Problem 1.1 *Properties of convolution*

Recall that 1D convolution between two signals $f, g \in \mathbb{R}^N$ is defined:

$$(f * g)[n] = \sum_{k=0}^{N-1} f[n-k]g[k]. \tag{1}$$

(a) Construct a matrix multiplication that produces the same result as a 1D convolution with a given filter. In other words, given a filter f, construct a matrix H such that $f * g = Hg$ for any input g. Here Hg denotes matrix multiplication between the matrix H and vector g.
(2 points)

Note: One option is to define H in "bracket" notation using "..." symbols, e.g., $H = \begin{bmatrix} 0 & 1 & \dots & N-1 \\ N-1 & N-2 & \dots & 1 \end{bmatrix}$. Another option is to explicitly define the entries H_{ij}.

a) $[Hg]_i = \sum_{j=0}^{N-1} H_{ij} g_i \quad , H_{ij} = f[i-j]$ #

(b) In class, we showed that convolution with a 2D Gaussian filter can be performed efficiently as a sequence of convolutions with 1D Gaussian filters. This idea also works with other kinds of filters. We say that a 2D filter $F \in \mathbb{R}^{N \times N}$ is separable if $F = uv^\top$ for some $u, v \in \mathbb{R}^N$, i.e., F is the *outer product* of u and v. Show that if F is separable, then the 2D convolution $G * F$ can be computed as a sequence of two one-dimensional convolutions **(2 points)**.

b) Given $F = uv^T \Rightarrow F[m,n] = u[m] \cdot v[n]$

$$(G * F)[i,j] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} G[i-m, j-n] F[m,n]$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} G[i-m, j-n] u[m] v[n]$$

$$= \sum_{m=0}^{N-1} u[m] \left(\sum_{n=0}^{N-1} G[i-m, j-n] v[n] \right)$$

let $G'[i,j] = (G * v^T)[i,j] = \sum_{n=0}^{N-1} G[i, j-n] v[n] = G * v^T$

$$(G * F)[i,j] = \sum_{m=0}^{N-1} G'[i-m, j] u[m] = G' * u = G * v^T * u$$

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