The combinatorial theory of single-elimination tournaments
by Christopher Todd Edwards

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Statistics
Montana State University
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Abstract:
This thesis is about single-elimination tournaments. These tournaments are popularly used in sporting events, but are also used in paired comparison procedures when the number of treatments is too large to use a round-robin tournament. The nature of this work is experimental design, and thus can help researchers to clarify pre-experiment issues.

Combinatorial methods are used to generate and count the distinct tournament structures that exist. Recursive formulas are derived. Various criteria for "best" are considered. A tournament draw is said to be ordered when the probabilities of teams winning the tournament are ordered by the relative strengths of the teams. Numerous theorems are given concerning ordered tournaments, subtournaments, and "best" tournaments. The results developed apply to any number of teams, not just when the number of teams is a power of two. Assuming a transitive preference structure for the pairwise probabilities of teams beating each other, results on ordered tournaments and "best" tournaments are given, some of which make use of optimization routines on the computer. Results show that there exist categories of ordered tournaments which are best under certain criteria. It is proven that ordered tournament draws can be generated by iteratively combining subtournaments in just two specific ways.

One important conclusion is that the popular "seeded" tournament is not optimal under several reasonable criteria, and hence many tournaments currently conducted, such as high school basketball tournaments, are perhaps inappropriate.

THE COMBINATORIAL THEORY OF SINGLE-ELIMINATION TOURNAMENTS

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Christopher Todd Edwards

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ABSTRACT

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CHAPTER 1

INTRODUCTION

Frequently, a researcher wants to choose the best treatment from a set of treatments. In situations where the treatments can only be compared in pairs, paired comparison techniques are well suited to choose the best treatment. A common example of such a situation is a consumer preference study, where people indicate pairwise preferences for a set of products. A similar situation occurs in sporting events where the best team or player is chosen from a set of teams or players. For convenience, we will use in this thesis the terminology of sports; however, the reader should be aware that all results apply to any situation where paired comparisons are appropriate.

A *tournament* is a rule which specifies how the teams or players are to be compared to choose a winner or winners. In the sports setting, it is customary to use the terms *games, competitions,* and *contests* interchangeably to represent experimental units. Similarly, the terms *teams, players, competitors,* and *contestants* are interchangeable and represent the treatments. We assume in this thesis that games always result in either a win or a loss; ties are not allowed. Further, the score or the size of the victory in a contest is not considered in future contests. The rule for a tournament may also include the method of awarding additional *places*, e.g. second place, third place, etc. Nevertheless, many tournaments are only concerned with choosing the winner, the first-place team.

The tournament most discussed in the statistical literature is the *round-robin tournament*, where every team competes once with every other team. The games specified in a round-robin tournament can be played in any order; the results of prior games do not influence the future games to be played. In *repeated round-robin tournaments*, each pair of

teams plays more than one game. *Knockout tournaments* differ from round-robin tournaments in that not all possible pairs of games occur. The games to be played depend on the results of prior games. In a knockout tournament, games are played until all but one player have been "knocked out", or eliminated. The games must be played in a specific order because the results of the earlier games designate the contestants in later games; a player who is eliminated does not compete again. The simplest example of a knockout tournament, and the topic of this thesis, is the single-elimination tournament.

Definition 1.1. A *single-elimination tournament* is a knockout tournament where teams are eliminated after losing one game, and game winners continue to play until all but one team have been eliminated. This single remaining team is the winner of the tournament. We will discuss only single-elimination tournaments in which the pairings of game winners do not depend upon which contestants win, and are determined in advance of the outcomes of the games.

An example of another knockout tournament is the *l-elimination* tournament, in which players are knocked out after l losses, such as in the well-known double-elimination tournament.

Knockout tournaments require fewer games than round-robin tournaments. For a round-robin tournament with t teams, $\binom{t}{2}$ games are played, while in a single-elimination tournament with t teams, only $t-1$ games are played. Thus, if comparisons, or games, are expensive or time-consuming, or the number of teams is large, a round-robin tournament may be infeasible. On the other hand, one upset in a knockout tournament may cause the strongest team to be eliminated. These two conflicting concerns must be balanced wisely. Often, *repeated knockout tournaments* are conducted, where each pair of teams plays more than one game against each other, for example, as in a "best-two-out-of-three" competition. In a repeated knockout tournament, an upset is less likely since more than one

victory is required before a team advances in the tournament. Once the decision has been made to use a single-elimination tournament instead of an alternative design, some criteria must be used to choose a desirable scheme.

The goals of this dissertation are to develop notation and terminology for tournaments, to count the number of elements in certain classes of tournaments, and to explore various properties of tournaments.

<div align="center">Terminology</div>

As in any subject, basic terminology must be addressed first. Some, but not all, of the terminology and notation used in the literature is standard. In this thesis, we have taken care to be as consistent as possible with the standard terminology. For convenience, and because the topic of this thesis is such tournaments, hereafter the term *tournament* will refer exclusively to a single-elimination tournament.

Tournaments are specified by the initial games to be played and the rules for pairing the winners in the following games. It is sometimes convenient to refer to the number of rounds in a tournament. The number of *rounds* in a tournament is the maximum number of games that any of the teams must play to win the tournament. In a tournament with k rounds, the last game is round k. The game(s) that produce the team(s) paired in the final round constitute the game(s) in round $k - 1$. A *bye* occurs when a team does not play in a round. In many tournaments, some teams will get a bye in earlier rounds. In tournaments with byes, not all teams will play in the first round. A popular type of tournament is the *classic* tournament, which we define below.

Definition 1.2. A *classic* tournament is a tournament where the number of teams is a power of two and there are no byes.

One of the most basic distinctions we make concerning tournaments is that between tournament structures and tournament draws.

Definition 1.3. A *tournament structure* is that part of a tournament rule which specifies *how* the contestants will be paired without specifying *which* contestants will be paired.

The tournament structure shows the method in which the winners of games will be paired with other winners or with teams receiving byes. An example of a tournament structure, displayed in Figure 1 below, is a five-team tournament where two teams play in the first round and the other three teams receive first-round byes. Two of the teams with byes will play in one of the second-round games. The other second-round game will be between the remaining team with a bye and the winner of the first-round game. The two winners of the second-round games will play the third-round game, which will determine the winner of the tournament. The first two horizontal lines on the left in Figure 1 indicate the first-round game. Each pair of horizontal lines connected by a vertical line represents a game. The horizontal line on the right represents the winner of the tournament. Horizontal lines with open left ends represent the initial locations of the teams.
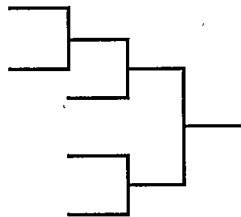
Figure 1. Example of a Tournament Structure for Five Teams.

Definition 1.4. In a tournament structure, a *slot* is one of the horizontal lines with an open left end, and represents the first appearance of a team in a tournament.

Definition 1.5. A *tournament draw* is a tournament structure where the players have been identified and assigned initial slots.

If the initial games of two or more teams or sets of teams play a symmetric role in the structure, such that all pairings would remain the same if the team labels in the slots were interchanged, then the different labelings are not regarded as determining different draws. Thus, distinct draws are distinguishable by the pairings resulting, regardless of their order in a tournament structure diagram.

Figure 2 below shows several different tournament draws. The first picture shows one possible tournament draw for the structure represented in Figure 1. The other two pictures show equivalent tournament draws.
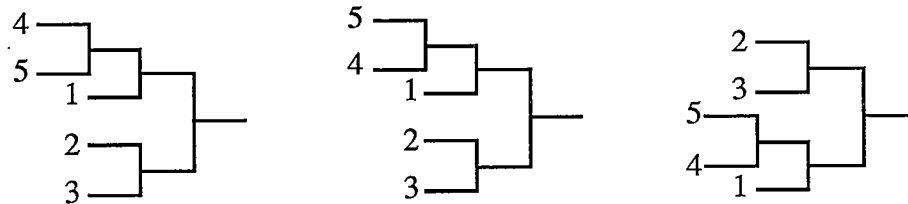
Figure 2. Examples of Tournament Draws.

Sometimes, we will want to refer to the slot a team occupies in a tournament structure; at other times we will identify a team by its *relative strength*. Relative strengths are integers between 1 and t, the number of teams in the tournament. The relative strength of a player represents the ranking of a player. The strongest player is assigned a relative strength of 1 and the weakest player is assigned a relative strength of t. Throughout this thesis, we will identify teams by their relative strengths. For example, using Figure 2 above, the team labeled "1" is the strongest team and the team labeled "5" is the weakest team.

Relative strengths, however, are not always easy or possible to assign. A related paradox associated with tournaments and relative strengths can be illustrated by considering

the following situation. Suppose team A has beaten team B, team B has beaten team C, and team C has beaten team A. This intransitivity would seem to contradict our intuition about relative strengths, but it is easy to invent an underlying probability structure for which this kind of intransitivity is expected. The following example shows four six-sided dice which have a cycle of probability dominance. Let die 1 have these sides: 4, 4, 4, 4, 4, 4. Let die 2 have these sides: 3, 3, 3, 3, 7, 7. Let die 3 have these sides: 2, 2, 2, 6, 6, 6. Let die 4 have these sides: 1, 1, 5, 5, 5, 5. Consider rolling two dice at a time and declaring the "winner" to be the die with the larger number. Then die 1 beats die 2 with probability $\frac{2}{3}$, die 2 beats die 3 with probability $\frac{2}{3}$, die 3 beats die 4 with probability $\frac{2}{3}$, and die 4 beats die 1 with probability $\frac{2}{3}$. For this example, there is no system of competition which will choose the "best" die, because each die is dominated by another die and thus there can be no "best" die.

To avoid situations such as the intransitive dice, it is customary to assume a particular stochastic structure for the outcomes. The stochastic structure can be summarized in a *preference matrix*, an array of elements that indicates the pairwise probabilities of teams beating each other. The probability that team i beats team j in a game is denoted by p_{ij}, and because ties are not allowed, $p_{ij} = 1 - p_{ij}$. For convenience, we have defined p_{ii} to be .5. The preference matrix for the example of the intransitive dice would be:

$$\mathbf{P} = \begin{bmatrix} .50 & .67 & .50 & .33 \\ .33 & .50 & .67 & .50 \\ .50 & .33 & .50 & .67 \\ .67 & .50 & .33 & .50 \end{bmatrix}.$$

A common assumption that is placed on the preference matrices associated with tournaments is the assumption of *strong stochastic transitivity* (SST). The version of SST we adopt was given by David (1963).

Definition 1.6. (David, 1963) A preference matrix **P** satisfies *strong stochastic transitivity* if and only if $p_{ik} \geq \max(p_{ij}, p_{jk})$ whenever $p_{ij} \geq .5$ and $p_{jk} \geq .5$ for each trio of players i, j, and k.

We assume throughout this thesis that the first row in a preference matirx represents the strongest team, the second row represents the second strongest team, etc. Under this assumption, the entries in a preference matrix are non-decreasing from left to right and non-increasing from top to bottom. Therefore, in the SST preference matrices that we use, the largest element represents the chance that the strongest team beats the weakest team, and is the element in the upper-right corner of the matrix. It is customary to assume independence between games so that mathematical results may be formulated. While this assumption may not be entirely warranted in real life, we feel it is an appropriate mathematical assumption.

Satisfaction of David's definition of transitivity allows us to use the preference matrix to rank the players and assign relative strengths. For the example of the intransitive dice, the preference matrix does not satisfy strong stochastic transitivity, and thus we cannot call any of the dice the "best" die.

An interesting question which might naturally arise at this point is how the initial placement of the teams (the tournament draw) affects the probabilities of the teams winning under a given tournament structure. This is addressed in Chapter 2. Another interesting but unrelated question is how many tournament structures and tournament draws exist for single-elimination tournaments with t teams. The number of structures is counted in Chapter 3. After the number of tournament structures and draws have been counted, it might be reasonable to ask which tournament structures and draws are "best." The definition of "best" will be more fully discussed later in this chapter, but many possibilities exist. One possible criterion for deciding if a tournament is "best" is the concept of *order*,

first used by Chung and Hwang (1978) using the equivalent term *monotone* and later more fully detailed by Horen and Riezman (1985) using the equivalent term *fair*. Because the terms monotone and fair have other connotations, we prefer to use the term *ordered*, and we give its definition next.

Definition 1.7. A tournament draw is *ordered* if for all preference matrices satisfying strong stochastic transitivity, the probabilities of the teams winning the tournament are ordered by their relative strengths. That is, if $i < j$ then Pr(team i wins the tournament) $\geq$ Pr(team j wins the tournament). A tournament draw is *unordered* if these probabilities are not completely ordered by the team relative strengths. More precisely, a tournament draw is unordered if there exists a preference matrix and two teams, $i < j$, such that Pr(team i wins the tournament) $<$ Pr(team j wins the tournament).

In an ordered tournament, the strongest team has the largest probability of winning the tournament, the second strongest team has the next largest probability of winning the tournament, etc. In an unordered tournament, all but two of the teams might be correctly ranked. It is important to recognize that this definition of ordered is a condition that holds for all preference matrices that satisfy SST. Showing that a tournament draw is unordered is much easier than showing that a tournament draw is ordered, because only one example of a preference matrix must be found to show that a tournament draw is unordered.

An intuitive discussion of this choice of order as a criterion for tournaments follows. The motivating concept is that of an intuitively "fair" tournament, but unfortunately the term "fair" cannot be fully described in one simple condition. The definition of ordered presented here can be regarded as the negation of a condition which would be contrary to our intuitive idea of "fair". It can be argued that the condition we have chosen for ordered is sufficient to be "fair", but it is by no means necessary. For example, if two teams are of

equal strength, we might expect that in a "fair" tournament their probabilities of winning would also be equal. Our condition of ordered does not address this aspect of "fairness".

Suppose teams i and j are of unequal strengths. To avoid being "unfair", should we require that Pr(team i wins the tournament) > Pr(team j wins the tournament), or is $\geq$ sufficient? The former is too strong a condition to obtain general results for the following reason. Imagine a large tournament with teams i and j in the same half such that their strengths relative to every team in their half are identical. Suppose also that team i is technically stronger than team j due to its strength versus the weakest team, team t, which plays in the other half of the tournament. If team t is beaten with certainty before reaching the finals, then Pr(team i wins the tournament) = Pr(team j wins the tournament), even though team i is stronger. To prevent the existence of a weak team destroying the generality of theorems, we use "$\geq$" in our definition instead of "$>$".

It is desirable to know which structures can produce ordered tournament draws. Some structures do not have draws which are ordered.

Definition 1.8. A tournament structure is *orderable* if there is a particular tournament draw using the tournament structure that is ordered. A tournament structure is *unorderable* if *all* tournament draws using the tournament structure are unordered.

Often researchers are not interested in the probabilities of the weakest teams winning the tournament. In such cases, it may be reasonable to only consider the criterion of order as it applies to the strongest teams. A tournament draw is *partially ordered of degree f* if the probabilities of the f strongest teams winning the tournament are ordered by their relative strengths and none of the $t - f$ weakest teams have a greater probability of winning the tournament than any of the f strongest teams. Thus, in a partially ordered tournament, the two weakest teams may have unordered probabilities of winning the

tournament, but both values may be so small as to be of no interest. Note that an ordered *t*-team tournament is always a partially ordered tournament of degree *t* or less.

Other questions which might be asked concerning tournaments are detailed in Table 1 below. This list is not meant to be exhaustive, but it does represent a collection of intriguing questions. Various authors have addressed some of the questions listed in Table 1, such as the issue of ranking contestants and the issue of the number of tournament structures. Many of these questions, however, are unanswered.

Table 1. Some Questions about Tournaments.

1. *	Is there a desirable method or notation for listing tournament structures?
2. *	Which tournament structures are "best"?
3. *	Which tournament draws are "best"?
4. *	What are some criteria for "best"?
5. *	If additional restrictions are imposed on a tournament, such as a limit on the number of rounds allowed, are there still "best" tournaments.
6. *	How many different tournament structures are orderable for a given number of contestants?
7. *	How many different tournament draws are ordered for a given tournament structure?
8. *	Is there always an ordered tournament draw or an orderable tournament structure for any number of contestants?
9.	How can the degree of a partially ordered tournament be determined?
10.	Is there a general notation for partially ordered tournaments?
11.	How would we define an "effective tournament"?
12.	How do incorrect *a priori* rankings influence "tournament effectiveness"?
13.	How does repetition affect "tournament effectiveness"?
14.	Can contestants be ranked from tournament results?

* Answered or partially answered in this dissertation.

To assist in defining the "best" tournament, Table 2 below was constructed. This table lists some possible ideas for how "best" could be defined when comparing two tournament draws.

Table 2. Some Definitions for "Best."

1. *	Highest probability of selecting strongest contestant.
2.	Highest probability of selecting a subset containing the strongest contestant.
3. *	Highest probability of the two strongest contestants meeting in the final round.
4. *	Ordered
5.	Partially ordered
6.	Fewest rounds
7.	Fewest expected number of contests (only appropriate in repeated single-elimination and l-elimination tournaments)

* Results provided in this dissertation.

Many authors have used the first three definitions of "best" in their works. Horen and Riezman (1985) detailed results using the fourth definition, that of *order*. Searls (1963) and Glenn (1960) compared the effectiveness of tournaments using the first definition and the sixth, the expected number of games in a tournament. In this thesis, the first, third, and fourth definitions will be examined.

Many of the above questions and ideas will be discussed in the following chapters. The question of notation will be addressed in Chapter 2 of this thesis, as well as methods of calculating probabilities of winning tournaments. Chapter 3 will deal with counting the number of distinct tournament structures and other combinatorial questions, and will include a proof of a recursive relationship stated without proof by Maurer (1975). The subject of order constitutes much of the material in Chapter 4. Chapter 5 will explore tournament effectiveness among the various criteria for "best," using some of the ideas expressed in Table 2.

Prior Work

Much work has been done in the fields of tournaments and paired comparisons. The work can be divided into two major categories: results concerning round-robin

tournaments, and results concerning knockout tournaments. David (1959), Glenn (1960), and Searls (1963) worked on the effectiveness of various types of tournaments, including round-robin tournaments, but, due to computer limitations at that time, they only looked at a few preference matrices among the contestants. Searls also discussed incorrect *a priori* rankings. Harary and Moser (1966) detailed many results for round-robin tournaments, and their work was extended by Moon (1968).

Kendall (1955), David (1963), and many others have worked on the problem of paired comparisons in general. These works are closely related to the work in round-robin tournaments. Davidson and Farquhar (1976) compiled a lengthy bibliography on the subjects of paired comparisons and tournaments. This is an excellent listing of articles and books on the topic of paired comparisons and the subtopics of round-robin tournaments and knockout tournaments.

Narayana (1968), Narayana and Zidek (1969a) and (1969b), Narayana and Hill (1974), Narayana and Agyepong (1979), and Handa and Maitri (1984) have done work on random knockout tournaments. In *random knockout tournaments* the structure of the knockout tournament is not deterministic; rather, contestants are chosen randomly for competitions from among all remaining contestants, regardless of previous pairings. The early work on knockout tournaments concentrated mostly on classic tournaments. Glenn (1960) also briefly discussed first-round byes.

Maurer (1975) worked on the problem of tournament structures. His work is important because it was the first attempt to analyze different tournament structures instead of using random or classic tournaments. Chung and Hwang (1978) and Hwang (1982) contributed to the theory by considering the assignment of relative strengths, or "seedings," in knockout tournaments. Horen and Riezman (1985) looked at ordered tournaments for four and eight contestants. Their work was significant because they considered *all* preference matrices, and not just specific examples.

Ford and Johnson (1959), David (1971), and others have worked on the problem of ranking the participants from the results of a tournament. These results are often concerned with round-robin tournaments, but Ford and Johnson invented an interesting method of ranking without examining all $\binom{t}{2}$ games.

Gilbert (1961), Freund (1956), Carroll (1947) and many others have worked on problems of scheduling round-robin and knockout tournaments. Carroll's work is interesting in that he focused his criticisms of knockout tournaments on their inability to select the second place team accurately. He designed an elaborate scheme to ensure that the second place team was chosen more equitably. Wiorkowski (1972) commented on this problem, and Fox (1973) pointed out that l-elimination tournaments are much better at choosing the other $l-1$ places than are single-elimination tournaments. The issue of tournament scheduling in round-robin tournaments is closely related to the topic of balanced incomplete block designs in experimental design. We feel that while tournament scheduling is interesting, it is not relevant to this discussion, and is better left to future considerations.

CHAPTER 2

NOTATION AND PRELIMINARY RESULTS

This chapter will outline some of the notation used in the rest of this dissertation and will include some preliminary results. The primary result in this chapter is Theorem 2.6, which gives a formula for calculating the probability of any given team winning a tournament. The results of Theorem 2.6 can also be written in matrix notation, which is often easier to write down, but more complicated in interpretation. This chapter will also detail a labeling system for tournaments and will give definitions for such terms as *subtournament*, *tournament half*, and *bracket*.

A labeling notation for the various tournament structures is particularly useful for determining the number of tournament structures. Other authors have devised notations for labeling tournament structures, among them Maurer (1975). Maurer's notation, however, was developed mostly for typographical reasons, and not necessarily for descriptive purposes.

Pictures such as those in Figures 1 and 2 of Chapter 1 are easy to understand and can be used to specify tournament structures. We want the labeling of tournament structures, however, to uniquely specify structures that are distinct. For example, we do not care if the first-round game is displayed at the bottom or the top of a figure. Figure 3 below shows two equivalent structures for three teams. In each structure, there is a first-round game and a team with a first-round bye.

Figure 3. Two Equivalent Structures for Three Teams.

To create uniqueness in our labeling scheme, structures will be of the first type in Figure 3, that is, the teams that play in the first round will be pictured above the teams with byes. Our notation also does not allow a team to have a bye after playing a game. As an example, consider the six-team tournament displayed to the left in Figure 4. From one point of view, there could be three first-round games, one second-round game, and a third-round game. With this viewpoint, a bye would follow one of the first-round games. To avoid this second-round bye, we will write the structure as displayed to the right in Figure 4. This tournament has only two first-round games, two second-round games, and a third-round game.
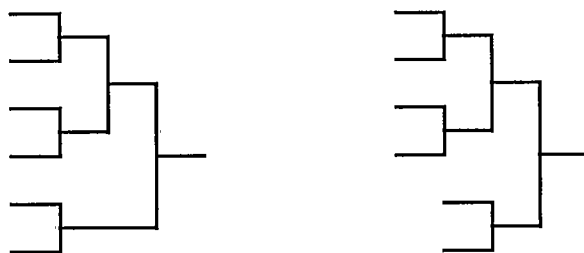
Figure 4. Example of Representations of Byes.

In using the labeling notation, we will have occasion to mention subtournaments and halves.

Definition 2.1. A *subtournament* is a subset of games from a tournament which is a tournament in its own right. It is determined by its final game.

Definition 2.2. A *half* of a tournament is one of the two subtournaments which directly precede the final game of the tournament.

The set of all tournament structures can be generated by combining subtournaments. Also, the method used in counting the distinct tournament structures in Chapter 3 will make use of tournament halves.

Definition 2.3. A *bracket of n rounds* is a subtournament of a classic tournament which has 2^n teams.

There are 2^{r-n} brackets of n rounds in a tournament with r rounds. Brackets can be divided into halves, called the first half and the second half. As an example of brackets, consider a classic tournament of eight teams and three rounds. In the first round, there are 2^{3-1}, or four brackets of one round, each with two teams. These first-round brackets are simply the first-round games. In the second round, there are two brackets of two rounds, each with four teams, and in the third round there is one bracket of three rounds with eight teams. This third-round bracket is precisely the entire tournament.

A labeling notation for tournament structures with t teams can be defined by the rules shown in Table 3 below. In Chapter 3, we will show a technique for counting tournament structures; in the course of this counting method, we will show the uniqueness of this labeling system. For immediate purposes, this labeling system is merely a shorthand for the pictures of tournament structures.

Table 3. Rules for Labeling Tournament Structures with r Rounds and t Teams

1. A label consists of a string of zeroes, ones, and twos of length 2^{r-1}.
2. The sum of the digits in a label is t.
3. Every label starts with a two.
4. A two cannot be followed by a zero.
5. The sum of the digits in the first half of a label is greater than or equal to the sum of the digits in the second half of a label. This rule also applies to each quarter, eighth, etc., that is, the sum of the digits in the first quarter of a label is greater than or equal to the sum of the digits in the second quarter of a label and the sum of the digits in the third quarter of a label is greater than the sum of the digits in the fourth quarter of a label.
6. If the sum of the digits is the same in both halves of a label, then the sum of the digits in the first quarter is greater than or equal to the sum of the digits in the third quarter. This rule also applies to eighths, sixteenths, etc., that is, if the sum of the digits is the same in all four quarters of a label, then the sum of the digits in the first eighth of a label is greater than or equal to the sum of the digits in the fifth eighth of a label and the sum of the digits in the first sixteenth of a label is greater than the sum of the digits in the ninth sixteenth of a label.

The label of digits is interpreted as follows. If the number of rounds in the tournament structure is r, then the length of the label is 2^{r-1}. Consider a classic tournament divided into 2^{r-1} brackets of two teams each. All tournament structures of at most r rounds can be derived from the classic tournament structure by letting "dummy" teams occupy appropriate initial slots. A "dummy" team is a team that can be beaten with certainty by any original team. In the label of digits, each digit represents one of the two-team brackets. Then, a "2" means there are two original teams (and no "dummy" teams) in the bracket, a "1" means there is one original team (and one "dummy" team) in the bracket, and a "0" means there are no original teams (and two "dummy" teams) in the bracket. To distinguish these labels from other numbers, labels will always be underlined. Examples using this notation are displayed in Figures 5, 6, and 7 below. In Figures 5 and 6 the picture on the left is our representation of the tournament structure, and the picture on the right demonstrates the labeling notation.

18
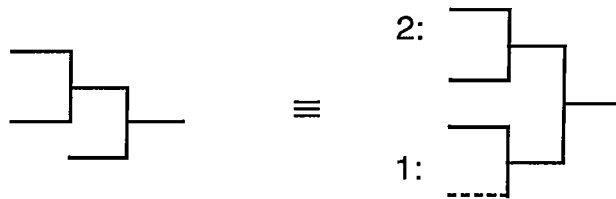
where ----- represents a dummy team.

Figure 5. Tournament Structure 21.

The picture to the left in Figure 5 shows the tournament structure for a tournament of two rounds and three teams, labeled 21. The two represents the bracket with two teams, and the one represents the bracket with one team. Note that the team in the bracket with one team received a first-round bye. The picture on the right in Figure 5 demonstrates the use of dummy teams to create the label 21.
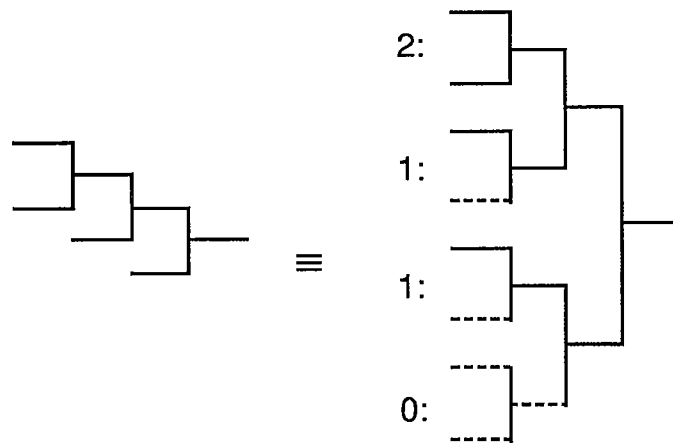
where ----- represents a dummy team.

Figure 6. Tournament Structure 2110.

Figure 6 represents the structure labeled 2110. The 21 half of the tournament structure is identical to the tournament structure for three teams. The 10 half of the tournament structure represents the team with the two-round bye.
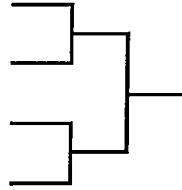
Figure 7. Tournament Structure <u>22</u>.

Figure 7 shows the tournament structure labeled <u>22</u>. This is a classic tournament because it has no byes, and therefore, all of the digits in the label are twos.

This labeling system has several desirable features. First, the sum of the digits is the number of teams in the tournament. Second, by looking at the length of the label, we can immediately see how many rounds the tournament requires. Third, the rules presented in Table 3 lend themselves to an iterative technique to generate the labels. Using this technique, all labels that satisfy the rules can be listed. In Chapter 3 we show, by way of our counting technique, that the above labeling system is unique in the sense that each distinct structure has only one label, and each label that satisfies the rules describes only one tournament structure.

We now introduce further notation en route to calculating the probability that any given team wins a tournament.

Classic Tournaments

The equations in this subsection will apply only to classic tournaments. Details for the case when the number of teams is not a power of two are discussed in the next subsection. Let q_{ij} be the probability that the team in slot i wins in round j. Given a tournament with r rounds, note that the probability that the team in slot i wins the tournament is q_{ir}. To calculate q_{ij}, we need to be able to identify the slot numbers of the potential opponents of a team. Proposition 2.5 below will give an equation for $S(i,n)$, the lowest slot number of the

potential opponents of the team in slot i in round n. The slot numbers for the other potential opponents will be the next $\left(2^{n-1}-1\right)$ consecutive integers. In the proof of the proposition, a function g will be created which identifies the initial team in a bracket, and Lemma 2.4 shows that g is non-increasing.

Lemma 2.4. Let i and n be positive integers. Then,

$$g(i,n) = 1 + \text{int}\left(\frac{i-1}{2^n}\right) \times 2^n \text{ is non-increasing in } n.$$

Proof: Let $j < k$. Then,

$$g(i,j) - g(i,k) = \left[1 + 2^j \times \text{int}\left(\frac{i-1}{2^j}\right)\right] - \left[1 + 2^k \times \text{int}\left(\frac{i-1}{2^k}\right)\right]$$

$$= 2^j \times \left[\text{int}\left(\frac{i-1}{2^j}\right) - 2^{(k-j)} \times \text{int}\left(\frac{(i-1) \times 2^{(j-k)}}{2^j}\right)\right]. \tag{1}$$

There are now two cases: $i - 1 < 2^j$, or $i - 1 \geq 2^j$.

Case I: If $i - 1 < 2^j$, then the first term in the brackets in equation (1) is zero and thus,

$$g(i,j) - g(i,k) = 2^j \times \left[-2^{(k-j)} \times \text{int}\left(\frac{(i-1) \times 2^{(j-k)}}{2^j}\right)\right] \leq 0.$$

Case II: If $i - 1 \geq 2^j$, then let $m = i - 2^j$, so that $i - 1 = m - 1 + 2^j$. Substituting in equation (1) gives:

$$g(i,j) - g(i,k) =$$

$$2^j \times \left[\text{int}\left(\frac{m-1+2^j}{2^j}\right) - 2^{(k-j)} \times \text{int}\left(\frac{(m-1+2^j) \times 2^{(j-k)}}{2^j}\right)\right]$$

$$= 2^j \times \left[\text{int}\left(\frac{m-1}{2^j}\right) + 1 - 2^{(k-j)} \times \text{int}\left(\frac{(m-1) \times 2^{(j-k)}}{2^j}\right) - 1\right]$$

$$= 2^j \times \left[\text{int}\left(\frac{m-1}{2^j}\right) - 2^{(k-j)} \times \text{int}\left(\frac{(m-1) \times 2^{(j-k)}}{2^j}\right)\right].$$

Now by replacing i with m in cases I and II, and by repeating the procedure if necessary, we will eventually have $m - 1 < 2^j$, and Case I will apply. Therefore, $g(i,j) - g(i,k) \leq 0$ in either case, and thus, $g(i,n)$ is non-increasing in n. $\square$

Proposition 2.5. In a classic tournament structure, the lowest slot number for the potential opponents of the team in slot i in round n is given by:

$$S(i,n) = 1 + 2^{n+1}\text{int}\left(\frac{i-1}{2^n}\right) + 2^{n-1} - 2^{n-1}\text{int}\left(\frac{i-1}{2^{n-1}}\right) . \tag{2}$$

Proof: In a classic tournament, there are 2^n teams in each bracket of n rounds. The lowest slot number of the teams in each of these brackets is one more than a multiple of the number of teams in each bracket, or $1 + k \times 2^n$, for some k. Define $g(i,n)$ to be the lowest slot number of the teams in the bracket containing slot i. Then, $k = \text{int}\left(\frac{i-1}{2^n}\right)$, and

$$g(i,n) = 1 + \text{int}\left(\frac{i-1}{2^n}\right) \times 2^n .$$

Lemma 2.4 shows that $g(i,n)$ is non-increasing in n, and therefore, either $g(i,n) = g(i,n-1)$, or $g(i,n) < g(i,n-1)$.

Case I: $g(i,n) = g(i,n-1)$.

In this case, the team in slot i is in the first half of its bracket, because the lowest slot number is the same for the bracket of n rounds and the bracket of $n-1$ rounds. Thus, the lowest slot number of teams in the second half of the bracket is $g(i,n) + 2^{n-1}$.

Case II: $g(i,n) < g(i,n-1)$.

In this situation, the team in slot i is in the second half of its bracket and thus, the lowest slot number of teams in the first half of the bracket is $g(i,n)$.

If team i is in the second half of its bracket, $S(i,n)$ will be the lowest slot number of teams in the first half, or $g(i,n)$. If team i is in the first half, $S(i,n)$ will be the lowest slot number of teams in the second half of the bracket, or $g(i,n) + 2^{n-1}$. In either case, $S(i,n) = g(i,n) + 2^{n-1} + \left[g(i,n) - g(i,n-1) \right]$. (The quantity in brackets is 0 in case I and equals $\left(-2^{n-1} \right)$ in case II.) Thus,

$$S(i,n) = 2 \times \left[1 + 2^n \, \text{int}\left(\frac{i-1}{2^n}\right) \right] + 2^{n-1} - \left[1 + 2^{n-1} \, \text{int}\left(\frac{i-1}{2^{n-1}}\right) \right]$$

$$= 1 + 2^{n-1} + 2^{n+1} \, \text{int}\left(\frac{i-1}{2^n}\right) - 2^{n-1} \, \text{int}\left(\frac{i-1}{2^{n-1}}\right) . \qquad \square$$

$S(i,n)$ gives the slot numbers for the teams that could face the team in slot i in round n. As an example, consider the classic tournament structure with eight teams and three rounds, displayed in Figure 8 below.

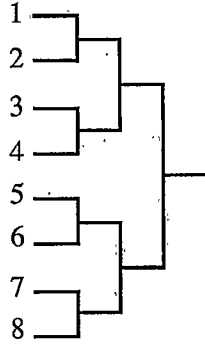Figure 8. Eight-team Classic Tournament Structure with Slot
Numbers Identified.

In Figure 8, the slot numbers, and not the relative strengths of the teams, have been labeled in the tournament structure. In this tournament, the team in slot one plays the team in slot two in the first round. In the second round, the team in slot one would play either the team in slot three or the team in slot four. From equation (2),

$S(1,1) = 1 + 0 + 1 - 0 = 2$, which agrees with the fact that the team in slot one plays the team in slot two in round one. Similarly, $S(1,2) = 1 + 0 + 2 - 0 = 3$, the lowest slot number of the two potential opponents of the team in slot one in the second round. For team six in round three, the potential opponents are the teams in slots one through four. Correspondingly, $S(6,3) = 1$. Table 4 below shows the values for $S(i,n)$ for tournament structure <u>2222</u>, that is, for $i = 1, 2, ..., 8$, and $n = 1, 2,$ and 3.

Table 4. Values of $S(i,n)$, Lowest Slot Number of Potential Opponents of the Team in Slot i in Round n for Tournament Structure <u>2222</u>.

	Values of n		
Values of i	1	2	3
1	2	3	5
2	1	3	5
3	4	1	5
4	3	1	5
5	6	7	1
6	5	7	1
7	8	5	1
8	7	5	1

The general notation for q_{ij}, the probability that the team in slot i wins in round j, can now be formulated. Let $P(i, k)$ be the probability that the team in slot i beats the team in slot k. For the first round then, $q_{i1} = P(i, S(i,1))$. For the other rounds, the probability of the team in slot i winning in round j is given by Theorem 2.6.

Theorem 2.6. For $j \geq 2$, $\quad q_{ij} = q_{i, j-1} \left[\sum_{k=l}^{u} P(i,k) q_{k, j-1} \right]$,

where $l = S(i, j-1)$, $u = l + 2^{j-1} - 1$, and $q_{i1} = P(i, S(i,1))$.

Proof: The chance of a team winning in any round is the chance that the team wins in the previous round times the chance that the team beats its opponent in the current round.

The team's opponent in the current round is conditional on which team wins the other subtournament. Thus we must sum over all possible opponents. The index for the summation is found by using Proposition 2.5 and using the next $\left(2^{n-1}-1\right)$ consecutive integers. $\qquad\square$

The notation for calculating the probability of a team winning the tournament can also be expressed using matrices. The matrix notation is simpler because we will not have to worry about the slot number a team occupies. Let $\mathbf{O}_{ij}$ be a square matrix identifying the potential opponents to team i in round j. $\mathbf{O}_{ij}$ has zeroes on the off-diagonals and ones and zeros on the diagonals. The ones correspond to teams that could possibly meet team i in round j. Note that in $P(i,j)$ the argument i refers to the slot number whereas in $\mathbf{O}_{ij}$ the argument i refers to the relative strength. This distinction makes $\mathbf{O}_{ij}$ easier to work with, because we often know the team numbers but do not care to know the specific slot numbers. As an example, consider Figure 9 below which shows the "seeded" eight-team classic tournament. This tournament is very popular in sporting events. The numbers in Figure 9 are the relative strengths, not the slot numbers.
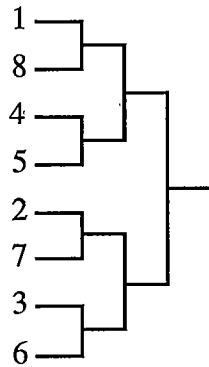

Figure 9. "Seeded" Eight-team Tournament.

In this tournament, the potential opponents for team five, for example, in the three rounds are team four in round one, either team one or team eight in round two, and either team two, team three, team six, or team seven in round three. Correspondingly,

$$
O_{51} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, O_{52} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and}
$$

$$
O_{53} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Note that the trace of O_{ij} is 2^{j-1}.

Let Q_j be the $t \times 1$ vector of probabilities of the teams winning in round j. That is, Q_j is composed of the q_{ij}'s, but not necessarily sequentially, because q_{ij} refers to the team in *slot i*, not the team with *relative strength i*. To deal with this notational problem, we define $L(i)$ to be a link function between slot number and relative strength. For example, $L(2)$ is the slot number of the second strongest team. Then $Q_j = [\, q_{L(1),j} \; q_{L(2),j} \; q_{L(3),j} \; \cdots \; q_{L(t),j} \,]'$. So $q_{L(i),r} = Q_r' e_i$, where e_i is a column indicator vector having all zeroes, except for a one in row i. Row i of the matrix P can then be written as $e_i' P$. The elements from this vector that correspond to the potential opponents of

team i are the non-zero elements in the vector $e_i'\mathbf{PO}_{ij}$. The scalar $e_i'\mathbf{PO}_{ij}\mathbf{Q}_{j-1}$ is the sum of the probabilities of team i beating its potential opponents times the probability that the potential opponents win in round $j-1$. Then, $q_{L(i),j} = e_i'\mathbf{PO}_{ij}\mathbf{Q}_{j-1}\mathbf{Q}_{j-1}'e_i$. Note that because $q_{L(i),j}$ depends on $\mathbf{O}_{ij}$, $\mathbf{Q}_j$ cannot be written conveniently in matrix notation. It is possible, however, to write a matrix expression for $\mathbf{Q}_j$ using Hadamard products, but the notation is cumbersome, so we have not pursued it here.

Non-classic Tournaments

If the number of teams is not a power of two, it is still possible to use the notation developed by creating "dummy" teams until the number of teams is a power of two. Both Searls (1963) and Chung and Hwang (1978) have used this technique. When dummy teams are created, the preference matrix must be supplemented with ones and zeroes. Each added dummy team has probability zero of beating any original team, and therefore, each original team has probability one of beating a dummy team. Note that the performance of dummy teams against other dummy teams is irrelevant to the outcome of the tournament, so any probabilities consistent with strong stochastic transitivity can be assigned to the meetings between dummy teams.

To see how these dummy teams would be created, consider the tournament draw for the tournament 2111 given in Figure 2 of Chapter 1. This tournament has five teams and thus requires three dummy teams to form a classic tournament. If these three dummy teams are given relative strengths of six, seven, and eight, the preference matrix is:

$$\begin{bmatrix} \frac{1}{2} & p_{12} & p_{13} & p_{14} & p_{15} & 1 & 1 & 1 \\ p_{21} & \frac{1}{2} & p_{23} & p_{24} & p_{25} & 1 & 1 & 1 \\ p_{31} & p_{32} & \frac{1}{2} & p_{34} & p_{35} & 1 & 1 & 1 \\ p_{41} & p_{42} & p_{43} & \frac{1}{2} & p_{45} & 1 & 1 & 1 \\ p_{51} & p_{52} & p_{53} & p_{54} & \frac{1}{2} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

This chapter has detailed some of the basic notation necessary to discuss tournaments, such as a labeling system and a formula to find the chance that a given team wins the tournament. Other notation used in this thesis will be introduced as necessary.

CHAPTER 3

THE NUMBER OF TOURNAMENT STRUCTURES

When a researcher uses a tournament to choose the best treatment, or when a tournament director schedules a sporting event, a question that must be addressed is which tournament structure to use. Historically, in sporting events, classic tournaments have been used extensively. Leagues and divisions are often specifically formed so that the number of teams is a power of two. There are situations, however, where non-classic tournaments have been used, the NCAA basketball tournament in the early eighties being an example. In that tournament, a number of higher ranked teams received first-round byes. The Pro Bowlers Tour uses a ladder-like tournament (as in Figure 6 of Chapter 2) to select their weekly champion. This chapter makes use of the labeling notation introduced in Chapter 2 to list and count the number of tournament structures. An equation for the number of tournament structures, including the number of tournaments in a restricted class of tournament structures, is given in Theorem 3.9. The proof of Theorem 3.9 proves one of the recursive relationships stated without proof by Maurer (1975).

Recall that the labeling notation introduced in Chapter 2 consists of strings of the digits zero, one, and two. The length of a label is a power of two, and labels can therefore be broken into halves. Throughout this chapter, capital Roman letters will denote sets, while capital Roman letters followed by a $ will indicate a label, a string of zeroes, ones, and twos.

Before deriving the formula to count tournament structures, a few definitions must be addressed. The *primary sum* of the character string $A\$$ is the sum of the digits in $A\$$. For example, the label 21101000 has a primary sum of 5. The primary sum of a string is

simply the number of teams in the tournament structure. To define the secondary sum of $A\$$, write $A\$$ as $A1\$A2\$$, where $A1\$$ is the first half and $A2\$$ is the second half. The *secondary sum* of $A\$$ equals the primary sum of $A1\$$. The secondary sum of $A\$$ gives the number of teams in the first half of the tournament. For example, the label $\underline{2110}$ would have $A1\$ = \underline{21}$ and $A2\$ = \underline{10}$. The secondary sum of $A\$$ is then the primary sum of $\underline{21}$, or 3.

The technique for listing the tournament structures will combine subtournament labels. At times in listing the tournament labels, illegal labels will be created; that is, some labels will not satisfy all of the rules given in Table 3 of Chapter 2. These illegal labels will be counted separately in a duplicate set and subtracted from the total count.

Definition 3.1. The *duplicate set* of A, D_A, is the set $\{A\$B\$$ such that $A\$ \in A$ and $B\$ \in A$ and the secondary sum of $A\$ <$ the secondary sum of $B\$\}$.

D_A is a set of duplicate tournament labels, and in fact contains only illegal labels. These illegal labels do not have the proper ordering of secondary sums, violating rule 6 in Table 3. In cases where the length of $A\$$ is greater than the length of $B\$$, the label $B\$$ is "expanded" to have the same length as $A\$$. Because the length of labels is a power of two, the expansion of a label will double its length. The expansion rules are as follows. A $\underline{2}$ becomes $\underline{11}$, a $\underline{1}$ becomes $\underline{10}$, and a $\underline{0}$ becomes $\underline{00}$. If necessary, expanding continues until the length of $B\$$ is the same as the length of $A\$$. Similarly, we expand $A\$$ if the length of $A\$$ is less than the length of $B\$$.

For $A \cap B = \phi$, define $A*B$ to be $\{A\$B\$$, where $A\$ \in A$ and $B\$ \in B\}$. Define $A*A$ to be $\{A\$B\$$ where $A\$ \in A$ and $B\$ \in A\} \sim D_A$, where $\sim$ represents the usual set difference operation. Thus, in $A*A$ we are excluding the duplicate tournament labels. As an example of illegal labels in D_A, consider $A = \{\underline{22}, \underline{2110}\}$. Then, using the expanded label $\underline{1111}$ for $\underline{22}$, $D_A = \{\underline{11112110}\}$. Note that $\underline{11112110}$ is equivalent to $\underline{21101111}$,

and because the leading digit is not a two, nor are the secondary sums properly ordered, <u>11112110</u> is not a legitimate label. Thus, $A*A = \{\underline{2222}, \underline{21101111}, \underline{21102110}\}$.

Now we are prepared to define sets of tournament structures.

Definition 3.2. $T(t, r)$ is the set of tournament structures of t teams in at most r rounds.

The set $T(t, r)$ will be enumerated by breaking each tournament structure into halves, and iteratively counting the number of tournament structures in each half. The final result of this counting is given in Theorem 3.7 below, but we first need some preliminary results. The following theorem details how the sets of tournament structures are constructed.

Theorem 3.3. For $r \geq 2$, $\displaystyle T(t, r) = \bigcup_{j = l}^{u} \big[T(t - j, r - 1) * T(j, r - 1)\big]$,

where $u = \text{int}\left(\frac{t}{2}\right)$ and $l = \max\left(1, t - 2^{r-1}\right)$. The initial conditions are $T(1,1) = \{\underline{1}\}$ and $T(2,1) = \{\underline{2}\}$.

Proof: We will break the t teams into two subtournaments and then take the union of all sets thus created. The union of the sets created will be

$$\bigcup_{j = l}^{u} \big[T(t - j, r - 1) * T(j, r - 1)\big].$$

To avoid duplication, we will place the larger number of teams in the first half of the set combining process. Thus, the upper limit in the union is $u = \text{int}\left(\frac{t}{2}\right)$. We also need to restrict the lower limit of the union if $t - j$ teams cannot be scheduled in $r - 1$ rounds. There are at most 2^{r-1} teams in round $r - 1$, so $t - 2^{r-1}$ is the smallest possible index. Of course, if this number is negative, the counting will start at one. Thus, the lower limit in the union is $l = \max\left(1, t - 2^{r-1}\right)$. Therefore,

$$T(t, r) = \bigcup_{j=l}^{u} [T(t-j, r-1) * T(j, r-1)], \qquad (1)$$

where $u = \text{int}\left(\frac{t}{2}\right)$ and $l = \max\left(1, t - 2^{r-1}\right)$. $\qquad \square$

Now, we will count the number of elements in $T(t, r)$. Define $N(A)$ to be the number of unique elements in A. In the following propositions, we assume $A = T(i, k)$ and $B = T(j, k)$, where $i \neq j$. Thus the primary sum is i for all elements in A.

Proposition 3.4. If $A \cap B = \phi$, then $N(A*B) = N(A) N(B)$.

Proof: Because the sets are disjoint, and because of the construction of $A*B$, the number of elements in $A*B$ is the product of the number of elements in A and the number of elements in B. $\qquad \square$

Proposition 3.5. $N(A*A) = N(A)^2 - N(D_A)$.

Proof: If we paired each of the elements of A with every other element of A, we would have $N(A)^2$ elements. Because the duplicate set D_A is contained in the set of pairs of elements in A, the number of elements in $A*A$ is $N(A)^2 - N(D_A)$. $\qquad \square$

Proposition 3.6. $N(D_A) = \binom{N(A)}{2}$.

Proof: First we note that we have assumed that all elements of A have the same primary sum. There are $N(A)^2$ ways to pair the elements of A together. However, some of these are violations of the labeling notation because the secondary sum of the second element is larger than the secondary sum of the first element. For each two elements in A, there is only one legitimate way to pair them. Consider placing the $N(A)^2$ elements in a square array. In the first row and column, there are $N(A) - 1$ pairs of elements that are violations of the labeling system. In the second row and the second column, there are

$N(A) - 2$ more pairs of elements that are violations of the labeling system. Continuing this argument, there are

$$\sum_{i=1}^{N(A)-1} i$$

pairs of elements where the secondary sums are not properly ordered. Thus,

$$N(D_A) = \sum_{i=1}^{N(A)-1} i = \frac{\left(N(A) - 1\right) * N(A)}{2} = \binom{N(A)}{2}. \qquad \square$$

Using Proposition 3.5, we see that $N(A*A) = N(A)^2 - \binom{N(A)}{2}$

$$= N(A)^2 - \left(\frac{N(A)^2 - N(A)}{2}\right) = \frac{N(A)(N(A) + 1)}{2}.$$

Because $T(t, r)$ in (1) is the union of disjoint sets, the number of elements in the set is the sum of the number of elements in each set. Then,

$$N(T(t, r)) = \sum_{j=l}^{u} N\left[T(t - j, r - 1) * T(j, r - 1)\right],$$

where $l = \max\left(1, t - 2^{r-1}\right)$ and $u = \text{int}\left(\frac{t}{2}\right)$. By using Proposition 3.4, Proposition 3.6, the definition of $A*B$, and some algebraic manipulation, we have Theorem 3.7, given below.

Theorem 3.7. In a tournament of t teams and at most r rounds, the number of tournament structures is given by

$$N(T(t, r)) =$$

$$\sum_{j=l}^{u} \left[N(T(t - j, r - 1))\right]\left[N(T(j, r - 1)) + \delta_{t-j, j}\right]\left[1 - \frac{\delta_{t-j, j}}{2}\right], \qquad (2)$$

where $\delta_{i,j}$ is Kronecker's delta, $l = \max\left(1, t - 2^{r-1}\right)$, and $u = \text{int}\left(\frac{t}{2}\right)$.

Equation (2) can be used to count the total number of tournament structures for a given number of teams, or it can be used to count tournament structures which are restricted by a maximum number of rounds. For instance, a basketball tournament may be required to be finished in four days, with only one game played per day by each team. Tournaments which have five or more rounds would then be unacceptable.

To count the total number of tournament structures for a given number of teams, note that there are at most $t - 1$ rounds in a tournament with t teams, and therefore $T(t, t - 1)$ is the set of all tournament structures for t teams. Let $a_t = N(T(t, t - 1))$. Note that the initial condition for this sequence is $a_1 = 1$. Then,

$$a_t = \sum_{j=l}^{u}\left[N(T(t - j, t - 2))\right]\left[N(T(j, t - 2)) + \delta_{t-j,j}\right]\left[1 - \frac{\delta_{t-j,j}}{2}\right],$$

from (2) above. A simpler notation can be derived for a_t and is given in Theorem 3.9 below. First, however, Lemma 3.8 shows that $N(T(t, j))$ can be simplified.

Lemma 3.8. $N(T(t, j)) = N(T(t, t - 1))$ for all $j \geq t - 1$, that is, when $j - t + 1 \geq 0$.

Proof: If $j \geq t - 1$, then there are more rounds than the tournament structure with the maximum number of rounds. Thus, these extra rounds are artificial and do not create additional tournament structures. Therefore, $N(T(t, j)) = N(T(t, t - 1))$ for all $j \geq t - 1$. □

Using Lemma 3.8, with $j \geq 1$, $N(T(t - j, t - 2)) = N(T(t - j, t - j - 1))$ $= a_{t-j}$, because $(t - 2) - (t - j) + 1 = j - 1 \geq 0$. Similarly, with $j \leq \text{int}\left(\frac{t}{2}\right)$,

$N(T(j, t-2)) = N(T(j, j-1)) = a_j$, because $(t-2) - j + 1 = t - j + 1$
$\geq t - \text{int}\left(\frac{t}{2}\right) - 1 \geq 0$.

Thus, $a_t = \displaystyle\sum_{j=1}^{\text{int}\left(\frac{t}{2}\right)} a_{t-j}\left[a_j + \delta_{t-j,j}\right]\left[1 - \frac{\delta_{t-j,j}}{2}\right]$.

To simplify a_t further, we prove Theorem 3.9.

Theorem 3.9. (Maurer, 1975) The number of tournament structures for t teams is given by

$$a_t = \frac{1}{2}\sum_{j=1}^{t-1} a_{t-j}\left(a_j + \delta_{t-j,j}\right), \tag{3}$$

where $a_1 = 1$.

Proof: t is either odd or even. If t is even, then $t = 2m$ for some m, and $\text{int}\left(\frac{t}{2}\right) = m$. Then,

$$a_t = \sum_{j=1}^{m} a_{t-j}\left[a_j + \delta_{t-j,j}\right]\left[1 - \frac{\delta_{t-j,j}}{2}\right]$$

$$= \sum_{j=1}^{m-1} a_{t-j}a_j + a_{t-m}\left(a_m + 1\right)\left(\frac{1}{2}\right)$$

$$= \sum_{j=1}^{m-1} a_{t-j}a_j + a_m\left(a_m + 1\right)\left(\frac{1}{2}\right)$$

$$= \sum_{j=1}^{m-1} \frac{a_{t-j}a_j}{2} + \sum_{j=1}^{m-1} \frac{a_{t-j}a_j}{2} + a_m\left(a_m + 1\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\left[\sum_{j=1}^{m-1} a_{t-j}a_j + \sum_{k=m+1}^{t-1} a_k a_{t-k} + a_m(a_m+1)\right]$$

$$= \frac{1}{2}\sum_{j=1}^{t-1} a_{t-j}(a_j + \delta_{t-j,j}).$$

If t is odd, then $t = 2m+1$ for some m, and hence $\text{int}\left(\frac{t}{2}\right) = m$.

$$a_t = \sum_{j=1}^{\text{int}\left(\frac{t}{2}\right)} a_{t-j}\left[a_j + \delta_{t-j,j}\right]\left[1 - \frac{\delta_{t-j,j}}{2}\right]$$

$$= \sum_{j=1}^{m} a_{t-j}a_j$$

$$= \frac{1}{2}\left[\sum_{j=1}^{m} a_{t-j}a_j + \sum_{j=1}^{m} a_{t-j}a_j\right]$$

$$= \frac{1}{2}\left[\sum_{j=1}^{m} a_{t-j}a_j + \sum_{k=m+1}^{2m} a_{t-k}a_k\right]$$

$$= \frac{1}{2}\left[\sum_{j=1}^{m-1} a_{t-j}a_j + \sum_{k=m+1}^{t-1} a_k a_{t-k} + a_m(a_m+1)\right]$$

$$= \frac{1}{2}\sum_{j=1}^{2m} a_{t-j}(a_j + \delta_{t-j,j}), \text{ because } t-j \text{ never equals } j. \text{ Then,}$$

$$a_t = \frac{1}{2}\sum_{j=1}^{t-1} a_{t-j}(a_j + \delta_{t-j,j}). \qquad \square$$

Theorem 3.7 is more useful than Theorem 3.9 for our purposes due to its capability of counting tournament structures which are restricted by the number of rounds. The following examples show the tournament structures for up to seven teams.

2   21

Figure 10. Tournament Structures for $t = 2$ and $t = 3$ Teams.

22   2110

2111   2210   21101000

Figure 11. Tournament Structures for $t = 4$ and $t = 5$ Teams.

2211

2121

22101000

21111000

21101010

2110100010000000

Figure 12. Tournament Structures for $t = 6$ Teams.

2221   22111000   21211000

21111010   2111100010000000   21101110

2110101010000000   2110100010001000   22101010

2210100010000000   2110100010000000100000000000000

Figure 13. Tournament Structures for $t = 7$ Teams.

Table 5 below shows the values for $N(T(t, r))$. The values in Table 5 are useful in situations where the tournament is constrained by time, i.e., the tournament is limited to less than $t - 1$ rounds. As an example from Table 5, for eight teams and at most four rounds, there are eight possible tournament structures. One of these tournament structures has only three rounds, so seven have exactly four rounds. The right most values in Table 5 for t less than or equal to eleven are the values reported by Maurer (1975), because such tournaments take at most ten rounds. For twelve or more teams, more columns are required before the values match Maurer's.

Table 5. The Number of Tournament Structures with t Teams and at Most r Rounds.

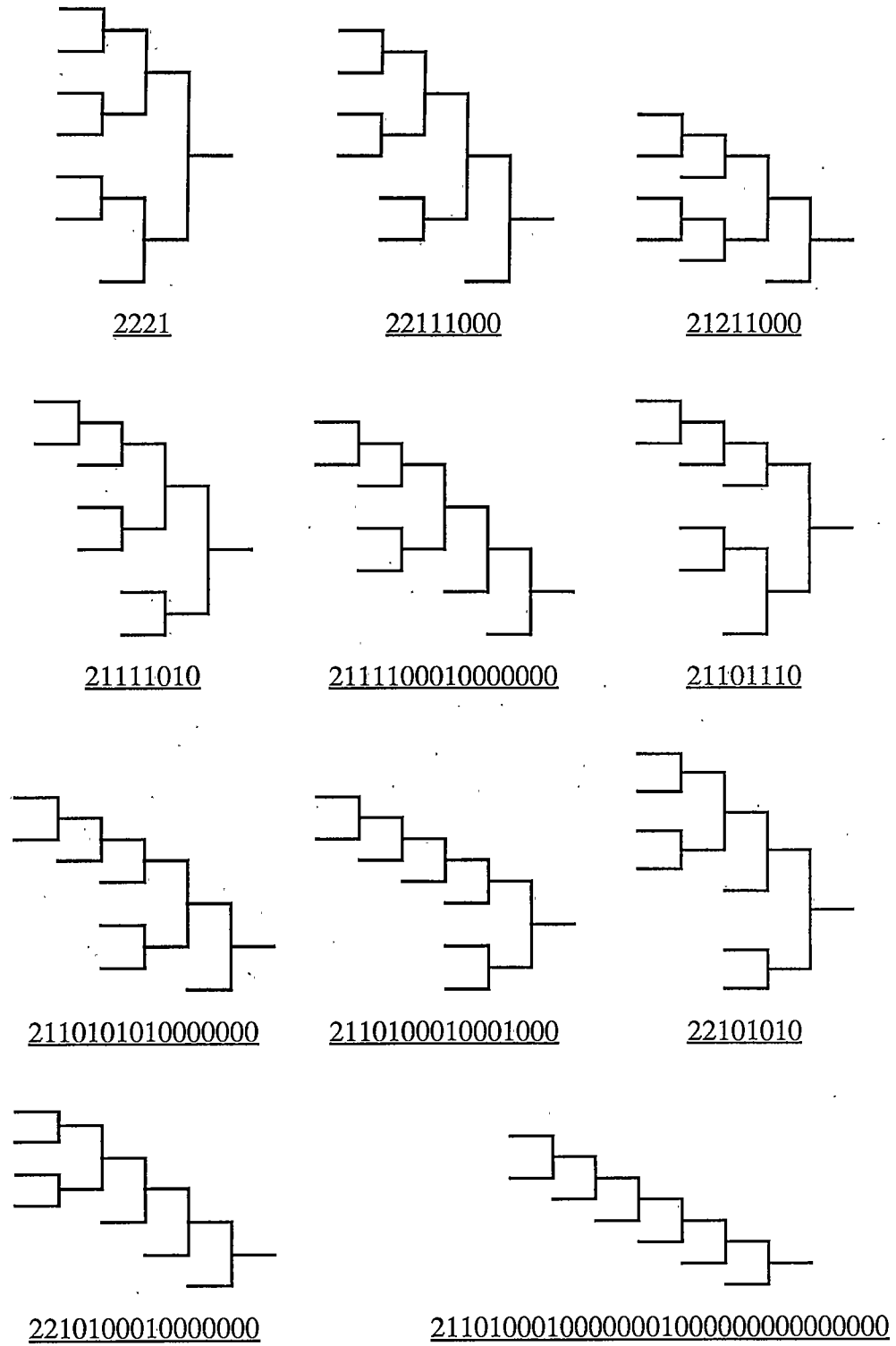| Values of t	Values of r								
	2	3	4	5	6	7	8	9	10
3		1	1	1	1	1	1	1	1
4	1	2	2	2	2	2	2	2	2
5		2	3	3	3	3	3	3	3
6		2	5	6	6	6	6	6	6
7		1	6	10	11	11	11	11	11
8		1	8	17	22	23	23	23	23
9			8	25	39	45	46	46	46
10			9	38	70	90	97	98	98
11			7	52	118	171	198	206	207
12			7	73	200	325	406	441	450
13			4	93	324	598	811	928	972
14			3	121	526	1,097	1,613	1,951	2,113
15			1	143	825	1,972	3,155	4,046	4,555
16			1	172	1,290	3,531	6,141	8,349	9,795

It is often convenient to know how many tournament structures of *exactly* r rounds exist. This can of course be calculated easily from Table 5 by subtracting adjacent columns. Table 6 below shows the number of tournament structures with t teams and exactly r rounds.

Table 6. The Number of Tournament Structures with t Teams and Exactly r Rounds.

Values of t	Values of r								
	2	3	4	5	6	7	8	9	10
3	1								
4	1	1							
5		2	1						
6		2	3	1					
7		1	5	4	1				
8		1	7	9	5	1			
9			8	17	14	6	1		
10			9	29	32	20	7	1	
11			7	45	66	53	27	8	1
12			7	66	127	125	81	35	9
13			4	89	231	274	213	117	44
14			3	118	405	571	516	338	162
15			1	142	682	1,147	1,183	891	509
16			1	171	1,118	2,241	2,610	2,208	1,446

We conclude this chapter by observing that $2^{t-4} < a_t < 2^{t-3}$. Thus, $\log_2(a_t)$ increases linearly.

At this point, we could count tournament draws. We feel, however, that because many of the possible tournament draws have unrealistic first-round games, such as the two best teams meeting, it is not worthwhile to count tournament draws. We will, however, count ordered tournament draws. In Theorem 4.24 in the next chapter, we show that the tournament draws associated with orderable tournament structures are unique, and thus if we count orderable tournament structures, we have then counted ordered tournament draws.

CHAPTER 4

ORDERED TOURNAMENTS

In many cases, a researcher wants to rank the treatments from the results of an experiment, or would like his method to choose the best treatment with high probability. If the *a priori* relative strengths are correctly assigned to the teams, then no tournament is necessary; we can just choose the strongest team and declare it to be the winner. This situation is of course mathematically uninteresting, so we restrict our attention to the case where a tournament must be held. We then assume that our *a priori* assignment of relative strengths is indeed correct when we calculate the properties of alternative tournament structures.

A reasonable condition that can be imposed on the tournament draws is that of order. Recall that an *ordered tournament draw* requires the probabilities of teams winning the tournament to be ordered by their relative strengths. Thus, the strongest team has the best chance of winning, the second strongest team has the second best chance, etc. This chapter will detail some results concerning ordered tournament draws. Horen and Riezman (1985) showed that the four-team classic tournament structure is orderable, while the eight-team classic tournament structure is unorderable. They also conjectured that for 16 and more teams, the classic tournament structures are unorderable. Later in this chapter we prove this conjecture and completely characterize the orderable tournament structures.

The first result we give concerning ordered tournaments is a simple fact about the strongest teams and their placement in an ordered tournament draw.

Theorem 4.1. If a weaker team can win by playing fewer games than a stronger team, then the tournament draw is unordered.

Proof: Let n equal the number of games the stronger team, team i, must play to win the tournament. Let m equal the number of games the weaker team, team $j > i$, must play to win the tournament. Consider a tournament draw where $m < n$. Because $\frac{m}{n} < 1$, and

$$\lim_{\varepsilon \to 0} \left[\frac{\ln(.5 + \varepsilon)}{\ln(.5 - \varepsilon)} \right] = 1, \text{ there exists an } \varepsilon_0, \text{ such that}$$

$$\frac{m}{n} < \left[\frac{\ln(.5 + \varepsilon_0)}{\ln(.5 - \varepsilon_0)} \right], \text{ or } (.5 + \varepsilon_0)^n < (.5 - \varepsilon_0)^m.$$

Let $p_{ij} = .5 + \frac{j - i}{t - 1} \varepsilon_0$. Then, $q_{L(i),r}$, the probability that team i wins the tournament, is less than $(.5 + \varepsilon_0)^n$ and $q_{L(j),r}$, the probability that team j wins the tournament, is greater than $(.5 - \varepsilon_0)^m$.

Now, $q_{L(i),r} > (.5 - \varepsilon_0)^m > (.5 + \varepsilon_0)^n > q_{L(i),r}$, and therefore the tournament draw is unordered. □

Theorem 4.1 says that in an ordered tournament draw, the strongest team will get the most byes, the second strongest team will get the second most byes, etc. As an example of the application of Theorem 4.1, consider the tournament structure 2111, shown in Figure 14 below.
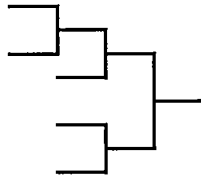

Figure 14. Tournament Structure 2111.

Theorem 4.1 says that the three strongest teams must be in the three slots receiving first-round byes. Teams four and five must play in the first round. The question of how to place the top three teams in this tournament structure will be answered by Theorem 4.3 below. Theorem 4.3 also will help us count the number of ordered tournament structures. The theorem involves the "*top-n*" tournament draw.

Definition 4.2. Let $n < t$ and consider a t-team tournament draw. Its associated *top-n* tournament draw is the n-team tournament draw which results from treating teams in the top n as automatic winners over teams not in the top n, and therefore does not include any games with a team not in the top n.

Theorem 4.3. If a t-team tournament draw is ordered, then its "top-n" tournament draw is also ordered.

Contrapositive: If the "top-n" tournament draw is unordered, then the t-team tournament draw is also unordered.

Proof of the Theorem: The "top-n" tournament draw can be represented with a preference matrix where $p_{ij} = 1$ for all $j > n$, where $j > i$. If $p_{ij} = 1$ for all $j > n$, then the tournament is essentially an n-team tournament. Because the tournament draw is ordered for any preference matrix, it is ordered for any preference matrix which uses probabilities where $p_{ij} = 1$ for all $j > n$. The probability that a team in the "top-n" wins the n-team tournament draw is therefore equal to the probability that the same team wins the t-team tournament draw. Thus, the "top-n" tournament draw must also be ordered. □

Theorem 4.1 and Theorem 4.3 taken together prove the following theorem.

Theorem 4.4. If a tournament draw is ordered, then the two strongest teams play in opposite halves of the tournament.

Proof: Suppose teams one and two are in the same half of the tournament. Let k be the strongest team in the other half. Then the "top-k" tournament draw would be unordered, because team k could win in only one game, whereas team one would require two games to win, in violation of Theorem 4.1. $\square$

Our next major goal is Theorem 4.9, which establishes that if a subtournament is unordered, then the tournament is also unordered. In a corollary to this theorem, we will be able to prove the conjecture of Horen and Riezman concerning classic tournaments with 16 or more teams. First, however, we will prove four lemmas. Lemma 4.5 allows us to assume that two teams causing a tournament to be unordered have consecutive relative strengths. Lemma 4.6 shows that if two teams causing a tournament draw to be unordered are in opposite halves of the tournament, then a preference matrix can be constructed which has equal rows for the two teams. Lemma 4.7 shows that if two teams causing a *subtournament* draw to be unordered have equal rows in the preference matrix, then a preference matrix for the *tournament* draw can be constructed such that the tournament draw is also unordered. Lemma 4.8 shows that either two teams causing a tournament to be unordered are in opposite halves of the tournament, or they are in the same unordered half.

Lemma 4.5. Let $j < k$ and Pr(team j wins) > Pr(team k wins). Then there exists i such that Pr(team i wins) > Pr(team $i + 1$ wins).

Proof: If Pr(team j wins) > Pr(team $j + 1$ wins) then let $i = j$. If Pr(team j wins) $\leq$ Pr(team $j + 1$ wins) then Pr(team $j + 1$ wins) > Pr(team k wins), and we use the above argument. Eventually, there exists some i such that Pr(team i wins) > Pr(team $i + 1$ wins). $\square$

Lemma 4.6. If a tournament draw is unordered with respect to teams i and $i+1$ using the preference matrix $\mathbf{P_0}$, and the two teams are in opposite halves of the tournament, then there exists a preference matrix $\mathbf{P_1}$ with rows i and $i+1$ equal, such that the tournament draw is unordered with respect to teams i and $i+1$.

Proof: Let $\mathbf{P_1} = \mathbf{P_0}$, except for the element $p_{1i,i+1}$ and the row $i+1$. Let the element $p_{1i,i+1} = 0.5$. This is possible under SST because $p_{1ii} = 0.5$, and $p_{1i+1,i+1} = 0.5$. Thus, we have not made $p_{1i,i+1}$ larger than $p_{0i,i+1}$. Let the elements in row $i+1$ equal the corresponding elements in row i. Note we are not decreasing these values from those in $\mathbf{P_0}$. Now, rows i and $i+1$ in $\mathbf{P_1}$ are equal.

Recall from Chapter 2 that $q_{L(i),j}$ is the probability that team i wins in round j. Let $q_{0L(i),j}$ and $q_{1L(i),j}$ represent the probabilities that team i wins in round j using the preference matrices $\mathbf{P_0}$ and $\mathbf{P_1}$, respectively. Also note that the winner of the subtournament wins the subtournament in round $r-1$. Then, given the method of constructing $\mathbf{P_1}$, we have the following four facts:

$$q_{0L(i),r-1} = q_{1L(i),r-1}, \tag{1}$$

because $p_{1i,i+1}$ is the only element concerning team i that has been changed, and team $i+1$ is in the other half. Similarly,

$$q_{0L(j),r-1} = q_{1L(j),r-1}, \tag{2}$$

where team j is in the same half as team i.

Further, $q_{0L(i+1),r-1} \leq q_{1L(i+1),r-1}, \tag{3}$

because team $i+1$ is better in $\mathbf{P_1}$ than in $\mathbf{P_0}$. Also, therefore,

$$q_{0L(k),r-1} \geq q_{1L(k),r-1}, \tag{4}$$

where team k is in the same half as team $i+1$.

Again using the notation of Chapter 2,

$$q_{1L(i),r} = q_{1L(i),r-1} \sum_k p_{1i,L(k)} q_{1L(k),r-1} \text{ and}$$

$$q_{1L(i+1),r} = q_{1L(i+1),r-1} \sum_j p_{1i+1,L(j)} q_{1L(j),r-1},$$

where team j is in the same half of the tournament as team i and team k is in the same half of the tournament as team $i + 1$. Thus,

$$q_{1L(i),r} = q_{1L(i),r-1} \sum_k p_{1i,L(k)} q_{1L(k),r-1}$$

$$\leq q_{0L(i),r-1} \sum_k p_{0i,L(k)} q_{0L(k),r-1} \qquad \text{(by (1) and (4) above)}$$

$$< q_{0L(i+1),r-1} \sum_j p_{0i+1,L(j)} q_{0L(j),r-1} \qquad \text{(by hypothesis)}$$

$$\leq q_{1L(i+1),r-1} \sum_j p_{1i+1,L(j)} q_{1L(j),r-1} \qquad \text{(by (2) and (3) above)}$$

$$= q_{1L(i+1),r} \qquad\qquad\qquad\qquad \square$$

The hypothesis that the two teams are in opposite halves is necessary for the following reason. If the two teams are in the same half, then by increasing the probabilities for team $i + 1$, we increase the probability of team $i + 1$ winning its half, as desired. But at the same time, we could also make the chance that team i wins the tournament larger, because opponents of team $i + 1$ have a smaller chance of winning the half containing team $i + 1$. To avoid the problem, we have required teams i and $i + 1$ to be in opposite halves.

We now state and prove Lemma 4.7, which shows how to extend the subtournament preference matrix to the tournament preference matrix such that the unorderedness still holds in the tournament.

Lemma 4.7. Let P_1 be a preference matrix for a subtournament draw which is a half of some tournament. If the subtournament draw is unordered with respect to teams i and j using the preference matrix P_1, and P_1 has rows i and j equal, then an expanded preference matrix P_2 can be constructed for the entire tournament draw for which:

1) rows i and j are equal, and

2) the tournament draw is unordered with respect to teams i and j.

Proof: We construct the expanded preference matrix P_2 by using the corresponding elements from P_1 and assigning the other elements as follows. First, some of the elements in rows i and j of P_2 correspond to teams not affected by P_1. Because the elements in rows i and j of P_1 satisfy SST, we can fill in these missing values in P_2 by using either of the two adjacent values from P_1. In a like manner, we can fill in row j such that row j equals row i. Then we let all rows between rows i and j be equal to row i. To complete the matrix, we fill in remaining entries with any numbers such that SST holds. Because P_1 satisfies SST, it is possible to set the intermediate missing values in P_2 such that SST still holds. The only difficulty in filling in the remaining elements is if there is some team in the other half of the tournament which can beat team i with certainty; then the unorderedness would not hold in the tournament. Because we are simply searching for any matrix which causes the unorderedness to hold, we avoid the problem by requiring non-zero probabilities in both P_1 and P_2. Thus, if P_1 has some zero elements, we replace them with a number ε small enough so that the unorderedness is still preserved. Such a number ε exists because the function q_{ij} is continuous with respect to the elements of P_1.

Now we show the tournament draw is unordered using P_2. Because the preference structure in the half for teams i and j is equal with respect to the teams in the other half, let λ be the common probability that team i or team j beats the winner of the other half, i.e.,

$$\lambda = \sum_k p_{2i,L(k)} q_{L(k),r-1} = \sum_k p_{2j,L(k)} q_{L(k),r-1} \, ,$$

where the possible values of k represent the teams from the other half of the tournament.

Then $q_{L(i),r} = \lambda q_{L(i),r-1}$ and $q_{L(j),r} = \lambda q_{L(j),r-1}$. By the hypothesis of an unordered subtournament draw, $q_{L(i),r-1} < q_{L(j),r-1}$, so $q_{L(i),r} < q_{L(j),r}$. $\square$

Lemma 4.8. If a tournament draw is unordered with respect to teams i and $i + 1$, then either

1) teams i and $i + 1$ are in opposite halves of the tournament, or

2) teams i and $i + 1$ are in the same unordered half.

Proof: If the two teams are in the same half of the tournament and that half is ordered, then $q_{L(i),r-1} \geq q_{L(i+1),r-1}$. Further, using the notation of Chapter 2,

$$q_{L(i),r} = q_{L(i),r-1} \sum_k p_{i,L(k)} q_{L(k),r-1} \text{ and}$$

$$q_{L(i+1),r} = q_{L(i+1),r-1} \sum_k p_{i+1,L(k)} q_{L(k),r-1},$$

where k represents teams in the other half of the tournament. Because $p_{i,L(k)} \geq p_{i+1,L(k)}$ for each k, we have $q_{L(i),r} \geq q_{L(i+1),r}$, a contradiction to the hypothesis that the tournament is unordered. Therefore, either the two teams are in opposite halves of the tournament, or the two teams are in an unordered half of the tournament. $\square$

We are now prepared to state and prove Theorem 4.9.

Theorem 4.9. If a subtournament draw is unordered, then the tournament draw is unordered.

Contrapositive: If a tournament draw is ordered, then every subtournament draw is ordered.

Proof of the Theorem: First, Lemma 4.5 shows that we may assume that two teams causing the unorderedness are teams i and $i + 1$. Now, consider a subtournament of k rounds that is unordered. Either the two teams causing the subtournament draw of k rounds to be unordered are in opposite halves of the subtournament or they are not. If they are in opposite halves, Lemma 4.6 shows that we can create a preference matrix with the two rows equal, and Lemma 4.7 lets us prove the theorem for the tournament of $k + 1$ rounds. If the two teams are in the same half of the subtournament with k rounds, we must examine that half, which has $k - 1$ rounds. By Lemma 4.8, we have that the subtournament of $k - 1$ rounds is unordered. We then repeat the above reasoning until the two teams are in opposite halves of some subtournament. Lemma 4.6 then gives us a preference matrix with the two rows equal. This preference matrix can be extended to become the preference matrix for the tournament of $k + 1$ rounds by iteratively applying Lemma 4.7, which then shows that the tournament of $k + 1$ rounds is unordered. $\square$

Horen and Riezman (1985) showed that the eight-team classic tournament structure is unorderable. Independently, Corollary 4.22 establishes this result also. Using Theorem 4.9, we can now show that classic tournament structures with sixteen or more teams are unorderable.

Corollary 4.10. Classic tournament structures for 16 or more teams are unorderable.

Proof: Every subtournament of a classic tournament is itself a classic tournament. Thus, for a classic tournament of 16 or more teams, there exists a subtournament of eight teams which has a classic tournament structure. Because the eight-team classic tournament structure is unorderable, Theorem 4.9 shows that the classic tournament structure of 16 or more teams is unorderable. $\square$

Before we list and count orderable tournament structures, we have one final comment on Theorem 4.9.

Theorem 4.11. The converse of Theorem 4.9 is false. That is, subtournaments of unordered tournaments do not have to be unordered themselves.

Proof: We later show in Proposition 4.17 that the tournament structure 2121 is unorderable. Both of the three-team subtournaments in 2121, however, are orderable, and threrefore it is possible to have ordered subtournament draws while the tournament draw is unordered. □

It is now appropriate to list and count the tournament structures which are orderable. To begin, Theorem 4.1 gives the strongest teams the most byes. We can create the following tournament draws which satisfy the necessary conditions of Theorem 4.1, but we have not yet shown these are ordered: 2, 21, and 2110, etc. Horen and Riezman (1985) stated without proof that 22 is orderable. We give the details in Proposition 4.12.

Proposition 4.12. (Horen and Riezman, 1985) Tournament structure 22 is orderable if and only if the ordered draw is the seeded draw.

Proof: Using the seeded draw, where team one meets team four and team two meets team three, the probabilities of the four teams winning the tournament are:

$$\text{Pr(Team 1 wins)} = q_{L(1),2} = p_{14}\big(p_{12}p_{23} + p_{13}p_{32}\big),$$
$$\text{Pr(Team 2 wins)} = q_{L(2),2} = p_{23}\big(p_{21}p_{14} + p_{24}p_{41}\big),$$
$$\text{Pr(Team 3 wins)} = q_{L(3),2} = p_{32}\big(p_{31}p_{14} + p_{34}p_{41}\big), \text{ and}$$
$$\text{Pr(Team 4 wins)} = q_{L(4),2} = p_{41}\big(p_{42}p_{23} + p_{43}p_{32}\big).$$

Now, $q_{L(1),2} - q_{L(2),2} = p_{14}p_{12}p_{23} + p_{13}p_{14}p_{32}$
$$- p_{14}p_{21}p_{23} - p_{23}p_{24}p_{41}$$

$$\geq 0.$$

This is greater than zero because $p_{12} \geq p_{21}, p_{13} \geq p_{23}, p_{14} \geq p_{24}$, and $p_{32} \geq p_{41}$ under strong stochastic transitivity. Similarly, $q_{L(2)2} - q_{L(3)2} \geq 0$ under SST.

Further, $q_{L(3)2} - q_{L(4)2} = p_{32}p_{31}p_{14} + p_{32}p_{34}p_{41}$

$$- p_{42}p_{41}p_{23} - p_{32}p_{43}p_{41}$$

$$\geq 0.$$

Thus the four-team seeded draw is ordered.

Horen and Riezman also showed that there are only two other possible draws for 22. For each draw, an application of the "top-n" theorem shows that the draw is unordered. □

Thus, each of the tournament structures for four or fewer teams satisfies the necessary conditions of Theorem 4.1. Theorems 4.13 and 4.16 below show a procedure for constructing ordered tournament draws that demonstrates that each of the tournament structures for four or fewer teams are indeed orderable.

Theorem 4.13. In the ordered tournament draw 2, if team two is replaced by an ordered t-team subtournament, the resulting $(t + 1)$-team tournament draw is ordered.

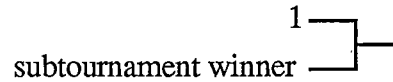**Proof:** Figure 15 below shows a representation of the $(t + 1)$-team tournament draw.

Figure 15. Representation of the $(t + 1)$-team Tournament
Draw for Theorem 4.13.

The probabilities of the teams winning the $(t + 1)$-team tournament draw are:

$$q_{L(1),r} = \sum_{k = 2}^{t + 1} p_{1k}q_{L(k),r - 1}, \text{ and}$$

$$q_{L(k),r} = q_{L(k),r-1}p_{k1}, \text{ for } k = 2, \ldots, t+1.$$

Now, $q_{L(1),r} - q_{L(2),r} = p_{12}q_{L(2),r-1} + \sum_{k=3}^{t+1} p_{1k}q_{L(k),r-1}$

$$- p_{21}q_{L(2),r-1}$$

$$\geq 0. \hspace{4cm} \text{(because } p_{12} \geq p_{21}\text{)}$$

Similarly, for $k = 2, \ldots, t,$

$$q_{L(k),r} - q_{L(k+1),r} = q_{L(k),r-1}p_{k1} - q_{L(k+1),r-1}p_{k+1,1}$$

$$\geq 0. \hspace{2cm} \text{(because of transitivity and by hypothesis)}$$

Because these differences are all non-negative, the $(t+1)$-team tournament draw is ordered. $\hspace{3cm}$ $\square$

Corollary 4.14. Tournament structure $\underline{21}$ is orderable.

Corollary 4.15. Tournament structure $\underline{2110}$ is orderable.

Theorem 4.16. In the seeded four-team tournament draw for the orderable tournament structure $\underline{22}$, if team four is replaced by an ordered t-team subtournament, the resulting $(t+3)$-team tournament draw is ordered.

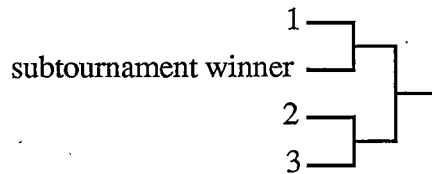**Proof:** Figure 16 below shows a representation of the $(t+3)$-team tournament draw.

Figure 16. Representation of the $(t+3)$-team Tournament
Draw for Theorem 4.16.

The probabilities of the teams winning the $(t + 3)$-team tournament draw are

$$q_{L(1),r} = \left[\sum_{k=4}^{t+3} p_{1k} q_{L(k),r-1} \right] [p_{12}p_{23} + p_{13}p_{32}],$$

$$q_{L(2),r} = p_{23} \left[p_{21} \sum_{k=4}^{t+3} p_{1k} q_{L(k),r-1} + \sum_{k=4}^{t+3} p_{2k}p_{k1} q_{L(k),r-1} \right],$$

$$q_{L(3),r} = p_{32} \left[p_{31} \sum_{k=4}^{t+3} p_{1k} q_{L(k),r-1} + \sum_{k=4}^{t+3} p_{3k}p_{k1} q_{L(k),r-1} \right], \text{ and}$$

$$q_{L(k),r} = p_{k1} q_{L(k),r-1} [p_{k2}p_{23} + p_{k3}p_{32}], \quad k = 4, \ldots, t+3.$$

Now,

$$
\begin{aligned}
q_{L(1),r} - q_{L(2),r} \quad &= \left[p_{12}p_{23} \sum_{k=4}^{t+3} p_{1k} q_{L(k),r-1} + p_{13} \sum_{k=4}^{t+3} p_{32}p_{1k} q_{L(k),r-1} \right] \\
&\quad - \left[p_{21}p_{23} \sum_{k=4}^{t+3} p_{1k} q_{L(k),r-1} + p_{23} \sum_{k=4}^{t+3} p_{k1}p_{2k} q_{L(k),r-1} \right] \\
&\geq 0.
\end{aligned}
$$

$$
\begin{aligned}
q_{L(2),r} - q_{L(3),r} \quad &= p_{23} \left[p_{21} \sum_{k=4}^{t+3} p_{1k} q_{L(k),r-1} + \sum_{k=4}^{t+3} p_{2k}p_{k1} q_{L(k),r-1} \right] \\
&\quad - p_{32} \left[p_{31} \sum_{k=4}^{t+3} p_{1k} q_{L(k),r-1} + \sum_{k=4}^{t+3} p_{3k}p_{k1} q_{L(k),r-1} \right] \\
&\geq 0.
\end{aligned}
$$

$$
\begin{aligned}
q_{L(3),r} - q_{L(4),r} \quad &= p_{32}p_{31}p_{14} q_{L(4),r-1} + p_{32}p_{34}p_{41} q_{L(4),r-1} \\
&\quad - p_{42}p_{41}p_{23} q_{L(4),r-1} - p_{32}p_{43}p_{41} q_{L(4),r-1} \\
&\quad + p_{32}p_{31} \sum_{k=5}^{t+3} p_{1k} q_{L(k),r-1} + p_{32} \sum_{k=5}^{t+3} p_{3k}p_{k1} q_{L(k),r-1} \\
&\geq 0.
\end{aligned}
$$

$$q_{L(k),r} - q_{L(k+1),r} = q_{L(k),r-1}p_{k1}\left(p_{k2}p_{23} + p_{k3}p_{32}\right)$$

$$- q_{L(k+1),r-1}p_{k+1,1}\left(p_{k+1,2}p_{23} + p_{k+1,3}p_{32}\right)$$

$$\geq 0, \text{ for } k = 4, \ldots, t+2.$$

Because these differences are all non-negative, the $(t+3)$-team tournament draw is ordered. $\square$

Corollaries 4.14 and 4.15, along with Proposition 4.12 show that all tournament structures for three and four teams are orderable. In addition, Theorems 4.13 and 4.16 show us how to generate more orderable tournament structures. Theorem 4.21 below will show that this sequence of tournament structures is indeed the set of orderable tournament structures. Before we state and prove that theorem, however, we need to show that the tournament structures 2121 and 2221 are unorderable.

Proposition 4.17. Tournament structure 2121 is unorderable.

Proof: To construct an ordered tournament draw, Theorem 4.1 requires teams one and two to be placed in slots three and six. Theorem 4.3 now gives us two candidates for an ordered tournament draw, because there are two orderable tournament structures for four teams. One case says that team three must be in the same half as team two, and team four must be in the same half as team one. The other case says that team four must be in the same half as teams two and three. The two tournament draws are shown in Figure 17.
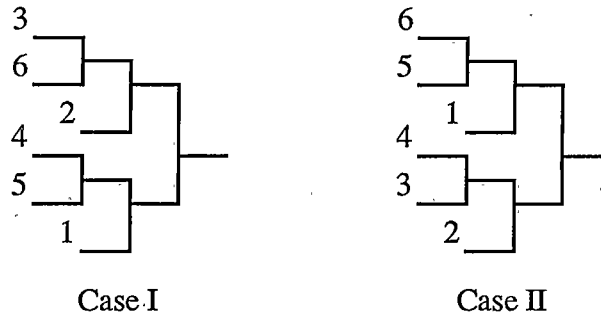
Figure 17. The Two Tournament Draws for Proposition 4.17.

However, Case II can be eliminated due to Theorem 4.3, which says that if team five can beat team six with certainty, then in the "top-5" tournament draw, team five would play fewer games to win the tournament than teams three or four. Thus, the "top-5" tournament draw for Case II is unordered.

For Case I, let the preference matrix be as follows:

$$\mathbf{P} = \begin{bmatrix} .5 & .5 & .5 & .5 & .5 & 1.0 \\ .5 & .5 & .5 & .5 & .5 & 1.0 \\ .5 & .5 & .5 & .5 & .5 & .5 \\ .5 & .5 & .5 & .5 & .5 & .5 \\ .5 & .5 & .5 & .5 & .5 & .5 \\ .0 & .0 & .5 & .5 & .5 & .5 \end{bmatrix}.$$

Using this preference matrix, the probabilities of winning the tournament are given by the following vector:

$$\mathbf{Q}_3 = \begin{bmatrix} .250 \\ .375 \\ .125 \\ .125 \\ .125 \\ .000 \end{bmatrix}.$$

Thus, team two has a better chance of winning the tournament than team one, because team two has a good chance of meeting the weakest team in the tournament. Team one does not have the chance to meet this weakest team, and the probability of team one winning the tournament is correspondingly lower. □

Proposition 4.18. Tournament structure 2221 is unorderable.

Proof: As in tournament structure 2121 of Proposition 4.17, Theorem 4.3 shows that there are two draws that might produce an orderable tournament structure, shown below in Figure 18.
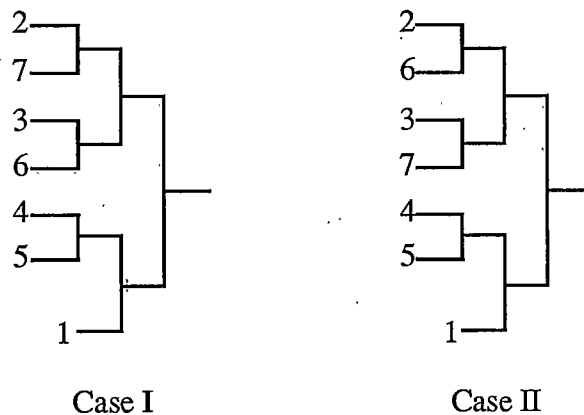

Case I　　　　　　　　Case II

Figure 18. The Two Tournament Draws for Proposition 4.18.

In either case, the "top-6" tournament draw gives tournament structure 2121. But, Proposition 4.17 shows that 2121 is unorderable, so 2221 is also unorderable. □

We are now prepared to characterize the set of orderable tournament structures.

Theorem 4.19. If A is a tournament structure generated by either Theorem 4.13 or Theorem 4.16, then A is an orderable tournament structure.

Theorem 4.20. If B is *not* a tournament structure that is generated by either Theorem 4.13 or Theorem 4.16, then B is an unorderable tournament structure.

Proof: If B is not generated by Theorem 4.13 or Theorem 4.16, and if we do not consider unordered subtournaments (violating Theorem 4.9) or replacing the strongest teams with subtournaments (violating Theorem 4.3), there are three possible cases.

Case I: Both teams in 2 are replaced by ordered subtournaments. (Note that neither of these ordered subtournaments can be 2; otherwise, Theorem 4.16 would be satisfied.)

Because of Theorem 4.3, we can assume the two ordered subtournaments are 21. The tournament then becomes 2121, which is shown in Proposition 4.17 to be unorderable.

Case II: Both teams three and four in 22 are replaced by ordered subtournaments.

Because of Theorem 4.3, we assume the ordered subtournaments are 2. Thus, the tournament becomes 2121, which is unorderable.

Case III: Teams two, three, and four in 22 are replaced by ordered subtournaments.

We assume in this case that all three subtournaments are 2. This reduces the tournament structure to 2221, which is shown in Proposition 4.18 to be unorderable. $\square$

Theorem 4.21. The set of tournament structures generated by Theorems 4.13 and 4.16 is the set of ordered tournament structures.

Proof: Theorem 4.19 shows one direction. The contrapositive of Theorem 4.20 shows the other direction. $\square$

Corollary 4.22. In an ordered tournament draw, team one plays either one or two games, and team two plays one, two, or three games.

Figure 19 below shows the ordered tournament draws for up to six teams. The arrows in the figure show what tournament structures are produced by applying

Theorem 4.3, the "top-n" theorem. Also, the two unorderable tournament structures for six teams are shown in the figure.
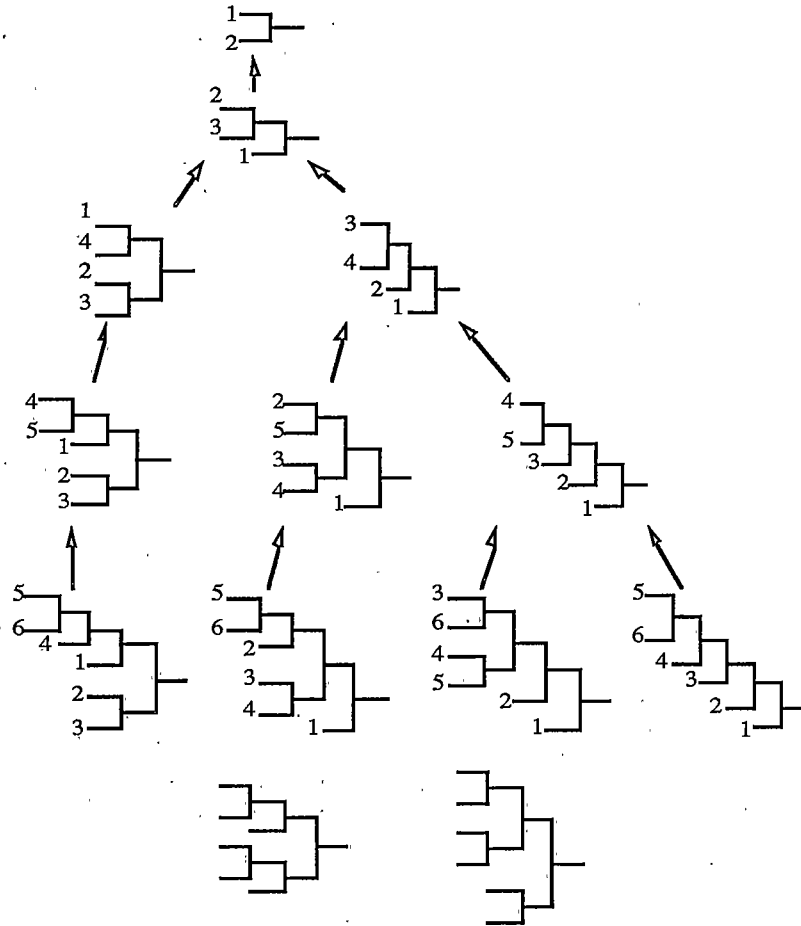

Figure 19. Ordered Tournament Draws for Two through Six
Teams and Unorderable Structures 2121 and 2211.

Finally, the formula for counting orderable tournament structures is given in the following theorem.

Theorem 4.23. The number of orderable tournament structures is given by

$$b_t = b_{t-1} + b_{t-3}. \tag{5}$$

Proof: Theorem 4.13 shows that each of the orderable tournament structures for $t-1$ teams will create an orderable tournament structure for t teams. Also, Theorem 4.16 shows that each of the orderable tournament structures for $t-3$ teams will create an orderable tournament structure for t teams. Because Theorem 4.21 shows that the set of tournament draws generated by Theorems 4.13 and 4.16 is the only set of ordered tournament draws, $b_t = b_{t-1} + b_{t-3}$. $\qquad\qquad\square$

Equation (5) is simple enough in nature to derive a closed-form solution. By methods used in the analysis of difference equations (see Tucker, 1984), the characteristic equation for the relationship given in (5) is $\alpha^3 - \alpha^2 - 1 = 0$. The three solutions to this characteristic equation are $\alpha_1 = 1.46557$, $\alpha_2 = -0.23279 + 0.79255\,i$, and $\alpha_3 = -0.23279 - 0.79255\,i$. By using the initial conditions of $b_0 = 0$, $b_1 = 1$, and $b_2 = 1$, we obtain the solution $b_t = ak^t - ab^t\cos(t\theta) + cb^t\sin(t\theta)$, where $a = 0.41724$, $k = 1.46557$, $b = 0.82603$, $\theta = 1.85648$, and $c = 0.36765$. Table 7 below shows the values for b_t for two through 31 teams.

Table 7. Values for b_t, the Number of OrderableTournament Structures.

Values of t	Values of b_t		Values of t	Values of b_t
2	1		17	277
3	1		18	406
4	2		19	595
5	3		20	872
6	4		21	1,278
7	6		22	1,873
8	9		23	2,745
9	13		24	4,023
10	19		25	5,896
11	28		26	8,641
12	41		27	12,664
13	60		28	18,560
14	88		29	27,201
15	129		30	39,865
16	189		31	58,425

Because cosine and sine are bounded by 1 and -1 and because $k > b$, $\ln(b_t) \approx \ln a + t \ln k$, for large t. Thus, $\ln(b_t)$ increases linearly.

We may now count the number of ordered tournament *draws* by using Theorem 4.24, which shows that the orderable structures have unique tournament draws that make them orderable.

Theorem 4.24. There exists only one unique, ordered tournament draw for each orderable tournament structure.

Proof: Because of the way orderable tournament structures are generated using Theorems 4.13 and 4.16, in each round there are either one or three teams receiving byes, or there are two or four teams in round one. Theorem 4.1 shows us that the strongest teams receive the most byes. Thus, we know how to place the teams in the various *rounds*. For rounds with only one seeded team, or two initial teams, it is obvious how to place the teams in the slots. For rounds with three seeded teams, Theorem 4.3 and

Theorem 4.9 show us that three teams receiving byes and the winner of the corresponding subtournament must conform to the seeded four-team draw. For the draws with four teams in round one, Proposition 4.12 shows that the teams must match the four-team seeded tournament draw. Thus, in each round there is only one way to place the seeded teams, which shows that for each orderable tournament structure, the only possible ordered draw is unique. $\square$

We conclude this chapter by noticing from Tables 5 and 7 that while the number of *orderable* tournament structures increases without bound, the number of tournament structures increases much faster. Many of these tournament structures may not be as undesirable as the results of this chapter would lead us to believe. We suspect that a great number of these tournaments are partially ordered, and thus are desirable as belonging to that class of tournament structures.

CHAPTER 5

TOURNAMENT EFFECTIVENESS

Once a decision has been made to restrict attention to ordered tournaments, other criteria must be used to make a final choice of a tournament structure. In Chapter 1, Table 2 listed several potential criteria one might use to determine a "best" tournament. This section will compare two definitions of "best" to determine tournament effectiveness.

The first definition we will examine is that of the strongest team having the highest chance of winning the tournament, or the *one highest* criterion.

Definition 5.1. Among several distinct orderable tournament draws for a given number of teams, a single one satisfies the *one highest* criterion if that draw maximizes the chance of the strongest team winning the tournament, regardless of the SST preference matrix.

This criterion for "best" is the one most used in the statistical literature, and was examined by Glenn (1960), Searls (1963), Maurer (1975), and Horen and Riezman (1985), among others.

Definition 5.2. A tournament draw, Draw 1, *dominates* another tournament draw, Draw 2, with respect to the *one highest* criterion, if for any SST preference matrix the probability of team one winning the tournament is greater using Draw 1 than it is using Draw 2.

The first result we will prove concerning the *one highest* criterion is that it does not produce uniformly best tournament structures. That is, depending on the particular

preference matrix, different tournament draws give the strongest team a higher chance of winning the tournament. To illustrate the non-uniformity of the criterion, consider the following proposition.

Proposition 5.3. For four and five teams, the *one highest* criterion does not produce a tournament structure that dominates all others. That is, each of the two orderable tournament structures for four teams and each of the three orderable tournament structures for five teams can give the highest probability of team one winning the tournament, with an appropriate choice for the preference matrix.

Proof: For four teams, consider the following preference matrices, P_1 and P_2:

$$P_1 = \begin{bmatrix} .5 & .5 & .5 & .5 \\ .5 & .5 & .5 & .5 \\ .5 & .5 & .5 & .5 \\ .5 & .5 & .5 & .5 \end{bmatrix}, \text{ and } P_2 = \begin{bmatrix} .5 & .5 & 1.0 & 1.0 \\ .5 & .5 & .5 & 1.0 \\ .0 & .5 & .5 & .5 \\ .0 & .0 & .5 & .5 \end{bmatrix}.$$

For the orderable tournament structure 22, consider the seeded draw. Under this draw, using P_1, the probability that team one wins is 0.25. Using P_2, the probability that team one wins is 0.75 (because team one has a certain first-round win over team four under P_2). For 2110, the other ordered tournament draw using four teams, teams three and four meet in the first round. The winner of the first round game plays team two, and the winner of that game meets team one. Thus, because team one only plays one game and under P_1 team one has an equal chance of beating any other team, under P_1 the probability of team one winning the tournament is 0.5. For any preference matrix, the probability of team one winning the tournament is $p_{12}(p_{23}p_{34} + p_{24}p_{43}) + p_{13}p_{34}p_{32} + p_{14}p_{43}p_{42}$. Using P_2, this reduces to $0.5 (0.5 \times 0.5 + 0.5) + 0.5 \times 0.5 = 0.625$. Thus, under P_1, 2110 gives team one a greater probability of winning than 22. Conversely, under P_2, 22 gives team one a greater probability. The explanation for this is that in 2110 team one is quite

likely to meet team two in the second round under P_2. In 22, team one is more likely to meet team three, a certain win for team one.

For five teams, consider the following three preference matrices:

$$P_3 = \begin{bmatrix} .5 & .5 & 1.0 & 1.0 & 1.0 \\ .5 & .5 & .5 & .5 & .5 \\ .0 & .5 & .5 & .5 & .5 \\ .0 & .5 & .5 & .5 & .5 \\ .0 & .5 & .5 & .5 & .5 \end{bmatrix}, P_4 = \begin{bmatrix} .5 & .5 & 1.0 & 1.0 & 1.0 \\ .5 & .5 & .5 & 1.0 & 1.0 \\ .0 & .5 & .5 & .5 & 1.0 \\ .0 & .0 & .5 & .5 & .5 \\ .0 & .0 & .0 & .5 & .5 \end{bmatrix}, \text{ and}$$

$$P_5 = \begin{bmatrix} .5 & .5 & .5 & .5 & .5 \\ .5 & .5 & .5 & .5 & .5 \\ .5 & .5 & .5 & .5 & .5 \\ .5 & .5 & .5 & .5 & .5 \\ .5 & .5 & .5 & .5 & .5 \end{bmatrix}.$$

For comparing 21101000 and 2210, under P_3, the chance of the strongest team winning the tournament is 0.75, while under P_4 the chance is 0.875. Other comparisons are displayed in Table 8.

Table 8. Probabilities of Team One Winning under Different Tournament Structures and Preference Matrices

	Tournament Structure		
Preference Matrix	21101000	2210	2111
P_3	.75	.875	.75
P_4	.6875	.625	.75
P_5	.5	.5	.25

From Table 8, we can see that for different preference matrices, different tournament structures give team one the greatest probability of winning the tournament. □

It is relatively simple to program a computer search routine to find the matrices for the counter-examples presented in Proposition 5.3. If we wanted to and had sufficient resources, we could continue doing this for six teams, seven teams, etc. Because we are considering maximizing the chance that the strongest team wins the tournament, we are essentially minimizing the chance that the strongest team meets the second strongest team. This thought leads to the following theorem.

Theorem 5.4. For any pair of distinct ordered tournament draws each with four or more teams, say Draw 1 and Draw 2, there exist two preference matrices P_1 and P_2 such that:

1) Under P_1, Draw 1 gives team one a higher probability of winning the tournament than Draw 2, and

2) Under P_2, Draw 2 gives team one a higher probability of winning the tournament than Draw 1.

Proof: We will prove this theorem by induction, using Proposition 5.3 for four and five teams. From Corollary 4.22, team one plays either one game or two games in any ordered tournament draw. There are then three cases to consider:

Case I: Team one plays one game in both Draw 1 and Draw 2.

Case II: Team one plays two games in both Draw 1 and Draw 2.

Case III: Team one plays one game in Draw 1 and two games in Draw 2.

Case I: Team one plays one game in both Draw 1 and Draw 2. Let $p_{1j} = 1.0$ for all $j > 2$, and let $p_{12} = 0.5$. Then to maximize the probability that team one wins, we should minimize the chance that team two meets team one in the final game. Consider the subtournaments (of five or more teams) that contain team two. By the induction hypothesis, there are two matrices such that the probability that team two wins

the subtournament is higher for one draw than for the other with one preference matrix, and vice versa for the other preference matrix. If we let these preference matrices fill in the lower-right portion of P_1 and P_2, the same ordering will exist for the tournament as for the subtournament.

Case II: Team one plays two games in both Draw 1 and Draw 2. We first note that $t = 7$ is the first t where this case applies. We will give only teams two, three, and four a non-zero chance of beating team one. Also, we note that when team one plays two games in a tournament, the highest ranked team which can meet team one in the next-to-the-last round is team four (Theorem 4.16). Using an argument similar to the one used in Case I, we show using Proposition 5.3 that there are two preference matrices such that under the first preference matrix, one draw gives team one a higher chance of winning the tournament, while under the second preference matrix, the other draw gives team one a higher chance of winning the tournament.

Case III: Team one plays one game in Draw 1 and two games in Draw 2. In this case, we will let $p_{1ij} = 0.5$ for all i and j, and we will let

$$
P_2 = \begin{bmatrix}
.5 & .5 & 1.0 & 1.0 & \cdots & 1.0 \\
.5 & .5 & .5 & 1.0 & \cdots & 1.0 \\
.0 & .5 & .5 & .5 & \cdots & .5 \\
.0 & .0 & .5 & .5 & \cdots & .5 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
.0 & .0 & .5 & .5 & \cdots & .5
\end{bmatrix}.
$$

Using P_1, the chance that team one wins Draw 1 is 0.5 while it is 0.25 for Draw 2. Using P_2, we will show that Draw 2 has a higher probability than Draw 1. In Draw 2 the best team that team one could play in its first game is team four (Theorem 4.16), and because in P_2 team one can beat all teams but team two with certainty, team one's opponent in the second game is either team two or team three. The chance that team one wins under

P_2 for Draw 2 is then 0.75. For Draw 1, team two will play at least team three before having the opportunity to meet team one. Because team three has a 0.5 chance of beating any opponent other than team one, and because team three must play at least one game before meeting team two, the chance that team two meets team one is at least 0.75. Consequently, the chance that team one wins the tournament is at most

$$0.5 \times 0.75 + 0.25 = 0.625 \qquad \square$$

While the concept of *one highest* seems reasonable and intuitive as a further criterion for studying ordered tournaments, we have shown that it does not identify dominating tournament structures. One might then wonder whether Theorem 5.4 would hold for all tournament structures, and not just for ordered tournament structures. A counter-example to Conjecture 5.5 shows that this is not the case.

Conjecture 5.5. For any pair of distinct tournament draws with t teams, ordered or unordered, say Draw 1 and Draw 2, there exist two preference matrices P_1 and P_2 such that:

1) Under P_1, Draw 1 gives team one a higher probability of winning the tournament than Draw 2, and

2) Under P_2, Draw 2 gives team one a higher probability of winning the tournament than Draw 1.

Counter-example to Conjecture 5.5: Consider the unorderable tournament structure <u>2211</u> and the orderable tournament structure <u>21111000</u>. Figure 20 below shows the two draws used in this counter-example.
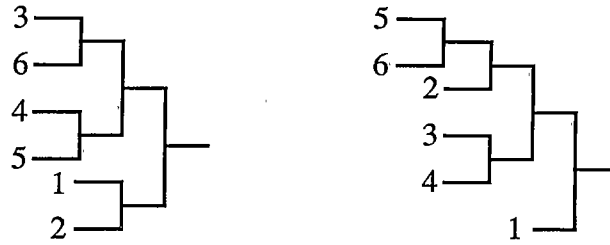
Figure 20. An Unordered Tournament Draw for 2211 and
the Ordered Tournament Draw 21111000.

For 2211, the probability of team one winning the tournament is less than the chance of team one beating team two, or p_{12}. Similarly, the chance that team one wins 21111000 is at least as large as the chance of meeting the toughest opponent in the subtournament, or p_{12}. Therefore, the chance of team one winning 21111000 is greater than the chance of team one winning 2211, no matter which preference matrix is used. This contradicts the conjecture. $\square$

If we have restricted our search for a "best" tournament draw to ordered tournament draws, then Theorem 5.4 shows that we have exhausted the criterion of *one highest*.

Another popular and reasonable criterion to consider is that of *one-two highest*, where the probability of team one meeting team two in the last round is maximized.

Definition 5.6. Among several distinct orderable tournament draws for a given number of teams, a single one satisfies the *one-two highest* criterion if that draw maximizes the chance of the two strongest teams meeting in the final round, regardless of the SST preference matrix.

In many sporting events, generating revenue is of great concern, and tournament directors would like the two strongest teams to meet in the championship game. Horen and Riezman (1985) showed that the seeded draw for 22, besides being ordered, also

maximized the chance of teams one and two meeting in the final game, among all draws for 22. We give several results concerning the *one-two highest* criterion, leading up to an existence theorem, Theorem 5.9.

Proposition 5.7. The ordered tournament draw for the orderable tournament structure 2110 dominates the ordered tournament draw for the orderable tournament structure 22 for any preference matrix under the criterion of *one-two highest*. That is, for any preference matrix, the chance that the two strongest teams meet in the final game is higher for 2110.

Proof: Under 2110, the probability that teams one and two meet in the final round is given by:

Pr(1 meets 2) = $A = p_{23}p_{34} + p_{24}p_{43} = p_{23}p_{34} + p_{24} - p_{24}p_{34}$.

Under 22, the probability is given by:

Pr(1 meets 2) = $B = p_{14}p_{23}$.

We show the difference in these two probabilities is non-negative. We then consider the difference:

$$
\begin{aligned}
A - B \quad &= p_{24} - p_{23}p_{14} + p_{34}(p_{23} - p_{24}) \\
&\geq p_{14}p_{24} - p_{23}p_{14} + p_{34}(p_{23} - p_{24}) \qquad \text{(because } p_{14} \leq 1\text{)} \\
&= -p_{14}(p_{23} - p_{24}) + p_{34}(p_{23} - p_{24}) \\
&= (p_{34} - p_{14})(p_{23} - p_{24}) \\
&\geq 0. \qquad\qquad\qquad\qquad \text{(due to strong stochastic transitivity)}
\end{aligned}
$$

Thus, no matter which SST preference matrix is used, 2110 has a greater chance of the two strongest teams meeting than 22. □

One might point out that because we are maximizing the probability of the two best teams meeting in the final round, the problem reduces to two applications of the *one highest*

criterion, but for team one in one subtournament and for team two in the other subtournament. Further, because Theorem 5.4 shows that the *one highest* criterion does not have uniform dominance, we might suspect this result could be applied to the criterion of *one-two highest*. This is in fact the case, as Theorem 5.8 shows.

Theorem 5.8. For any pair of distinct ordered tournament draws with t teams where team one plays the same number of games, say Draw 1 and Draw 2, there exist two preference matrices P_1 and P_2 such that:

1) Under P_1, Draw 1 gives a higher probability of team one meeting team two in the final game than Draw 2, and

2) Under P_2, Draw 2 gives a higher probability of team one meeting team two in the final game than Draw 1.

Proof: If the two draws have team one playing only one game, then we need only examine the subtournaments with respect to team two, the strongest team in the subtournament. Theorem 5.4 gives the existence of two preference matrices such that neither draw dominates the other with respect to the *one highest* criterion for the subtournament containing team two. Because team one is already playing in the final game, the two matrices given by Theorem 5.4 can be appended with any values consistent with strong stochastic transitivity. If team one plays two games, then by Theorem 4.16 teams two and three play in the opposite half. If we let $p_{15} = 1.0$, then the chance of team one playing in the final round depends only on whether team four wins the subtournament. Theorem 5.4 shows that there are two preference matrices (for the subtournament) which we can append under SST such that the above argument when team one plays only one game holds. □

Theorem 5.8 is discouraging in the sense that uniform dominance does not exist using the *one-two highest* criterion. Nevertheless, the lack of dominance is not universal, as shown by Theorem 5.9.

Theorem 5.9. For any number of teams, there exist two ordered tournament draws such that under the *one-two highest* criterion, one draw dominates the other draw.

Proof: Each of the tournament structures in Proposition 5.7 can be used to generate tournament structures of t teams by adding ordered subtournaments of the appropriate size, according to Theorems 4.13 and 4.16. The strongest team in the added subtournament is team four. Because the criterion worked in Proposition 5.7 using team four, it follows that if we use a preference matrix from a "top-4" tournament draw, we will get the same results. □

We characterize the tournament draws which satisfy Theorem 5.9 by noticing that one draw has team one playing one game and the other has team one playing two games. Nevertheless, not all pairs of tournament draws where in one draw team one plays one game and in the other draw team one plays two games yield a *one-two highest* dominance. Proposition 5.10 details this fact.

Proposition 5.10. There exists a pair of tournament draws, with team one playing one game in one draw and two games in the other draw, such that the *one-two highest* criterion does not produce a dominance. That is, we can find two matrices, as in Theorem 5.4, such that one matrix makes one draw look better and the other matrix makes the other draw look better.

Proof: Consider the ordered draws for <u>2210</u> and <u>2111</u>. Let

$$P_1 = \begin{bmatrix} .5 & .5 & .5 & 1.0 & 1.0 \\ .5 & .5 & .5 & .5 & .5 \\ .5 & .5 & .5 & .5 & .5 \\ .0 & .5 & .5 & .5 & .5 \\ .0 & .5 & .5 & .5 & .5 \end{bmatrix}, \text{ and } P_2 = \begin{bmatrix} .5 & .5 & .5 & 1.0 & 1.0 \\ .5 & .5 & .5 & 1.0 & 1.0 \\ .5 & .5 & .5 & .5 & .5 \\ .0 & .0 & .5 & .5 & .5 \\ .0 & .0 & .5 & .5 & .5 \end{bmatrix}.$$

Under P_1, team one can beat teams four and five with certainty, so in <u>2111</u> the chance of teams one and two meeting in the final round is simply p_{23}, or 0.5. In <u>2210</u>, because team one is already in the final game, the chance of the two strongest teams meeting in the final game is the chance of team two winning its half of the tournament, or 0.25. Thus under P_1, <u>2111</u> looks better than <u>2210</u>. Under P_2, again, team one can beat teams four and five with certainty, so in <u>2111</u> the chance is again 0.5. In <u>2210</u>, team two can beat team five with probability one, so the chance of team two winning its half is $p_{23}p_{34} + p_{24}p_{43} = 0.5 \times 0.5 + 0.5 = 0.75$. Using this preference matrix, <u>2210</u> appears to be the best draw. $\square$

While Theorem 5.9 gives the existence of tournament draws which give a dominance with respect to *one-two highest*, Theorem 5.8 and Proposition 5.10 show that there are pairs of tournament draws where no dominance exists. We find this result discouraging, but not surprising in light of the complexity of the tournaments involved.

We conclude this section by reminding the reader that Table 2 of Chapter 1 gives a number of other criteria which could be used to determine "best" tournaments. We have explored two particular definitions of "best" in this chapter, showing that neither of them give the uniform results of ordered tournaments in Chapter 4.

CHAPTER 6

SUMMARY

Conclusions

Prior work on knockout tournaments has been either preliminary in nature or primarily concerned with random rather than deterministic structures. Random tournament structures do not specify ahead of time how the winners will compete against each other. Hwang (1982) used the concept of ordered, or monotone tournaments to show that if a classic tournament is "reseeded" after each round, then the tournament is ordered. Horen and Riezman (1985) showed fair, or ordered, tournament draws for four teams and determined that the eight-team classic tournament structure is unorderable. Maurer (1975) stated without proof recursive formulas similar to Theorem 3.9 for counting tournament structures. Maurer also gave results for the *one highest* criterion, but he examined only selected classes of preference matrices, such as the case with one strongest player and $t - 1$ equal-strength players, and the case of intransitive preference matrices.

This thesis has given the following types of results:

1) Results concerning counting tournament structures, including the ability to count tournaments restricted by the number of rounds.

2) Results concerning ordered tournaments, including generating and counting them.

3) Limited results on "best" tournaments.

The statistical and mathematical literature has seen a great deal of work in the area of round-robin tournaments, and also a large number of articles and book chapters on other types of tournaments. The subject of knockout tournaments is important not only for its

mathematical properties, but also in industrial settings where smaller sample sizes might be used to select the "best" treatment with confidence.

Future Work

It seems likely that some results can be formulated for partially ordered tournament draws. When adding "dummy" teams, we are restricting the preference matrix due to inclusion of the zeroes and ones, but at the same time we are considering a classic tournament structure. On the other hand, we have shown that for eight teams or more, classic tournament draws are unordered using SST preference matrices.

We also feel that some work should be done to check the influence of the *a priori* rankings. In sports situations, a regular season composed of repeated round-robin competitions may be an adequate way to measure team strengths. On the other hand, due to the stochastic nature of these competitions, some incorrect seeding may still occur. We would be interested in seeing how robust knockout tournaments are to the misclassification issue. This issue is also closely related to the Bayesian viewpoint where we assume some prior distribution for the preference matrix.

As a further extension of knockout tournaments, we would like to consider double-elimination tournaments. These tournaments are simply a combination of two single-elimination tournaments, one tournament for the winners and one tournament for the losers. When a player in the winner's tournament loses, he continues play in the loser's tournament. When a player in the loser's tournament loses, he is eliminated from further play. The complications involved with double-elimination tournaments are the placement of the losers in the second single-elimination tournament. We suspect that definitions such as ordered should be modified for the double-elimination tournament and that fewer tournaments will satisfy the criterion than for single-elimination tournaments.

This thesis has not addressed the issue of using an estimated preference matrix; we have only restricted our matrices by strong stochastic transitivity. If a believable estimate for the preference matrix is available, perhaps certain unorderable tournament structures would become orderable under the particular preference matrix that is estimated. We feel that in certain situations where the prior information is reliable, hand-tailored tournaments can be constructed.

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