Reevaluating $\zeta(2)$

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Continuing the work of Robin Chapman, I list three proofs of the famous identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

or its odd version :

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

They were shared on Internet. They add geometric and probabilistic aspects to the demonstrations. I tried to be as concise as possible while being understandable.

Proof 15 : Suggested by Josef Hofbauer in "The American Mathematical Monthly : A Simple Proof of and Related Identities".

Repeated application of the identity :

$$\frac{1}{\sin^2 x} = \frac{1}{\sin^2 \frac{x}{2} \cos^2 \frac{x}{2}} = \frac{1}{4}\left[\frac{1}{\sin^2 \frac{x}{2}} + \frac{1}{\cos^2 \frac{x}{2}}\right] = \frac{1}{4}\left[\frac{1}{\sin^2 \frac{x}{2}} + \frac{1}{\sin^2 \frac{\pi+x}{2}}\right],$$

yields :

$$1 = \frac{1}{\sin^2 \frac{\pi}{2}} = \frac{1}{4^n} \sum_{k=0}^{2^n-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}}.$$

By symmetry of sin with respect to $\frac{\pi}{2}$, we have :

$$\frac{1}{4^n} \sum_{k=0}^{2^n-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}} = \frac{2}{4^n} \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}}.$$

The kth term is bounded by $2/(2k+1)^2$ since $\sin x > 2x/\pi$ holds for $0 < x < \pi/2$. Then, by Lebesgue's dominated convergence theorem, the summation converges, and yields :

$$1 = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2},$$

using $\lim_{n\to\infty} 2^n \sin(x/2^n) = x$.

This proof has an interesting geometric and physical interpretation described by Johan Wästlund in "Summing inverse squares by euclidean geometry" and popularized by 3Blue1Brown on Youtube.

Here is the idea : in the Euclidean plane, consider a circle $\mathcal{C}_0$ on which lie two points O and P diametrically opposed. To fit with the problem, consider that $\mathcal{C}_0$ has perimeter 2. Thus $1/\mathrm{OP}^2 = \pi^2/4$.

Consider $\mathcal{C}_1$ as the homothety of factor 2 from O of $\mathcal{C}_0$. Following the perpendicular line to (OP), project P on $\mathcal{C}_1$ into Q and R. The triangle OQR is rectangle in O and by the Inverse Pythagorean Theorem :

$$\frac{1}{\mathrm{OP}^2} = \frac{1}{\mathrm{OQ}^2} + \frac{1}{\mathrm{OR}^2}.$$

Furthermore, by the inscribed angle theorem, Q and R are equally distributed in $\mathcal{C}_1$.

At the next step, consider $\mathcal{C}_2$ has the homothety of factor 2 from O of $\mathcal{C}_1$. Following the perpendicular line to (OQ) (resp. OR), project Q (resp. R) on $\mathcal{C}_2$ into S and T (resp. U and V). Those four points are equally distributed on $\mathcal{C}_2$ and verify :

$$\frac{1}{\mathrm{OP}^2} = \frac{1}{\mathrm{OS}^2} + \frac{1}{\mathrm{OT}^2} + \frac{1}{\mathrm{OU}^2} + \frac{1}{\mathrm{OV}^2}.$$

Repeat the process. Notice that on each circle, the generated points remain equally distributed, separated by circle arcs of length 2, and symmetrically with respect to (OP). At the limit, the circle degenerates into a line passing through O. On that line, the projected points are at distance $2k+1$ for $k \in \mathbb{N}$ from O, providing the result:

$$\frac{\pi^2}{4} = 2\sum_{n=0}^{\infty} \frac{1}{(2k+1)^2}.$$

As a physical interpretation, we can consider each point as a source of light and the inverse square of its distance to O as its apparent brightness from O.

Proof 16: Probabilistically proving that $\zeta(2) = \frac{\pi^2}{6}$, by Luigi Pace.

Consider two random variables X and Y that are Cauchy distributed. Their densities are :

$$f_X(x) = \frac{2}{\pi(1+x^2)} \text{ and } f_Y(y) = \frac{2}{\pi(1+y^2)}.$$

Let $T = Y/X$. As a ratio of two similar Cauchy distributions, T is distributed as follows:

$$f_T(t) = \frac{4}{\pi^2}\frac{\log(\frac{1}{t})}{1-t^2}.$$

By symmetry, $\mathbb{P}[t < 1] = 1/2$. But we can also compute this probability :

$$\begin{aligned}
\mathbb{P}[t<1] &= \frac{4}{\pi^2}\int_0^1 \frac{\log(\frac{1}{t})}{1-t^2}dt \\
&= \frac{4}{\pi^2}\sum_{k=0}^{\infty}\int_0^1 \log\left(\frac{1}{t}\right)t^{2k}dt \\
&= \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{t^{2k+1}}{2k+1}\left(\frac{1}{2k+1} - \log(t)\right)\Bigg|_0^1 \\
&= \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}.
\end{aligned}$$

By identification, we get : $\frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = 1/2$.

Proof 17 : This proof is due to Ash Malyshev on the forum math.stackexchange.com. We compute the expected number of nonzero lattice points in a given disk in two different ways.

A lattice in $\mathbb{R}^2$ is a set of the form $\Lambda := \{av + bw, (a, b) \in \mathbb{Z}^2\}$, where v and w are linearly independant vectors in $\mathbb{R}^2$. A lattice is unimodular if v and w span of a parallelogram of area 1. Equivalently, a unimodular lattice in $\mathbb{R}^2$ is an image under some element of SL(2,R) of the lattice $\mathbb{Z}^2$.

A unimodular lattice is spanned by vectors of the form $v = t^{-1/2}(1, 0)$ and $w = t^{-1/2}(s, t)$ with $t > 0$, where v is the shortest vector in the lattice and w the second (independant) vector. We use the complex number $z = s + it$ to represent this lattice. With that representation, every unimodular lattice in $\mathbb{R}^2$, modulo rotation, corresponds to exactly one complex number in the fundamental region:

$$\mathcal{F} = \{z \in \mathbb{C} \text{ such that } \mathrm{Im}(z) > 0, -\tfrac{1}{2} < \mathrm{Re}(z) < \tfrac{1}{2}, |z| > 1\}.$$

For the distribution of complex numbers in $\mathcal{F}$ to fit with the distribution of unimodular lattice, the measure must be preserved by the action of the groupe SL(2,R) on the upper half-plane. That means we must consider the half-plane as the Poincaré half-plane, in which the hyperbolic area of a small ds by dt rectangle at (s, t) is $t^{-2}ds\ dt$.

We pick a random unimodular lattice in SL(2,R) by picking a random complex number in $\mathcal{F}$. The area A of $\mathcal{F}$ is given by :

$$A = \int_{-1/2}^{1/2} \int_{\sqrt{1-s^2}}^{\infty} t^{-2} dt ds = \frac{\pi}{3}.$$

A hyperbolic argument on angles gives straightforwardly the same result.

Now, we compute the probability that a random unimodular lattice has a nonzero vector in the ball $B_r = \{p \in \mathbb{R}^2; |v| \leq r\}$, for a given $r \leq 1$. If the length of the shortest vector is $t^{-1/2}$, the event corresponds to $t^{-1/2} \leq r$, thus :

$$\mathbb{P}\left[B_r \cap \Lambda \setminus \{0\} \neq \emptyset\right] = \frac{1}{A} \int_{-1/2}^{1/2} \int_{r^{-2}}^{\infty} t^{-2} dt ds = \frac{r^2}{A}.$$

A primitive element of a lattice is a nonzero element which is not a natural number multiple of any other element. Let Δ_1 denote the set of primitive elements of Δ. If Δ is unimodular and $r < 1$, then the ball B_r either contains no element of Δ_1 or exactly two, v and $-v$. Thus, the expected number of primitive vectors of a random unimodular lattice in B_r is :

$$\mathbb{E}\left[|B_r \cap \Lambda_1|\right] = 2\frac{r^2}{A} = \frac{2}{A} r^2.$$

On the other hand, every nonzero vector in a lattice Δ is either a primitive vector, twice a primitive vector, three times a primitive vector, and so on. So the expected number of nonzero vectors of Δ in B_r is the expected number of primitive vectors in B_r, plus the expected number of primitive vectors in $B_{r/2}$, plus the expected number of primitive vectors in $B_{r/3}$, and so on :

$$\begin{aligned}
\mathbb{E}\left[|B_r \cap \Lambda \setminus \{0\}|\right] &= \frac{2}{A} r^2 + \frac{2}{A}(r/2)^2 + \frac{2}{A}(r/3)^2 + ... \\
&= \frac{2}{A}(1 + 1/2^2 + 1/3^2 + ...) r^2 \\
&= \frac{2\zeta(2)}{A} r^2.
\end{aligned}$$

But we can compute the expectation in a more straightforward way. Heuristically, generating a random unimodular lattice boils down to starting with $\mathbb{Z}^2$ and applying a random element of SL(2,R) to it. This smears out the nonzero lattice points uniformly over all of $\mathbb{R}^2$. Since the lattice have density 1, the expected number of nonzero lattice points that land in a given set is precisely the area of that set. In other words :

$$\mathbb{E}\left[|B_r \cap \Lambda \setminus \{0\}|\right] = Area(B_r) = \pi r^2.$$

Comparing this to the previous expression, we have $2\zeta(2)/A = \pi$ and hence :

$$\zeta(2) = \frac{A}{2}\pi = \frac{\pi^2}{6}.$$