1. (10 points) Given the following vectors:

$$p = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}, \quad q = \begin{bmatrix} -4 \\ -2 \\ 7 \end{bmatrix}, \quad r = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

Determine the following **by hand** (remember to show your work):

a. $p + 2q$

b. $p \cdot r$ and $r \cdot p$ where "$\cdot$" denotes the dot product

c. $q \times r$ and $r \times q$ where "$\times$" denotes the cross product

d. $||p||$, and $||q||$ where $||\cdot||$ denotes the Euclidean norm of a vector

e. distance between the tips of p and q

a. $p+2q = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -4 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 15 \end{bmatrix}$ #

b. $p \cdot r = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = 5 \cdot -1 + -1 \cdot -2 + 1 \cdot 3 = 0$ #

$r \cdot p = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} = -1 \cdot 5 + -2 \cdot -1 + 3 \cdot 1 = 0$ #

c. $q \times r = \begin{bmatrix} -4 \\ -2 \\ 7 \end{bmatrix} \times \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \cdot 3 - (-2) \cdot 7 \\ 7 \cdot -1 - (3) \cdot (-4) \\ -4 \cdot -2 - (-1)(-2) \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 6 \end{bmatrix}$ #

$r \times q = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} \times \begin{bmatrix} -4 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \cdot 7 - (-2) \cdot 3 \\ 3 \cdot -4 - 7 \cdot (-1) \\ -1 \cdot -2 - (-4)(-2) \end{bmatrix} = \begin{bmatrix} -8 \\ -5 \\ -6 \end{bmatrix}$ #

d. $||p|| = \sqrt{p_1^2 + p_2^2 + p_3^2} = \sqrt{5^2 + (-1)^2 + 1^2} = \sqrt{27}$ #

$||q|| = \sqrt{q_1^2 + q_2^2 + q_3^2} = \sqrt{(-4)^2 + (-2)^2 + 7^2} = \sqrt{69}$ #

e. $d(p,q) = ||p-q|| = \sqrt{(5-(-4))^2 + (-1-(-2))^2 + (1-7)^2} = \sqrt{118}$ #

2. (10 points) Find all k such that $p = \begin{bmatrix} -2 \\ 1 \\ -k \end{bmatrix}$ and $q = \begin{bmatrix} 2 \\ -3k \\ -k \end{bmatrix}$ are orthogonal **by hand**.

If two vectors are orthogonal, their dot product should be 0.

solve: $p \cdot q = 0$

$\Rightarrow -2 \cdot 2 + 1 \cdot -3k + (-k) \cdot (-k) = 0$

$\Rightarrow k^2 - 3k - 4 = 0$

$\Rightarrow (k-4)(k+1) = 0$

$\Rightarrow k = 4$ or -1 #

Matrices

3. (5 points) Partition the following matrix into submatrices (i.e. find W, X, Y, and Z) **by hand**:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

where $W \in \mathbb{R}^{2\times 1}$ and $Z \in \mathbb{R}^{2\times 3}$.

$W = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $X = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \end{bmatrix}$, $Y = \begin{bmatrix} 9 \\ 13 \end{bmatrix}$, $Z = \begin{bmatrix} 10 & 11 & 12 \\ 14 & 15 & 16 \end{bmatrix}$

4. (10 points) Perform the following matrix multiplication **by hand**:

$$\begin{bmatrix} 3 & -1 & -3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 9 & 10 \\ 0 & -5 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \end{bmatrix} \#$$

5. (12 points) Solve the following systems of equations using `numpy`. Recall that there can be a unique solution, no solution, or infinitely many solutions.

a) $\begin{bmatrix} 0 & 0 & -1 \\ 4 & 1 & 1 \\ -2 & 2 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 0 & -2 & 6 \\ -4 & -2 & -2 \\ 2 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

c) $\begin{bmatrix} 2 & -2 \\ -4 & 3 \end{bmatrix} x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(a) $x = [0.4, 2.4, -3.0]^T$

(b) x has no solution

(c) $x = [-2.5, -4]^T$

6. (14 points) Given the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & -2 \\ 4 & -3 \end{bmatrix}$$

Use `numpy` to calculate the following.

a. $A + 2B$

b. AB and BA

c. A^T, transpose of A

d. B^2

e. $A^T B^T$ and $(AB)^T$

f. $\det(A)$, determinant of A

g. B^{-1}, inverse of B

a. $A+2B = \begin{bmatrix} -3 & -2 \\ 11 & -7 \end{bmatrix}$

b. $AB = \begin{bmatrix} 6 & -8 \\ -10 & -3 \end{bmatrix}$, $BA = \begin{bmatrix} -8 & -2 \\ -5 & 11 \end{bmatrix}$

c. $A^T = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

d. $B^2 = \begin{bmatrix} -4 & 10 \\ -20 & 1 \end{bmatrix}$

e. $A^T B^T = \begin{bmatrix} -8 & -5 \\ -2 & 11 \end{bmatrix}$, $(AB)^T = \begin{bmatrix} 6 & -10 \\ -8 & -3 \end{bmatrix}$

f. $\det(A) = -7$

g. $B^{-1} = \begin{bmatrix} -0.214 & 0.143 \\ -0.286 & -0.143 \end{bmatrix}$

7. (10 points) A rotation matrix R is defined by the following sequence of basic rotations:

   i. A rotation of $\pi/2$ about the z-axis
   ii. A rotation of $-\pi/5$ about the new y-axis
   iii. A rotation of π about the new z-axis

   Compute the rotation matrix R.

Calculated using Python

$$R = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) & 0 \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-\frac{\pi}{5}) & 0 & \sin(-\frac{\pi}{5}) \\ 0 & 1 & 0 \\ -\sin(-\frac{\pi}{5}) & 0 & \cos(-\frac{\pi}{5}) \end{bmatrix} \begin{bmatrix} \cos(\pi) & -\sin(\pi) & 0 \\ \sin(\pi) & \cos(\pi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \doteq \begin{bmatrix} 0 & 1 & 0 \\ -0.809 & 0 & -0.588 \\ -0.588 & 0 & 0.809 \end{bmatrix} \#$$

8. (20 points) You would like to point a mobile robot so that it is looking at a target point. The camera of the robot is aligned with x in the robot's coordinate frame. The robot is assumed to be on flat ground ($z = 0$). The robot's pose in the world frame is:

$$T_r^0 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1.7 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 2.1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   a. The target point is at $p = [-0.4, 0.9, 0]^T$ in the world frame. Assuming the robot stays at its current position, calculate the vector v in the world frame that the camera should align with.
   b. Use v to calculate the desired pose of the robot, again assume the robot does not change position. The robot must remain upright on the ground; i.e. its z axis must not change.
   c. Prove that the rotation component of the pose meets **all** conditions for being a valid rotation matrix.

a.

$$T_r^0 = \begin{bmatrix} \boxed{\begin{matrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{matrix}}^{\text{rotation}} & \boxed{\begin{matrix} 1.7 \\ 2.1 \\ 0 \end{matrix}}^{\text{translation}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}$$

position of robot is $[1.7, 2.1, 0]^T$

$v = p - [1.7, 2.1, 0]^T = [-0.4, 0.9, 0]^T - [1.7, 2.1, 0]^T$
$= [-2.1, -1.2, 0]^T$

$$\frac{v}{\|v\|} = \frac{[-2.1, -1.2, 0]^T}{\sqrt{(-2.1)^2 + (-1.2)^2 + 0^2}} = [-0.8682, -0.4961, 0]^T \#$$

b. Now camera direction (x-dir) points to $\frac{v}{\|v\|} = [-0.8682, -0.4961, 0]^T$

robot remain upright $\Rightarrow$ z-dir points to $\vec{z}$ $[0, 0, 1]$

y-dir be orthogonal to x-dir & z-dir, thus y-dir $= x$-dir $\times$ z-dir

$\Rightarrow y\text{-dir} = [-0.8682, -0.4961, 0]^T \times [1, 0, 0]^T = [0.4961, -0.8682, 0]^T$

desired pose of robot:

$$\begin{bmatrix} \boxed{\begin{matrix} -0.8682 & 0.4961 & 0 \\ -0.4961 & -0.8682 & 0 \\ 0 & 0 & 1 \end{matrix}}^{R} & \boxed{\begin{matrix} 1.7 \\ 2.1 \\ 0 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \#$$

doesn't change position

C. ① Columns of R are orthogonal :

$[-0.8682, -0.4961, 0]^T \cdot [0.4961, -0.8682, 0]^T = -0.8682 \cdot 0.4961 - 0.4961 \cdot -0.8682 + 0 = 0$

$[0.4961, -0.8682, 0]^T \cdot [0, 0, 1]^T = 0 + 0 + 0 = 0$

$[0, 0, 1]^T \cdot [-0.8682, -0.4961, 0]^T = 0 + 0 + 0 = 0$

② Columns of R are unit vectors

$\| [-0.8682, -0.4961, 0]^T \| = \sqrt{(-0.8682)^2 + (-0.4961)^2 + 0^2} = 1$

$\| [0.4961, -0.8682, 0]^T \| = \sqrt{(0.4961)^2 + (-0.8682)^2 + 0^2} = 1$

$\| [0, 0, 1]^T \| = 1$

③ $\det(R) = 1$

$= -0.8682 \cdot \det\left(\begin{bmatrix} -0.8682 & 0 \\ 0 & 1 \end{bmatrix}\right) - 0.4961 \cdot \det\left(\begin{bmatrix} -0.4961 & 0 \\ 0 & 1 \end{bmatrix}\right) + 0 \cdot \det\left(\begin{bmatrix} -0.4961 & -0.8682 \\ 0 & 0 \end{bmatrix}\right)$

$= -0.8682 \cdot -0.8682 - 0.4961 \cdot -0.4961$

$= 1$

④ $R^{-1} = R^T$

Least Squares

9. (30 points) You are testing one of the joints of a new robot arm and you notice there is some error when moving to a target position. To investigate, you command the joint to move through a series of positions and measure where the joint moves for each command. The recorded data is in `calibration.txt`, where the first column is the commanded position and the second column is the measured position. You see that there is significant error in the commanded vs. measured positions. Note: You are strongly advised to consult the linear algebra book, section 13.1 for this problem, especially part (c).

a. Define problem as $Ax=b$, where $A = \begin{bmatrix} c_1 & 1 \\ c_2 & 1 \\ \vdots & \vdots \\ c_n & 1 \end{bmatrix}$, $x = [x_0, x_1]^T$, $b = [m_1, m_2, \ldots, m_n]^T$,

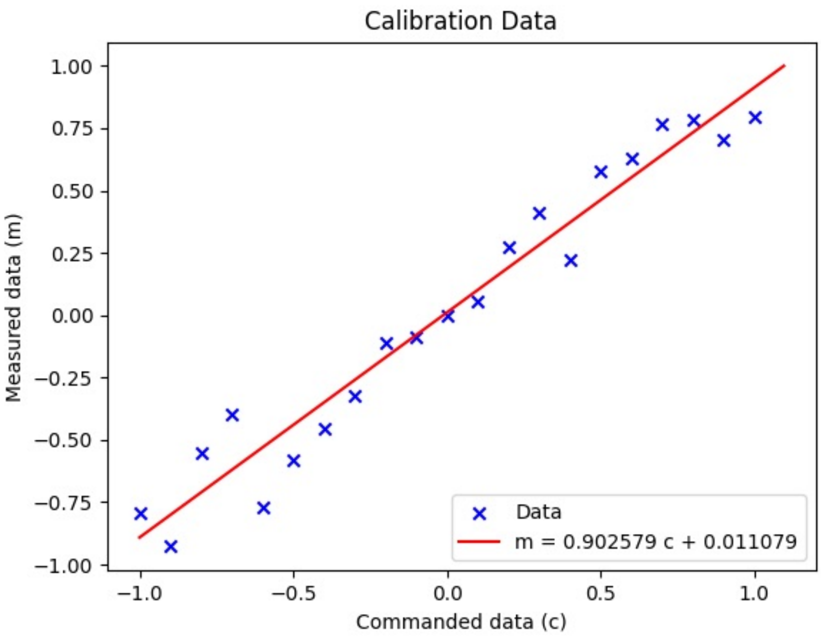
c_n & m_n denotes the n^{th} command data and measurement data respectively.

Solve for x: Use $A^+ = (A^TA)^{-1}A^T \Rightarrow x = A^+b$

① least-squares fit parameters:

$x_0 = 0.902579$, $x_1 = 0.011079$

② Sum of squared error: 0.310597



b. (5 points) Is this least-squares problem *underdetermined* or *overdetermined*? Explain your answer.

It's overdetermined ∵ there's >1 equations but only 2 unknowns to solve.

C. Define problem as $Ax = b$

where $A = \begin{bmatrix} (c_1-0.5)\,c\_nott & (c_1-(-0.5))\,c\_not & c_1 & 1 \\ (c_2-0.5)\,c\_nott & (c_2-(-0.5))\,c\_not & c_2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ (c_n-0.5)\,c\_nott & (c_n-(-0.5))\,c\_not & c_n & 1 \end{bmatrix}$, $c\_nott = \begin{cases} 0, & c_n < 0.5 \\ 1, & c_n \geq 0.5 \end{cases}$

$c\_not = \begin{cases} 0, & c_n < -0.5 \\ 1, & c_n \geq -0.5 \end{cases}$

$x = [x_0, x_1, x_2, x_3]^T$, $b = [m_1, m_2, \dots, m_n]^T$

c_n & m_n denotes the n^{th} command data and measurement data respectively.

Solve for x : Use $A^+ = (A^TA)^{-1}A^T \Rightarrow x = A^+ b$

① least-squares fit parameters :

$x_0 = -0.536132$, $x_1 = 0.563752$, $x_2 = 0.524522$, $x_3 = -0.272770$

② Sum of squared error $= 0.243304$

③ Measurement prediction with command 0.68 : 0.652628