CSC 119: Mathematics for Computer Science

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Outline

Limits and Continuity (10 CH)
This module addresses calculation of limits of simple functions as well as fairly complex functions. It involves the ϵ - σ calculation of limits of functions using limit laws. It also involves establishment of limits to infinity and horizontal asymptotes. It also covers differentiability of functions using continuity tests as well as generating derivatives of functions, applying chain rules and application of derivatives into real life applications like rates of change.

Progressions and Sequences (10CH)
This module covers computation of standard progressions (AP and GP), special progressions and sequences as well as computing their limits and Mathematical upper / lower bounds. Techniques like breaking sequences into separate progressions will be emphasized.

Matrix Algebra (10 CH)
This module will address the fundamental skills in matrix algebra (matrix algebra rules, transposes, inverses, determinants, eigine vectors and kernels) as well as how matrices are used in mathematical operations like transformations in vector spaces and solving of equations.

Vectors and Vector Spaces (15 CH)
This module will cover computations in vectors and vector spaces, vector operations, linear combination of vectors, spanning sets , linear dependence and Independence and orthogonality.

Vectors and Planes (15 CH)
This module will cater for the analytical interpretation of vectors and planes. This will include vector equations of lines in n- space, equations of planes, as well as characteristics between planes and vectors like angle between them, distance between them, etc

Part I

Limits and Continuity (Week 5, Week 6, Week 7)

1. Definition of a limit
2. Investigation of simple limit values
3. Laws of limits
4. One-sided limits
5. Infinite/Undefined Limits
6. Limits for Composite functions
7. Inverse Functions
8. Continuity and discontinuity
9. Function differentiability
10. Trigonometric functions

1 Limits and Continuity

1.1 Definition of Limit

From elementary mathematics we evaluated functions. If we have a function $f(x) = 2x^2 - 1$, then we can evaluate $f(a)$ where a is any number in the domain of $f(x)$. It can be assumed that a has to be real but it is not always the case as you will find out in other courses in the program. But assuming it is real, we can evaluate the value of $f(x)$ when x is a. For example $f(1) = 1$.

We may, however, be interested in the value of $f(x)$ as x approaches 1 but not necessarily reach 1. So we are interested in *the value f(x) approaches as x approaches 1.* Such a value is called the limit. If such a value is A, then we can write that

$$\lim_{x \to \infty} f(x) = A \tag{1}$$

Finding limits quite often involves an "investigation" not merely a substitution. Much as a substitution often works, but it also often do not work.

Example 1.1 *Find* $\lim_{x \to 1} \frac{x}{x^2+1}$

Solution 1.1 *Lets investigate the value to which* $\frac{x}{x^2+1}$ *tend to as* x *tends to* 1.

x	$f(x)$
2	*0.4*
1.5	*0.46153846*
1.1	*0.49773756*
1.05	*0.49940547*
1.01	*0.49997525*
1.001	*0.49999975*

It can be deduced from the trend that $\lim_{x \to 1} \frac{x}{x^2+1} = 0.5$

From Example 1.1 it can be observed that the limit is the same as $f(1)$. It is a common temptation to assume that the two are always equal and hence just substitute. This is not always correct.

Example 1.2 *Find* $\lim_{x \to \infty} \frac{x^2+5}{x^2-1}$

Solution 1.2 *This is a typical case where even an attempt to substitute is not viable. It is easy to assume the output is undefined, If we investigate for different values of* x*, we get the table below*

x	$f(x)$
2	*3*
10	*1.0606061*
100	*1.00060006*
1000	*1.000006*
10000	*1.00000006*

We can therefore convince ourselves that $\lim_{x \to \infty} \frac{x^2+5}{x^2-1} = 1$

Much as we could not substitute in Example 1.2, the limit comes up to be finite. In fact, substitution is possible if only we tweak the function a bit. We can divide numerator and denominator by x^2 and get the function to be $\frac{1+\frac{5}{x^2}}{1-\frac{1}{x^2}}$ which evaluates to 1 when we substitute $x = \infty$

1.2 The σ - ϵ definition of limit

The limit of a function $\lim_{x \to a} f(x) = L$ if for any number $\epsilon > 0$, there exists a number $\sigma > 0$ such that $|f(x) - L| < \sigma \quad \forall x : \quad 0 < |x - a| < \sigma$

Generally, it implies that if the values of x are assured to be in the range $a \pm \sigma$, then the values of $f(x)$ are assumed to be in the range $L \pm \epsilon$ for positive values of σ, ϵ

2 Laws of Limits

2.1 Statements (part 1)

Law 2.1 *Constant Law*

$\lim_{x \to a} C = C$ *where* C *is a constant*

Law 2.2 *Addition Law*

If $\lim_{x \to a} F(x) = L$ *and* $\lim_{x \to a} G(x) = M$*, then* $\lim_{x \to a} (F(x) \pm F(x)) = L \pm M$

Law 2.3 *Product Law*

If $\lim_{x \to a} F(x) = L$ *and* $\lim_{x \to a} G(x) = M$, *then* $\lim_{x \to a}(F(x)F(x)) = LM$

Law 2.4 *Quotient Law*

If $\lim_{x \to a} F(x) = L$ *and* $\lim_{x \to a} G(x) = M$, *then* $\lim_{x \to a} \frac{(F(x)}{F(x)}) = \frac{L}{M}$

Law 2.5 *Root Law*

So long as n *is a positive integer and* a *is positive whenever* n *is even, then* $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$

2.2 Applications

Applying the laws of limits helps in simplifying the limit value generation process. however, care has to be taken that the application does not generate meaningless / undefined expressions. Undefined expressions have to be handled with care. The limit may actually be undefined or it may be finite. A deeper investigation therefore has to be made

Example 2.1 *Find* $\lim_{x \to 2} \frac{1}{(x-2)^2}$

Solution 2.1 *The Problem can be solved by application of the quotient law as well as the constant law. It then "simplifies" to* $\frac{1}{(\lim_{x \to 2} x-2)^2}$. *As* x *tends* 2, $(x - 2)$ *tends to zero. Being a reciprocal the the quotient is undefined. Therefore* $\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty$

Example 2.2 *Find* $\lim_{x \to 1} \frac{x^2+2x-3}{(x^2-1)}$

Solution 2.2 *This can be solved by the quotient rule. We simplify it to* $\frac{\lim_{x \to 1}(x^2+2x-3)}{\lim_{x \to 1}(x^2-1)}$ *This gets to* $\frac{0}{0}$ *which is indeterminate*[1]

We can first simplify the quotient before we apply the quotient law of limits. $\lim_{x \to 1} \frac{x^2+2x-3}{(x^2-1)} = \lim_{x \to 1} \frac{(x-1)(x+3)}{(x-1)(x+1))} = \lim_{x \to 1} \frac{(x+3)}{(x+1))}$ *which simplifies to* 2

[1] Evaluation of an indeterminate function can be tricky. You can cancel by 0 and get 1, you can argue that any number by 0 is ∞ and can also argue that 0 by any number is 0. Overall, it does not make sense!

Law 2.6 *Squeeze Law If three functions are in such a way that* $g(x) \leq f(x) \leq h(x)$ *for a certain neighborhood of* a *and it so happens that* $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} h(x) = L$*, then* $\lim\limits_{x \to a} f(x) = L$

Example 2.3 *Find* $\lim\limits_{x \to 0} \frac{x^2}{1+\sqrt{|x|}}$

Solution 2.3 *The denominator is always positive and always greater than 1. This implies that* $\frac{x^2}{1+\sqrt{|x|}} \leq x^2$*. At the same time, the expression* $\frac{x^2}{1+\sqrt{|x|}}$ *is always positive. Hence* $0 \leq \frac{x^2}{1+\sqrt{|x|}}$*. This implies* $0 \leq \frac{x^2}{1+\sqrt{|x|}} \leq x^2$

We know that $\lim\limits_{x \to 0} 0 = 0$ *and* $\lim\limits_{x \to 0} x^2 = 0$ *Therefore from Squeeze Law,* $\lim\limits_{x \to 0} \frac{x^2}{1+\sqrt{|x|}} = 0$

Exercise

Evaluate the following limits

$\lim\limits_{x \to -1} (2x - x^5)$	$\lim\limits_{t \to 2} \frac{t^2+2t-5}{t^3-2t}$	$\lim\limits_{t \to 9} \frac{3-\sqrt{x}}{9-x}$	$\lim\limits_{x \to 8} \frac{x^{2/3}-4}{x-8}$	$\lim\limits_{x \to 0} \frac{3^x-1}{x}$
$\lim\limits_{x \to 2} (8 - 3x - 12x^2)$	$\lim\limits_{x \to -3} \frac{6+4x}{x^2+1}$	$\lim\limits_{x \to -5} \frac{x^2-25}{x^2+2x-15}$	$\lim\limits_{x \to 8} \frac{2x^2-17x+8}{8-x}$	$\lim\limits_{x \to 7} \frac{x^2-4x-21}{3y^2-17y-28}$
$\lim\limits_{h \to 6} \frac{(6+h)^2-36}{h}$	$\lim\limits_{z \to 4} \frac{\sqrt{z}-2}{z-4}$	$\lim\limits_{x \to -3} \frac{\sqrt{2x+22}-4}{x+3}$	$\lim\limits_{x \to 0} \frac{x}{3-\sqrt{x+9}}$	$\lim\limits_{x \to 0} \lvert x \rvert$

3 One sided limits and Limit existence

3.1 One sided limits

When we say $\lim\limits_{x \to a} f(x)$, we do not specify how a is approached. Is it approached from the left or from the right. Quite often it does not matter but sometimes it matters.

The right hand limit of $f(x)$, written as $\lim\limits_{x \to a+} f(x) = L$ if the value of L is got by approaching a from the right. i.e. values greater than it when approaching a.

The left hand limit of $f(x)$, written as $\lim_{x \to a-} f(x) = L$ if the value of L is got by approaching a from the left. i.e. values less than it when approaching a.

Overall, if a function $f(x)$ has a limit L at a, then it implies that $\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = L$ We then can simply state that $\lim_{x \to a} f(x) = L$

Example 3.1 *Find* $\lim_{x \to 0+} \frac{1}{x}$ *and* $\lim_{x \to 0-} \frac{1}{x}$

Solution 3.1 *We can investigate the trend of* $\frac{1}{x}$ *as it is approached from the right and the left and summarized in the table below*

x	f(x)	x	f(x)
1	1	-1	-1
0.5	2	-0.5	-2
0.1	10	-0.1	-10
0.01	100	-0.01	-100
0.001	1000	-0.001	-1000
0.0001	10000	-0.0001	-10000

We observe that $\lim_{x \to 0+} \frac{1}{x} = \infty$ *while* $\lim_{x \to 0-} \frac{1}{x} = -\infty$

3.2 Existence of Limit

Theorem 3.2 *Existence of a limit*
For any function $f(x)$ defined in the neighborhood of a, the limit of a $f(x)$ at a exists and if equal to L if and only if (iff) $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = L$. *Otherwise* $\lim_{x \to a} f(x)$ *does not exist.*

From this theorem, we can be able to observe that what seem to be straight forward functions may actually lack limits at certain critical points. For example a function like $\frac{1}{x}$ lacks a limit at 0, $\sqrt{x-3}$ lacks a limit at 3, $\frac{1}{x^2-2}$ and $\frac{1}{\sqrt{x^2-2}}$ lack limits at 2. Students should convince themselves of these facts.

3.3 Infinite Limits

If we investigate a function like $h(x) = \frac{1}{x-1}$, we can observe that $\lim_{x \to 1+} h(x) = \infty$ while $\lim_{x \to 1-} h(x) = -\infty$. Therefore the limit does not exist at 1. However, if the function was $h(x) = \frac{1}{(x-1)^2}$, then we can observe that the left and right hand limits are ∞ and therefore $\lim_{x \to 1} h(x) = \infty$ A limit being infinity is not an admission that there exists a number called infinity. It just implies that while x can be adjusted boundedly as close to 1 as possible, the function increases unboundedly.

As an exercise, If we have a function $\psi(x) = (2-x)^\gamma$ where gamma is any real number (written as $\gamma \in \Re$), what values of γ give us a limit in $\psi(x)$?

Exercise

Investigate the existence of the following limits and find them where they do exist

$$\lim_{x \to 0+} (3 - \sqrt{x}) \quad \lim_{x \to 1-} \sqrt{x-1} \quad \lim_{x \to 5-} \frac{x-5}{|x-5|} \quad \lim_{x \to 3+} \frac{\sqrt{x^2-6x+9}}{x-3}$$

$$\lim_{x \to 5+} \frac{\sqrt{(5-x)^2}}{5-x} \quad \lim_{x \to -4-} \frac{4+x}{\sqrt{(4+x)^2}} \quad \lim_{x \to 2} \frac{1-x^2}{x+2} \quad \lim_{x \to 1} \frac{|1+x|}{(1-x)^2}$$

4 Composite and Inverse Functions

4.1 Composite functions

A function is normally made up of a combination of expressions. A function like $f(x) = 2x^3 + x^2$ is made up of sub functions like doubling, squaring, cubing, adding etc. The functions are therefore made by composing other functions. If we have two functions f and g and a constant c we define

$(cf)(x) = cf(x)$ - A scalar multiple of the function f

$(f \pm g)(x) = f(x) \pm g(x)$ the sum and difference of functions

$(f.g)(x) = f(x).g(x)$ the product of functions

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{f(x)}$ quotient of functions

We quite often need to find limits for composite functions. As a rule of thumb, if we have two functions f and g where $\lim\limits_{x\to a} g(x) = b$, then $\lim\limits_{x\to a} fg(x) \equiv f(\lim\limits_{x\to a} g(x))$

It is a strong simplification rule that is often taken for granted. For instance, $\lim\limits_{x\to 1}(x+2)^3 log(x+2) \equiv \lim\limits_{x\to 1}(x+2)^3 \lim\limits_{x\to 1} log(x+2)$

4.2 Continuity

A function $f(x)$ is continuous at a if $\lim\limits_{x\to a} f(x) = f(a)$ and is finite. In case we look at an interval $[a, b]$, then $f(x)$ is continuous over $[a, b]$ if it is continuous for any point in the interval $[z, b]$. From the composite function expression we saw earlier, $\lim\limits_{x\to a} fg(x) \equiv f(\lim\limits_{x\to a} g(x)) = f(g(a))$ if $g(x)$ is continuous at a.

Theorem 4.1 *Intermediate Value Theorem Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between in $[f(a), f(b)]$. Then there exists a number c such that $a < c < b$ and $f(c) = M$*

The intermediate value Theorem is quite basic and can be used to deduce solutions for simple problems (like you have done in school). When the interval is a bit big and the function is not necessarily monotonic[2]

Take an example, $f(x) = 20sin(x+3)cos(\frac{x^2}{2})$. Does it have a value of x in the range $[0, 5]$ where it is equal to 10? How about -5? Get a sketch of the function in the said range and deduce the necessity and sufficiency of the IVT. You can sketch the curve your self or look for a curve sketching application on line.

4.3 Differentiability

From school maths, we did a lot of differentiation. Some functions were differentiated from first principles. The derivative of a function $f(x)$ is given by $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$. A curve is considered differentiable at a point $x = a$ if two conditions exists. (i) It is continuous at a and $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists [what does it mean to exist?].

A function can be continuous but not differentiable at a certain point. For example, the curve $y = |x|$ is continuous at $x = 0$ but not differentiable at that point since the limit does not exist.

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[2]A monotonic function is one which is either increasing or decreasing over a certain range but not doing both

Part II

Progressions and Sequences

Part III

Matrix Algebra

Part IV

Vectors and Vector Spaces

Part V

Vectors and Planes