma^{-1} = e$. In summary, the problem of finding a route between arbitrary locations α and β is equivalent to finding a route from $\alpha \circ \beta^{-1}$ to the identity permutation.

A route from a permutation $\gamma = \beta \circ \alpha^{-1}$ to e can be viewed as a way to sort the permutation γ^{-1} by prefix reversals. The problem of finding the shortest sorting sequence using prefix reversals is precisely the Pancake Problem. The Deterministic Pancake Problem is a possible deterministic sorting strategy. That is, given a permutation expressed (for instance) as a string γ, one may use the deterministic strategy of choosing the prefix reversal whose size is given by the first integer in the string γ. At each step in the sorting process, the size of the prefix reversal is determined by the first integer in the permutation γ. The procedure stops when the number 1 is brought to the front of the permutation. If at this point the sequence is sorted, then the sorting is done. If the sequence is not sorted, then continue by recursively performing the deterministic process on the unsorted suffix with the initial sorted portion (conceptually) removed. For example, consider the permutation on $\{1,...,7\}$ expressed by the string 3124765. The initial steps of the deterministic strategy are 124765 $\mapsto$ 2134765 $\mapsto$ 1234765. At this point, the number 1 is in front, but the sequence is not sorted. So we

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continue recursively by sorting the unsorted suffix 765 and leaving the sorted prefix 1234 untouched. To do this, we renumber the suffix by subtracting the length of the sorted prefix from each symbol in the suffix. In this example, we would subtract the number 4. Thus, 765 becomes 321 and therefore, as its first element is 3, a prefix reversal of size 3 (of this suffix) results in a sorted sequence. That is, the entire sorting sequence is 124765 $\mapsto$2134765 $\mapsto$1234765 $\mapsto$1234567. Observe that an arbitrary substring reversal corresponds to three steps in the pancake network. That is, a substring reversal operation, say xyz $\mapsto$ xy^Rz, can be accomplished by three prefix reversals: xyz $\mapsto$ y^Rx^Rz $\mapsto$ yx^Rz $\mapsto$ xy^Rz. In this way the deterministic strategy can be viewed as an algorithm for routing messages in the pancake network.

However, how good is this routing strategy? If one were to combine the conjecture of Knuth and the conjecture of Berman and Klamkin, one might conjecture that the long-winded (or hardest) permutations on n symbols terminate with the identity *e* and involve O(n) steps. So, for hardest permutations, there would be no need for the recursive process in the algorithm because when 1 comes to the front the sequence is sorted. Hence, the length of the sorting procedure would be linear in the length of the permutation. Were these conjectures true, the deterministic strategy might be reasonable for routing. However, as we shall see, both conjectures are false. The number of steps needed for 1 to come to the front can be quadratic in the size of the permutation and there are examples of integers n where no long-winded permutation on n objects terminates with the identity.

On the other hand, there are reasonable deterministic routing algorithms for the pancake network. Some of these are described in [15] and [19]. These algorithms are based on a different strategy. That is, they are based on techniques

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to achieve adjacencies as quickly as possible, where an adjacency is formed when consecutive integers are made contiguous.

1.1 Definitions

We begin with some basic definitions. A *graph*, G, is a pair (V,E), where V is a set of distinct *vertices* or *nodes*, and E is a set of *edges*. In the case of a *directed graph*, $E \subseteq V \times V$ is a set of ordered pairs of vertices. For an *undirected graph,* E is a set of two element subsets of V. In this thesis, we use the term *network* synonymously with the term undirected graph. A *path* is a sequence of vertices $v_1,\ldots,v_i,\ldots, v_r$, such that for all i, $1 \le i \le r-1$, (v_i, v_{i+1}), if G is a directed graph,(or $\{v_i, v_{i+1}\}$ if G is an undirected graph) is an edge in E. The *distance* between two vertices v_i and v_j, denoted by *distance*(v_i, v_j), is the length of the shortest path between v_i and v_j. The *diameter* of a graph G, denoted by *diameter*(G) is defined by:

$$\text{diameter}(G) = \max_{v_i, v_j \in G} \left\{ \text{distance}(v_i, v_j) \right\}.$$

A *group* is a finite set $\mathcal{G}$ paired with a binary operation • such that the following *group axioms* hold:

i. the set is closed under the operation •,
ii. the operation • is associative,
iii. there is an element *e* in $\mathcal{G}$, called the *identity*, such that for all elements x in $\mathcal{G}$, $e \cdot x = x \cdot e = x$,
iv. every element has an *inverse*, *i.e.*, for every element x in $\mathcal{G}$, there is an element x^{-1} in $\mathcal{G}$ such that $x^{-1} \cdot x = x \cdot x^{-1} = e$.

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The *order* of a group is the cardinality of the set $\mathcal{G}$. When the binary operation is clear from context, we follow the common practice of referring to $\mathcal{G}$ as the group, with the operation implicitly understood. The binary operation • is often called the *group operation* or the *multiplication operation,* and the result of its application, a *product.* A set of *generators* for a group $\mathcal{G}$ is a subset Γ of $\mathcal{G}$, such that every element of $\mathcal{G}$ can be expressed as a product of elements of Γ.

Let X be a set of n distinct objects. A *permutation* is a one-to-one function that maps X onto itself. Let Σ be the set of all permutations on the set X. The *symmetric group of dimension n,* denoted by Σ_n, is the group $(\Sigma, \circ)$, where $\circ$ denotes function composition. The *dimension* of the symmetric group is the number of elements in the set X. Unless otherwise noted, we shall assume that permutations on n objects are permutations on the symbols $\{1,\ldots,n\}$.

There are many distinct sets of generators for Σ_n. For example:

$$A_n = \left\{ \begin{pmatrix} i & j \end{pmatrix} \middle| 1 \le i < j \le n \right\},$$

$$B_n = \left\{ \begin{pmatrix} 1 & j \end{pmatrix} \middle| j \ge 2 \right\},$$

$$C_n = \left\{ c_{ij} = \begin{pmatrix} 1 & 2 & \ldots & i-1 & i & i+1 & \ldots & j-1 & j & j+1 & \ldots & n \\ 1 & 2 & \ldots & i-1 & j & j-1 & \ldots & i+1 & i & j+1 & \ldots & n \end{pmatrix} \middle| 1 \le i < j \le n \right\},$$

$$D_n = \left\{ d_j = \begin{pmatrix} 1 & 2 & \ldots & j & j+1 & \ldots & n \\ j & j-1 & \ldots & 1 & j+1 & \ldots & n \end{pmatrix} \middle| j \ge 1 \right\}, \; (D_n' = D_n - \{d_1\}),$$

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$$F_n = \left\{ f_i = \begin{pmatrix} 1 & 2 & \dots & i-1 & i & i+1 & \dots & n \\ 2 & 3 & \dots & i & 1 & i+1 & \dots & n \end{pmatrix} \middle| \; i \geq 1 \right\}, \text{ and}$$

$$F_n' = \left\{ f_i, f_i^{-1} \middle| \; f_i \in F_n \right\}.$$

Following the usual group theoretic notation, (i j) denotes the *transposition* that exchanges the i^{th} and j^{th} symbols, and in particular, (1 j) denotes a *special transposition* that exchanges the first and j^{th} symbols. Also, c_{ij} denotes a *substring reversal* of the substring of size j-i+1 that starts at the i^{th} position. A *prefix reversal* of size j, denoted by d_j, is equivalent to the substring reversal of size j that starts in the first position. That is, d_j reverses the order of first j symbols. Finally, f_i denotes a *cycle prefix operation* of size i, which moves the first symbol to the i^{th} position and shifts the second through i^{th} symbols one position to the left. Similarly, its inverse f_i^{-1} moves the i^{th} symbol to the first position and shifts the first through $(i-1)^{th}$ symbols one position to the right.

A *Cayley graph,* defined by a group $\mathcal{G}$ and a set of generators Γ, is a graph with one vertex for each group element x, and an edge between two vertices x and y when y is the product of x with a generator in Γ. That is, the Cayley graph defined by $G(\mathcal{G},\Gamma)$ is isomorphic to the graph G(V,E), where $V= \mathcal{G}$ and $E = \{(x,y) \mid y = \gamma \circ x, \text{ where } \gamma \in \Gamma\}$. If Γ is closed under the inverse operation, then the Cayley graph $G(\mathcal{G},\Gamma)$ is an undirected graph; it is a directed graph otherwise. That is, if Γ is closed under the inverse operation, then one replaces each pair of directed edges (the edge from x to y and the edge from y to x) by a single undirected edge between x and y. All of the Cayley graphs studied in this thesis are undirected graphs.

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We are particularly interested in Cayley graphs defined on the symmetric group. A Cayley graph defined by Σ_n and a set of generators Γ_n, and denoted by $G(\Sigma_n, \Gamma_n)$, is the graph with one node for each permutation in Σ_n and an edge between two nodes (permutations) x and y when y is the composition of x with a generator in Γ_n. Thus,

$G(\Sigma_n, A_n)$ is the transposition network of dimension n,

$G(\Sigma_n, B_n)$ is the star network of dimension n, denoted by S_n,

$G(\Sigma_n, C_n)$ is the substring reversal network of dimension n,

$G(\Sigma_n, D_n')$ is the pancake network of dimension n, denoted by P_n, and

$G(\Sigma_n, F_n')$ is the cycle prefix network of dimension n, denoted by CP_n.

Let Z_2^n denote all n-tuples of 0's and 1's, and let the symbol $\Diamond$ denote concatenation. Let $\Delta_n = \{\delta_i \Diamond d_i \mid 1 \le i \le n\}$ be a set of generators, where $\delta_i \in Z_2^n$ denotes the n-tuple whose first i coordinates are 1 and all others are 0, and $\delta_i \Diamond d_i$ denotes the (n+1)-tuple whose first n coordinates form δ_i and whose last coordinate is $d_i \in D_n$. The n-dimensional burnt pancake network, denoted by BP_n, is defined by the Cayley graph with group $Z_2^n \times \Sigma_n$ and set of generators Δ_n. An object $\pi_n = (s_1, s_2, \ldots, s_n, p) \in Z_2^n \times \Sigma_n$ denotes a stack of burnt pancakes of n distinct sizes, where each pancake has one side burnt, the stack is in the order described by the permutation p and, for all i ($1 \le i \le n$), the i^{th} pancake in the stack

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has its burnt side up if and only if $s_i = 1$. The multiplication operation on objects $\pi_n = (s_1, s_2, \ldots , s_n, p)$ and $\pi_n' = (s_1', s_2', \ldots , s_n', p')$ yields $(s_1 \oplus s_1', s_2 \oplus s_2', \ldots , s_n \oplus s_n', p \circ p')$, where $\oplus$ is modulo 2 addition and $\circ$ is functional composition. Thus, the multiplication of the generator $\delta_i \lozenge b_i$ with the object $\pi_n = (s_1, s_2, \ldots , s_n, p)$ yields an object denoting the stack obtained from π_n by flipping the top i pancakes, thereby reversing the orientation of the flipped ones' burnt sides. Signed symbols can be used for a more compact notation in the following way. Each burnt pancake is denoted by a distinct signed symbol, and for all i ($1 \le i \le n$), the i^{th} symbol is negative if and only if the i^{th} pancake has its burnt side up. The i^{th} symbol is positive otherwise. When necessary for clarity, negative symbols are enclosed in parentheses. In a similar manner, one could, define n-dimensional burnt star networks, burnt substring reversal networks, and burnt transposition networks.

The *hypercube of dimension n*, denoted by Q_n, is the graph $G(Z_2^n, E)$, where E is the set of all edges connecting nodes labeled by binary strings of length n that differ in exactly one bit. Note that the set Z_2^n together with the operation $\oplus_n$ forms a group, where $\oplus_n$ denotes bitwise modulo 2 addition of binary strings of length n. Let $T_n \subseteq Z_2^n$ be the set of binary strings of length n such that for all i ($1 \le i \le n$), $t_i \in T_n$ if and only if the i^{th} bit of t_i is 1 and every other bit of t_i is zero. Observe that T_n is a set of generators for Z_2^n, hence Q_n can also be defined as the Cayley graph with group Z_2^n and set of generators T_n.

Let $\wp(X)$ denote the power set of a set X. Given two networks, $G = (V,E)$ and $H = (V',E')$ and an integer d, $f: V \rightarrow \wp(V')$ is a *one-to-many, dilation d embedding* [14,28] of G into H if, for every pair of vertices x, y in V:

1) $f(x) \cap f(y) = \varnothing$, and

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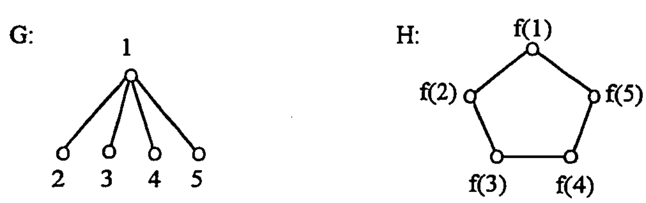
2) if x and y are adjacent in G, then for each $x' \in f(x)$ there corresponds at least one vertex $y' \in f(y)$ for which x' and y' are joined in H by a path of length at most d.

A one-to-many embedding f is a one-to-one embedding if and only if f is a one-to-one function. The symbols $\overset{d}{\Rightarrow}$ and $\overset{d}{\rightarrow}$ denote one-to-many and one-to-one dilation d embeddings, respectively.

The literature often defines a *routing* ρ *for a one-to-one embedding f* as an assignment, to each pair of adjacent processors x and x' in G, a path in H connecting the images of f(x) and f(x'). Such paths in H are called *routes*. The *traffic of an edge e in the host network under a routing* ρ, denoted by *traffic*$_\rho$(e), is the number of routes that pass through e. The *congestion of f under a routing* ρ is defined by max{ traffic$_\rho$(e) | e is an edge in H }. For example, consider the networks G and H shown in Figure 1.3, and the one-to-one dilation two embedding f defined by the node labels for H.



neighbors in G	a more congested routing	a less congested routing
1,2	f(1), f(2)	f(1), f(2)
1,3	f(1), f(2), f(3)	f(1), f(2), f(3)
1,4	f(1), f(2), f(3), f(4)	f(1), f(5), f(4)
1,5	f(1), f(2), f(3), f(4), f(5)	f(1), f(5)

Figure 1.3. A one-to-one dilation two embedding f of a guest network G into a host network H and two possible routings for f.

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There are many routings for f. One possibility (shown in the second column of the table in Figure 1.3) is to choose paths that traverse H in a counter-clockwise manner. The congestion of f under this routing is 4 because four routes pass through the edge {f(1), f(2)}. The second embedding, shown in the third column of the table in Figure 1.3, is less congested. Its congestion is 2 because the edge {f(1), f(2)} (for example) is used by two routes. Note also that the first routing has a route of length 5, which exceeds the dilation of the embedding. The example illustrates that a poorly chosen routing can result in high congestion. In reality, the length of a route need never exceed the dilation of the embedding. Moreover, a route whose length *does* exceed the dilation of the embedding is undesirable and violates the intuition of the embedding.

The definitions for routing and congestion given above are insufficient for our purposes since they do not take the notion of timing into account. That is, the traffic at an edge e describes the number of routes that use the edge e, but messages routed along these paths may not arrive at edge e at the same time. Clearly, the real issue of congestion corresponds to the problem of several messages arriving at a common link at the same instant.

We define a *timed routing* ρ *for a one-to-one embedding f from G to H* as a sequence of pairs, $(e_1,\theta_1), (e_2,\theta_2), \ldots, (e_k,\theta_k)$, for each pair of adjacent processors x and x' in G, such that $e_1,e_2,\ldots,e_k$ is a path from f(x) to f(x') in H, and $\theta_1,\theta_2,\ldots,\theta_k$ is a strictly increasing sequence of positive integers. The sequence $(e_1,\theta_1), (e_2,\theta_2), \ldots, (e_k,\theta_k)$ is called a *timed route*. That is, a timed routing ρ specifies that a communication from f(x) to f(x') is to proceed through edge e_i at time θ_i for all i, $(1 \leq i \leq k)$. The *traffic of edge e in H at time* θ *under a timed routing* ρ, denoted by $traffic_\rho(e,\theta)$, is the number of routes that contain (e,θ). The *congestion of an embedding f from G to H under a timed routing* ρ, is

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defined by max{ $traffic_\rho(e,\theta)$ | e is an edge in H, and θ is a positive integer}. If the embedding f has dilation d, we assume that that the time slots θ are 1,2,...,d. In the case of one-to-many embeddings, the definitions are analogous except that for each pair of adjacent processors x and x', we specify a timed route from each processor y in f(x) to the closest processor y' in f(x').

1.2 Previous Work and New Results

A survey of the literature yields a rich diversity of research pertaining to embeddings of networks, for example, see [5-9,14,20,25,27-30]. The motivations for such studies are numerous. The need to design efficient parallel processing algorithms inspires the study of embeddings of parallel programming branching structures. For example, Leighton [25] describes static and dynamic embeddings of binary trees into hypercubes. Also, Bouabdallah, Heydemann, Opatrny and Sotteau [8], embed binary trees into star and pancake networks. Structural comparisons between networks are often studied through network simulations. Embeddings of networks are one means of investigating such simulations. Notions such as the dilation of an embedding and the congestion of an embedding under a given routing allow us to quantify the degree of similarity between networks being compared. The relatively low diameter and degree of pancake and related networks motivates studies of embedding hypercubes into these networks [14,27,28]. Simulations of star networks by hypercubes are studied by Bettayeb, Cong, Girou and Sudborough [5]. Embeddings of networks also allow us to compare the computational power of networks, with the goal of identifying possible candidates for a *universal network*, i.e., a network capable of efficiently simulating all other networks [29].

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Bhatt, Chung, Leighton and Rosenberg [7] describe dilation 10 embeddings of arbitrary trees into hypercubes. Embeddings of meshes into hypercubes are the subject of several studies. For example, Chan [9] describes a dilation 2 embedding of 2-dimensional meshes into optimal hypercubes, and a dilation 4d+1 embedding of d-dimensional meshes into optimal hypercubes. Improvements in dilation are due to Bettayeb, Miller, Peng and Sudborough [6]. They describe embeddings of d-dimensional meshes into optimum hypercubes with dilation 2d-1. Embeddings of hamiltonians and hypercubes into star networks are discussed by Nigam, Sahni and Krishnamurthy [30]. Improvements and extensions of this work, including a one-to-many dilation one embedding of hypercubes into star networks are given by Miller, Pritikin and Sudborough [27,28]. Structural studies of the star network and embeddings of grids into the star network are described by Jwo, Lakshmivarahan and Dhall [20].

Low dilation embeddings of hypercubes into pancake and related networks are described by Gardner, Miller, Pritikin and Sudborough [14]. Their one-to-many dilation one embedding of hypercubes into pancake networks is summarized by $Q_r \overset{\text{dil } 1}{\Rightarrow} P_k$, where k is defined by the congruence class r(mod3) by $Q_{3n} \overset{\text{dil } 1}{\Rightarrow} P_{11n^2-n}$, when r = 3n, $Q_{3n+1} \overset{\text{dil } 1}{\Rightarrow} P_{11n^2+21n-14}$, when r = 3n+1, and $Q_{3n+2} \overset{\text{dil } 1}{\Rightarrow} P_{11n^2+13n+5}$, when r = 3n+2. These embeddings use a technique quite similar to method used in this thesis to embed star networks into pancake networks with dilation one. We give a qualitative description of this technique in the next paragraph. The idea is to encode the position *and* identity of each symbol in unary notation, and then represent this quantity by the size of a block of host symbols. Appropriate encodings allow us to use the same technique to design embeddings of two dissimilar networks, namely hypercubes and star networks, into pancake networks. This same technique is likely to be useful for

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other dilation one embeddings into pancake networks. Several other low dilation embeddings are described, such as, $Q_n \overset{\text{dil } 2}{\Rightarrow} P_{2n}$, $Q_{m(k-2^{m+1})} \overset{\text{dil } 3}{\Rightarrow} P_k$, when $k \geq 2$ and $m \leq \lfloor \log_2 k \rfloor - 1$, and $Q_{(k-2)2^k+2} \overset{\text{dil } 4}{\rightarrow} P_{2^k-1}$. Embeddings of hypercubes into cycle prefix networks, substring reversal networks, and burnt pancake networks are also described [14].

We begin our current study with low dilation embeddings of star networks into pancake networks with the goal of comparing the computational capabilities of these networks. Our results can be summarized as follows:

dilation one:	$S_n \overset{\text{dil } 1}{\Rightarrow} P_{(n^3-4n^2+5n+4)/2}$
dilation two:	$S_n \overset{\text{dil } 2}{\Rightarrow} P_{2n-2}$
dilation four:	$S_n \overset{\text{dil } 4}{\rightarrow} P_n$
dilation six:	$S_n \overset{\text{dil } 6}{\rightarrow} BP_{n-1}$

These embeddings have several noteworthy characteristics. Regarding our dilation one embedding, $S_n \overset{\text{dil } 1}{\Rightarrow} P_{(n^3-4n^2+5n+4)/2}$, the problem of efficiently simulating an arbitrary special transposition by a *single* prefix reversal has no easy solution. A special transposition (1 i) changes the position of the first and the i^{th} symbols only, leaving all other symbols, but significantly, the second through the $(i-1)^{th}$ symbols stationary. A prefix reversal, on the other hand, alters the position of the first and the i^{th} symbols, as well as the positions of all symbols in between. Hence, for any dilation one embedding into the pancake network, information about the position of a guest symbol must be immune to the effects of prefix reversals. In particular, positional information must be encoded in a manner that is independent of the actual location of the symbol's image in a host permutation.

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This requirement in effect guarantees that there is no general one-to-one dilation one embedding of star networks into pancake networks. Our one-to-many dilation one result uses an intricate construction that represents the symbols {1,...,n} and their positions in an unusual manner. We represent each symbol *and* its position in the label of a guest processor by a block of symbols in the label of a corresponding host processor. The identity and position of a symbol in a guest label is deduced from the size of a corresponding specified block. Hence, the position of a symbol in a guest permutation is in no way related to the position of its image in the host. As we shall see in detail in Chapter 2, a single prefix reversal is sufficient to simulate an arbitrary special transposition. Among the consequences is that each guest permutation is represented by more than one host permutation, hence, the embedding is a one-to-many embedding.

When we allow ourselves the use of two prefix reversals to simulate a single special transposition, the construction of an efficient embedding is somewhat easier. Notice that our dilation two embedding is considerably more efficient asymptotically than our dilation one result. That is, the size of the host is not so large as for dilation one. (In fact, when we allow ourselves four prefix reversals to simulate one special transposition, we are able to construct a one-to-one embedding. This embedding uses the identity map and has dilation four.) Our dilation two result, $S_n \overset{\text{dil } 2}{\Rightarrow} P_{2n-2}$, has a relatively straightforward construction. We use two host symbols to represent one guest symbol in the following manner. For a guest symbol i, one host symbol, say i' gives the guest symbol's identity, and another host symbol, say a_i, gives the guest symbol's position. As we shall see, this representation allows us to simulate an arbitrary special transposition with at most two prefix reversals.

We describe several embeddings of pancake (and related) networks into the star network. Our results can be summarized as follows:

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dilation one : $P_n \overset{\text{dil } 1}{\Rightarrow} S_{2n}$

$BP_n \overset{\text{dil } 1}{\Rightarrow} S_{2n}$

dilation three : $P_n \overset{\text{dil } 3}{\Rightarrow} S_{2n-1}$

In the dilation one embedding of the pancake network into the star network, we use the familiar model of a linked list to design an efficient mapping. That is, we represent the positions of symbols by an adjacency list. This allows us to create an embedding that is linear in the number of host symbols. The dilation one embedding of the burnt pancake network into the star network is an extension of this result and uses the additional technique of representing each burnt pancake by a sequence of two unburnt pancakes. We show that the technique used for the dilation one embedding of pancakes into stars can be improved to decrease the dimension of the host network, but at the cost of increasing the dilation. For example, we can decrease the dimension of the host network by one symbol in the embedding $P_n \overset{\text{dil } 3}{\Rightarrow} S_{2n-1}$, but the dilation rises to 3.

Embeddings with cycle prefix network are constructed by techniques similar to those described above. We give a summary of these results below.

dilation two: $S_n \overset{\text{dil } 2}{\rightarrow} CP_n$

$CP_n \overset{\text{dil } 2}{\rightarrow} P_n$

$P_n \overset{\text{dil } 2}{\Rightarrow} CP_{2n}$

$CP_n \overset{\text{dil } 2}{\Rightarrow} S_{2n}$

The first two embeddings are one-to-one embeddings based on the identity map. The first embedding uses the idea that an arbitrary special transposition can be simulated by two cycle prefix operations. Likewise, the second embedding uses

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the fact that an arbitrary cycle prefix operation can be simulated by two prefix reversals. Details are given in Chapter 4. We use a two step process to construct the third embedding. The first step embeds the pancake network of dimension n into the star network of dimension 2n via a one-to-many dilation one embedding. The second step uses the identity map to embed the star network into the cycle prefix network of like dimension. The composition of these two embeddings gives us the final result. The fourth embedding uses the linked list representation described in the previous paragraph to embed the cycle prefix network of dimension n into the star network of twice the dimension.

We describe the results of further investigations into the structure of pancake and related networks. Jwo, Lakshmivarahan and Dhall [20] showed that the star network of dimension n has all cycles of even length between 6 and n! We show that the pancake network of dimension n has cycles of *all* lengths between 6 and n!, and that the burnt pancake network of dimension n has cycles of *all* lengths between 8 and $2^n(n!)$.

In Figure 1.4 we give a graphical summary of our network comparisons.

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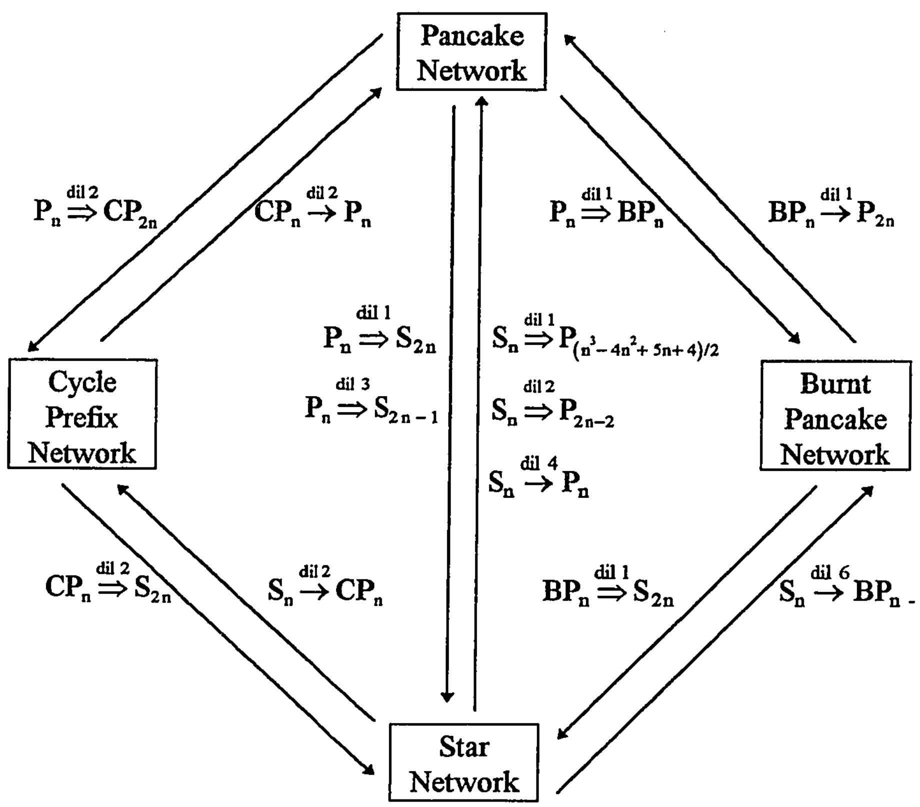

Figure 1.4. Comparisons of four Cayley networks on the symmetric group: the pancake network, the star network, the burnt pancake network and the cycle prefix network.

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We discuss the Deterministic Pancake Problem. As mentioned earlier, we are interested in the asymptotic behavior of the function h(n), which is defined as $|\mathrm{run}(\pi)|$ for long-winded permutations. We show that h(n) is $\Omega(n^2)$.
Our study of the Deterministic Pancake Problem arises, in part, from our interest in routing in the Pancake Network. The Pancake Problem asks for the shortest sorting sequence for an arbitrary permutation on n symbols using prefix reversals. This is equivalent to asking for bounds on the diameter of the pancake network. We are interested in proving that the *deterministic* pancake process is an inefficient routing algorithm in the pancake network.

We show that the deterministic pancake process has a quadratic lower bound by describing permutations of a special form. Let $\Pi_n^{(k)}$ denote the family of all permutations π on $\{1,\ldots,n\}$ such that $\pi(i) = i$, for all i, $(2 \le i \le n\text{-}k)$. As an example when k=12 and n=21, the permutation σ_{21}, defined by

$$\sigma_{21} = 21,2,3,4,5,6,7,8,9,11,20,12,17,13,19,14,10,1,16,18,15$$

is in Π_{21}^{12}, as the integers 2,...,9 are in positions 2,...,9, respectively, and it is a permutation on $\{1,\ldots,21\}$. More generally, for $n \ge 21$, define σ_n in $\Pi_n^{(12)}$ to be the permutation

$$\sigma_n = n,2,3,\ldots,n\text{-}12,n\text{-}10,n\text{-}1,n\text{-}9,n\text{-}4,n\text{-}8,n\text{-}2,n\text{-}7,n\text{-}11,1,n\text{-}5,n\text{-}3,n\text{-}6.$$

(Note that substituting 21 for n gives the permutation σ_{21}.)

Our motivation for studying the special families of permutations denoted by $\Pi_n^{(k)}$, for various values of k and n, is that for small values of k the number of distinct permutations in $\Pi_n^{(k)}$ is small compared to the total number of permutations in Σ_n. For example, there are n! permutations on the symbols $\{1,\ldots,n\}$ and hence, there are more than 10^{12} permutations for n =15. In other words, the growth rate of the function n! prohibits extensive computer search for permutations with desirable properties. Thus, one needs to restrict the search to

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some seemingly fruitful subspace of the space of all permutations and this subspace needs to be reasonably small.

No matter how large the integer n, the number of permutations in the subspace $\Pi_n^{(k)}$ is (k+1)! For example, Π_n^{12} contains $13! \cong 6*10^9$ permutations, for all $n \geq 13$. And, although large, a space of roughly six billion objects is a reasonable size for computer search.

One might now ask, why should Π_n^{12} be a fruitful subspace? And more fundamentally, what is it that one hopes to find? Some intuition for the choice of search space can be provided with the following observation. Let π_n be any permutation on $\{1,...,n\}$ such that run(π_n) terminates with the identity permutation, *i.e.*, run(π_n) ends in sorted order. Consider the process of creating a permutation π_{n+2} from π_n by replacing 1 with n+2, placing 1 in position n+1, and placing n+1 in position n. For example, for n = 7, consider the permutation given by π_7= 3146752, for which run(π_7) has length 17 and ends with the identity permutation. That is, run(π_7) is

$$3146752 \mapsto 4136752 \mapsto 6314752 \mapsto 5741362 \mapsto 3147562$$
$$\mapsto 4137562 \mapsto 7314562 \mapsto 2654137 \mapsto 6254137 \mapsto 3145267$$
$$\mapsto 4135267 \mapsto 5314267 \mapsto 2413567 \mapsto 4213567 \mapsto 3124567$$
$$\mapsto 2134567 \mapsto 1234567.$$

Next form the permutation π_9 from π_7 as indicated. One obtains 394675218. As the element 1 comes to the front only at the end of the run sequence described above, and 1 has been replaced in π_9 by the number 9, the sequence run(π_9) will be similar to the sequence run(π_7). The difference is that in each permutation in the sequence, the symbol 1 is replaced everywhere by the symbol 9, and the symbols 1,8 are added as a suffix. In fact, the seventeenth element of run(π_9),

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corresponding to the last element in run(π_7) described above, is 923456718. Note that this permutation is in Π_9^7, as 2,...,7 are in positions 2,...,7, respectively.

Run(π_9) does not terminate with the seventeenth step since the permutation in this step is 923456718. Instead, the process continues with the sequence

$$923456718 \mapsto 817654329 \mapsto 234567189 \mapsto 324567189$$
$$\mapsto 423567189 \mapsto 532467189 \mapsto 642357189 \mapsto 753246189$$
$$\mapsto 164235789.$$

Note that after the nineteenth step, the permutation is 234567189, and run(π_9) continues through a sequence of flips indicated by consecutively larger integers before termination. Similarly, in general, the permutation π_{n+2}, formed from π_n as indicated, will exhibit a sequence of permutations very similar to run(π_n). This will be followed, after two more steps, by the permutation 23...n1(n+1)(n+2). Run(π_{n+2}) will then proceed through a sequence of flips described by the consecutively larger integers 2,...,n. In other words, given any permutation π_n on {1,...,n} whose run sequence terminates with the identity, one can create a permutation π_{n+2} on {1,...,n+2} *whose run sequence has* n+1 *more permutations than run*(π_n). This suggests that the length of run sequences does not grow linearly with the size of the permutation.

In fact, this observation would be sufficient to disprove Knuth's conjecture, if run(π_{n+2}) also ended with the identity, as one could then iterate the procedure and describe permutations with run sequences of quadratic length. Unfortunately, run(π_{n+2}) does not end with the identity permutation. We designed a computer search for permutations created from π_n, similar to π_{n+2}, which do terminate with the identity. The search was conducted in an appropriate

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subspace, namely the subspace $\Pi_n^{(k)}$, for suitably small values of k. In particular, the computer was programmed to search for any permutation in $\Pi_n^{(k)}$ whose run sequence is long, and which terminates with the identity permutation.

The result of this search was the permutation σ_n described earlier. The permutation σ_n was one of the best permutations found, as $|run(\sigma_n)|$ is large. Observe that any computer search of $\Pi_n^{(12)}$ is limited to a finite number of instances of n and yet our goal is to prove a statement about infinitely many values of n. That is, we want to prove a statement about the asymptotic performance of $|run(\sigma_n)|$. Fortunately, an analysis of the sequence $run(\sigma_n)$ uncovers a regular pattern and on the basis of this regular pattern, we prove a lemma for infinitely many values of n. The lemma is stated as follows: For all $n\geq 21$, such that $n\equiv 9 \pmod{12}$, $|run(\sigma_n)|=(21n-77)/4$ and the last element of $run(\sigma_n)$ is the identity permutation.

Observe that this lemma does not accomplish what we set out to do, namely to show an $\Omega(n^2)$ lower bound. It simply shows that $run(\sigma_n)$ has length that is linear in n. However, σ_n will play the role of the permutation π_{n+2} constructed from π_n in our previous discussion and, as such, it can be used iteratively. For example, since $run(\sigma_{21})$ ends with the identity, one can create a different permutation on $\{1,\ldots,33\}$, *i.e.*, a permutation on 12 *more* objects, using σ_{21} and σ_{33} as building blocks. That is, take σ_{21}, replace 1 with the first element of σ_{33}, namely 33, and add the last 12 elements of σ_{33} as a suffix to σ_{21}. The result is a permutation, denoted by $\sigma_{21}\oplus\sigma_{33}$, whose run sequence is essentially that of $run(\sigma_{21})$ followed by $run(\sigma_{33})$. This can be repeated as many times as desired with other members of the family $\{\sigma_n\}$. That is, one can chain together as many of these permutations as desired and thereby obtain a run sequence whose

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length is the sum of the lengths of each permutation's run sequence. This is expressed in the following corollary: For all $m \geq 1$, $\sigma_{21} \oplus \sigma_{33} \oplus \ldots \oplus \sigma_{21+12m}$ is a permutation on $n=21+12m$ symbols and

$$|\mathrm{run}(\sigma_{21} \oplus \sigma_{33} \oplus \ldots \oplus \sigma_{21+12m})| = |\mathrm{run}(\sigma_{21})| \oplus |\mathrm{run}(\sigma_{33})| \oplus \ldots \oplus |\mathrm{run}(\sigma_{21+12m})|$$

$$= \frac{7}{96}\left(3n^2 + 14n - 369\right).$$

This accomplishes our goal of showing that there are permutations of length n whose run sequences have length quadratic in n.

We describe another family of permutations, denoted by $\{\tau_n\}$ for $n \geq 69$ such that $n \equiv 5 \pmod 8$, for which $|\mathrm{run}(\tau_n)|$ has a quadratic lower bound. This result is summarized in the next lemma: For all $n \geq 69$, such that $n \equiv 5 \pmod 8$,

$$|\mathrm{run}(\tau_n)| \geq \frac{1}{12}(n^2 - 39n + 750).$$

This means that for every permutation in the family $\{\tau_n\}$, the deterministic pancake process takes $\Omega(n^2)$ steps to terminate. However, the process does not terminate with the identity permutation, so these permutations cannot be chained together. Nevertheless, we can use a permutation from $\{\tau_n\}$ as the last permutation in the earlier chaining process. The result, summarized in the following theorem, is an improvement on the earlier corollary: For all even $m \geq 4$, the permutation $\sigma_{21} \oplus \sigma_{33} \oplus \ldots \oplus \sigma_{21+12(m-1)} \oplus \tau_{21+12m}$ is a permutation on $n=21+12m$ symbols, and $|\mathrm{run}(\sigma_{21} \oplus \sigma_{33} \oplus \ldots \oplus \sigma_{21+12(m-1)} \oplus \tau_{21+12m})|$

$$= |\mathrm{run}(\sigma_{21})| \oplus |\mathrm{run}(\sigma_{33})| \oplus \ldots \oplus |\mathrm{run}(\sigma_{21+12(m-1)})| \oplus |\mathrm{run}(\tau_{21+12m})|$$

$$\geq \frac{1}{96}\left(29n^2 - 718n + 5265\right).$$

This last result is better in the sense that the coefficient for n^2 has increased from $\frac{21}{96}$ to $\frac{29}{96}$.

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Chapter 2

Embedding Stars into Pancakes

We begin with a review of pertinent definitions from Chapter 1. Let $\wp(X)$ denote the power set of a set X. Given two networks, $G = (V,E)$ and $H = (V',E')$ and an integer d, $f: V \rightarrow \wp(V')$ is a *one-to-many dilation d embedding* [14,28] of G into H if, for every pair of vertices x, y in V:

1) $f(x) \cap f(y) = \varnothing$, and

2) if x and y are adjacent in G, then for each $x' \in f(x)$ there corresponds at least one vertex $y' \in f(y)$ for which x' and y' are joined by a path of length at most d.

A one-to-many embedding f is a one-to-one embedding if and only if f is a one-to-one function. The symbols $\overset{d}{\Rightarrow}$ and $\overset{d}{\rightarrow}$ denote one-to-many and one-to-one dilation d embeddings, respectively.

Let $G=(V,E)$ be a *guest* network and $H=(V',E')$ be a *host* network, and let $\varnothing \neq R \subseteq V'$. A *co-embedding of dilation d* is a pair (R,g), where $g:R \rightarrow V$ is an onto function such that for every $y \in R$, and any neighbor x' of g(y) in G, there exists some $y' \in R$ such that $g(y') = x'$ and $\text{distance}_H(y,y') \leq d$. We shall often talk about the co-embedding g, its domain R understood implicitly. By the comments above, we see that there exists a one-to-many dilation d embedding of G into H if and only if there exists an appropriate dilation d co-embedding g. Let x and x' be adjacent vertices in G. Let α be a generator of G such that $x \circ \alpha = x'$. Throughout Chapters 2, 3 and 4, we routinely and implicitly specify a one-to-many embedding of G into H by specifying a co-embedding g. In order to

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demonstrate dilation d, we show how to simulate the action of α on x by the action of an appropriate sequence of d generators $\alpha_1, \alpha_2, \ldots, \alpha_d$ in H on any vertex y such that g(y) = x. That is, we demonstrate that there is a path of length at most d in H from y to a vertex y' such that g(y') = x'.

We define a *timed routing* ρ *for a one-to-one embedding f from G to H* as a sequence of pairs, $(e_1,\theta_1), (e_2,\theta_2), \ldots, (e_k,\theta_k)$, for each pair of adjacent processors x and x' in G, such that $e_1,e_2,\ldots,e_k$ is a path from f(x) to f(x') in H, and $\theta_1,\theta_2,\ldots,\theta_k$ is a strictly increasing sequence of positive integers. The sequence $(e_1,\theta_1), (e_2,\theta_2), \ldots ,(e_k,\theta_k)$ is called a *timed route*. That is, a timed routing ρ specifies that a communication from f(x) to f(x') is to proceed through edge e_i at time θ_i for all i, $(1 \le i \le k)$. The *traffic of edge e in H at time* θ *under a timed routing* ρ, denoted by *traffic*$_\rho(e,\theta)$, is the number of routes that contain (e,θ). The *congestion of an embedding f from G to H under a timed routing* ρ, is defined by $\max\{\ \mathrm{traffic}_\rho(e,\theta) \mid e$ is an edge in H, and θ is a positive integer$\}$. If the embedding f has dilation d, we assume that that the time slots θ are 1,2,...,d. In the case of one-to-many embeddings, the definitions are analogous except that for each pair of adjacent processors x and x', we specify a timed route from each processor y in f(x) to the closest processor y' in f(x').

In the next three chapters, a permutation $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma_1 & \sigma_2 & \ldots & \sigma_n \end{pmatrix}$ will be written as the string $\sigma_1\sigma_2 \ldots \sigma_n$ or as the sequence $\sigma_1,\sigma_2, \ldots, \sigma_n$ as is convenient for the particular embedding being described. That is, we will refer to the permutation $\sigma=\sigma_1\sigma_2 \ldots \sigma_n$ or the permutation $\sigma=\sigma_1,\sigma_2, \ldots, \sigma_n$.

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2.1 A one-to-one dilation four embedding.

In Figure 2.1, we show a one-to-one, dilation 2 embedding of the star network of dimension 4 into the pancake network of the same dimension. Each node in the illustration has two labels. The unparenthesized label is a permutation in the pancake network, and the parenthesized label beneath it gives the corresponding permutation in the star network. Note that star network nodes 1243 (1423, 4132, 4312) and their neighbors 2143 (4123, 1432, 3412) and 4213 (2413, 3142, 1342), respectively, are assigned to pancake network nodes at distance 2. All other pairs of adjacent nodes in S_4 are assigned to adjacent nodes in P_4.

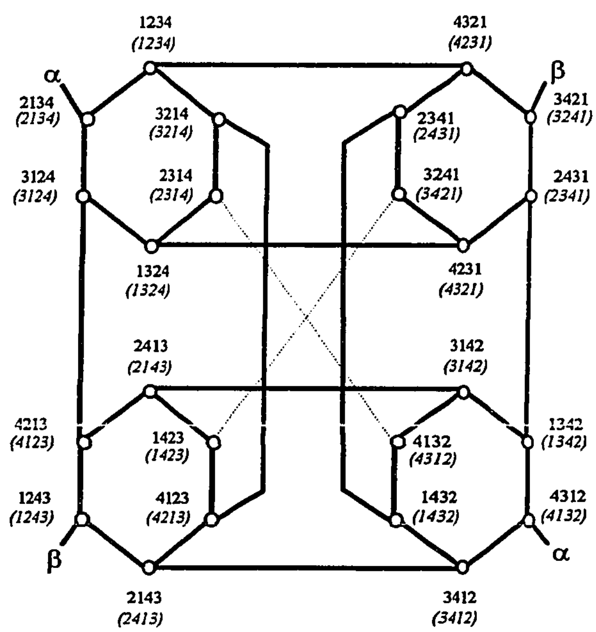

Figure 2.1. A one-to-one, dilation 2 embedding of S_4 into P_4.

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This embedding is not readily generalized to higher dimensions. In fact, any reasonable generalization would probably not have fixed dilation. As we want fixed, low dilation embeddings, we turn to an embedding of stars into pancakes via the identity map. This embedding has dilation 4, and is described in Theorem 2.1.

Theorem 2.1: $S_n \xrightarrow{\text{dil } 4} P_n$

Proof: Let $f: S_n \to P_n$ be the identity map. Let σ and σ' be permutations on $\{1,\ldots,n\}$ such that σ' is at distance 1 from σ in the star network. It follows that there is a special transposition (1 i), for some i, $2 \le i \le n$, such that $\sigma' = \sigma \circ (1\ i)$. We simply observe that the special transpositions (1 2) and (1 3) are identical to the prefix reversals of sizes 2 and 3 respectively, and for all i>3, any special transposition (1 i) can be simulated by the sequence of four prefix reversals of sizes: i, i-1, i-2, i-1, as demonstrated for the transformation of σ into σ' below:

$$
\begin{aligned}
\sigma &= \sigma_1\sigma_2\ldots\sigma_{i-1}\sigma_i\sigma_{i+1}\ \cdots\ \sigma_n \\
&\mapsto \sigma_i\sigma_{i-1}\ldots\sigma_2\sigma_1\sigma_{i+1}\ \cdots\ \sigma_n \\
&\mapsto \sigma_2\ldots\sigma_{i-1}\sigma_i\sigma_1\sigma_{i+1}\ \cdots\ \sigma_n \\
&\mapsto \sigma_{i-1}\ldots\sigma_2\sigma_i\sigma_1\sigma_{i+1}\ \cdots\ \sigma_n \\
&\mapsto \sigma_i\sigma_2\ldots\sigma_{i-1}\sigma_1\sigma_{i+1}\ \cdots\ \sigma_n \\
&= \sigma'.
\end{aligned} \tag{1}
$$

Hence, the embedding, f, has dilation 4.

□

The edge congestion of this embedding is also small with respect to a model of synchronous computation. The proof of Theorem 2.1 specifies an exact sequence of prefix reversals to simulate the action of a transposition (1 i). This

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corresponds to a specification for routing messages. Suppose that σ and σ' are permutations such that $\sigma' = \sigma\circ(1\ i)$. We refer to σ and σ' as *i-dimensional neighbors* in the star network, the edge joining them as an *i-dimensional edge*, and a message traveling along this edge an *i-dimensional message*. In the discussion that follows, we use the notation for prefix reversals introduced on page 15, and we label each edge in the pancake network with a representation for the appropriate prefix reversal. For example if $\gamma = \sigma\circ d_j$, then the edge between γ and σ in the pancake network has label d_j, where d_j denotes the prefix reversal of size j.

Suppose that each of the n! processors in the star network of dimension n were simultaneously to send a message to each of its adjacent processors. For the sake of our analysis, we assume that at any stage in a simulation of this computation on the pancake network, all traffic competing for an edge is allowed to propagate to the appropriate endpoint of the edge before the next stage in the simulation. In addition, we assume that each edge transmits at most one message in one time interval. That is, if k messages compete for the same edge at the same time, all k messages will be transmitted before the next stage of messages is propagated. Note that this requires k time intervals to elapse before the next stage in the simulation. We show that, following our stated routing, the traffic in the pancake network results in no more than two messages per edge, per unit of time. We allow all messages to reach their destination before starting another step in the simulation, hence our dilation 4 embedding implies a simulation with a slowdown factor of 8. Furthermore, any two messages competing for an edge in the same time interval must be traveling in opposite directions along the edge, as we shall see.

We show that for all i, all time intervals t, and all nodes σ in P_n, there is at time interval θ, one i-dimensional message at node σ. This is clearly true in the

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first time interval. Suppose it were not true for some node σ at some later time interval. Let t be the earliest time interval for which it is not true. Then for some node σ and some i, there would be two i-dimensional messages at node σ at time interval θ. As these messages are both i-dimensional, and all i-dimensional messages are routed through edges with labels d_i, d_{i-1}, d_{i-2}, d_{i-1}, it follows that both messages must enter σ through the same edge. Hence they are at the same node at time interval θ-1, a contradiction. We conclude that there is one i-dimensional message at each node at any time interval, and consequently two messages traversing each edge, *in opposite directions*, at any time interval. So, for the model of synchronous computation we use, the edge congestion is relatively low.

To introduce the burnt pancake network, we provide an illustration of BP_3 in Figure 2.2 below, followed by a table defining a one-to-one dilation 3 embedding of S_4 into BP_3 in Figure 2.3.

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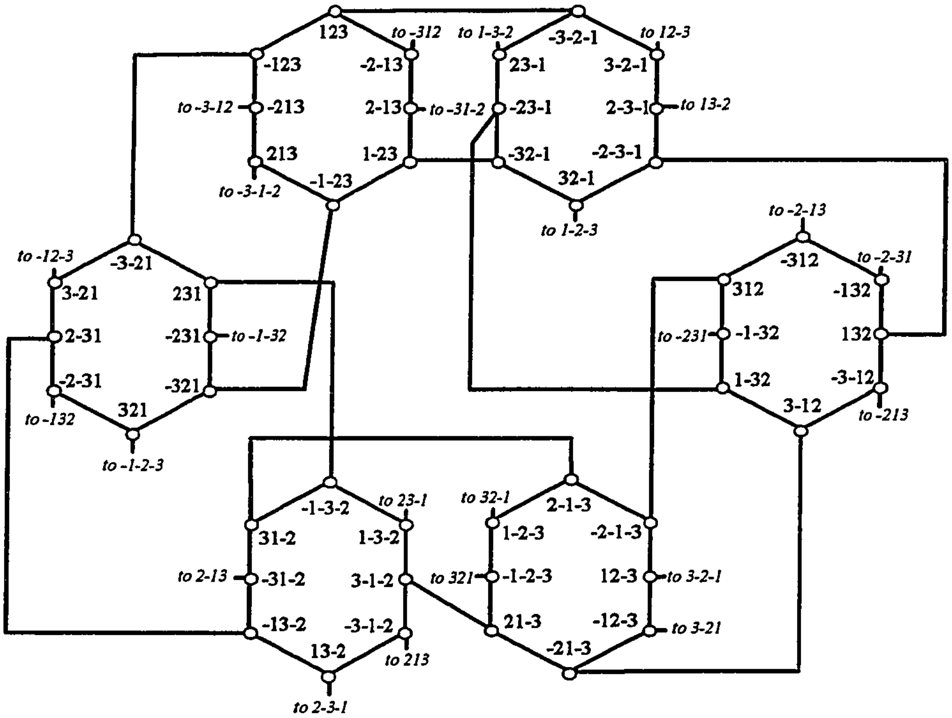

Figure 2.2. BP_3, the burnt pancake network of dimension 3.

node in S_4	image in BP_3	node in S_4	image in BP_3
1234	123	2143	3-21
2134	-123	4123	-3-21
3214	2-13	1243	2-31
3124	213	1423	-231
2314	1-23	4213	-321
1324	-1-23	2413	231
4231	-3-2-1	3142	13-2
2431	23-1	4132	-3-1-2
3241	2-3-1	1342	-13-2
3421	-23-1	1432	1-3-2
2341	-2-3-1	4312	-31-2
4321	-32-1	3412	-1-32

Figure 2.3. A one-to-one, dilation 3 embedding of S_4 into BP_3.

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The embedding defined in Figure 2.3 is not readily generalized to higher dimensions. However, Theorem 2.1 can be generalized to yield $S_n \xrightarrow{\text{dil } 6} BP_n$. That is, the identity map can be used together with the observation that six prefix reversals in the burnt pancake network are sufficient to simulate one special transposition. (The process is basically the same as in (1) in the proof of Theorem 2.1, but two extra steps are needed to change the signs of the first and the i^{th} elements). In fact, we can do better. We now show that $S_n \xrightarrow{\text{dil } 6} BP_{n-1}$. The idea is to represent a permutation on n symbols by an induced permutation on n-1 of these symbols, each symbol signed, but with at most one symbol being negative at any one time. That is, n-1 symbols are placed in the same relative order as in the represented permutation on n symbols; the location of the missing symbol is determined by the symbol with a negative sign, if there is one. Specifically, the missing symbol is located immediately to the right of the negative symbol, if there is a negative symbol; otherwise, the missing symbol is in the initial position. For example, using this interpretation, the permutation 21(-5)34 of five signed symbols on {1,2,3,4,5} represents the permutation 215634 on {1,2,3,4,5,6}. We show in the following theorem that this representation can be maintained throughout a simulation of an arbitrary sequence of special transpositions, using at most six prefix reversals to simulate each special transposition.

Theorem 2.2: $S_n \xrightarrow{\text{dil } 6} BP_{n-1}$

Proof: Let f be the one-to-one function mapping permutations on {1,...,n} into permutations of signed symbols on {1,...,n-1} with at most one negative symbol, defined by

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$$f(\sigma_1\sigma_2\ldots\sigma_n) = \begin{cases} \sigma_1\sigma_2\ldots\sigma_{i-1}(-\sigma_i)\sigma_{i+2}\ldots\sigma_n, & \text{if } \sigma_{i+1} = n, \text{ for some } i\ (1\le i\le n-1), \\ \sigma_2\ldots\sigma_n, & \text{otherwise.} \end{cases}$$

We show that any special transposition can be simulated by a sequence of at most six prefix reversals, by considering four cases based on the position of the symbol 'n' in σ. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_{i-1}\sigma_i\sigma_{i+1}\ldots\sigma_n$ and σ' be permutations on $\{1,\ldots,n\}$ such that σ' is at distance 1 from σ in the star network. It follows that there is a special transposition (1 i) for some i, $2\le i\le n$, such that $\sigma' = \sigma\circ(1\ i)$.

Case 1: In σ, the symbol 'n' is either to the left of σ_i (but not in the first position) or to the right of σ_{i+1}. That is, $\sigma_j = n$ for some j, where either $1<j<i$, or $i+1<j<n$. By the definition of f this means that in $f(\sigma)$, σ_j is missing and the negative symbol, $(-\sigma_{j-1})$, is either to the left of σ_i or to the right of σ_{i+1}, respectively. In either case, the special transposition (1 i) can be simulated by the sequence of six prefix reversals of sizes: 1, i, i-1, i-2, i-1, 1, as demonstrated below for the case in which $(-\sigma_{j-1})$ is to the right of σ_{i+1}:

$$\begin{aligned} f(\sigma) &= \sigma_1\sigma_2\ldots\sigma_{i-1}\sigma_i\sigma_{i+1}\ldots(-\sigma_{j-1})\sigma_{j+1}\ldots\sigma_n \\ &\mapsto (-\sigma_1)\sigma_2\ldots\sigma_{i-1}\sigma_i\sigma_{i+1}\ldots(-\sigma_{j-1})\sigma_{j+1}\ldots\sigma_n \\ &\mapsto (-\sigma_i)(-\sigma_{i-1})\ldots(-\sigma_2)\sigma_1\sigma_{i+1}\ldots(-\sigma_{j-1})\sigma_{j+1}\ldots\sigma_n \\ &\mapsto \sigma_2\ldots\sigma_{i-1}\sigma_i\sigma_1\sigma_{i+1}\ldots(-\sigma_{j-1})\sigma_{j+1}\ldots\sigma_n \\ &\mapsto (-\sigma_{i-1})\ldots(-\sigma_2)\sigma_i\sigma_1\sigma_{i+1}\ldots(-\sigma_{j-1})\sigma_{j+1}\ldots\sigma_n \\ &\mapsto (-\sigma_i)\sigma_2\ldots\sigma_{i-1}\sigma_1\sigma_{i+1}\ldots(-\sigma_{j-1})\sigma_{j+1}\ldots\sigma_n \\ &\mapsto \sigma_i\sigma_2\ldots\sigma_{i-1}\sigma_1\sigma_{i+1}\ldots(-\sigma_{j-1})\sigma_{j+1}\ldots\sigma_n \\ &= f(\sigma'). \end{aligned}$$

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Case 2: In σ, the symbol 'n' occupies the $(i+1)^{st}$ position, *i.e.*, $\sigma_{i+1}= n$. By the definition of f this means that in $f(\sigma)$, σ_{i+1} is not present, and the negative symbol is σ_i. The special transposition (1 i) can be simulated by the sequence of four prefix reversals of sizes: i, i-1, i-2, i-1, demonstrated below:

$$
\begin{aligned}
f(\sigma) &= \sigma_1\sigma_2\ldots\sigma_{i-1}(-\sigma_i)\sigma_{i+2}\ldots\sigma_n \\
&\mapsto \sigma_i(-\sigma_{i-1})\ldots(-\sigma_2)(-\sigma_1)\sigma_{i+2}\ldots\sigma_n \\
&\mapsto \sigma_2\ldots\sigma_{i-1}(-\sigma_i)(-\sigma_1)\sigma_{i+2}\ldots\sigma_n \\
&\mapsto (-\sigma_{i-1})\ldots(-\sigma_2)(-\sigma_i)(-\sigma_1)\sigma_{i+2}\ldots\sigma_n \\
&\mapsto \sigma_i\sigma_2\ldots\sigma_{i-1}(-\sigma_1)\sigma_{i+2}\ldots\sigma_n \\
&= f(\sigma').
\end{aligned}
$$

Case 3: In σ, the symbol 'n' occupies the i^{th} position, *i.e.*, $\sigma_i= n$. By the definition of f this means that in $f(\sigma)$, σ_i is not present, and the negative symbol is σ_{i-1}. The special transposition (1 i) can be simulated by the sequence of four prefix reversals of sizes: 1, i-1, 1, i-2, demonstrated below:

$$
\begin{aligned}
f(\sigma) &= \sigma_1\sigma_2\ldots\sigma_{i-2}(-\sigma_{i-1})\sigma_{i+1}\ldots\sigma_n \\
&\mapsto (-\sigma_1)\sigma_2\ldots\sigma_{i-2}(-\sigma_{i-1})\sigma_{i+1}\ldots\sigma_n \\
&\mapsto \sigma_{i-1}(-\sigma_{i-2})\ldots(-\sigma_2)\sigma_1\sigma_{i+1}\ldots\sigma_n \\
&\mapsto (-\sigma_{i-1})(-\sigma_{i-2})\ldots(-\sigma_2)\sigma_1\sigma_{i+1}\ldots\sigma_n \\
&\mapsto \sigma_2\ldots\sigma_{i-2}\sigma_{i-1}\sigma_1\sigma_{i+1}\ldots\sigma_n \\
&= f(\sigma').
\end{aligned}
$$

Case 4: In σ, the symbol 'n' occupies the first position, *i.e.*, $\sigma_1= n$. By the definition of f this means that in $f(\sigma)$, σ_1 is missing, and all of the symbols are positive. The special transposition (1 i) can be simulated by

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the sequence of four prefix reversals of sizes: i-2, 1, i-1, 1, demonstrated below: (Note that this is simply the reverse of the sequence of steps performed for Case 3.)

$$\begin{aligned}
f(\sigma) &= \sigma_2 \ldots \sigma_{i-2}\sigma_{i-1}\sigma_i\sigma_{i+1}\ldots\sigma_n \\
&\mapsto (-\sigma_{i-1})(-\sigma_{i-2})\ldots(-\sigma_2)\sigma_i\sigma_{i+1}\ldots\sigma_n \\
&\mapsto \sigma_{i-1}(-\sigma_{i-2})\ldots(-\sigma_2)\sigma_i\sigma_{i+1}\ldots\sigma_n \\
&\mapsto (-\sigma_i)\sigma_2\ldots\sigma_{i-2}(-\sigma_{i-1})\sigma_{i+1}\ldots\sigma_n \\
&\mapsto \sigma_i\sigma_2\ldots\sigma_{i-2}(-\sigma_{i-1})\sigma_{i+1}\ldots\sigma_n \\
&= f(\sigma').
\end{aligned}$$

These four cases enumerate all possibilities for the position of n in σ, and prove that a special transposition in the star network of dimension n can be simulated by at most six prefix reversals in the burnt pancake network of dimension n-1. Hence, the embedding has dilation 6.

□

We conjecture that the embedding of Theorem 2.1 has the best dilation possible for any one-to-one embedding of star networks into pancake networks of the same dimension. There is a fairly obvious lower bound for the dilation of one-to-one embeddings of pancakes into stars. That is, there is no dilation one embedding of P_n into S_n, for n>3. This is so, as P_n has cycles of odd length and S_n does not. So P_n is not a subgraph of S_n. Also, there does not exist a one-to-one, dilation one embedding of S_n into P_n. If there were, then its inverse would be a one-to-one, dilation one embedding of P_n into S_n (as P_n and S_n have the same number of nodes and edges). That is, P_n and S_n would be isomorphic, which they are not.

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2.2 A one-to-many dilation two embedding.

We now design embeddings with lower dilation than the embedding of Theorem 2.1. We do this with embeddings suggested by non-standard representations of permutations. There are two pieces of information about a symbol in a permutation that must be represented in any embedding: the symbol's identity and its position. With the identity embedding, this is done in the direct and obvious way, *i.e.,* each symbol represents itself, and its position is exactly where it resides. However, there are other possibilities. Each symbol in a permutation can be represented by *two* objects, one describing the symbol's position and the other describing its identity. This approach is used in Theorem 2.3, where we describe a dilation 2, one-to-many embedding of the star network of dimension n into the pancake network of dimension 2n-2.

Consider a set of symbols $\Psi=\{1,\dots,n\}$, and let Σ_n be the set of permutations over Ψ. Let Θ be a set of n-2 new symbols, say $\Theta = \{a_2, a_3, \dots, a_{n-1}\}$. We represent permutations in Σ_n with permutations over $\Psi \cup \Theta$. Specifically, the permutations over $\Psi \cup \Theta$ used to represent permutations in Σ_n will be strings in the regular set $R = \Psi(\Psi\Theta \cup \Theta\Psi)^{n-2}\Psi$. These are permutations whose first and last symbols are from Ψ, and in between have n-2 pairs of symbols, each pair having one symbol from Ψ and one symbol from Θ. A permutation $r = \pi_1\lambda_2 \dots \lambda_{n-1}\pi_n$ in R, with π_1 and π_n in Ψ and λ_i in $\Psi\Theta \cup \Theta\Psi$, for all i ($2 \le i \le n-1$), denotes the permutation g(r) in Σ_n, where the co-embedding g is defined by

$$g(r) = \sigma_1\sigma_2\dots\sigma_n, \text{ where } \sigma_i = \begin{cases} \pi_i, & \text{for } i = 1 \text{ and } i = n, \\ j, & \text{if there is a pair } \lambda_k, \text{ for some k, containing the two symbols } a_i \text{ and } j, \text{ for } i,\ 2 \le i \le n-1. \end{cases}$$

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The idea is to represent each symbol σ_i, ($2 \le i \le n-1$), with two objects. That is, σ_i is denoted by the pair λ_k, where $\lambda_k = a_i j$ or where $\lambda_k = j a_i$. The object a_i gives the symbol's position, and the object j identifies the symbol itself. For example, the string $3a_2 1 a_4 5 2 a_3 4$ denotes the permutation 31254, as 1 is paired with a_2, 2 is paired with a_3, 5 is paired with a_4, and 3 and 4 are the beginning and ending symbols, respectively.

Observe that a permutation in Σ_n has more than one representative in R. That is, if $\pi_1 \lambda_2 \ldots \lambda_{n-1} \pi_n$ in R denotes the permutation $\sigma_1 \sigma_2 \ldots \sigma_n$ in Σ_n, then so does the string $\pi_1 \lambda_{\rho(2)} \ldots \lambda_{\rho(n-1)} \pi_n$ in R, where ρ is any permutation of $\{2,\ldots,n-1\}$. In other words, changing the order of the pairs in the string does not change the permutation (in Σ_n) denoted. This is true, as each pair contains both a representation of a position i, by the existence of a symbol a_i, and a representation of the i^{th} symbol in the permutation in Σ_n. So, if the order is changed, the result still denotes the same permutation. Also, note that the order of items within a pair is unimportant.

Theorem 2.3: For all n>3, $S_n \overset{\text{dil } 2}{\Rightarrow} P_{2n-2}$.

Proof: Let $g : R \rightarrow \Sigma_n$ be the co-embedding described above. We show that the co-embedding has dilation 2. Let $\sigma = \sigma_1 \sigma_2 \ldots \sigma_{i-1} \sigma_i \sigma_{i+1} \ldots \sigma_n$ be an arbitrary permutation on $\{1,\ldots,n\}$ and let (1 i), $2 \le i \le n$, be one of the special transpositions that define the edges of the star network. Let σ' denote the permutation obtained from σ by applying the special transposition (1 i), *i.e.*, $\sigma' = \sigma_i \sigma_2 \ldots \sigma_{i-1} \sigma_1 \sigma_{i+1} \ldots \sigma_n$. Let $r = \pi_1 \lambda_2 \ldots \lambda_{n-1} \pi_n$ be a permutation in R such that $g(r) = \sigma$. We show that with at most two prefix reversals we can obtain from r a string r' such that $g(r') = \sigma'$.

First observe that if the special transposition is (1 n), then the reversal r^R of the entire string (permutation) r is such that $g(r^R) = \sigma'$. That is, $g(r^R)$ is the

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permutation with $\pi(1)$ and $\pi(n)$ exchanged, *i.e.* the permutation $\sigma' = \sigma_n\sigma_2 \ldots \sigma_{n-1}\sigma_1$. Note that this reversal exchanges the first and last elements in r^R, but the list of pairs between them is unchanged except for order.

Next consider a special transposition of the form (1 i), where $2 \leq i \leq n-1$. We consider two cases, based on the permutation $r = \pi_1\lambda_2 \ldots \lambda_{n-1}\pi_n$ in R, and depending on whether the pair λ_j that contains a_i has a_i as the first or second object in the pair.

If λ_j contains a_i first, then the i^{th} element of the permutation in Σ_n that r denotes, say σ_i, must be second. As the special transposition (1 i) exchanges the first element of the permutation, say σ_1, with the i^{th} element, σ_i, we obtain a string in R that denotes this permutation by first doing a reversal of the prefix that includes everything up to, but not including, the symbol a_i, and then doing a reversal of the prefix of everything up to and including the symbol σ_i. These two prefix reversals have the cumulative effect of moving σ_i to the front, and putting σ_1 in a pair with a_i. Thus, the result is an appropriate representative.

If λ_j contains a_i second, then the i^{th} element of the permutation in Σ_n that r denotes, say σ_i, must be first in this pair. As the special transposition (1 i) exchanges the first element of the permutation, say σ_1, with the i^{th} element, σ_i, we obtain a string in R that denotes this permutation by a reversal of the prefix that includes everything up to and including the symbol σ_i. This prefix reversal has the effect of moving σ_i to the front, putting σ_1 in a pair with a_i, and simply reversing the order of the pairs $\lambda_2, \ldots, \lambda_{i-1}$. Thus, the result is an appropriate representative.

□

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2.3 A one-to-many dilation one embedding.

Theorem 2.3 gives a one-to-many, dilation 2 embedding of the star network of dimension n into the pancake network of dimension 2n-2. We can use an entirely different technique which results in an embedding with dilation one. To achieve this result, we represent each symbol by a *block* of symbols, where the particular symbol being represented is denoted by the number of symbols in the block. In addition, the position of each symbol is uniquely deduced from the size of the block representing it. In Theorem 2.4 we describe a dilation one, one-to-many embedding of stars into pancakes, in which distinct symbols are represented (in unary notation) by the size of an associated block of symbols in the host network and the sizes of blocks are such that the position of each symbol is encoded uniquely as well. As we shall see below, this embedding requires $O(n^3)$ symbols to represent permutations on $\{1,...,n\}$.

It is helpful to describe our mapping as one from permutations on $\{1,...,n\}$ into permutations defined over the union of two sets of symbols, $\mathbf{A}=\{A_1,A_2,A_3,...,A_t\}$, for some $t\geq 1$, and $\mathbf{B}=\{B_1,B_2,B_3,...,B_{n-1}\}$. The elements of **A** are called *A-symbols*, or simply A's, and the elements of **B**, similarly, *B-symbols*, or simply B's. A^x denotes the concatenation of x many symbols from **A**, and is called an *A-block of size x*. The B-symbols serve only as dividers between A-blocks and will be treated as indistinguishable, so we omit subscripts on the B's. Also, as we use only the *size* of the A-blocks in our mapping, we drop subscripts on the A's as well.

Let m=n+t-1, and P_m be the set of permutations defined on the symbols in $\mathbf{A} \cup \mathbf{B}$. In particular, each permutation in P_m will be viewed as a string of A-blocks interleaved with B-symbols. We are interested only in *well-formed* permutations, which are permutations for which every pair of B's is separated by

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at least one A. These permutations have the form $A^{x_1}BA^{x_2}B...BA^{x_i}B...A^{x_{n-1}}BA^{x_n}$, where $x_i \geq 1$ for $1 \leq i \leq n$.

Given a permutation $\sigma = \sigma_1\sigma_2...\sigma_n$ in Σ_n, our intention is to associate with each integer i ($1 \leq i \leq n$), an A-block in the corresponding permutation π. Moreover, integers i and j, $i \neq j$, will have disjoint size ranges for their associated A-blocks. Thus, the size, x_i, of the A-block representing σ_i gives us the position i and the symbol occupying the i^{th} position, namely σ_i. Specifically, the size of the A-block that represents σ_i is:

(2) $$x_i = (i-1)n + \sigma_i.$$

So, the permutation $\sigma = 216453$ can be represented, for example, by the permutation $\pi = A^2BA^7BA^{18}BA^{22}BA^{29}BA^{33}$. By defining a distinct range of values for σ_i and σ_j, $i \neq j$, the order of the A-blocks becomes irrelevant. Thus, the well-formed permutations $\pi = A^2BA^7BA^{18}BA^{22}BA^{29}BA^{33}$ and $\pi' = A^2BA^{33}BA^{18}BA^{29}BA^7BA^{22}$ both represent the permutation $\sigma = 216453$. However, not every well-formed permutation $\pi = A^{x_1}BA^{x_2}B ... BA^{x_i}B ... A^{x_{n-1}}BA^{x_n}$ represents a permutation in Σ_n, as one needs to ensure that distinct A-blocks represent distinct elements of $\{1,...,n\}$, and that the size of each A-block is in one of the permitted ranges.

To define the set of well-formed permutations that *do* represent permutations on $\{1,...,n\}$, we describe a useful shorthand notation. First note that once the size of each A block is known, the A's and B's themselves are no longer necessary. We can simply denote a well-formed permutation $\pi = A^{x_1}BA^{x_2}B...BA^{x_i}B...A^{x_{n-1}}BA^{x_n}$ by the list $x_1,x_2,...,x_i,...,x_{n-1},x_n$. Call this a *size list* for π. The size list in which the sizes are arranged in nondecreasing order is referred to as the *canonical size list.* Now define a function s, that maps sizes of A-blocks into $\{1,...,n\}$. Let x be the size of an A-block. Define s(x) by:

$$s(x) = x - (n * \lfloor (x-1)/n \rfloor).$$

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For a canonical size list $x_1,x_2,\ldots,x_n$, we get the corresponding list of integers $s(x_1), s(x_2),\ldots s(x_n)$. If the integers in the corresponding list of integers are distinct elements of $\{1,\ldots,n\}$, then π represents a permutation, namely the permutation σ, where $\sigma_i=s(x_i)$. We call a well-formed permutation $\pi=A^{x_1}BA^{x_2}B\ldots BA^{x_i}B\ldots A^{x_{n-1}}BA^{x_n}$ *proper,* if it represents a permutation σ on $\{1,\ldots,n\}$ in this way. Note that any re-ordering of A-blocks yields the same canonical size list, due to the distinct ranges of values.

Continuing the previous example, where $\pi=A^2BA^7BA^{18}BA^{22}BA^{29}BA^{33}$ represented the permutation $\sigma=216453$ on $\{1, 2,\ldots,6\}$, we get, for the block A^{33} that $s(33)=3$. This tells us that the symbol "3" is represented by this A-block, and it occupies the 6^{th} position in the corresponding permutation (according to equation 2). That is, $\sigma_6=3$. Note that $\pi=A^2BA^7BA^{18}BA^{22}BA^{29}BA^{33}$ and $\pi'=A^2BA^{33}BA^{18}BA^{29}BA^7BA^{22}$, both of which represent the same permutation $\sigma=216453$, have the same canonical size list, namely, 2,7,18,22,29,33.

We now compute the dimension of the pancake network needed for the embedding. In order to maintain the required size ranges for the A-blocks, we need zero extra A's in the block that represents the first symbol, n extra A's in the block that represents the second symbol, and, more generally, (i-1)n extra A's in the block that represents the i^{th} symbol. In addition, we need one A to represent the symbol "1", two A's to represent the symbol "2", and more generally, i A's to represent the i^{th} symbol. Finally we need (n-1) B's to delimit the n A-blocks. This gives us a grand total of m symbols, where m is given by

$$m=\sum_{i=1}^{n}(i-1)n+\sum_{i=1}^{n}i+n-1=(n+1)\sum_{i=1}^{n}i-n^2+n-1$$
$$=\frac{n(n+1)^2}{2}-n^2+n-1=\frac{1}{2}(n^3+3n-2).$$

The lemma that follows completes our formal description of the dilation one embedding of stars into pancakes. In the proof of the lemma, one must show

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how to simulate a transposition in the star network with a prefix reversal on a proper permutation in the pancake network. Transpositions in the star network are of the form (1 i), for some i>1. Our construction ensures that the first A-block in any proper permutation representation of a permutation σ on {1,...,n} represents the symbol σ_1. Thus, to simulate the transposition (1 i) we move the entire contents of the first A-block into the A-block representing σ_i. It follows that this reversal is possible only when the proper permutation $\pi=A^{x_1}BA^{x_2}B...BA^{x_i}B...A^{x_{n-1}}BA^{x_n}$ representing σ has $x_1 < x_i$. So, our embedding will map S_n into the set $R \subseteq P_m$, where R is defined by:

$$R=\{\pi \mid \pi = A^{x_1}BA^{x_2}B...BA^{x_i}B...A^{x_{n-1}}BA^{x_n}, \text{ and } x_1<x_i \text{ for all i, } 2\le i\le n, \text{ and } \pi \text{ is proper}\}.$$

Lemma: $S_n \overset{\text{dil } 1}{\Rightarrow} P_m$, where $m=\frac{1}{2}(n^3+3n-2)$.

Proof: Let P_m be the pancake network of size m, and let R be the subset of P_m defined above. Let π be a permutation in R and let $x_1,x_2,...,x_i,...,x_{n-1},x_n$ be its canonical size list. We define the co-embedding $g:R\to S_n$ by

$$g(\pi)=\begin{pmatrix} 1 & 2 & \dots & n \\ s(x_1) & s(x_2) & \dots & s(x_n) \end{pmatrix},$$

where the function s is defined above. (Note that since π is proper, $g(\pi)$ is a permutation on {1,...,n}, as desired.)

A transposition of the form (1 i) in the star network can be simulated by a single pancake move. Suppose that π represents a permutation σ in the star network. To simulate the transposition (1 i), locate the A-block whose size is in the range {(i-1)*n+1, ..., i*n}, and perform the prefix reversal that exchanges σ_i symbols from this block, say block p, with the first block. That is, if we start with a proper permutation whose first A-block has size σ_1 and whose p^{th} A-block has

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size $(i-1)*n+\sigma_i$, then after the appropriate prefix reversal, we would have a permutation π' whose first A-block has size σ_i and whose p^{th} A-block has size $(i-1)*n+\sigma_1$. Although this reverses the order of the intervening A-blocks, the representation is still correct as the order of the A-blocks is irrelevant. With a small amount of algebra, it is easily confirmed that since π is proper, π' is also proper, and that the prefix reversal does indeed simulate the desired transposition.

□

We improve this result by taking advantage of a few special features of the embedding. First, observe that the first symbol, σ_1, of any permutation σ in S_n is always represented by the first A-block of a corresponding permutation in R. Since its location is known, the size range of the first A-block need not be disjoint from others. For example, one can make the size ranges for the first two blocks identical. Similarly, the last symbol, σ_n, is always represented by the last A-block of a corresponding permutation in R. So the last A-block need not have extra symbols because its size range need not be distinct either. That is, its range can coincide exactly with the ranges of σ_1 and σ_2. This means that the size ranges for the remaining A-blocks (the blocks representing the symbols $\sigma_3, \ldots, \sigma_{n-1}$) can be more economical. That is, n extra A's can be used in the block that represents σ_3, 2n extra A's can be used in the block that represents σ_4, and in general, $(i-2)n$ extra A's can be used in the block that represents σ_i, for all i, $1 \leq i \leq n$. These observations allow us to decrease the size of the set **A** by $2n^2-3n$ symbols.

With a bit more work, further improvements are possible. One can make the size ranges of consecutive *pairs* of A-blocks in the canonical size list overlap in exactly one value. That is, if the range of one block is $\{(i-2)n+1, \ldots, (i-1)n\}$, the range of the next one can be $\{(i-1)n, \ldots, in-1\}$. In other words, the largest valid value for the block representing σ_i can coincide with the smallest valid value

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for the block representing σ_{i+1}. The appropriate values for σ_i and σ_{i+1} can still be correctly deduced. To see why, suppose that two adjacent sizes in the canonical size list for a permutation π in R are equal. Then in the corresponding list of integers, two adjacent integers will be identical. The first of these integers represents σ_i, for some i, and has its maximum value. So it must certainly represent n. That is, σ_i=n. The second represents σ_{i+1}, and has its minimum value. So it must represent 1. That is, σ_{i+1}=1. So one has an unambiguous interpretation of the canonical size list that yields a permutation in S_n. Consider the simulation of the transposition (1 i). Now, the A-blocks representing σ_i and σ_{i+1} have the same size, so one doesn't know which of these A-blocks to change. Observe that one can choose *either* block, because whether one changes the size of the A-block representing σ_i *or* the size of the A-block representing σ_{i+1}, the result is a permutation π' whose canonical size list accurately represents the appropriate new permutation. This improvement allows us to 'delete' one symbol from each A-block (excluding the first two A-blocks and the last A-block), which adds up to a savings of n-3 A's. Finally, we can use the symbol set {0,...,n-1} instead of {1,...,n} for S_n, saving n additional A-symbols.

These observations allow us to decrease the size of the set A by $2n^2$-3n+ n-3+n = $2n^2$-n-3 symbols. Thus, the number of symbols in the host network is

$$m=\frac{1}{2}(n^3+3n-2)-(2n^2-n-3)=\frac{1}{2}(n^3-4n^2+5n+4).$$

The improved results are summarized in the following theorem.

Theorem 2.4: $S_n \overset{\text{dil }1}{\Rightarrow} P_m$, where $m=\frac{1}{2}(n^3-4n^2+5n+4)$.

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Chapter 3

Embedding Pancakes into Stars

3.1. A one-to-many dilation one embedding.

There is no dilation 1 embedding of P_n into S_n for n>3 because P_n has cycles of odd length, whereas S_n does not. There is, however, a one-to-one dilation 2 map of P_4 into S_4. This is illustrated in Figure 3.1 below. Note that this mapping is simply the inverse of the mapping shown in Figure 2.1 in Chapter 2.

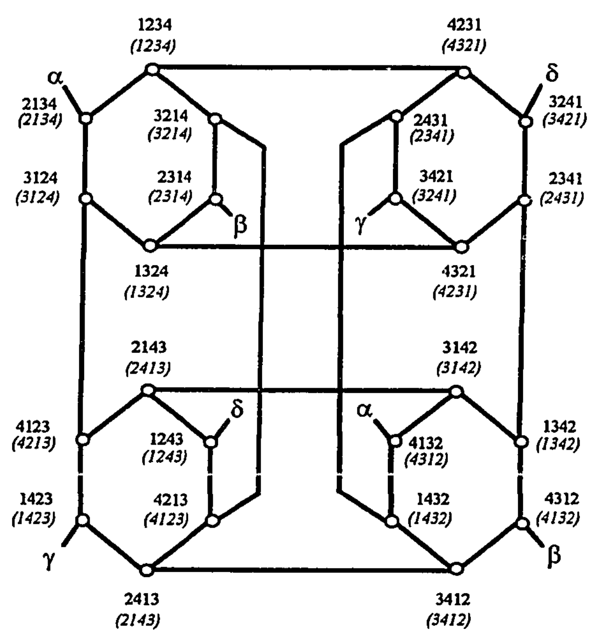

Figure 3.1. A one-to-one, dilation 2 embedding of P_4 into S_4.

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Consider a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$. We wish to develop a representation of σ suggestive of a linked list, in which adjacent symbols σ_i and σ_{i+1} are indicated by an *undirected edge* $\{\sigma_i, \sigma_{i+1}\}$. For example, the permutation σ=2,1,3,6,4,5 can be represented as the list 2,{2,1},{1,3},{3,6},{6,4},{4,5},5, where the first symbol, namely 2, represents the *head* of the list, the last symbol, 5, represents the *tail* of the list, and the pairs between the head and the tail represent undirected edges describing the adjacencies in σ. Note that 2n symbols are required to represent a permutation on {1,...,n} in this way. In fact, we shall represent a permutation σ on {1,...,n} by a permutation on the 2n symbols {1,...,n,**1,...,n**}. So, $\sigma_1,\sigma_2,\ldots,\sigma_n$ may be represented, for instance, by $\sigma_1,\boldsymbol{\sigma}_1,\sigma_2,\boldsymbol{\sigma}_2,\ldots,\boldsymbol{\sigma}_{n-1},\sigma_n,\boldsymbol{\sigma}_n$, where $\boldsymbol{\sigma}_i$ denotes σ_i, for all i ($1 \le i \le n$), and the length two subsequence σ_i,σ_{i+1} represents the edge $\{\sigma_i,\sigma_{i+1}\}$. In the discussion that follows, we will use the notation $\sigma_1,\boldsymbol{\sigma}_1,\sigma_2,\ldots,\boldsymbol{\sigma}_{n-1},\sigma_n,\boldsymbol{\sigma}_n$ interchangeably with $\sigma_1,\{\boldsymbol{\sigma}_1,\sigma_2\},\ldots,\{\boldsymbol{\sigma}_{n-1},\sigma_n\},\boldsymbol{\sigma}_n$, the latter notation being useful when we wish to emphasize adjacency relationships.

Note that a permutation σ is represented by more than one permutation on {1,...,n,**1,...,n**}. That is, any reordering of the edge list or of the symbols in any edge represents the same permutation. For example,

2,{**2**,1},{3,**1**},{**3**,6},{**6**,**4**},{4,5},**5** and 2,{**2**,1},{6,**3**},{4,5},{**1**,3},{**4**,**6**},**5**

represent the permutation 2,1,3,6,4,5. In addition, any interchanging of $\boldsymbol{\sigma}_i$ and σ_i also represents the same permutation. So, the permutation

2,{2,1},{**6**,**3**},{**4**,**5**},{**1**,3},{6,4},5 also represents 2,1,3,6,4,5. However, not all permutations on {1,...,n,**1,...,n**} represent permutations on {1,...,n} in the desired way. For example, a permutation containing an edge list that denotes a cycle does not represent a permutation on {1,...,n}, and neither does a permutation that contains a pair {j,**j**} or {**j**,j}. Let R denote the set of permutations on

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$\{1,\ldots,n,\mathbf{1,\ldots,n}\}$ that *do* represent permutations on $\{1,\ldots,n\}$ in the manner described. For any permutation π in R, we shall always choose the head to be π_1 and the tail to be π_{2n}. Note that π uniquely determines an edge list E. The edges in E will be pairs of symbols $\{\pi_i, \pi_{i+1}\}$, where i is an even integer. We now give an iterative algorithm to determine which permutation σ on $\{1,\ldots,n\}$, if any, is represented by a permutation π in R.

Algorithm A:

Initially, set $\sigma_1 = \pi_1$ and t=1. Repeat the following until t = n:

Each symbol in π appears twice: once in bold form and once in regular form, so *either* there is an edge $\{\sigma_t, \pi_j\}$ in E, *or* σ_t is not part of any other edge in E, *i.e.*, $\sigma_t = \pi_{2n}$ is the tail. In the latter case, there is no corresponding permutation σ on $\{1,\ldots,n\}$, and the algorithm fails. Otherwise, let $\{\sigma_t, \pi_j\}$ be this edge in E. Set $\sigma_{t+1} = \pi_j$, delete the edge $\{\sigma_t,\pi_j\}$ from E, and continue with t=t+1.

□

The set R described earlier is exactly that set of permutations that yield a permutation on $\{1, \ldots, n\}$ by the above algorithm. We now use this adjacency list representation as the basis of a one-to-many, dilation one embedding of pancakes into stars.

Theorem 3.1: $P_n \overset{\text{dil } 1}{\Rightarrow} S_{2n}$

Proof: Let $g{:}R \to P_n$ be the co-embedding defined by $g(\pi) = \sigma$, where σ is the permutation produced by Algorithm A. We show that g has dilation 1. Let σ be an arbitrary permutation on $\{1,\ldots,n\}$ and let π be a permutation in R such that

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$g(\pi) = \sigma$. Consider a prefix reversal of length i, for any i ($2 \le i \le n$), and let σ' be the permutation obtained from σ by this reversal. We show that there is a permutation π' at distance one from π in S_{2n} such that $g(\pi') = \sigma'$.

If $i = n$, then $\sigma' = \sigma^R$. We simulate a prefix reversal of length n by exchanging $\pi_1 = \sigma_1$ and $\pi_{2n} = \sigma_n$. That is, we perform the special transposition (1 2n) on π. The result is a permutation π' with head σ_n and tail σ_1, but otherwise identical to π. Thus, π and π' represent exactly the same adjacencies, so $g(\pi') = \sigma'$ as desired. This result is achieved with one special transposition.

For all other permitted values of i, the prefix reversal of length i brings σ_i to the front, and in doing so, breaks a single adjacency, *i.e.*, the adjacency between σ_i and σ_{i+1}, and creates a single new adjacency, *i.e.*, the adjacency between σ_1 and σ_{i+1}. This adjacency between σ_i and σ_{i+1} in the permutation σ is represented in π as an edge between σ_i and σ_{i+1}, and there is also a representation of σ_1 as π_1. Therefore, to simulate in the permutation π this prefix reversal, it is sufficient to exchange π_1 with the representative of σ_i in the edge in π representing the adjacency between σ_i and σ_{i+1}. This has the effect of replacing the adjacency between σ_i and σ_{i+1} with an adjacency between $\pi_1 = \sigma_1$ and σ_{i+1}, and of moving a representative of σ_i to the front. Consequently this does, in fact, result in a representation of σ', as desired. This interchange can be done by a single special transposition, as it is an exchange of the first symbol with another, hence the embedding has dilation 1.

$\square$

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3.2 A one-to-many dilation three embedding.

In the embedding of Theorem 3.1, each adjacency in σ is represented explicitly as an edge, *i.e.*, a pair of symbols in π. This requires that each symbol in σ has two representatives in π, hence, the representation needs a total of 2n symbols. One can use less than 2n symbols by representing some adjacencies by singletons rather than pairs. For some $m \le n$, let $D=\{d_1,\ldots,d_m\}$ be the set of symbols we wish to represent by single symbols. We call D the set of *designated symbols*. Each designated symbol will have only one representative in π, the other occurrence being *implied* rather than explicit. For each implied occurrence, we reserve a position in π. That is, suppose d_k is a designated symbol and the j^{th} position in π is reserved to indicate the symbol adjacent to d_k. Then a correct representation of σ will have in position j a description of the symbol before or after d_k in the permutation σ. For example, consider the permutation $\sigma=2,1,3,6,4,5$. Suppose the symbol "3" is a designated symbol and the second position in π is reserved for a symbol adjacent to 3 in σ. Two possible representations of σ are the lists $2,6,\{\mathbf{2},\mathbf{1}\},\{1,3\},\{\mathbf{6},4\},\{\mathbf{4},\mathbf{5}\},5$ and $2,\mathbf{1},\{\mathbf{2},\mathbf{1}\},\{6,4\},\{\mathbf{4},\mathbf{5}\},\{\mathbf{6},3\},\mathbf{5}$.

Define Δ to be the set $\{1,\ldots,n,\mathbf{1},\ldots,\mathbf{n}\}\setminus \mathbf{D}$, where $\mathbf{D}=\{\mathbf{d}_1,\ldots,\mathbf{d}_m\}$ is the set of designated symbols in bold form. Our improved mapping takes permutations on $\{1,\ldots,n\}$ into permutations on the set Δ. Once again, a permutation σ on $\{1,\ldots,n\}$ is represented by more than one permutation on Δ, however, not all permutations on Δ represent permutations on $\{1,\ldots,n\}$ in the way desired. Let R denote the set of permutations on Δ that *do* represent permutations on $\{1,\ldots,n\}$. We present an iterative algorithm to determine which permutation σ on $\{1,\ldots,n\}$, *if any*, is represented by a permutation π in R.

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Algorithm B:

Initially, set $\sigma_1 = \pi_1$ and $t = 1$. Repeat the following until $t = n$:

If $\sigma_t \in D$ (that is, σ_t is a designated symbol), then there is a position in π, say the p^{th} position, reserved for a symbol adjacent to σ_t. If π_p has not been marked, set $\sigma_{t+1} = \pi_p$, mark π_p, and continue with $t = t+1$. Otherwise, there is an edge $\{\sigma_t, \pi_j\}$ in E. Set $\sigma_{t+1} = \pi_j$, delete the edge $\{\sigma_t, \pi_j\}$ from E, and continue with $t = t+1$.

If σ_t is not a designated symbol, it follows that this symbol appears twice in π: once in bold form and once in regular form. It also follows that one of the following two cases must hold:

Case 1: There is an edge $\{\sigma_t, \pi_j\}$ in E. Set $\sigma_{t+1} = \pi_j$, delete the edge $\{\sigma_t, \pi_j\}$ from E, and continue with $t = t+1$.

Case 2: σ_t is not part of any other edge in E. If σ_t is the tail of π then there is no corresponding permutation σ on $\{1,...,n\}$, and the algorithm fails. Otherwise, σ_t must occupy a position, say the p^{th} position, reserved for a symbol adjacent to a designated symbol. Let d_k be this designated symbol. Set $\sigma_{t+1} = d_k$, mark π_p, and continue with $t = t+1$.

$\square$

The set R is exactly that set of permutations on Δ that yield a permutation on $\{1, ..., n\}$ by the above algorithm. We use the ideas described above for an embedding of pancakes into stars that uses one less symbol, but at the cost of an

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increase in dilation. In Theorem 3.2 below, we use one designated symbol to achieve an embedding of the pancake network of dimension n into the star network of dimension 2n-1. The embedding has dilation 3.

Theorem 3.2: $P_n \overset{\text{dil } 3}{\Rightarrow} S_{2n-1}$

Proof: Let D be the set consisting of a single designated symbol, and let Δ and R be as defined above. Without loss of generality, suppose that D={1}. Let g:R→P_n be the co-embedding defined by $g(\pi) = \sigma$, where σ is the permutation produced by Algorithm B. We show that g has dilation 3. Let σ be an arbitrary permutation on {1,...,n} and let π be a permutation in R such that $g(\pi) = \sigma$. Without loss of generality, suppose that the second position in π is reserved for a symbol adjacent to the designated symbol "1". (Note that any position in π except the head can be reserved for this purpose.) Consider a prefix reversal of length i, for any i ($2 \leq i \leq n$), and let σ' be the permutation obtained from σ by this reversal. We show that there is a permutation π' at distance (at most) three from π in S_{2n-1} such that $g(\pi') = \sigma'$.

If i = n, then $\sigma' = \sigma^R$. We simulate a prefix reversal of length n by exchanging $\pi_1 = \sigma_1$ and $\pi_{2n-1} = \sigma_n$. That is, we perform the special transposition (1 2n-1) on π. The result is a permutation π' with head σ_n and tail σ_1, but otherwise identical to π. Thus, π and π' represent exactly the same adjacencies, so $g(\pi') = \sigma'$ as desired. This result is achieved with one special transposition.

For all other permitted values of i, the prefix reversal of length i brings σ_i to the front, and in doing so, breaks a single adjacency, *i.e.,* the adjacency between σ_i and σ_{i+1}, and creates a single new adjacency, *i.e.,* the adjacency between σ_1 and σ_{i+1}. In all but one case, the prefix reversal can be simulated by a single special transposition, as described in Theorem 3.1. However there is one

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case that requires, at worst, three special transpositions, namely the case in which σ_i is a designated symbol and the adjacency between σ_i and σ_{i+1} is implied, *i.e.*, $\sigma_i = 1$ and $\pi_2 = \sigma_{i+1}$. In this case there is no edge that represents the adjacency we wish to break. Moreover, the representative of σ_i that we would normally move to the head cannot be moved, due to the fact that it is an implied representative. However, there is an explicit edge in E that represents the adjacency between σ_i and σ_{i-1}. We alter this edge in our simulation of the prefix reversal as follows.

$$
\begin{aligned}
\pi = {} & \pi_1, \sigma_{i+1}, \{\pi_3,\pi_4\},\ldots,\{\sigma_i,\sigma_{i-1}\}, \ldots, \pi_{2n-1} \\
\mapsto {} & \sigma_{i-1},\sigma_{i+1},\{\pi_3,\pi_4\},\ldots,\{\sigma_i,\pi_1\}, \ldots, \pi_{2n-1} \\
\mapsto {} & \sigma_{i+1},\sigma_{i-1},\{\pi_3,\pi_4\},\ldots,\{\sigma_i,\pi_1\}, \ldots, \pi_{2n-1} \\
\mapsto {} & \sigma_i, \quad \sigma_{i-1},\{\pi_3,\pi_4\},\ldots,\{\sigma_{i+1},\pi_1\},\ldots,\pi_{2n-1} \\
= {} & \pi'
\end{aligned}
$$

Note that π' represents the permutation $\sigma_i,\sigma_{i-1},\ldots,\sigma_1\sigma_{i+1},\ldots,\sigma_n$, hence, $g(\pi')=\sigma'$, as desired. We use at most three special transpositions to transform π into π', hence, the embedding has dilation three.

□

As an example of the worst case, consider the permutation σ=2,6,3,1,4,5. Suppose the symbol '1' is a designated symbol and the second position in π is reserved for a symbol adjacent to 1 in σ. A valid representation of σ is the permutation π= 2,4,$\{\mathbf{2,6}\}$,{6,3},$\{\mathbf{3},1\}$,$\{\mathbf{4,5}\}$,5. A prefix reversal of size 4 would bring 1 to the front, resulting in the permutation σ'=1,3,6,2,4,5. According to Theorem 3.2, we simulate this prefix reversal on π as follows:

$$
\begin{aligned}
\pi = {} & 2,4,\{\mathbf{2,6}\},\{6,3\},\{\mathbf{3},1\},\{\mathbf{4,5}\},5 \\
\mapsto {} & \mathbf{3},4,\{\mathbf{2,6}\},\{6,3\},\{2,1\},\{\mathbf{4,5}\},5 \\
\mapsto {} & 4,\mathbf{3},\{\mathbf{2,6}\},\{6,3\},\{2,1\},\{\mathbf{4,5}\},5
\end{aligned}
$$

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$\mapsto 1,\mathbf{3},\{\mathbf{2},\mathbf{6}\},\{6,3\},\{2,4\},\{\mathbf{4},\mathbf{5}\},5$

$= \pi'$

The transformation of π into π' takes 3 special transpositions, and π' is, in fact, a valid representation of σ'.

Note that the decrease in the dimension of the host network comes at the price of an increase in dilation. It can be shown that for each additional designated symbol, the dilation increases by an additive factor of 2. That is, for m designated symbols, the dilation becomes 1+2m.

We now extend the result of Theorem 3.1 by showing that $BP_n \overset{\text{dil } 1}{\Rightarrow} S_{2n}$. The mapping is based on the notion that a sequence of n burnt pancakes can be represented by a sequence of 2n unburnt pancakes. That is, the technique is to represent the i^{th} burnt pancake by a combination two unburnt pancakes, say the $(2i-1)^{th}$ and the $(2i)^{th}$ unburnt pancakes, which are adjacent and never separated. The sign of the i^{th} burnt pancake is represented by the relative order of these two unburnt pancakes. That is, the sign is positive if the $(2i-1)^{th}$ unburnt pancake is before the $(2i)^{th}$ unburnt pancake, and is negative otherwise. As the pair of unburnt pancakes always remains adjacent, it need not be represented in our list of adjacencies. That is, our edge list needs to describe adjacencies between even numbered and odd numbered unburnt pancakes only. So, the edge list consists of n-1 undirected pairs.

The construction resembles the mapping of Theorem 3.1. In this case, the co-domain is a subset of the permutations on {1,...,2n}. As noted, adjacencies between certain pairs of unburnt pancakes are implicit and are not explicitly represented. For example, the permutation σ =2,3,(-1),4,5,6 is represented by the list 3,{4,5},{6,2},{1,7},{8,9},{10,11},12, or equivalently by the list, 3,{5,4},{8,9},{2,6},{7,1},{10,11},12. That is, with 3=2*2-1 at the head of the list, the burnt pancake sequence must begin with +2. Then, as the other half of

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the burnt pancake +2 is 4, and 4 is paired with 5=2*3-1, the next element of the burnt pancake sequence is +3, and so on. Observe that a signed permutation may be represented by more than one unsigned permutation because any reordering of the edge list or of the symbols in any edge represents the same signed permutation. However, not all permutations on {1,...,2n} represent signed permutations on {1,...,n}. For example, a permutation with a pair representing an adjacency between the $(2i-1)^{th}$ and the $(2i)^{th}$ unburnt pancakes does not correspond to *any* permutation in the domain. This is so, as the $(2i-1)^{th}$ and $(2i)^{th}$ pancakes always remain adjacent, hence our representation never represents an edge between them explicitly. And, if such an edge were listed, then it would have to be at the expense of omitting a needed adjacency.

Let R denote the set of permutations on {1,...,2n} that *do* represent permutations on {1,...,n} in the manner described. For any permutation $\pi = \pi_1\pi_2...\pi_{2n}$ in R, we shall always choose the head to be π_1 and the tail to be π_{2n}. Note that π uniquely determines an edge list E. That is, the edges in E will be pairs of symbols $\{\pi_{2i-1}, \pi_{2i}\}$, where i is an integer. We now give an iterative algorithm to determine which permutation σ on {1,...,n}, if any, is represented by a permutation π in R.

Algorithm C:

Initially, set t=1. If π_1 is even, set $\sigma_1 = -\left\lceil \frac{\pi_1}{2} \right\rceil$, and set $j=\pi_1-1$.

Otherwise set $\sigma_1 = +\left\lceil \frac{\pi_1}{2} \right\rceil$, and set $j=\pi_1+1$. Repeat the following until t = n:

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Locate the subset $\{j, \pi_k\}$ in the edge list E. If $\{j, \pi_k\}=\{2i-1, 2i\}$, for some i, $1\le i\le n$, there is no corresponding signed permutation σ on the symbols $\{1,...,n\}$, and the algorithm fails. Otherwise, if π_k is even, set $\sigma_{t+1}= -\left\lceil \frac{\pi_k}{2} \right\rceil$, and set $j=\pi_k-1$. On the other hand, if π_k is odd, set $\sigma_{t+1} = +\left\lceil \frac{\pi_k}{2} \right\rceil$, and set $j=\pi_k+1$. Continue with $t=t+1$.

□

The set R described earlier is exactly that set of permutations that yield a signed permutation on $\{1, ..., n\}$ by the above algorithm. We now present a one-to-many, dilation one embedding of the burnt pancake network of dimension n into the star network of dimension 2n.

Theorem 3.3: $BP_n \overset{\text{dil } 1}{\Rightarrow} S_{2n}$.

Proof: Let $g:R\rightarrow BP_n$ be the co-embedding defined by $g(\pi) = \sigma$, where σ is the signed permutation produced by Algorithm C. We show that g has dilation 1. Let σ be an arbitrary signed permutation on the symbols $\{1,...,n\}$ and let π be a permutation in R such that $g(\pi) = \sigma$. Consider a prefix reversal of length i, for any i ($1\le i\le n$), and let σ' be the permutation obtained from σ by this reversal. We show that there is a permutation π' at distance one from π in S_{2n}, that is, that $g(\pi') = \sigma'$.

If $i = n$, then σ' is the reversal of σ with all signs switched. We simulate a prefix reversal of length n by exchanging π_1 and π_{2n}. That is, we perform the special transposition (1 2n) on π. The result is a permutation π' with head π_{2n} and tail π_1, but otherwise identical to π. In particular, π and π' represent exactly the same adjacencies. The effect is that the edge list is now in reverse order, and the

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output of Algorithm C is a permutation that is the reversal of σ, with the signs of all symbols changed. Note that the sign change for each symbol is accomplished simply by the fact that the edge list is now read in reverse order. That is, $g(\pi') = \sigma'$, as desired. This result is achieved with one special transposition.

Next suppose that $i = 1$. The effect of a prefix reversal of length 1 is to change the sign of σ_1, leaving $\sigma_2,\ldots,\sigma_n$ unaltered. If σ_1 is positive, then $\pi_1=2|\sigma_1|-1$, and the adjacency between σ_1 and σ_2 is represented in π by a subset $\{\pi_m, \pi_{m'}\}$ in E, where $\pi_m = 2|\sigma_1|$. We simulate the prefix reversal of length i by exchanging π_1 with π_m. The result is a permutation π' with head $2|\sigma_1|$, and in which the edge representing the adjacency between σ_1 and σ_2 contains the symbol $2|\sigma_1|-1$. The permutation π' is otherwise identical to π. Thus, π and π' represent exactly the same adjacencies, however, π' is now even. Hence, Algorithm C assigns a negative sign to σ_1 and yields the same output for the remaining symbols. So $g(\pi') = \sigma'$ as desired. This is achieved with one transposition. (The case in which σ_1 is negative is proved in a similar fashion, noting however that $\pi_1= 2|\sigma_1|$ and $\pi_m = 2|\sigma_1|-1$.)

For all other permitted values of i, the prefix reversal of length i brings σ_i to the front, and in doing so, breaks a single adjacency, *i.e.*, the adjacency between σ_i and σ_{i+1}, and creates a single new adjacency, *i.e.*, the adjacency between σ_1 and σ_{i+1}. In the process, the signs of $\sigma_1,\ldots,\sigma_i$ are changed in the same manner as noted earlier. The adjacency between σ_i and σ_{i+1} is represented in π by a subset $\{\pi_m, \pi_{m'}\}$ in E, where

$$\{\pi_m, \pi_{m'}\} = \begin{cases} \{2|\sigma_i|,\ 2|\sigma_{i+1}|-1\}, & \text{if both } \sigma_i \text{ and } \sigma_{i+1} \text{ are positive,} \\ \{2|\sigma_i|-1,\ 2|\sigma_{i+1}|\}, & \text{if both } \sigma_i \text{ and } \sigma_{i+1} \text{ are negative,} \\ \{2|\sigma_i|-1,\ 2|\sigma_{i+1}|-1\}, & \text{if } \sigma_i \text{ is negative and } \sigma_{i+1} \text{ is positive,} \\ \{2|\sigma_i|,\ 2|\sigma_{i+1}|\}, & \text{if } \sigma_i \text{ is positive and } \sigma_{i+1} \text{ is negative.} \end{cases}$$

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We simulate a prefix reversal of length i by exchanging π_1 and π_m. The result is a permutation π' with head π_m and in which the edge $\{\pi_m, \pi_{m'}\}$ is replaced by $\{\pi_1, \pi_{m'}\}$, but otherwise π' is identical to π. The effect is that the portion of the edge list that represents the adjacencies between $\sigma_1,...,\sigma_i$ is now processed by Algorithm C in reverse order. Hence the output of Algorithm C is a permutation in which the order of $\sigma_1,...,\sigma_i$ is reversed, and the signs of these symbols are changed. So $g(\pi') = \sigma'$ as desired. This is achieved with one transposition.

□

For example, the permutation σ =2,3,-1,4,5,6 can be represented by π =3,{4,5},{6,2},{1,7},{8,9},{10,11},12. We simulate a prefix reversal of size four by performing the special transposition (1 8) on π as illustrated below:

$$\begin{aligned}\pi &= 3,\{4,5\},\{6,2\},\{1,7\},\{8,9\},\{10,11\},12\\ &\mapsto 8,\{4,5\},\{6,2\},\{1,7\},\{3,9\},\{10,11\},12\\ &= \pi'.\end{aligned}$$

It is easily verified that $g(\pi') = -4,1,-3,-2,5,6 = \sigma'$, as desired.

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Chapter 4

Cycles and Cycle Prefix Networks

4.1 Comparing Cycle Prefix Networks with Pancakes and Stars.

Observe that there is a one-to-one, dilation 2 embedding of the cycle prefix network of dimension n into the pancake network of the same dimension via the identity map. To simulate a cycle prefix operation that brings the i^{th} symbol to the front, we simply perform a prefix reversal of the symbols up to, but not including, the i^{th} symbol, followed by a prefix reversal of the symbols up to, and including, the i^{th} symbol. Hence, an arbitrary cycle prefix operation can be simulated by two prefix reversals, and the embedding has dilation 2. The inverse of this cycle prefix operation, *i.e.*, one that moves the first symbol to position i and shifts to the left the symbols in positions 1, ..., i-1, is simulated by the same two prefix reversals, applied in reverse order. This result is stated in Theorem 4.1 below.

Theorem 4.1: $CP_n \xrightarrow{\text{dil } 2} P_n$.

There is also a one-to-one, dilation 2 embedding of the star network of dimension n into the cycle prefix network of the same dimension, again via the identity map. To simulate a special transposition of the form (1 i), we first perform a cycle prefix operation that puts the first symbol into the i^{th} position and shifts the second through i^{th} symbols to the left. We follow this with a cycle prefix

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operation that moves the symbol now occupying the $(i-1)^{th}$ position to the front. The result of the consecutive application of these two cycle prefix operations is a special transposition of the form (1 i). That is, we perform the following operations to simulate the special transposition (1 i): $a\alpha b\beta \mapsto \alpha b a \beta \mapsto b \alpha a \beta$, where α is a string of i-2 symbols, and β is a string of n-i symbols. We state this result in Theorem 4.2.

Theorem 4.2: $S_n \overset{\text{dil } 2}{\rightarrow} CP_n$.

The embedding of Theorem 4.2, combined with the embedding of Theorem 3.1 gives us a one-to-many, dilation 2 embedding of the pancake network of dimension n into the cycle prefix network of dimension 2n, which we describe in the next theorem.

Theorem 4.3: $P_n \overset{\text{dil } 2}{\Rightarrow} CP_{2n}$

Proof: Let $f_1: P_n \overset{\text{dil } 1}{\Rightarrow} S_{2n}$ be the embedding of Theorem 3.1, and let $f_2: S_n \overset{\text{dil } 2}{\rightarrow} CP_n$ be the embedding of Theorem 4.2. Define $f: P_n \overset{\text{dil } 2}{\Rightarrow} CP_{2n}$ by $f_1 \circ f_2$, where $\circ$ denotes function composition.

□

We now describe a one-to-many, dilation two embedding of the cycle prefix network of dimension n into the star network of twice the dimension. The construction is similar that of Theorem 3.1, which describes a dilation one embedding of pancakes into stars based on a list of adjacent pairs. In this case, however, *two* edges in the list must be modified to simulate a cycle prefix operation. We use a representation of σ suggestive of a list of adjacent pairs, in

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which adjacent nodes σ_i and σ_{i+1} are joined by an undirected edge $\{\sigma_i,\sigma_{i+1}\}$. As discussed in Theorem 3.1, we use the 2n symbols $\{1,\ldots,n,\mathbf{1},\ldots,\mathbf{n}\}$ to represent a permutation on $\{1,\ldots,n\}$. So, $\sigma = \sigma_1,\sigma_2,\ldots,\sigma_n$ may be represented, for instance, by $\sigma_1,\boldsymbol{\sigma}_1,\sigma_2,\boldsymbol{\sigma}_2,\ldots,\sigma_n,\boldsymbol{\sigma}_n$, where $\boldsymbol{\sigma}_i$ denotes σ_i, for all i ($1 \le i \le n$), and the length two subsequence σ_i,σ_{i+1} represents the edge $\{\sigma_i,\sigma_{i+1}\}$.

Once again, we note that a permutation σ is represented by more than one permutation on $\{1,\ldots,n,\mathbf{1},\ldots,\mathbf{n}\}$. That is, any reordering of the edge list or of the symbols in any edge represents the same permutation. However, not all permutations on $\{1,\ldots,n,\mathbf{1},\ldots,\mathbf{n}\}$ represent permutations on $\{1,\ldots,n\}$. For example, a permutation containing an edge list that denotes a cycle does not represent a permutation on $\{1,\ldots,n\}$, and neither does a permutation that contains a pair $\{j,\mathbf{j}\}$ or $\{\mathbf{j},j\}$. Let R denote the set of permutations on $\{1,\ldots,n,\mathbf{1},\ldots,\mathbf{n}\}$ that *do* represent permutations on $\{1,\ldots,n\}$ in the manner described. For any permutation, π, in R, we shall always choose the head to be π_1 and the tail to be π_{2n}. Note that π uniquely determines an edge list E. The edges in E will be pairs of symbols $\{\pi_i,\pi_{i+1}\}$, where i is an even integer. The iterative Algorithm A given for Theorem 3.1 determines which permutation σ on $\{1,\ldots,n\}$, if any, is represented by a permutation π in R.

Theorem 4.4: $CP_n \overset{\text{dil 2}}{\Rightarrow} S_{2n}$.

Proof: Let $g{:}R \to CP_n$ be the co-embedding defined by $g(\pi) = \sigma$, where σ is the permutation produced by Algorithm A. We show that g has dilation 2. Let σ be an arbitrary permutation on $\{1,\ldots,n\}$ and let π be a permutation in R such that $g(\pi) = \sigma$. Consider a cycle prefix operation on the first i symbols in σ, for any i

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($2 \leq i \leq n$), and let σ' be the permutation obtained from σ by this operation. We show that there is a permutation π' at distance two from π in S_{2n} such that $g(\pi') = \sigma'$.

Suppose $i = n$. The cycle prefix operation of length n brings σ_n to the front, and in doing so, breaks one adjacency, *i.e.*, the adjacency between σ_{n-1} and σ_n. In addition, one new adjacency is created, namely the adjacency between σ_1 and σ_n. Adjacencies in the permutation σ are represented in π as edges between appropriate symbols, and there is also a representation of σ_1 as π_1 and of σ_n as π_{2n}. Therefore, to simulate this cycle prefix operation, we first exchange π_1 with the representative of σ_{n-1} in the edge in π representing the adjacency between σ_{n-1} and σ_n. This has the effect of replacing the adjacency between σ_{n-1} and σ_n with an adjacency between $\pi_1=\sigma_1$ and σ_n. Next, we exchange σ_{n-1}, which is now in the first position, with $\pi_{2n}=\sigma_n$. This has the effect of making σ_n the head and σ_{n-1} the tail. Consequently these two special transpositions do, in fact, result in a representation of σ', as desired. We have just shown that the transformation can be accomplished with two special transpositions, as each step is an exchange of the first symbol with another.

For all other permitted values of i, the cycle prefix operation of length i brings σ_i to the front, and in doing so, breaks two adjacencies, *i.e.*, the adjacency between σ_i and σ_{i-1} and the one between σ_i and σ_{i+1}. In addition, two new adjacencies are created, namely the adjacency between σ_1 and σ_i, and the adjacency between σ_{i-1} and σ_{i+1}. Adjacencies in the permutation σ are represented in π as edges between appropriate symbols, and there is also a representation of σ_1 as π_1. Therefore, to simulate the cycle prefix operation, we first exchange π_1 with the representative of σ_{i-1} in the edge in π representing the adjacency between σ_{i-1} and σ_i. This has the effect of replacing the adjacency between σ_{i-1} and σ_i with

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an adjacency between $\pi_1=\sigma_1$ and σ_i. Next, we exchange σ_{i-1}, which is now in the first position, with the representative of σ_i in the edge in π representing the adjacency between σ_i and σ_{i+1}. This has the effect of replacing the adjacency between σ_i and σ_{i+1} with an adjacency between σ_{i-1} and σ_{i+1}, and of moving a representative of σ_i to the front. Consequently these two special transpositions do, in fact, result in a representation of σ', as desired. We have just shown that the transformation can be accomplished with two special transpositions, as each step is an exchange of the first symbol with another.

In both cases, a cycle prefix operation can be simulated by two special transpositions, hence the embedding has dilation 2.

$\square$

For example, suppose σ is the identity permutation. A valid representation of σ is the permutation π=1,{2,3},{**3**,4},{**4**,5},{**1**,**2**},{**5**,6},**6**. A cycle prefix operation on the prefix of length 4 results in the permutation σ'=4,1,2,3,5,6. The following sequence of two special transpositions transforms π into π', such that $g(\pi') = \sigma'$.

$$\pi = 1,\{2,3\},\{\mathbf{3},4\},\{\mathbf{4},5\},\{\mathbf{1},\mathbf{2}\},\{\mathbf{5},6\},\mathbf{6}$$
$$\mapsto \mathbf{3},\{2,3\},\{1,4\},\{\mathbf{4},5\},\{\mathbf{1},\mathbf{2}\},\{\mathbf{5},6\},\mathbf{6}$$
$$\mapsto \mathbf{4},\{2,3\},\{1,4\},\{\mathbf{3},5\},\{\mathbf{1},\mathbf{2}\},\{\mathbf{5},6\},\mathbf{6} = \pi'.$$

Other embeddings, such as $BP_n \overset{\text{dil } 2}{\Rightarrow} CP_{2n}$ and $CP_n \overset{\text{dil } 2}{\Rightarrow} BP_n$, can be obtained from these results by function composition.

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4.2 Constructing cycles in pancake networks.

Many more structural comparisons can be made between the various Cayley graphs on the symmetric group. For example, it is known that for $n \geq 3$, the star network of dimension n contains all cycles of even lengths between 6 and n! [20]. Is this true for the pancake network? Interestingly, a stronger statement can be made for the pancake network of dimension n for $n \geq 3$, namely, that it contains cycles of all lengths between 6 and n! We give a proof of this. The proof uses the notion of *vertex symmetry* [1], which is defined as follows. Let x and y be vertices of a graph G. G is *vertex symmetric* if there exists an *automorphism* that maps x into y for every pair of vertices x and y. An *automorphism* of a graph is a permutation of the vertices that preserves adjacencies, that is, an automorphism is an isomorphism of a graph to itself. It has been shown that every Cayley graph is vertex symmetric [1]. This means that the symbols in vertex labels can be permuted in a uniform way without altering the structural properties of the graph. Furthermore, for any i, $1 \leq i \leq n$, all i-dimensional edges are equivalent to each other. In particular, this means that if x and y are i-dimensional neighbors, the dimension of their common edge is preserved under any relabeling of the vertices. For example, consider the adjacent vertices in P_5 with labels x = 34521 and y = 25431. Observe that x and y are joined by a 4-dimensional edge. Suppose we uniformly permute the symbols labeling the vertices by (for example) exchanging the symbols 3 and 5. That is, we relabel the vertices according to the permutation $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix}$. The new labels are $\delta \circ x = 54321$ and $\delta \circ y = 23451$. Observe that $\delta \circ x$ and $\delta \circ y$ are 4-dimensional neighbors. That is, the dimensionality of the edge is preserved by the mapping δ.

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Our proof also uses the property that P_n contains n disjoint copies of P_{n-1} interconnected via n-dimensional edges. Akers and Krishnamurthy [1] refer to this as the *recursive decomposition property*, a property the pancake network has in common with many other Cayley graphs. We use a sequence of n-dimensional edges and (n-1)-dimensional edges to form a cycle in our proof.

Specifically, we describe a cycle of length 2n consisting of n-dimensional edges alternating with (n-1)-dimensional edges. Basically, this is the result of transforming a standard cycle prefix sequence into a prefix reversal sequence. That is, let π be an arbitrary permutation on n symbols. By definition, a cycle prefix operation of size n moves π's first symbol to the n^{th} position and shifts the second through n^{th} symbols one position to the left. Hence, a sequence of n cycle prefix operations of size n applied to π generates the following sequence of permutations:

$$
\begin{aligned}
\pi &= \pi_1\pi_2 \ldots \pi_{n-1}\pi_n \\
&\mapsto \pi_2 \ldots \pi_{n-1}\pi_n\pi_1 \\
&\qquad \ldots \\
&\mapsto \pi_{n-1}\pi_n \ldots \pi_1\pi_2 \\
&\mapsto \pi_n\pi_1\pi_2 \ldots \pi_{n-1} \\
&\mapsto \pi_1\pi_2 \ldots \pi_{n-1}\pi_n \\
&= \pi.
\end{aligned}
$$

Note that the first and last permutations in the sequence are identical, that is, the sequence describes a cycle of length n. The proof of Theorem 4.1 showed that a cycle prefix operation of size i can be simulated by two prefix reversals: one of size i followed by one of size i-1, or these two prefix reversals in reverse order. In fact, the above sequence of permutations is equivalent to a sequence of 2n prefix reversals alternating between size n and size n-1. That is, starting at an arbitrary vertex in P_n, we can describe a cycle of length 2n by performing the sequence of

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prefix reversals n, n-1, n, n-1, ..., n, n-1 (a total of 2n prefix reversals). Let such a cycle in P_n be denoted by CYC_{2n}. Since P_n is vertex symmetric, a cycle isomorphic to CYC_{2n} exists for every starting vertex π in P_n.

Theorem 4.5: For $n \geq 3$ and for all i, $6 \leq i \leq n!$, P_n has cycles of length i.

Proof: (By induction on n). The statement is trivially true for n = 3. For n = 4, a cycle of length i, for each i, $6 \leq i \leq 4!$, is described in Figure 4.1. Since P_4 is vertex symmetric, every vertex in P_4 can be included in at least one cycle of each length i.

Assume that the statement is true for P_{n-1} and now consider P_n. Since P_{n-1} is a subgraph of P_n, all cycles of length i, $6 \leq i \leq (n-1)!$, exist in P_n by the induction hypothesis. To construct cycles for i in the range $(n-1)! < i \leq n!$, observe that any i in the required range can be decomposed into a sum involving integers $a_1, a_2, \ldots, a_k$, such that

$$i = (2n-k) + (a_1-1) + (a_2-1) + \ldots + (a_j-1) + \ldots + (a_k-1),$$

where $k \leq n$ and $(n-2)! < a_j \leq (n-1)!$ for all j, $1 \leq j \leq k$. The resulting sequence of integers $a_1, a_2, \ldots, a_k$, gives the lengths of a sequence of cycles that can be merged with CYC_{2n} to form a cycle of length i. The idea is as follows. Let $CYCa_j$ be a cycle of length a_j in the j^{th} copy of P_{n-1} such that $CYCa_j$ and CYC_{2n} share an (n-1)-dimensional edge. $CYCa_j$ clearly exists by the induction hypothesis. Furthermore, $CYCa_j$ must contain an (n-1)-dimensional edge due to its size. That is, there are no cycles of length a_j in any copy of the (n-2)-dimensional pancake P_{n-1}. Merge the two cycles by deleting the shared edge. Continue this process for all j in the described range to create a cycle of length i.

□

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CYC_6: 1234, 2134, 3124, 1324, 2314, 3214

CYC_7: 2134, 3124, 4213, 1243, 2143, 3412, 4312

CYC_8: 1234, 4321, 3421, 2431, 4231, 1324, 3124, 2134

CYC_9: 1234, 3214, 4123, 1423, 2413, 4213, 1243, 3421, 4321

CYC_{10}: 1234, 2134, 3124, 4213, 2413, 3142, 1342, 2431, 3421, 4321

CYC_{11}: 1234, 2134, 4312, 3412, 2143, 1243, 4213, 3124, 1324, 2314, 3214

CYC_{12}: 1234, 2134, 3124, 4213, 1243, 2143, 3412, 4312, 1342, 2431, 3421, 4321

CYC_{13}: 1234, 2134, 4312, 3412, 2143, 4123, 1423, 2413, 4213, 3124, 1324, 2314, 3214

CYC_{14}: 1234, 2134, 3124, 4213, 1243, 2143, 3412, 4312, 1342, 2431, 4231, 3241, 2341, 4321

CYC_{15}: 1234, 2134, 4312, 1342, 3142, 4132, 1432, 3412, 2143, 1243, 4213, 3124, 1324, 2314, 3214

CYC_{16}: 1234, 2134, 3124, 4213, 1243, 2143, 3412, 1432, 4132, 3142, 1342, 2431, 4231, 3241, 2341, 4321

CYC_{17}: 1234, 2134, 4312, 1342, 3142, 4132, 1432, 3412, 2143, 4123, 1423, 2413, 4213, 3124, 1324, 2314, 3214

CYC_{18}: 1234, 2134, 3124, 4213, 2413, 1423, 4123, 2143, 3412, 1432, 4132, 3142, 1342, 2431, 4231, 3241, 2341, 4321

CYC_{19}: 1234, 2134, 3124, 1324, 2314, 4132, 3142, 1342, 4312, 3412, 2143, 1243, 3421, 2431, 4231, 3241, 1423, 4123, 3214

CYC_{20}: 1234, 3214, 2314, 1324, 3124, 4213, 2413, 1423, 4123, 2143, 3412, 1432, 4132, 3142, 1342, 2431, 4231, 3241, 2341, 4321

CYC_{21}: 1234, 3214, 2314, 4132, 3142, 2413, 1423, 3241, 4231, 2431, 1342, 4312, 2134, 3124, 4213, 1243, 2143, 3412, 1432, 2341, 4321

Figure 4.1. Sequences of vertices describing cycles in P_4.

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CYC_{22}: 1234, 3214, 2314, 1324, 3124, 4213, 2413, 1423, 4123, 2143, 3412, 4312, 1342, 3142, 4132, 1432, 2341, 3241, 4231, 2431, 3421, 4321

CYC_{23}: 1234, 2134, 3124, 4213, 2413, 3142, 1342, 4312, 3412, 2143, 1243, 3421, 2431, 4231, 1324, 2314, 4132, 1432, 2341, 3241, 1423, 4123, 3214

CYC_{24}: 1234, 2134, 3124, 1324, 2314, 3214, 4123, 1423, 2413, 4213, 1243, 2143, 3412, 4312, 1342, 3142, 4132, 1432, 2341, 3241, 4231, 2431, 3421, 4321

Figure 4.1 (continued). Sequences of vertices describing cycles in P_4.

We illustrate Theorem 4.5 by constructing some of the indicated cycles in P_5. Suppose we wish to find a cycle of length 26 in P_5. In this case, n = 5. We decompose the number 26 into, for example, $26 = (2n-1) + (a_1-1) = 9 + 17$, so by the strategy indicated in the proof of Theorem 4.5, we choose $a_1 = 18$. Let $CYC_{2n} = CYC_{10}$ be the cycle consisting of the sequence of vertices 12345, 43215, 51234, 32154, 45123, 21543, 34512, 15432, 23451, 54321}. Due to the recursive decomposition property, P_5 contains 5 disjoint subgraphs isomorphic to P_4. Hence cycles isomorphic to those described in Figure 4.1 can be found in each copy of P_4. For this particular example, we use the copy of P_4 denoted by P_5^5, *i.e.*, the subgraph consisting of all vertices whose labels have the symbol 5 in the last position. In this subgraph, we can find a cycle of length 18 (isomorphic to CYC_{18} in Figure 4.1). This cycle consists of the sequence of vertices 12345, 21345, 31245, 42135, 24135, 14235, 41235, 21435, 34125, 14325, 41325, 31425, 13425, 24315, 42315, 32415, 23415, 43215. Observe that CYC_{18} shares the 4-dimensional edge labeled {12345, 43215} with CYC_{10}. We construct a cycle of length 26 in P_5 by deleting the common edge {12345, 43215} which

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merges CYC_{10} and CYC_{18}. Figure 4.2 illustrates the merging process. The common edge is shown as the heavy line connecting the vertices labeled 12345 and 43215.

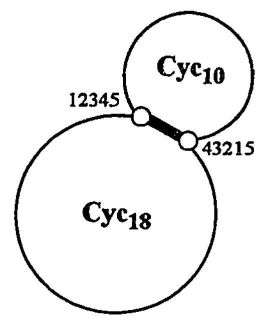

Figure 4.2. Merging CYC_{10} and CYC_{18} in P_5

Now suppose we wish to find a cycle of length 111 in P_5. We decompose the number 111 into, for example, $111 = (2n\text{-}5) + (a_1\text{-}1) + (a_2\text{-}1) + (a_3\text{-}1) + (a_4\text{-}1) + (a_5\text{-}1) = 5 + 23 + 23 + 23 + 23 + 14$. By the strategy of Theorem 4.5, we may choose $a_1 = 24$, $a_2 = 24$, $a_3 = 24$, $a_4 = 24$, and $a_5 = 15$. Again let $CYC_{2n} = CYC_{10}$ be the cycle consisting of the sequence of vertices 12345, 43215, 51234, 32154, 45123, 21543, 34512, 15432, 23451, 54321. For CYC_{a_1} we can again use the copy of P_4 denoted by P_5^5, *i.e.*, the subgraph consisting of all vertices whose labels have the symbol 5 in the last position. In this subgraph, we can find a cycle of length 24 (isomorphic to CYC_{24} in Figure 4.1) consisting of the sequence of vertices 12345, 21345, 31245, 13245, 23145, 32145, 41235, 14235, 24135, 42135, 12435, 21435, 34125, 43125, 13425, 31425, 41325, 14325, 23415, 32415, 42315, 24315, 34215, 43215. This cycle shares the 4-dimensional edge labeled {12345, 43215} with CYC_{10}. For CYC_{a_2} we can use the copy of P_4 denoted by P_5^4, *i.e.*, the subgraph consisting of all vertices whose labels have the symbol 4 in the last position. In this subgraph, we can find a cycle of length 24

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(isomorphic to CYC_{24} in Figure 4.1) consisting of the sequence of vertices 12354, 21354, 31254, 13254, 23154, 32154, 51234, 15234, 25134, 52134, 12534, 21534, 35124, 53124, 13524, 31524, 51324, 15324, 23514, 32514, 52314, 25314, 35214, 53214. This cycle shares the 4-dimensional edge labeled {32154, 51234} with CYC_{10}. We construct a similar cycle of length 24 for CYC_{a_3} (resp., CYC_{a_4}) in the copy of P_4 denoted by P_5^3 (resp., P_5^2), *i.e.*, the subgraph consisting of all vertices whose labels have the symbol 3 (resp., 2) in the last position. Finally, for CYC_{a_5} ($a_5 = 15$) we can use the copy of P_4 denoted by P_5^1, *i.e.*, the subgraph consisting of all vertices whose labels have the symbol 1 in the last position. In this subgraph, we can find a cycle of length 15 (isomorphic to CYC_{15} in Figure 4.1) consisting of the sequence of vertices 32541, 23541, 45321, 35421, 53421, 43521, 34521, 54321, 23451, 32451, 42351, 53241, 35241, 25341, 52341. This cycle shares the 4-dimensional edge labeled {23451, 54321} with CYC_{10}. We construct a cycle of length 111 in P_5 by deleting the 5 edges that CYC_{10} has in common with CYC_{a_1}, CYC_{a_2}, CYC_{a_3}, CYC_{a_4}, and CYC_{a_5}, respectively. The cycles CYC_{a_1}, ..., CYC_{a_5} are derived from Figure 4.1 by an appropriate isomorphism, which exists in every case because the pancake network is vertex symmetric. Figure 4.3 illustrates the merging process. The common edges are indicated by heavy lines.

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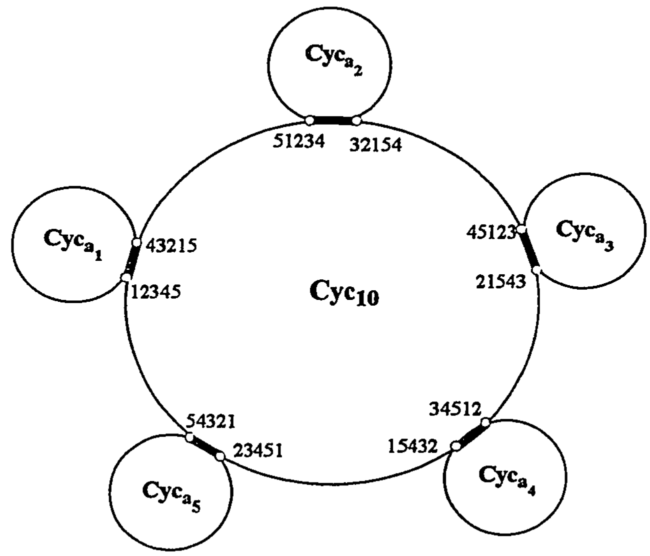

Figure 4.3. Merging CYC_{10}, CYC_{a_1}, CYC_{a_2}, CYC_{a_3}, CYC_{a_4}, and CYC_{a_5} in P_5.

The burnt pancake network has a similar cycle structure. It has cycles of all lengths between 8 and $2^n(n!)$, as stated in Theorem 4.6 below. The proof uses the same technique as the proof of Theorem 4.5 and is left to the reader.

Theorem 4.6: For $n \geq 3$ and for all i, $8 \leq i \leq 2^n(n!)$, BP_n has cycles of length i.

The cycles in BP_3 of length i, for all i, $8 \leq i \leq 2^3(3!)$, are listed in Figure 4.4 below. Note that these cycles constitute the basis for the inductive proof of Theorem 4.6 in the same way that the cycles of Figure 4.1 constituted the basis for cycles in the (unburnt) pancake network.

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CYC$_8$: 123, -2-13, 2-13, 1-23, -1-23, 213, -213, -123

CYC$_9$: 123, -3-2-1, 23-1, -23-1, 1-32, -1-32, 312, -312, -2-13

CYC$_{10}$: 123, -2-13, 2-13, 1-23, -32-1, 32-1, -2-3-1, 2-3-1, 3-2-1, -3-2-1

CYC$_{11}$: 123, -2-13, -312, -132, 132, -3-12, 3-12, 1-32, -23-1, 23-1, -3-2-1

CYC$_{12}$: 123, -3-2-1, 3-2-1, 2-3-1, -2-3-1, 32-1, -32-1, 1-23, -1-23, 213, -213, -123

CYC$_{13}$: 123, -2-13, -312, 312, -1-32, 1-32, -23-1, -32-1, 32-1, -2-3-1, 2-3-1, 3-2-1, -3-2-1

CYC$_{14}$: 123, -3-2-1, 3-2-1, 2-3-1, -2-3-1, 32-1, -32-1, 1-23, -1-23, -321, -231, 231, -3-21, -123

CYC$_{15}$: 123, -2-13, -312, 312, -1-32, 1-32, 3-12, -3-12, 132, -2-3-1, 32-1, -32-1, -23-1, 23-1, -3-2-1

CYC$_{16}$: 123, -3-2-1, 3-2-1, 2-3-1, -2-3-1, 32-1, -32-1, 1-23, -1-23, -321, 321, -2-31, 2-31, 3-21, -3-21, -123

CYC$_{17}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, -132, 132, -3-12, 3-12, 1-32, -23-1, 23-1, -3-2-1

CYC$_{18}$: 123, -3-2-1, 23-1, -23-1, -32-1, 1-23, -1-23, -321, -231, 231, -1-3-2, 31-2, -31-2, -13-2, 2-31, 3-21, -3-21, -123

CYC$_{19}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, 312, -1-32, 1-32, -23-1, -32-1, 32-1, -2-3-1, 2-3-1, 3-2-1, -3-2-1

CYC$_{20}$: 123, -3-2-1, 23-1, -23-1, -32-1, 1-23, -1-23, -321, -231, 231, -1-3-2, 1-3-2, 3-1-2, -3-1-2, 13-2, -13-2, 2-31, 3-21, -3-21, -123

CYC$_{21}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, -132, 132, -3-12, 3-12, 1-32, -23-1, -32-1, 32-1, -2-3-1, 2-3-1, 3-2-1, -3-2-1

CYC$_{22}$: 123, -3-2-1, 3-2-1, 2-3-1, -2-3-1, 32-1, -32-1, 1-23, -1-23, -321, 321, -2-31, 2-31, -13-2, 13-2, -3-1-2, 3-1-2, 1-3-2, -1-3-2, 231, -3-21,-123

Figure 4.4. Sequences of vertices describing cycles in BP_3.

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CYC$_{23}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, 312, -2-1-3, 2-1-3, 1-2-3, -1-2-3, 21-3, -21-3, 3-12, -3-12, 132, -2-3-1, 2-3-1, 3-2-1, -3-2-1

CYC$_{24}$: 123, -123, -3-21, 3-21, 2-31, -13-2, -31-2, 31-2, -1-3-2, 231, -231, -321, -1-23, 1-23, -32-1, 32-1, -2-3-1, 132, -3-12, 3-12, 1-32, -23-1, 23-1, -3-2-1

CYC$_{25}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, 312, -2-1-3, 2-1-3, 31-2, -1-3-2, 1-3-2, 3-1-2, 21-3, -21-3, 3-12, -3-12, 132, -2-3-1, 2-3-1, 3-2-1, -3-2-1

CYC$_{26}$: 123, -123, -3-21, 3-21, 2-31, -13-2, 13-2, -3-1-2, 3-1-2, 1-3-2, -1-3-2, 231, -231, -321, -1-23, 1-23, -32-1, 32-1, -2-3-1, 132, -3-12, 3-12, 1-32, -23-1, 23-1, -3-2-1

CYC$_{27}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, 312, -2-1-3, 2-1-3, 31-2, -1-3-2, 1-3-2, 3-1-2, 21-3, -21-3, 3-12, -3-12, 132, -2-3-1, 32-1, -32-1, -23-1, 23-1, -3-2-1

CYC$_{28}$: 123, -123, -3-21, 3-21, 2-31, -13-2, 13-2, -3-1-2, 3-1-2, 1-3-2, -1-3-2, 231, -231, -321, -1-23, 1-23, -32-1, 32-1, -2-3-1, 132, -132, -312, 312, -1-32, 1-32, -23-1, 23-1, -3-2-1

CYC$_{29}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, 312, -2-1-3, 2-1-3, 31-2, -31-2, -13-2, 13-2, -3-1-2, 3-1-2, 21-3, -21-3, 3-12, -3-12, 132, -2-3-1, 32-1, -32-1, -23-1, 23-1, -3-2-1

CYC$_{30}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, 312, -1-32, 1-32, 3-12, -3-12, 132, -132, -2-31, 2-31, 3-21, -3-21, 231, -231, -321, 321, -1-2-3, 21-3, 3-1-2, 1-3-2, 23-1, -3-2-1

CYC$_{31}$: 123, -123, -3-21, 231, -231, -321, -1-23, 1-23, 2-13, -2-13, -312, 312, -2-1-3, 2-1-3, 31-2, -31-2, -13-2, 13-2, -3-1-2, 3-1-2, 21-3, -21-3, 3-12, -3-12, 132, -2-3-1, 32-1, -32-1, -23-1, 23-1, -3-2-1

Figure 4.4 (continued). Sequences of vertices describing cycles in BP_3.

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CYC_{32}: 123, -123, -3-21, 3-21, 2-31, -13-2, 13-2, -3-1-2, 3-1-2, 21-3, -21-3, -12-3, 12-3, -2-1-3, 2-1-3, 31-2, -1-3-2, 231, -231, -321, -1-23, 1-23, -32-1, -23-1, 1-32, 3-12, -3-12, 132, -2-3-1, 2-3-1, 3-2-1, -3-2-1

CYC_{33}: 123, -123, -3-21, 3-21, 2-31, -2-31, 321, -321, -1-23, 1-23, 2-13, -2-13, -312, 312, -2-1-3, 2-1-3, 31-2, -31-2, -13-2, 13-2, -3-1-2, 3-1-2, 21-3, -21-3, 3-12, -3-12, 132, -2-3-1, 32-1, -32-1, -23-1, 23-1, -3-2-1

CYC_{34}: 123, -123, -3-21, 3-21, 2-31, -13-2, 13-2, -3-1-2, 3-1-2, 21-3, -21-3, -12-3, 12-3, -2-1-3, 2-1-3, 31-2, -1-3-2, 231, -231, -321, -1-23, 1-23, -32-1, -23-1, 1-32, -1-32, 312, -312, -132, 132, -2-3-1, 2-3-1, 3-2-1, -3-2-1

CYC_{35}: -123, -3-21, 3-21, 2-31, -2-31, 321, -1-2-3, 21-3, -21-3, -12-3, 12-3, -2-1-3, 312, -312, -2-13, 2-13, 1-23, -32-1, 32-1, 1-2-3, 2-1-3, 31-2, -31-2, -13-2, 13-2, -3-1-2, 3-1-2, 1-3-2, -1-3-2, 231, -231, -321, -1-23, 213, -213

CYC_{36}: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, 312, -1-32, 1-32, 3-12, -3-12, 132, -132, -2-31, 2-31, 3-21, -3-21, 231, -231, -321, 321, -1-2-3, 1-2-3, 2-1-3, -2-1-3, 12-3, -12-3, -21-3, 21-3, 3-1-2, 1-3-2, 23-1, -3-2-1

CYC_{37}: 213, -213, -123, -3-21, 3-21, 2-31, -13-2, 13-2, 2-3-1, -2-3-1, 132, -3-12, 3-12, -21-3, 21-3, -1-2-3, 1-2-3, 2-1-3, -2-1-3, 312, -1-32, 1-32, -23-1, 23-1, -3-2-1, 123, -2-13, 2-13, 1-23, -1-23, -321, -231, 231, -1-3-2, 1-3-2, 3-1-2, -3-1-2

CYC_{38}: -123, -3-21, 3-21, 2-31, -2-31, 321, -1-2-3, 21-3, -21-3, -12-3, 12-3, -2-1-3, 312, -1-32, 1-32, 3-12, -3-12, 132, -132, -312, -2-13, 123, -3-2-1, 23-1, 1-3-2, 3-1-2, -3-1-2, 13-2, -13-2, -31-2, 31-2, -1-3-2, 231, -231, -321, -1-23, 213, -213

Figure 4.4 (continued). Sequences of vertices describing cycles in BP_3.

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CYC$_{39}$: -2-13, 123, -123, -213, 213, -1-23, 1-23, 2-13, -31-2, -13-2, 13-2, -3-1-2, 3-1-2, 1-3-2, -1-3-2, 31-2, 2-1-3, -2-1-3, 12-3, -12-3, -21-3, 21-3, -1-2-3, 1-2-3, 32-1, -32-1, -23-1, 23-1, -3-2-1, 3-2-1, 2-3-1, -2-3-1, 132, -3-12, 3-12, 1-32, -1-32, 312, -312

CYC$_{40}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, -132, -2-31, 2-31, 3-21, -3-21, 231, -231, -321, 321, -1-2-3, 1-2-3, 2-1-3, -2-1-3, 12-3, -12-3, -21-3, 21-3, 3-1-2, 1-3-2, -1-3-2, 31-2, -31-2, -13-2, 13-2, 2-3-1, -2-3-1, 32-1, -32-1, -23-1, 23-1, -3-2-1

CYC$_{41}$: -123, -3-21, 3-21, 2-31, -2-31, 321, -1-2-3, 21-3, -21-3, -12-3, 12-3, -2-1-3, 312, -312, -2-13, 2-13, 1-23, -32-1, -23-1, 23-1, -3-2-1, 3-2-1, 2-3-1, -2-3-1, 32-1, 1-2-3, 2-1-3, 31-2, -31-2, -13-2, 13-2, -3-1-2, 3-1-2, 1-3-2, -1-3-2, 231, -231, -321, -1-23, 213, -213

CYC$_{42}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, 312, -1-32, 1-32, 3-12, -3-12, 132, -132, -2-31, 2-31, 3-21, -3-21, 231, -231, -321, 321, -1-2-3, 1-2-3, 2-1-3, -2-1-3, 12-3, -12-3, -21-3, 21-3, 3-1-2, -3-1-2, 13-2, -13-2, -31-2, 31-2, -1-3-2, 1-3-2, 23-1, -3-2-1

CYC$_{43}$: -213, -123, -3-21, 3-21, 2-31, -2-31, 321, -1-2-3, 1-2-3, 2-1-3, 31-2, -31-2, -13-2, 13-2, -3-1-2, 3-1-2, 1-3-2, -1-3-2, 231, -231, -321, -1-23, 1-23, 2-13, -2-13, -312, -132, 132, -2-3-1, 2-3-1, 3-2-1, -3-2-1, 23-1, -23-1, 1-32, -1-32, 312, -2-1-3, 12-3, -12-3, -21-3, 3-12, -3-12

CYC$_{44}$: -123, -3-21, 3-21, 2-31, -2-31, 321, -1-2-3, 21-3, -21-3, -12-3, 12-3, -2-1-3, 312, -1-32, 1-32, 3-12, -3-12, 132, -132, -312, -2-13, 123, -3-2-1, 3-2-1, 2-3-1, -2-3-1, 32-1, -32-1, -23-1, 23-1, 1-3-2, 3-1-2, -3-1-2, 13-2, -13-2, -31-2, 31-2, -1-3-2, 231, -231, -321, -1-23, 213, -213

Figure 4.4 (continued). Sequences of vertices describing cycles in BP_3.

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CYC$_{45}$: 123, -123, -213, 213, -1-23, -321, -231, -1-32, 1-32, 3-12,
-3-12, 132, -132, -312, 312, -2-1-3, 12-3, -12-3, -21-3, 21-3,
3-1-2, -3-1-2, 13-2, -13-2, -31-2, 31-2, 2-1-3, 1-2-3, -1-2-3,
321, -2-31, 2-31, 3-21, -3-21, 231, -1-3-2, 1-3-2, 23-1, -23-1,
-32-1, 32-1, -2-3-1, 2-3-1, 3-2-1, -3-2-1

CYC$_{46}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, 312, -1-32,
1-32, 3-12, -3-12, 132, -132, -2-31, 2-31, 3-21, -3-21, 231,
-231, -321, 321, -1-2-3, 1-2-3, 2-1-3, -2-1-3, 12-3, -12-3, -21-3,
21-3, 3-1-2, 1-3-2, -1-3-2, 31-2, -31-2, -13-2, 13-2, 2-3-1,
-2-3-1, 32-1, -32-1, -23-1, 23-1, -3-2-1

CYC$_{47}$: -123, -3-21, 3-21, 2-31, -2-31, 321, -1-2-3, 21-3, -21-3, -12-3,
12-3, -2-1-3, 312, -1-32, 1-32, 3-12, -3-12, 132, -132, -312,
-2-13, 2-13, 1-23, -32-1, -23-1, 23-1, -3-2-1, 3-2-1, 2-3-1,
-2-3-1, 32-1, 1-2-3, 2-1-3, 31-2, -31-2, -13-2, 13-2, -3-1-2,
3-1-2, 1-3-2, -1-3-2, 231, -231, -321, -1-23, 213, -213

CYC$_{48}$: 123, -123, -213, 213, -1-23, 1-23, 2-13, -2-13, -312, 312, -1-32,
1-32, 3-12, -3-12, 132, -132, -2-31, 2-31, 3-21, -3-21, 231,
-231, -321, 321, -1-2-3, 1-2-3, 2-1-3, -2-1-3, 12-3, -12-3, -21-3,
21-3, 3-1-2, -3-1-2, 13-2, -13-2, -31-2, 31-2, -1-3-2, 1-3-2,
23-1, -23-1, -32-1, 32-1, -2-3-1, 2-3-1, 3-2-1, -3-2-1

Figure 4.4 (continued). Sequences of vertices describing cycles in BP_3.

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Chapter 5

The Deterministic Pancake Problem

5.1. Introduction

The Deterministic Pancake Problem asks for the maximum number of steps, for all permutations on n symbols, for 1 to be placed in the first position according to the deterministic pancake process. Knuth included this problem on a take-home examination in 1974 in the following form [4]. (His solution, taken from [4], showing an exponential upper bound, is contained in our Appendix 1.):

> **Problem**: *Let* $\pi = \pi[1]\pi[2] \ldots \pi[n]$ *be a permutation of* $\{1,2,\ldots,n\}$ *and consider the following algorithm (called topcard):*
>
> ```
>          begin integer array A[1:n]; integer k;
>          (A[1], ... ,A[n]) ← (π[1], ... ,π[n]);
> loop:    print (A[1], ... ,A[n]);
>          k ← A[1];
>          if k=1 then go to finish;
>          (A[1], ... ,A[k]) ← (A[k], ... ,A[1]);
>          go to loop;
> finish:  end
> ```
>
> *For example, when* n=9 *and* π=314592687, *the algorithm will print*
>
> 3 1 4 5 9 2 6 8 7
> 4 1 3 5 9 2 6 8 7
> 5 3 1 4 9 2 6 8 7
> 9 4 1 3 5 2 6 8 7
> 7 8 6 2 5 3 1 4 9
> 1 3 5 2 6 8 7 4 9

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> *and then it will stop.*
>
> *Let* m=m(π) *be the total number of permutations printed by the above algorithm. Prove that* m *never exceeds the Fibonacci number* F_{n+1}. *(In particular, the algorithm always halts.)*
>
> *Extra credit problem. Let* $M_n = \max\{m(\pi) \mid \pi \text{ a permutation of } \{1, \dots, n\}\}$. *Find the best upper and lower bounds on* M_n *that you can.*

Knuth in [4] conjectures that $M_n = O(n)$. As mentioned earlier, we show that the O(n) conjecture is false. We do this by defining an infinite family of permutations so that, for a given constant c>0, a permutation on n symbols in this family causes **topcard** to write at least $c \cdot n^2$ permutations, *i.e.* $M_n = \Omega(n^2)$.

The Deterministic Pancake Problem is just one of several problems about sequences with significant open questions. For example, the notoriously difficult *3x+1 problem*, also known as the *Collatz* problem, asks about termination of the sequence n, p(n), $p^2(n)$, ... , for all positive integers n, where p is given by:

$$p(n) = \begin{cases} \dfrac{3n+1}{2}, & \text{if n is odd,} \\ \dfrac{n}{2}, & \text{if n is even,} \end{cases}$$

(*Terminates* means "eventually achieves the value 1.") (See [24] for an extensive survey of the literature about this problem.) As a counterpoint, Clark and Lewis [10] consider sequences, motivated by the above, where, for a given x_1 and x_2, the sequence $x_1, x_2, x_3, x_4, \dots$ is defined by:

$$x_n = \begin{cases} \dfrac{x_{n-1} + x_{n-2}}{2}, & \text{if } x_{n-1} + x_{n-2} \text{ is even,} \\ x_{n-1} - x_{n-2}, & \text{if } x_{n-1} + x_{n-2} \text{ is odd.} \end{cases}$$

For these sequences, unlike the one attributed to Collatz, conditions for convergence are known [10].

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Another interesting issue concerns the sequence f(1), f(2), f(3), ... , defined by:

$$f(n) = \min \{ \max(S) \mid |S| = n \text{ and } S \text{ has distinct subset sums} \}.$$

Erdös conjectures that $f(n) \geq c \cdot 2^n$, for some constant $c>0$. Conway and Guy describe a sequence of integers [16] and conjecture that one can always obtain sets from this sequence with distinct subset sums. Moreover, they conjecture that this *Conway-Guy sequence* is the best possible with respect to defining sets with small maximum element and subset sum distinctness. Lunnon [26] shows that the *Conway-Guy sequence* is indeed best possible, for all $n \leq 8$, but that there are other sequences with better asymptotic growth rates that also appear to give in the same way sets with distinct subset sums.

As mentioned in Chapter 1, the Deterministic Pancake Problem seems to be most closely related to the Pancake Problem [12] and the Burnt Pancake Problem [15], which ask for the number of prefix reversals required to sort permutations on n unsigned (*resp.* signed) symbols, for each positive integer n.

We use the following notation for the Deterministic Pancake Problem. In this chapter, a permutation $\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$ will be written as the sequence $\pi(1), \pi(2), \dots, \pi(n)$, when context requires that each symbol be explicitly listed. When we wish to refer to the entire permutation without making specific reference to individual symbols, we will write π_n, or simply π when n is clear. Permutations will be written in bold face type to distinguish them from other sequences of symbols that are not permutations.

For any permutation π, let *run(π)* denote the sequence of permutations printed by the program **topcard** when started on input π, and let *|run(π)|* denote the number of permutations in the sequence run(π). Let *front(run(π))* denote the derived sequence of first elements from each successive permutation in run(π). It

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follows, from the basic definition of the Deterministic Pancake procedure, that front(run(π)) describes the sequence of flips (or prefix reversals) executed on input π. Let h(n) be defined by:

$$h(n) = \max\{\ |run(\pi)|\ \mid \pi \text{ is a permutation of } \{1, \ldots, n\}\}.$$

A permutation π such that $|run(\pi)| = h(n)$ is called a *long-winded* permutation. Let H(n) be the number of long-winded permutations on {1, ... ,n}. We show that $h(n) = \Omega(n^2)$. In addition, computer calculations reveal the exact values of h(n), for n≤15, indicated in Figure 5.1. (The values of h(n), for n≤9, were given in [4].)

n	h(n)	H(n)
2	1	1
3	2	2
4	4	2
5	7	1
6	10	5
7	16	2
8	22	1
9	30	1
10	38	1
11	51	1
12	65	1
13	80	1
14	101	4
15	113	6

Figure 5.1. Exact values of h(n) and H(n), for n≤15.

The long-winded permutations for n≤9 were given previously [4]. Here are the long-winded permutations for 10≤n≤15:

<u>n=10:</u> 5, 9, 1, 8, 6, 2, 10, 4, 7, 3

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<u>n=11:</u> 4, 9, 11, 6, 10, 7, 8, 2, 1, 3, 5

<u>n=12:</u> 2, 6, 1, 10, 11, 8, 12, 3, 4, 7, 9, 5 (*)

<u>n=13:</u> 2, 9, 4, 5, 11, 12, 10, 1, 8, 13, 3, 6 ,7

<u>n=14:</u> 2, 4, 9, 3, 11, 1, 8, 13, 6, 5, 10, 14, 12, 7
9, 4, 11, 3, 1, 8, 13, 6, 2, 5, 10, 14, 12, 7
3, 13, 4, 9, 2, 1, 8, 11, 6, 5, 10, 14, 12, 7
3, 9, 4, 2, 11, 1, 8, 13, 6, 5, 10, 14, 12, 7

<u>n=15:</u> 2, 9, 4, 11, 7, 1, 13, 5, 3, 14, 12, 15, 8, 10, 6 (*)
3, 1, 7, 11, 4, 9, 2, 5, 13, 14, 12, 15, 8, 10, 6 (*)
3, 1, 7, 11, 15, 12, 9, 4, 10, 2, 13, 8, 14, 5, 6 (*)
3, 11, 8, 15, 12, 10, 5, 9, 2, 13, 1, 4, 14, 7, 6 (*)
5, 15, 12, 11, 8, 10, 3, 9, 2, 13, 1, 4, 14, 7, 6 (*)
7, 12, 15, 10, 13, 4, 2, 11, 1, 8, 9, 5, 14, 3, 6 (*)

In addition, Figure 5.2 gives lower bounds for h(n), for $16 \leq n \leq 30$, and permutations exhibiting the stated run lengths.

(*) This symbol denotes that the run sequence for this permutation does not end with the identity.

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n	h(n)≥	permutation exhibiting run length indicated in column 2
16	139	9,12,6,7,2,14,8,1,11,13,5,4,15,16,10,3 (*)
17	159	2,10,15,11,7,14,5,16,6,4,17,13,1,3,8,9,12
18	191	6,14,9,2,15,8,1,3,4,12,18,5,10,13,16,17,11,7
19	221	12,15,11,1,10,17,19,2,5,8,9,4,18,13,16,7,3,14,6 (*)
20	243	4,10,19,3,18,1,2,15,14,11,13,20,6,16,7,8,5,9,12,17 (*)
21	266	6,16,5,2,12,9,4,7,3,1,20,17,8,10,18,13,11,14,21,15,19 (*)
22	292	10,4,9,13,19,16,5,11,6,3,8,7,2,15,14,18,1,21,20,22,12,17 (*)
23	332	2,4,9,3,11,17,8,13,6,5,10,14,12,7,20,1,21,23,16,22,15,18,19 (*)
24	348	2,4,10,3,7,14,15,11,21,9,5,13,6,8,12,19,16,24,1,23,10,17,18,22 (*)
25	380	6,14,9,2,15,8,19,3,4,12,18,5,10,13,16,17,11,7, 21,24,23,25,20,1,22 (*)
26	428	6,14,9,2,15,8,23,3,4,12,18,5,10,13,16,17,11,7,22,26,1,25,20,19,21,24 (*)
27	475	4,10,19,3,18,25,2,15,14,11,13,20,6,16,7,8,5,9,12,17,26,24,21,27,22,1,23 (*)
28	549	6,14,7,13,17,12,8,22,1,26,10,2,5,27,9,4,3,24,16,19,15,11,28,25,20,21,18,23 (*)
29	578	2,10,15,11,7,14,5,16,6,4,17,13,18,3,8,9,12,26,23,19,27,28,25,20,29,21,22,1,24 (*)
30	620	12,15,11,21,10,17,19,2,5,8,9,4,18,13,16,7,3,14,6,23,25,30,22,29,27,24,20,1,26,28 (*)

Figure 5.2. Lower bounds for h(n), for 16≤n≤30.

5.2 An $\Omega(n^2)$ lower bound

Our objective is to describe a family Π of permutations $\{\pi_n\}$, such that, for all n>0, π_n is a permutation on the integers $\{1, \ldots ,n\}$, and, for some constant c>0, and infinitely many n, $|\mathrm{run}(\pi_n)| \geq c \cdot n^2$. For each natural number k>1, let $\Pi^{(k)}$ denote the infinite family of permutations containing, for n>k, all permutations π_n on the integers $\{1, \ldots ,n\}$ such that $\pi_n(j)=j$, for all $2 \leq j \leq n-k$. That is, the permutations in $\Pi^{(k)}$, although strictly speaking permutations on an arbitrary number of symbols, may be viewed as permutations on only k+1 symbols. A permutation π_n in $\Pi^{(k)}$ on n symbols has the symbols 2,...,n-k fixed in positions 2,...,n-k, respectively. We call the symbols 2,...,n-k the *fixed* symbols, while

(*) This symbol denotes that the run sequence for this permutation does not end with the identity.

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1, n-k+1, ..., n are the *non-fixed* symbols. In fact, the reason we have focused on permutations in the family $\Pi^{(k)}$, for small values of k, is that this allows us to perform a computer search for permutations on an arbitrary number of symbols, say n, but which are in effect permutations on a smaller number, *i.e.*, k+1. For small values of k, the number of permutations on k+1 symbols, of course, is small. This allows the search to terminate in reasonable time. Our computer search, for any particular value n, looks for permutations in $\Pi^{(k)}$ on n symbols for which the procedure **topcard** takes a large number of steps. We are particularly interested in those that terminate with the identity permutation.

As it turns out, if such a permutation π_n exists, it often represents a regular event, in the sense that, for many m>n, there are permutations on m symbols in $\Pi^{(k)}$, all with their non-fixed symbols in the same relative order as in π_n, and for which **topcard** writes out a sequence of permutations exhibiting a certain pattern. This can be exploited to derive an infinite family of permutations satisfying the property that every permutation in the family has a run sequence satisfying the same pattern. We describe such an infinite family $\{\sigma_n\}$ in the following. All permutations in $\{\sigma_n\}$ are in $\Pi^{(12)}$. It should be noted, in fact, that many such regular families exist. Our computer search found several other families in $\Pi^{(12)}$ and many others in $\Pi^{(9)}$, $\Pi^{(10)}$, $\Pi^{(11)}$, $\Pi^{(13)}$, and in $\Pi^{(14)}$. We have chosen $\{\sigma_n\}$, as the asymptotic results obtained are as good as for any of the others and, in fact, are better than most. The permutations σ_n in $\Pi^{(12)}$, for n>13, are defined by:

$$\sigma_n = \mathbf{n, (2, 3, \ldots, n\text{-}12), n\text{-}10, n\text{-}1, n\text{-}9, n\text{-}4, n\text{-}8, n\text{-}2, n\text{-}7, n\text{-}11, 1, n\text{-}5, n\text{-}3, n\text{-}6.}$$

We shall see that both front(run(σ_n)) and run(σ_n) are sequences exhibiting considerable regularity. For explicitness, we have given the run sequence for σ_{33} in Appendix 2.

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Lemma 5.1. For all $n \geq 21$, such that $n \equiv 9 \pmod{12}$, $|run(\sigma_n)|=(21n-77)/4$ and the last element of $run(\sigma_n)$ is the identity permutation.

Proof: To begin with we claim that, for all $n \geq 21$, such that $n \equiv 9 \pmod{12}$, $front(run(\sigma_n))$ is:

(*) $n, n-6, \{7+6j\}_{0 \leq j \leq (n-21)/6}, \mathbf{A}, \{\mathbf{B(j)}\}_{0 \leq j \leq (n-33)/12}, \mathbf{C}, \{4+2j\}_{0 \leq j \leq (n-11)/2}, \{\mathbf{D(j)}\}_{0 \leq j \leq (n-17)/4}, 2, 5, 3, 4, 2, 3,$

where:

(1) **A** = n-4, 5, n-5, n-9, 5, n-2, 3, n-4, n-11, n-9, n-6, 3, n-3, n-7, n-1, 2, n-2, n-10, 4, n- 11, n-12, 6, n-8, n-7, n-9, n-10, n-8, n-4, n-3, 2, n-4, n-5, 2, n-9, 5, 6, n-14, n-13, n-12, n-11, n-10, 2, n-11, n-12,

(2) **B(j)** = 2, (n-13)-12j, (n-14)-12j, 2, (n-20)-12j, (n-15)-12j, 5, 6, (n-20)-12j, (n-26)-12j, 2, (n-21)-12j, 5, 6, (n-26)-12j, (n-25)-12j, (n-24)-12j, (n-23)-12j, (n-22)-12j, 2,(n-23)-12j, (n-24)-12j,

(3) **C** = 2, 8, 7, 2, 6, 5, 2, n-6, 3, n-8, and

(4) **D(j)** = 2, (n-8)-4j, 3, (n-12)-4j, (n-13)-4j, (n-11)-4j, (n-10)-4j, (n-9)-4j, 2, (n-10)-4j, (n-11)-4j.

It is straightforward to verify that $front(run(\sigma_n))$ begins with the natural numbers n, n-6, and that (after these two flips) the resulting permutation in $run(\sigma_n)$ is (*e.g.*, see line number 2 in Appendix 2.):

7, 8, ..., (n-12), (n-10), (n-1), (n-9), (n-4), (n-8), (n-2), (n-7), (n-11), 1,

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(n-5), (n-3), (n-6), 6, 5, 4, 3, 2, n.

The next (n-15)/6 elements of front(run(σ_n)) form the sequence $\{7+6j\}_{0\leq j\leq (n-21)/6}$, as shown below. After the first flip of size 7 one obtains:

[13, 12, 11, 10, 9, 8],
7,
14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32 ..., (n-12),
(n-10), (n-1), (n-9), (n-4), (n-8), (n-2), (n-7), (n-11), 1,(n-5), (n-3), (n-6),
6, 5, 4, 3, 2, n,

Then a flip of size 13 yields:

[19, 18, 17, 16, 15, 14],
7,
[8, 9, 10, 11, 12, 13],
20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, ..., (n-12),
(n-10), (n-1), (n-9), (n-4), (n-8), (n-2), (n-7), (n-11), 1,(n-5), (n-3), (n-6),
6, 5, 4, 3, 2, n,

Then a flip of size 19 yields:

[25, 24, 23, 22, 21, 20], [13, 12, 11, 10, 9, 8],
7,
[14, 15, 16, 17, 18, 19],
26, 27, 28, 29, 30, 31, ..., (n-12),
(n-10), (n-1), (n-9), (n-4), (n-8), (n-2), (n-7), (n-11), 1,(n-5), (n-3), (n-6),
6, 5, 4, 3, 2, n,

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Then a flip of size 25 yields:

[31, 30, 29, 28, 27, 26], [19, 18, 17, 16, 15, 14],

7,

[8, 9, 10, 11, 12, 13], [20, 21, 22, 23, 24, 25],

32, ..., (n-12),

(n-10), (n-1), (n-9), (n-4), (n-8), (n-2), (n-7), (n-11), 1,(n-5), (n-3), (n-6),

6, 5, 4, 3, 2, n,

...

And similarly, after the remaining flips in the sequence $\{7+6j\}_{0\leq j\leq (n-21)/6}$, one obtains:

(n-4), (n-9), (n-1), (n-10), (n-12), (n-13),

[(n-20), (n-21), (n-22), (n-23), (n-24), (n-25)],

[(n-32), (n-33), (n-34), (n-35), (n-36), (n-37)],

..., [13, 12, 11, 10, 9, 8],

7,

[14, 15, 16, 17, 18, 19], ...,

[(n-31), (n-30), (n-29), (n-28), (n-27), (n-26)],

[(n-19), (n-18), (n-17), (n-16), (n-15), (n-14)],

(n-8), (n-2), (n-7), (n-11), 1, (n-5), (n-3), (n-6),

6, 5, 4, 3, 2, n.

As seen above, the first flip in the series creates a block of length 6 consisting of the symbols 13, 12, 11, 10, 9, 8 followed by the symbol 7. The second flip creates the next consecutive block of length 6, with the symbols 19, 18, 17, 16, 15, 14 and these two blocks (the first one now reversed) bracket the

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symbol 7. This process continues for (n-15)/6 flips and the resulting permutation has (n-15)/12 pairs of blocks arranged like balanced parentheses around the symbol 7.

When n=21, the variable j in $\{7+6j\}_{0\le j\le (n-21)/6}$ takes on the single value 7, hence contributing the single element 7 to front(run(σ_{21})). Furthermore, since the symbol 17 is equal to n-4, **topcard** generates *no* permutations with bracketing blocks of length 6, but immediately generates the permutation **(n-4), (n-9), (n-1), (n-10), (n-12), (n-13), 7, (n-8), (n-2), (n-7), (n-11), 1, (n-5), (n-3), (n-6), 6, 5, 4, 3, 2, n**, as illustrated below:

7, 8, 9, 11, 20 12, 17, 13, 19, 14, 10, 1, 16, 18, 15, 6, 5, 4, 3, 2, 21.

$\mapsto$ **17, 12, 20, 11, 9, 8, 7, 13, 19, 14, 10, 1, 16, 18, 15, 6, 5, 4, 3, 2, 21.**

However, as we have seen for larger values of n such that n≡9(mod 12), the next (n-15)/6 elements of front(run(σ_n)) is the sequence $\{7+6j\}_{0\le j\le (n-21)/6}$, and after this sequence of flips, the resulting permutation is

(n-4), (n-9), (n-1), (n-10), (n-12), (n-13),
[(n-20), (n-21), ..., (n-25)], [(n-32), (n-33), ..., (n-37)], ... [13, 12, ..., 8],
7,
[14, 15, ..., 19] ... [(n-31), (n-30), ..., (n-26)], [(n-19), (n-18), ..., (n-14)],
(n-8), (n-2), (n-7), (n-11), 1, (n-5), (n-3), (n-6), 6, 5, 4, 3, 2, n.

It is then straightforward to verify that the next 44 elements of front(run(σ_n)) are the elements in the described sequence **A** and that afterwards the resulting permutation in run(σ_n) is (*e.g.* see line numbers 5-49 in Appendix 2.):

2, (n-13),
[(n-20), (n-21), ..., (n-25)], [(n-32), (n-33), ..., (n-37)], ..., [13, 12, ..., 8],

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7,

[14, 15, ..., 19], [26,27, ...,31], ..., [(n-31), (n-30), ..., (n-26)],

[(n-19), (n-18), ..., (n-15)],

6, (n-6), 4, 1, 5, (n-14), [(n-12), (n-11), ..., (n-7)], 3,

[(n-5), (n-4), ..., n].

In fact, the above permutation can be written in a more general form as follows with j=0:

2, (n-13-12j),

[(n-20-12k), (n-21-12k), ..., (n-25-12k)]$_{j\leq k\leq (n-33)/12}$,

7,

[(14+12k), (15+12k), ..., (19+12k)]$_{0\leq k\leq (n-45-12j)/12}$,

[(n-19-12j), (n-18-12j), ..., (n-15-12j)],

6, (n-6), 4, 1, 5, (n-14-12j), [(n-12-12j), (n-11-12j), ..., (n-7)], 3,

[(n-5), (n-4), ..., n].

By induction on j similar to that done above, it can be shown that for each j (0≤j≤(n-33)/12), the next 22 elements of front(run(σ_n)) form the sequence **B(j)**, and that, after each such sequence of flips, the resulting permutation in run(σ_n) is as indicated (above) for j+1. (For example, where n=33, (n-33)/12 is zero, so the next 22 elements of front(run(σ_{33})) form **B(0)** and result in the permutations shown in lines 49-71 of Appendix 2.) In general, substituting j = [(n-33)/12] we obtain the following permutation in run(σ_n), which occurs before the last of these flip sequences, namely **B(j)**, is executed:

2, 20,

[13, 12, ..., 8],

7,

[14, 15, ..., 18],

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6, (n-6), 4, 1, 5, 19, [21, 22, ..., (n-7)], 3,
[(n-5), (n-4), ..., n].

And, after this flip sequence **B(j)**, *i.e.* 2, 20, 19, 2, 13, 18, 5, 6, 13, 7, 2, 12, 5, 6, 7, 8, 9, 10, 11, 2, 10, 9, the permutation in run(σ_n) is (see line number 71 in Appendix 2):

2, 8, 6, (n-6), 4, 1, 5, 7,
[9, 10, ..., (n-7)],
3,
[(n-5), (n-4), ..., n].

It is then straightforward to verify that the next ten elements of front(run(σ_n)) forms the sequence **C** and, after this sequence of flips, the resulting permutation in run(σ_n) is (see lines 71-81 in Appendix 2):

4, 1,
[5, 6, ..., (n-9)],
3, (n-7), (n-8), 2,
[(n-6), (n-5), (n-4), ..., n].

Again, by an inductive argument on j, it can be verified that, for each j ($0 \leq j \leq (n-11)/2$), the next element in front(run(σ_n)) is 4+2j. For example, see line numbers 81-93 in Appendix 2. Generally, after each odd number j>0, and corresponding two flips, *i.e.* 4+2(j-1) and 4+2j, the resulting permutation in run(σ_n) is:

[(6+2j, 5+4j-1), (2+2j, 1+2j), ..., (8,7)],
4, 1,

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[(5,6), (9,10), ..., (3+2j, 4+2j)],
[7+2j, 8+2j, ..., n-9],
3, (n-7), (n-8), 2,
[(n-6), (n-5), (n-4), ..., n].

So, after j =[(n-11)/2]-2, the resulting permutation in run(σ_n) is (by substitution of j = [(n-11)/2]-2 in the above):

[((n-9), (n-10)), ((n-13),(n-14)), ..., (8,7)]
4,1,
[(5,6), (9,10), ..., ((n-12),(n-11))],
3, (n-7), (n-8), 2,
[(n-6), (n-5), (n-4), ..., n].

and the next two flips are (n-9) = 4+2((n-11)/2-1) and (n-7) = 4+2((n-11)/2), resulting in the permutation (see, for example, line 93 in Appendix 2):

2, (n-8), [((n-9),(n-10)), ((n-13),(n-14)), ..., (8,7)],
4,1,
[(5,6), (9,10), ..., ((n-12),(n-11))],
3, [(n-7), (n-6), ..., n],

which is equal to the following permutation, for j = 0:

2, (n-8-4j), [((n-9-4j,(n-10-4j)), ((n-13-4j),(n-14-4j)), ..., (8,7)],
4,1,
[(5,6), (9,10), ..., ((n-12-4j),(n-11-4j))],
3, [(n-7-4j), (n-6-4j), ..., n].

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By induction on j, it can be seen that, for each j ($0 \le j \le (n-13)/4$), the next 11 elements of front(run(σ_n)) form the sequence **D(j)** and that, after the j^{th} such sequence of 11 flips, the permutation in run(σ_n) is as indicated above for j+1. (For example, see lines 93-104, lines 104-115, lines 115-126, lines 126-137, and lines 137-148 in Appendix 2.) At the start of the last of these sequences, namely **D(j)**, for j =[(n-13)/4]-1 , the permutation in run(σ_n) (obtained by substituting j = [(n-13)/4]-1 in the permutation indicated) is:

2, 5,

4, 1,

3, [6, 7, ..., n].

Thus, as can easily be seen, the last 6 elements in front(run(σ_n)) are 2, 5, 3, 4, 2, 3 and, after this sequence of flips, the sequence run(σ_n) terminates with the identity permutation.

By a simple counting of the length of the sequence (*), it follows that $|run(\sigma_n)| = (21n-77)/4$.

□

Lemma 5.1 exhibits a family of permutations $\{\sigma_n\}$ such that $|run(\sigma_n)|$ is $\Omega(n)$. Recall that our objective is to describe a family Π of permutations $\{\pi_n\}$, such that, for all n>0, π_n is a permutation on the integers {1, ... ,n}, and, for some constant c>0, and infinitely many n, $|run(\pi_n)| > c \cdot n^2$. The objective is now achieved by chaining permutations in the family $\{\sigma_n\}$ together so that the sequence of flips **topcard** executes on the chain is the juxtaposition of the sequences for each individual permutation in the chain. This will suffice to establish our objective, as we show.

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For a permutation ρ_n in $\Pi^{(t)}$ and a permutation ρ_{n+k} in $\Pi^{(k)}$, define the permutation $\rho_n \oplus \rho_{n+k}$ in $\Pi^{(t+k)}$ by:

$$(\rho_n \oplus \rho_{n+k})(i) = \begin{cases} \rho_n(i), & \text{if } 1 \leq i \leq n \text{ and } \rho_n(i) \neq 1, \\ \rho_{n+k}(1), & \text{if } \rho_n(i) = 1, \\ \rho_{n+k}(i), & \text{if } n+1 \leq i \leq n+k. \end{cases}$$

For example, consider σ_{21} =21, (2,3,...,9), 11, 20, 12, 17, 13, 19, 14, 10, 1, 16, 18, 15 and σ_{33}=33, (2,3,...,21), 23, 32, 24, 29, 25, 31, 26, 22, 1, 28, 30, 27. The permutation $\sigma_{21} \oplus \sigma_{33}$ is: 21, (2,3,...,9), 11, 20, 12, 17, 13, 19, 14, 10, 33, 16, 18, 15, 23, 32, 24, 29, 25, 31, 26, 22, 1, 28, 30, 27.

Lemma 5.2. For any t>0 and any permutations ρ_n in $\Pi^{(t)}$ and ρ_{n+k} in $\Pi^{(k)}$, $\rho_n \oplus \rho_{n+k}$ is a permutation on n+k symbols in $\Pi^{(t+k)}$ and $|run(\rho_n \oplus \rho_{n+k})| = |run(\rho_n)| + |run(\rho_{n+k})|$.

Proof: It follows easily from the definition of $\oplus$ that $\rho_n \oplus \rho_{n+k}$ is a permutation on n+k symbols and has the integers 2,3,...,(n-t) = (n+k) - (t+k) fixed in their respective positions. Thus, $\rho_n \oplus \rho_{n+k}$ is in $\Pi^{(t+k)}$.

By definition $\rho_n \oplus \rho_{n+k}(i)$ is identical to $\rho_n(i)$, for all $1 \leq i \leq n$, except for the integer i such that $\rho_n(i)=1$, and for this value of i, $\rho_n \oplus \rho_{n+k}(i) = \rho_{n+k}(1)$, hence it follows that, if $|run(\rho_n)| = L$, then the first L elements of front(run($\rho_n \oplus \rho_{n+k}$)) form the same sequence as front(run(ρ_n)). That is, the two permutations execute the same initial sequence of L flips, until run(ρ_n) terminates with the identity permutation and run($\rho_n \oplus \rho_{n+k}$) terminates with the permutation ρ_{n+k}. (To see this, the reader should observe the following: Let n-run($\rho_n \oplus \rho_{n+k}$) denote the sequence obtained by restricting each permutation in run($\rho_n \oplus \rho_{n+k}$) to its first n elements, and replacing $(\rho_n \oplus \rho_{n+k})$ (1), wherever it occurs with 1. Then, n-run($\rho_n \oplus \rho_{n+k}$) is

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easily seen to be exactly the same sequence of permutations as the sequence run(ρ_n).) As the L^{th} element of run($\rho_n \oplus \rho_{n+k}$) is the permutation ρ_{n+k}, it follows that

$$\begin{aligned}|\mathrm{run}(\rho_n \oplus \rho_{n+k})| &= |\mathrm{front}(\mathrm{run}(\rho_n \oplus \rho_{n+k}))| \\ &= L + |\mathrm{run}(\rho_{n+k})| = |\mathrm{run}(\rho_n)| + |\mathrm{run}(\rho_{n+k})|.\end{aligned}$$

□

One can, of course, chain together more than two permutations. In general, for permutations $\rho_n, \rho_{n+k}, \rho_{n+2k}, \ldots, \rho_{n+mk}$, for $m \geq 2$, let $\rho_n \oplus \rho_{n+k} \oplus \rho_{n+2k} \ldots \oplus \rho_{n+mk}$ denote the permutation $(\ldots((\rho_n \oplus \rho_{n+k}) \oplus \rho_{n+2k}) \ldots \oplus \rho_{n+mk})$. The following is derived directly from Lemma 5.2.

Corollary 5.1. For all $m \geq 1$,

$$|\mathrm{run}(\rho_n \oplus \rho_{n+k} \oplus \rho_{n+2k} \ldots \oplus \rho_{n+mk})| = \sum_{i=0}^{m} |\mathrm{run}(\rho_{n+ik})|$$

Corollary 5.2. For all $m \geq 1$, $\sigma_{21} \oplus \sigma_{33} \oplus \sigma_{45} \ldots \oplus \sigma_{21+12m}$ is a permutation on $n = 21 + 12m$ symbols and

$$|\mathrm{run}(\sigma_{21} \oplus \sigma_{33} \oplus \sigma_{45} \ldots \oplus \sigma_{21+12m})| = \frac{7}{96}\left(3n^2 + 14n - 369\right).$$

Proof: By Corollary 5.1, $|\mathrm{run}(\sigma_{21} \oplus \sigma_{33} \oplus \sigma_{45} \ldots \oplus \sigma_{21+12m})| = \sum_{i=0}^{m} |\mathrm{run}(\sigma_{21+12i})|$. By Lemma 5.1, $|\mathrm{run}(\sigma_{21+12i})| = (21(21+12i) - 77)/4$,

for all $i \geq 0$. Thus, $|\mathrm{run}(\sigma_{21} \oplus \sigma_{33} \oplus \sigma_{45} \ldots \oplus \sigma_{21+12m})| = \sum_{i=0}^{m} (63i + 91)$

$$= 31.5m^2 + 122.5m + 91 = 31.5\left(\frac{n-21}{12}\right)^2 + 122.5\left(\frac{n-21}{12}\right) + 91$$

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$$= \frac{7}{96}\left(3n^2 + 14n - 369\right).$$

□

The above establishes the following theorem:

Theorem 5.1. $h(n) = \Omega(n^2)$.

For example for m =10, $\sigma_{21} \oplus \sigma_{33} \oplus \sigma_{45} \ldots \oplus \sigma_{141}$ is the following permutation:

21, 2, 3, 4, 5, 6, 7, 8, 9,
11, 20, 12, 17, 13, 19, 14, 10, 33, 16, 18, 15,
23, 32, 24, 29, 25, 31, 26, 22, 45, 28, 30, 27,
35, 44, 36, 41, 37, 43, 38, 34, 57, 40, 42, 39,
47, 56, 48, 53, 49, 55, 50, 46, 69, 52, 54, 51,
59, 68, 60, 65, 61, 67, 62, 58, 81, 64, 66, 63,
71, 80, 72, 77, 73, 79, 74, 70, 93, 76, 78, 75,
83, 92, 84, 89, 85, 91, 86, 82, 105, 88, 90, 87,
95, 104, 96, 101, 97, 103, 98, 94, 117, 100, 102, 99,
107, 116, 108, 113, 109, 115, 110, 106, 129, 112, 114, 111,
119, 128, 120, 125, 121, 127, 122, 118, 141, 124, 126, 123,
131, 140, 132, 137, 133, 139, 134, 130, 1, 136, 138, 135,

and by Corollary 5.2,

$$|\mathrm{run}(\sigma_{21} \oplus \sigma_{33} \oplus \sigma_{45} \ldots \oplus \sigma_{141})| = \frac{7}{96}\left(3(141)^2 + 14(141) - 369\right) = 4466.$$

In the preceding we used permutations whose run sequence terminates with the identity permutation. In fact, the proof of Lemma 5.2 *requires* that each of the permutations σ_n, σ_{n+k}, σ_{n+2k}, ..., $\sigma_{n+(m-1)k}$ be such. However, the last

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permutation in the sequence, namely σ_{n+mk}, need not terminate with the identity. Using this, the constant in the lower bound can be improved. That is, one can replace σ_{n+mk} with another permutation τ_{n+mk} whose run sequence is longer, but does not terminate with the identity. To this end we now describe a family of permutations $\{\tau_n\}$ in $\Pi^{(11)}$ also found by our computer search. We actually found several families exhibiting the same type of regularity as $\{\tau_n\}$, whose run lengths are asymptotically better than those for $\{\sigma_n\}$ and do not end with the identity. We have chosen $\{\tau_n\}$, as it as good as any of the others and is better than several.

For all $n \geq 69$, let τ_n be the permutation on $\{1,2, \ldots ,n\}$ defined by:

τ_n = n-4, (2,3, ... , n-11), n-9, n-3, n-10, n, n-5, n-7, n-8, n-1, 1, n-6, n-2.

Lemma 5.3. For all $n \geq 69$, such that $n \equiv 5 \pmod 8$,

$$|\mathrm{run}(\tau_n)| \geq \frac{1}{12}(n^2 - 39n + 750).$$

Proof: We claim that, for all $n \geq 69$, such that $n \equiv 5 \pmod 8$, front(run(τ_n)) *begins* with:

(**) n-4, n-8, $\{5+4j\}_{0 \leq j \leq (n-17)/4}$, **A**, $\{\mathbf{B}(j)\}_{0 \leq j \leq (n-29)/8}$, **C**, $\{\mathbf{Z}(k)\}_{0 \leq k \leq \lfloor (n-69)/2 \rfloor}$

where:

(1) **A** = n-10, n-5, 2, 3, 4, n-8, n-13, n-10, 2, 4, n-13, n-11, n-3, n-1, n-6, n-8, n-3, n-5, n-13, n-11

(2) **B**(j) = 3, (n-13)-8j, (n-17)-8j, 2, 4, (n-15)-8j, 3, (n-17)-8j, 2, 4, (n-21)-8j, (n-19)-8j,

(3) **C** = 3, 8, 6, 3, 2, 9, 5, 7, 3, 5, 11, 17, 13, 7, 11, 19, 25, 21, 9,

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(4) $\mathbf{Z(k)} = \{\ 5, 11, 7, 5, 9, 3, \{\mathbf{D(j)}\}_{0\le j\le 1+4k}, \mathbf{E(k)}, \{\mathbf{F(j,k)}\}_{0\le j\le 4k},$
$3, 7, 11, 5, 7, 3, 5,\ \{\mathbf{G(j)}\}_{0\le j\le 2+4k}, \mathbf{H(k)}, \{\mathbf{I(j,k)}\}_{0\le j\le 2+4k},$
$\{\mathbf{N(j)}\}_{0\le j\le 3+4k}, \mathbf{P(k)}, \{\mathbf{Q(j,k)}\}_{0\le j\le 2+4k}, 3, 5, 7, 11, 15, 9, 3\ \},$

where:

(i) **D(j)** = 7+6j,

(ii) **E(k)** = 15+24k, 19+24k, 27+24k, 33+24k, 29+24k, 17+24k, 11+24k,
13+24k, 19+24k, 15+24k,

(iii) **F(j,k)** = 9+24k-6j, 13+24k-6j, 17+24k-6j, 11+24k-6j, 7+24k-6j,
13+24k-6j, 9+24k-6j,

(iv) **G(j)** = 9+6j,

(v) **H(k)** = 23+24k, 27+24k, 35+24k, 41+24k, 37+24k, 25+24k, 19+24k,

(vi) **I(j,k)** = 17+24k-6j, 21+24k-6i, 25+24k-6j, 19+24k-6j, 15+24k-6j,
21+24k-6j, 17+24k-6j,

(vii) **N(j)** = 11+6j,

(viii) **P(k)** = 31+24k, 35+24k, 43+24k, 49+24k, 45+24k, 33+24k,
27+24k, 29+24k, 35+24k, 31+24k,

(ix) **Q(j,k)** = 25+24k-6j, 29+24k-6j, 33+24k-6j, 27+24k-6j, 23+24k-6j,
29+243k-6j, 25+24k-6j.

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It is straightforward to verify that front(run(τ_n)) begins with the natural numbers n-4, n-8, and that (after these two flips) the third element of run(τ_n) is:

(5, 6, 7, 8, 9, 10, 11, ... ,n-11), n-9, n-3, n-10, n, n-5, n-7, n-8, 4, 3, 2, n-4, n-1, 1, n-6, n-2.

The reader can verify, by induction on n similar to that done in the proof of Lemma 5.1, that the next (n-13)/4 elements of front(run(τ_n)) are $\{5+4j\}_{0\leq j\leq (n-21)/6}$, and that, after this sequence of flips, the resulting permutation in run(τ_n) is:

(n-12, n-13, n-14, n-15), (n-20, n-21, n-22, n-23), ... (9, 8, 7, 6), 5,
(10, 11,12,13), (18, 19, 20, 21), ... (n-19, n-18, n-17, n-16),
n-11, n-9, n-3, n-10, n, n-5, n-7, n-8, 4, 3, 2, n-4, n-1, 1, n-6, n-2.

The next twenty elements of front(run(τ_n)) are **A** = n-10, n-5, 2, 3, 4, n-8, n-13, n-10, 2, 4, n-13, n-11, n-3, n-1, n-6, n-8, n-3, n-5, n-13, n-11 and after this sequence the resulting permutation in run(τ_n) is:

3, n-10, n-13, n-7, 2, 4,
(n-14, n-15),(n-20, n-21, n-22, n-23), ... (9, 8, 7, 6),
5,
(10, 11,12,13), (18, 19, 20, 21), ... (n-19, n-18, n-17, n-16),
n-11, n-8, 1, n-6, n-12, n, n-5, n-9, n-3, n-4, n-1, n-2.

The next 12 elements of front(run(τ_n)) form the sequence **B**(0), *i.e.* 3, n-13, n-17, 2, 4, n-15, 3, n-17, 2, 4, n-21, n-19, and, after this sequence, the resulting permutation in run(τ_n) is indicated below:

3, n-18, (n-21,n-20), 2, 4, (n-22,n-23),
(n-28, n-29, n-30, n-31) ... (9,8,7,6),

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5,
(10, 11, 12, 13), (18, 19, 20, 21), ... (n-27, n-26, n-25, n-24),
(n-19, n-7, n-17, n-14, n-15, n-10, n-13, n-16)
n-11, n-8, 1, n-6, n-12, n, n-5, n-9, n-3, n-4, n-1, n-2.

By induction on j, it can be shown, that for each j ($1 \leq j \leq (n-37)/8$), the next 12 elements of front(run(τ_n)) form the sequence **B**(j), and that, after each such sequence of flips, the resulting permutation in run(τ_n) is the one indicated below:

3, n-18-8j, (n-21-8j, n-20-8j), 2, 4, (n-22-8j, n-23-8j)
(n-28-8j, n-29-8j, n-30-8j, n-31-8j), ... , (9, 8, 7, 6), 5,
(10, 11,12,13), (18, 19, 20, 21), ... (n-27-8j, n-26-8j, n-25-8j n-24-8j),
[(n-19-8j, n-12-8j, n-17-8j, n-14-8j, n-15-8j, n-10-8j, n-13-8j, n-16-8j), ...
(n-27, n-20, n-25, n-22, n-23, n-18, n-21, n-24)],
(n-19, n-7, n-17, n-14, n-15, n-10, n-13, n-16),
n-11, n-8, 1, n-6, n-12, n, n-5, n-9, n-3, n-4, n-1, n-2.

Substituting j=(n-37)/8 in the permutation above, one obtains what is indicated below:

3, 19, (16,17), 2, 4, (15, 14), (9, 8, 7, 6), 5, (10, 11, 12, 13)
[(18, 25, 20, 23, 22, 27, 24, 21), (26, 33, 28, 31, 30, 35, 32, 29), ... ,
(n-27, n-20, n-25, n-22, n-23, n-18, n-21, n-24)],
(n-19, n-7, n-17, n-14, n-15, n-10, n-13, n-16),
n-11, n-8, 1, n-6, n-12, n, n-5, n-9, n-3, n-4, n-1, n-2.

The next 12 elements of front(run(τ_n)) form the sequence **B**(j), for j = n-29/8, *i.e.*, 3, 16, 12, 2, 4, 14, 3, 12, 2, 4, 8, 10, and, after this sequence, the permutation in run(τ_n) is the one indicated below:

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3, 11, 8, 9, 2, 4, 7, 6, 5,

[(10, 17, 12, 15, 14, 19, 16, 13), (18, 25, 20, 23, 22, 27, 24, 21),

(n-27, n-20, n-25, n-22, n-23, n-18, n-21, n-24)],

(n-19, n-7, n-17, n-14, n-15, n-10, n-13, n-16),

n-11, n-8, 1, n-6, n-12, n, n-5, n-9, n-3, n-4, n-1, n-2.

The next 19 elements of front(run(τ_n)) are **C** = 3, 8, 6, 3, 2, 9, 5, 7, 3, 5, 11, 17, 13, 7, 11, 19, 25, 21, 9 and after this sequence of flips the permutation in run(τ_n) is the one indicated below:

5, 4, 3, 8, 11, 16, 13, 2, 9, 6, 7, 12, 15, 14, 19, 20, 23, 22, 27, 24, 21, 10, 17, 18, 25,

$\{26+8j, 33+8j, 28+8j, 31+8j, 30+8j, 35+8j, 32+8j, 29+8j\}_{0\le j\le (n-53)/8}$,

(n-19, n-7, n-17, n-14, n-15, n-10, n-13, n-16),

n-11, n-8, 1, n-6, n-12, n, n-5, n-9, n-3, n-4, n-1, n-2.

Then, for each k ($0 \le k \le \lfloor (n-69)/24 \rfloor$), it can be shown that the next 96k+108 elements of front(run(τ_n)) form the sequence **Z(k)** and afterwards the permutation in run(τ_n) is the appropriate one of those indicated below:

k≡1(mod 4): **5, 2, 3, 6, 11, 16, 13, 4, 9, 8, 7,** Y(k), X(k), W(k),
(n-19, n-7, n-17, n-14, n-15, n-10, n-13, n-16),
n-11, n-8, 1, n-6, n-12, n, n-5, n-9, n-3, n-4, n-1, n-2

k≡2(mod 4): **5, 6, 3, 4, 11, 16, 13, 8, 9, 2, 7,** Y(k), X(k), W(k),
(n-19, n-7, n-17, n-14, n-15, n-10, n-13, n-16),
n-11, n-8, 1, n-6, n-12, n, n-5, n-9, n-3, n-4, n-1, n-2

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k≡3(mod 4): **5, 4, 3, 8, 11, 16, 13, 2, 9, 6, 7,** Y(k), X(k), W(k),
(n-19, n-7, n-17, n-14, n-15, n-10, n-13, n-16),
n-11, n-8, 1, n-6, n-12, n, n-5, n-9, n-3, n-4, n-1, n-2

k≡0(mod 4): **5, 8, 3, 2, 11, 16, 13, 6, 9, 4, 7,** Y(k), X(k), W(k),
(n-19, n-7, n-17, n-14, n-15, n-10, n-13, n-16),
n-11, n-8, 1, n-6, n-12, n, n-5, n-9, n-3, n-4, n-1, n-2

where **Y(k)** = **{ 14+24j, 19+24j, 12+24j, 15+24j, 10+24j, 17+24j, 18+24j, 25+24j, 24+24j, 21+24j, 20+24j, 23+24j, 28+24j, 31+24j, 22+24j, 27+24j, 32+24j, 29+24j, 40+24j, 37+24j, 26+24j, 33+24j, 30+24j, 35+24j}**$_{0\le j\le k}$,

X(k) = **[36+24k, 39+24k, 38+24k, 43+24k, 44+24k, 47+24k, 46+24k, 51+24k, 48+24k, 45+24k, 34+24k, 41+24k, 42+24k, 49+24k],** and

W(k) = **{50+24k+8j, 57+24k+8j, 52+24k+8j, 55+24k+8j, 54+24k+8j, 59+24k+8j, 56+24k+8j, 53+24k+8j}**$_{0\le j\le (n-77)/8}$

As **Z(k)** has length 96k+108 and is done for k=0,1, ... $\lfloor (n-69)/24 \rfloor$, the length of this portion of front(run(τ_n)) is

$$\sum_{k=0}^{\lfloor (n-69)/24 \rfloor}(96k+108) = \frac{1}{12}\left(n^2 - 60n + 675\right).$$ The initial portion of front(run(τ_n)), before any of the **Z(k)**'s, has length

$$2 + \left(\frac{n-17}{4}+1\right) + 20 + 12\left(\frac{n-29}{8}+1\right) + 19 = \frac{1}{4}(7n+25).$$

Therefore, the sequence (**) has total length $\frac{1}{12}\left(n^2 - 39n + 750\right)$.

□

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For explicitness, we observe that τ_{141} is the permutation

137, (2, 3, 4, ..., 129, 130), 132, 138, 131, 141, 136, 134, 133, 140, 1, 135, 139,

and from Lemma 5.3, $|\mathrm{run}(\tau_{141})| \geq \frac{1}{12}\left(141^2 - 39(141) + 750\right) = 1261$. Notice that Lemma 5.3 is another way to show Theorem 5.1, namely $h(n) = \Omega(n^2)$.

We also observe that $\mathrm{front}(\mathrm{run}(\tau_n))$ does not *terminate* with the sequence (**). (This is apparent as in all four cases indicated in the proof of Lemma 5.3 the first symbol in the final permutation shown is 5 and not 1.) In fact, $\mathrm{front}(\mathrm{run}(\tau_n))$ often continues after (**) for quite a long time. However, we have been unable to discern a pattern in this remaining portion and consequently have not included it.

We now combine the results of Corollary 5.2 and Lemma 5.3 to improve the coefficient of n^2 from $\frac{21}{96}$ in Corollary 5.2 to $\frac{29}{96}$.

Theorem 5.2. For all even $m \geq 4$, $\sigma_{21}\oplus\sigma_{33}\oplus\sigma_{45}\ldots\oplus\sigma_{21+12(m-1)}\oplus\tau_{21+12m}$ is a permutation on $n=21+12m$ symbols and

$$|\mathrm{run}(\sigma_{21}\oplus\sigma_{33}\oplus\sigma_{45}\ldots\oplus\sigma_{21+12(m-1)}\oplus\tau_{21+12m})| \geq \frac{1}{96}\left(29n^2 - 718n + 5265\right).$$

Proof: By Corollary 5.2, $|\mathrm{run}(\sigma_{21}\oplus\sigma_{33}\oplus\sigma_{45}\ldots\oplus\sigma_{21+12r}| = \frac{7}{96}\left(3s^2 + 14s - 369\right)$, where $s = 21+12r$. Letting $r = m-1$, we get $s = 21+12(m-1) = (21+12m)-12 = n-12$ and $|\mathrm{run}(\sigma_{21}\oplus\sigma_{33}\oplus\sigma_{45}\ldots\oplus\sigma_{21+12(m-1)})| = \frac{7}{96}\left(3(n-12)^2 + 14(n-12) - 369\right)$

$= \frac{7}{96}\left(3n^2 - 58n - 105\right)$. By Lemmas 2.2.2 and 2.2.3,

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$$|\mathrm{run}(\sigma_{21}\oplus\sigma_{33}\oplus\sigma_{45}\ldots\oplus\sigma_{21+12(m-1)}\oplus\tau_{21+12m})|$$
$$= |\mathrm{run}(\sigma_{21}\oplus\sigma_{33}\oplus\sigma_{45}\ldots\oplus\sigma_{21+12(m-1)})| + |\mathrm{run}(\tau_{21+12m})|$$
$$\geq \frac{7}{96}\left(3n^2 - 58n - 105\right) + \frac{1}{12}\left(n^2 - 39n + 750\right),$$

giving us a total length of at least $\frac{1}{96}\left(29n^2 - 718n + 5265\right)$.

□

To illustrate Theorem 5.2 for m=10, $\sigma_{21}\oplus\sigma_{33}\oplus\sigma_{45}\ldots\oplus\sigma_{129}\oplus\tau_{141}$ is the permutation

21, 2, 3, 4, 5, 6, 7, 8, 9,
11, 20, 12, 17, 13, 19, 14, 10, 33, 16, 18, 15,
23, 32, 24, 29, 25, 31, 26, 22, 45, 28, 30, 27,
35, 44, 36, 41, 37, 43, 38, 34, 57, 40, 42, 39,
47, 56, 48, 53, 49, 55, 50, 46, 69, 52, 54, 51,
59, 68, 60, 65, 61, 67, 62, 58, 81, 64, 66, 63,
71, 80, 72, 77, 73, 79, 74, 70, 93, 76, 78, 75,
83, 92, 84, 89, 85, 91, 86, 82, 105, 88, 90, 87,
95, 104, 96, 101, 97, 103, 98, 94, 117, 100, 102, 99,
107, 116, 108, 113, 109, 115, 110, 106, 129, 112, 114, 111,
119, 128, 120, 125, 121, 127, 122, 118, 137, 124, 126, 123,
130, 132, 138, 131, 141, 136, 134, 133, 140, 1, 135, 139.

and $|\mathrm{run}(\sigma_{21}\oplus\sigma_{33}\oplus\sigma_{45}\ldots\oplus\tau_{141})| \geq \frac{1}{96}\left(29(141)^2 - 718(141) + 5265\right) = 5006.$

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Figure 5.3 compares results from Corollary 5.2 and Theorem 5.2.

n	Lower bound for length of $run(\sigma_{21} \oplus \sigma_{33} \oplus \sigma_{45} \ldots \oplus \sigma_{n-12} \oplus \tau_n)$	Length of $run(\sigma_{21} \oplus \sigma_{33} \oplus \sigma_{45} \ldots \oplus \sigma_{n-12} \oplus \sigma_n)$
69	977	1,085
93	1,972	1,960
117	3,315	3,087
141	5,006	4,466
165	7,045	6,097
189	9,432	7,980
213	12,167	10,115
237	15,250	12,502
261	18,681	15,141
285	22,460	18,032
309	26,587	21,175
...	...	...
789	182,207	136,955
...	...	...

Figure 5.3. A comparison of results from Theorem 5.2 and Corollary 5.2.

5.3 Conclusions

The permutations described here are among the best we have found by substantial computer search. In fact, several searches using permutations chosen at random did not find any with run sequences as long as those described. This suggests that long-winded permutations are scarce. Furthermore, when the random search is limited to permutations in $\Pi^{(k)}$ for various small values of k such as k=16 and k=24, the computer rarely finds permutations with long run sequences that end with the identity. Interestingly enough, however, the computer finds many with *the same* flip sequence pattern as our previously described τ_n (which like τ_n do not end with the identity permutation). This

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suggests that permutations similar to τ_n are *not* sparsely populated in $\Pi^{(k)}$ for small values of k, but permutations similar to σ_n are. Significantly, the random search for long-winded permutations in $\Pi^{(16)}$ and $\Pi^{(24)}$ did not find any permutations with longer run sequences than τ_n. Clearly new techniques are needed to find permutations with run sequences growing faster asymptotically than $\Omega(n^2)$.

In our exhaustive computer searches of $\Pi^{(k)}$ for various small values of k, such as k=9, 10, 11, 12, 13, and 14, we found several permutations with behavior similar to σ_n and τ_n. For completeness we list some of these permutations in the following figures.

$\Pi^{(k)}$, k=	general form of permutation on {1,2,...,n}	good values of n
12	n, (2, 3, ... , n-12), n-1, n-7, n-3, n-2, n-6, n-4, n-5, n-11, n-9, 1, n-8, n-10	n≡16(mod 20)
12	n, (2, 3, ... , n-12), n-10, n-7, n-2, n-9, n-4, n-11, n-1, n-3, 1, n-5, n-8, n-6	n≡2(mod 12)
12	n, (2, 3, ... , n-12), n-9, n-2, n-7, n-11, n-10, n-1, n-3, n-5, n-4, 1, n-6, n-8	n≡6(mod 8)
13	n,(2,3,...,n-13),n-10, 1,n-2, n-5, n-7, n-12, n-9, n-11, n-4, n-3, n-1, n-8,n-6	n≡3(mod 12)
14	n,(2,3,...,n-14),n-10,n-13,n-8,n-5,n-4,n-3,n-12,n-11,n-7,n-2,n-1,1,n-9,n-6	n≡7(mod 20)

Figure 5.4. Other Permutations with Long Run Sequences that Terminate with the Identity.

general form of permutation on {1,2,...,n}	good values of n
n, (2, 3, ... , n-12), n-6, n-10, n-1, n-3, n-9, n-8, n-5, 1, n-7, n-2, n-11, n-4	n≡2 (mod 4)
n, (2, 3, ... , n-12), n-6, n-11, n-1, n-8, n-10, n-9, n-2, 1, n-7, n-5, n-3, n-4	n≡2 (mod 4)
n, (2, 3, ... , n-12), n-10, 1, n-6, n-5, n-11, n-8, n-7, n-1, n-2, n-9, n-3, n-4	n≡0 (mod 4)
n, (2, 3, ... , n-12), n-7, n-1, n-5, 1, n-2, n-3, n-10, n-6, n-11, n-8, n-9, n-4	n≡3 (mod 4)
n, (2, 3, ... , n-12), n-8, n-3, n-9, n-10, n-5, 1, n-11, n-6, n-1, n-7, n-2, n-4	n≡2 (mod 4)
n, (2, 3, ... , n-12), 1, n-5, n-3, n-10, n-7, n-1, n-8, n-6, n-2, n-9, n-11, n-4	n≡1 (mod 4)

Figure 5.5. Other Permutations in $\Pi^{(12)}$ with Long Run Sequences that Do Not Terminate with the Identity.

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Chapter 6

Open Questions and Concluding Remarks

In Theorem 3.1, we describe a relatively simple dilation 1 embedding of pancakes into stars. This construction indicates how to simulate the pancake network of dimension n on the star network of dimension 2n. We have so far been unable to find a dilation one embedding into a smaller dimension star network. In the other direction, we conjecture that there is no dilation one embedding of star networks into pancake networks unless the dimension of the pancake network is substantially larger than that of the star network. Note that our result in Theorem 2.4 uses $O(n^3)$ symbols for a dilation one embedding of stars into pancakes. It is entirely possible that one may be able to construct a one-to-many dilation one embedding of stars into pancakes that uses, say, $O(n^2)$ symbols. This remains an open question.

These and other embedding results described in this thesis suggest that significant structural differences exist between pancake and star networks, and that the networks' computational capabilities may be quite different. Furthermore, we described other structural differences in Chapter 4. For example, the star network of dimension n contains all cycles of even length between 6 and n!, but no cycles of odd length. On the other hand, we have shown that the pancake network of dimension n contains cycles of *all* lengths between 6 and n!. This means that a dilation one embedding of the pancake network of dimension n into the star network of dimension n is impossible.

There is further evidence in the literature to strengthen the claim that pancakes and stars are less similar than commonly assumed. Another possible

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view of their structural differences can be seen by comparing embeddings of hypercubes into star networks [28,30] with embeddings of hypercubes into pancake networks [14]. In [28], Miller, Pritikin and Sudborough describe a one-to-many dilation one embedding of k-dimensional hypercubes into star networks of dimension 3k+1. The result is considerably different when the host network is the pancake network. Gardner, Miller, Pritikin and Sudborough [14] describe a one-to-many dilation one embedding of the hypercube of dimension 3n into the pancake network with $O(n^2)$ as many symbols. The complete result is defined for hypercube dimensions r in congruence classes r(mod 3), given by $Q_r \overset{\text{dil } 1}{\Rightarrow} P_k$, where:

$$Q_{3n} \overset{\text{dil } 1}{\Rightarrow} P_{11n^2-n}, \text{ when } r = 3n;$$

$$Q_{3n+1} \overset{\text{dil } 1}{\Rightarrow} P_{11n^2+21n-14}, \text{ when } r = 3n + 1; \text{ and}$$

$$Q_{3n+2} \overset{\text{dil } 1}{\Rightarrow} P_{11n^2+13n+5}, \text{ when } r = 3n + 2.$$

In fact, we have improved each of the above results by decreasing the dimension of the host network by 8(n-1) symbols. The new results are:

$$Q_{3n} \overset{\text{dil } 1}{\Rightarrow} P_{11n^2-9n+8}, \text{ when } r = 3n;$$

$$Q_{3n+1} \overset{\text{dil } 1}{\Rightarrow} P_{11n^2+13n+6}, \text{ when } r = 3n + 1; \text{ and}$$

$$Q_{3n+2} \overset{\text{dil } 1}{\Rightarrow} P_{11n^2+5n+13}, \text{ when } r = 3n + 2.$$

Observe that these embeddings require $O(n^2)$ host symbols whereas only O(n) symbols are required when the host is the star network. For completeness, we note improved embeddings (over those given in [14]) specifically for Q_2 and Q_3. The host network is the burnt pancake network, and the results easily generalized

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to embeddings into the pancake network. The improved embeddings are:

$$Q_2 \overset{\text{dil } 1}{\Rightarrow} BP_3, \text{ (hence } Q_2 \overset{\text{dil } 1}{\Rightarrow} P_4\text{), and}$$

$$Q3 \overset{\text{dil } 1}{\Rightarrow} BP_6, \text{ (hence } Q_3 \overset{\text{dil } 1}{\Rightarrow} P_7\text{).}$$

We believe that these and other results indicate important structural differences between pancakes and stars.

However, some of these structural differences are overcome when we allow greater dilation. For dilation two embeddings, O(n) symbols are required when the host is *either* the pancake *or* the star network. Specifically, the embeddings are:

$$Q_n \overset{\text{dil } 2}{\Rightarrow} S_{13n+2} \text{ presented in [28], and}$$

$$Q_n \overset{\text{dil } 2}{\Rightarrow} P_{2n} \text{ presented in [14].}$$

When the dilation is four, the embeddings of hypercubes into stars in [30] requires *the same number of host symbols* as the embedding of hypercubes into pancakes [14]. The specific results are:

$$Q_{(k-2)2^k+2} \overset{\text{dil } 4}{\rightarrow} S_{2^k-1} \text{ and}$$

$$Q_{(k-2)2^k+2} \overset{\text{dil } 4}{\rightarrow} P_{2^k-1}.$$

It is interesting to note that despite their superficial similarity, these embeddings have radically different constructions. The embedding into the star network is based on the recursive decomposition property of hypercubes and stars. That is, the following implication is shown in [30]:

$$\text{If } Q_t \overset{\text{dil } 4}{\rightarrow} S_n \text{ then } Q_{t+\lfloor \log_2(n+1) \rfloor} \overset{\text{dil } 4}{\rightarrow} S_{n+1}.$$

That is, if Q_t can be embedded with dilation 4 in S_n, then one can create a copy of a hypercube of dimension $\lfloor \log_2(n+1) \rfloor$ larger in S_{n+1}. That is, let $k = \lfloor \log_2(n+1) \rfloor$. Consider the set $P = \{\alpha_1, \alpha_2, \ldots, \alpha_{2k}\}$ of all $m = 2^k$ binary strings of length k. As

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$2^{\lfloor \log_2(n+1) \rfloor} < n+1$, one can label m different copies of the subgraph S_n in S_{n+1} with a unique string in the set P. Then, the idea is as follows: We embed Q_t in the copy of S_n labeled with, say, α_i. A node in this copy of Q_t, labeled, say, with the binary string β, will be labeled $\beta\alpha$ in $Q_{t+\lfloor \log_2(n+1) \rfloor}$. It is shown in [30] that adjacent hypercube nodes embedded in this way are at distance at most 4 in S_{n+1}.

The dilation four embedding into the pancake network has a very different construction. An example will perhaps serve to give the intuition behind the construction. To embed Q_{10} into P_7, define P_7 over symbols in the set $X=\{A_1, A_2, A_3, A_4, B_1, B_2, B_3\}$. Viewing a permutation on the set X as a string of symbols, the B's are used mark off four blocks in the string. We assign distinct two bit addresses to these blocks. Let a binary string labeling an arbitrary node in Q_{10} be denoted by $t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}$. The embedding associates the pancake symbol A_1 with hypercube bits t_1t_2, pancake symbol A_2 with bits t_3t_4, pancake symbol A_3 with bits t_5t_6, and pancake symbol A_4 with bits t_7t_8. The value of each bit pair is determined by the block address of the corresponding A symbol. Take for example the permutation $A_4A_1B_1B_2A_3B_3A_2$, where the block to the left of B_1 has address 00, the block between B_1 and B_2 has address 01, the block between B_2 and B_3 has address 10, and the block to the right of B_3 has address 11. According to this addressing scheme, hypercube bits t_1t_2 have the value 00 since A_1 is in the leftmost block; hypercube bits t_3t_4 have the value 11 since A_2 is in the rightmost block; hypercube bits t_5t_6 have the value 10 since A_3 is in the third block; and hypercube bits t_7t_8 have the value 00 since A_4 is in the leftmost block. Notice that so far, the location of the A symbols alone provides bit information. A bit is toggled by moving the corresponding A symbol to a block with the appropriate block address.

So far, the B symbols have been used only to demarcate blocks, and for this purpose, the B's might as well be indistinguishable from each other. The

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construction goes one step further and uses the uniqueness of the B symbols to represent two more bits. To understand how this is done, we imagine that the A symbols are invisible, and focus on the position of the B symbols. There are three of these, so we view the B's as a permutation on the 3 symbols $\{B_1, B_2, B_3\}$. We use one symbol, say B_1 as a block marker, and assign a bit address of 0 to the block to the left of B_1 symbol, and a bit address of 1 to the block on its right. We now associate hypercube bit t_9 with B_2 and hypercube bit t_{10} with B_3. The bit value of t_9 is determined by the block address of B_2, and the bit value of t_{10} is determined by the block address of B_3. Hence, the permutation $A_4A_1B_1B_2A_3B_3A_2$ represents a hypercube node with bit $t_9 = 1$ and bit $t_{10} = 1$ as well, because B_2 and B_3 are both to the right of B_1. By this interpretation, the permutation $A_4A_1B_1B_2A_3B_3A_2$ represents a hypercube node labeled 0011100011.

For larger pancake networks, more blocks can be created, so that each A symbol represents a larger number of bits. This has the additional effect of increasing the number of bits represented by each B symbol. Furthermore, the strategy of permuting the block dividers can be repeated recursively using the same general idea. A formal presentation of this embedding can be found in [14].

It is not clear how to account for the apparent similarity between these dilation four embeddings into stars and pancakes. The constructions are significantly different. It would be interesting to know if a dilation four embedding exists that uses the same technique for both host networks. We have not been able to find such an embedding.

We have shown a quadratic lower bound for the deterministic pancake problem. How tight is this lower bound? As noted in Chapter 5, the permutations in the family $\{\tau_n\}$ have run sequences that do not terminate after the number of steps described in Lemma 5.3, but in fact, continue for a large number of steps. Unfortunately, there is no readily discernible pattern in the latter portion

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of front(run(τ_n)) for any n. The significance of this seemingly random portion diminishes asymptotically, because the number of steps in the regular portion seems to grow faster. A quadratic lower bound is probably the best that can be found by the technique described in this thesis. Any improvement would most likely require a different approach.

Knuth's exponential upper bound [4] is the best known upper bound to date, although it can be improved slightly using our known values for $n \leq 15$. The intuition we have gained leads us to conjecture that the upper bound can be improved asymptotically. We conjecture a polynomial upper bound. Proving this conjecture is no easy task. A considerable amount of further and insight seems to be necessary to close the gap between these upper and lower bounds.

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Appendix 1

Knuth's Solution to the Deterministic Pancake Problem

Knuth's proof that $M_n \leq F_{n+1}$, where F_n denotes the n^{th} Fibonacci number [4]:

If array element A[1] takes on k distinct values during the (possibly infinite) execution of the algorithm, we will show that $m \leq F_{k+1}$ (hence m is finite). This is obvious for k=1, since k = 1 can occur only when $\pi[1]=1$.

If $k \geq 2$, let the distinct values assumed by A[1] be $d_1 < d_2 \ \ldots < d_k$. Suppose that A[1] occurs first on the rth permutation, and let $t = \pi[d_k]$. Then the $(r+1)^{st}$ permutation will have A[1] = t and $A[d_k] = d_k$. All subsequent permutations will also have $A[d_k] = d_k$ (they leave A[j] untouched for $j \geq d_k$), hence at most k-1 values are assumed by A[1] after the rth permutation has been passed. By induction, $m-r \leq F_k$, so m is finite and $d_1 = 1$.

Interchanging d_k with 1 in π produces a permutation π' such that $m(\pi')=r$, and for which the values d_k and t never appear in position A[1] unless t = 1. If t = 1 we have $r \leq F_k$, since A[1] assumes at most k-1 values when processing π', hence $m = r+1 \leq F_{k+1}$. If t >1 we have $r \leq F_{k-1}$ since A[1] assumes at most k-2 values when processing π' (note that $t = d_j$ for j<k) hence $m \leq F_k + r \leq F_{k+1}$.

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Appendix 2

Run(σ_{33})

The following is run(σ_{33}), where σ_{33} itself is given as line number 0:

0	33 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 23 32 24 29 25 31 26 22 1 28 30 27
1	27 30 28 1 22 26 31 25 29 24 32 23 21 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 33
2	7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 23 32 24 29 25 31 26 22 1 28 30 27 6 5 4 3 2 33
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