Differentiation-part 2

Content to be covered:

- Implicit Differentiation
- Higher Order Derivatives
- Logarithmic Differentiation
- Interpretations of Derivatives:
    - Rate of Change
    - Slope of a tangent line
    - Velocity
    - Related Rates

Implicit Differentation.

<u>Example 1:</u>

Find y' for each of the following

a) $x^2 \tan(y) + y^{10} \sec(x) = 2x$

b) $e^{(2x+3y)} = x^2 - \ln(xy^3)$

Higher Order Derivatives

Higher order derivatives are second, third, fourth etc derivatives.

If $p(x)$ is a polynomial of degree n (i.e the largest exponent in the polynomial) then,

$$P^{(k)}(x) = 0 \text{ for } K \geq n + 1$$

Notation:

a) $f^{(2)}(x) = f''(x)$ ....Denotes Differentiation

b) $f^{2}(x) = [f(x)]^2$ ....Denotes Exponentiation

Higher Order Derivatives

Examples:

1. Find the third order derivative of the following function:

$$f(y) = \sin(3y) + e^{-2y} + \ln(7y)$$

2. Find y'' for $x^2 + y^4 = 10$

Logarithmic Differentation

- If you are taking derivatives of some complicated functions, you can simplify your work by introducing logarithms in your equation.
- This is called **Logarithmic differentiation.**

Logarithmic Differentation

Examples:

Differentiate the following functions

a) $y = \dfrac{x^5}{(1-10x)\left(\sqrt{x^2+2}\right)}$

b) $y = (1-3x)^{\cos(x)}$

Interpretations of Derivatives

1. <u>Rate of Change:</u>

   - If $f(x)$ represents a quantity that varies as x varies, then the derivative $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x = a$.
   - With instantaneous rate of change- we determine how fast $f(x)$ is changing at some point, say $x = a$.

<u>Note:</u>

- If the rate of change is positive then the quantity is increasing.
- If the rate of change is negative then the quantity is decreasing.

Interpretations of Derivatives

<u>Example:</u>

Suppose that the amount of water in a holding tank at $t\ minutes$

is given by $v(t) = 2t^2 - 16t + 35$. Determine each of the following.

i. The rate of change of the volume of water in the tank.
ii. Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?
iii. Is the volume of water in the tank increasing or decreasing at $t = 5$ minute?
iv. Is the volume of water in the tank ever not changing? If So, when?

Interpretations of Derivatives

Example 2:

Determine where the following function is increasing and decreasing.

$$A(t) = 27t^5 - 45t^4 - 130t^3 + 150$$

Interpretations of Derivatives

2. **Slope of Tangent Line:**

- The slope of the tangent line of $f(x)$ at $x = a$ is $f'(a)$.
- The tangent line is given by;

$$y = f(a) + f'(a)(x - a)$$

Example:

Find the tangent line to the following function at $z = 3$.

$$z = \sqrt{5z - 8}$$

Interpretations of Derivatives

3. **Velocity:**

- If the position of an object is given by $f(t)$ after t units of time, the velocity of the object at $t = a$ is given by $f'(a)$.
- Given $f(t)$ as the position function, then velocity, $V(t) = f'(t)$. First derivative of $f(t)$, and Acceleration, $A(t) = V'(t) = S'(t)$. Second derivative of $f(t)$

Interpretations of Derivatives

Example: Suppose that the position of an object after t hours is given by,

$$g(t) = \frac{t}{t+1}$$

Answer the following questions:

a. Is the object moving to the right or the left at $t = 10$ hours?

b. Does the object ever stop moving?

Interpretations of Derivatives

4. Related rates:

How to relate a formula to its change according to time.

For example:

- How fast is cost changing according to time.
- How volume is changing over time.
- How profit is changing over time etc.

We relate a formulae to time.

Interpretations of Derivatives

4. **Related Rates:**

Examples

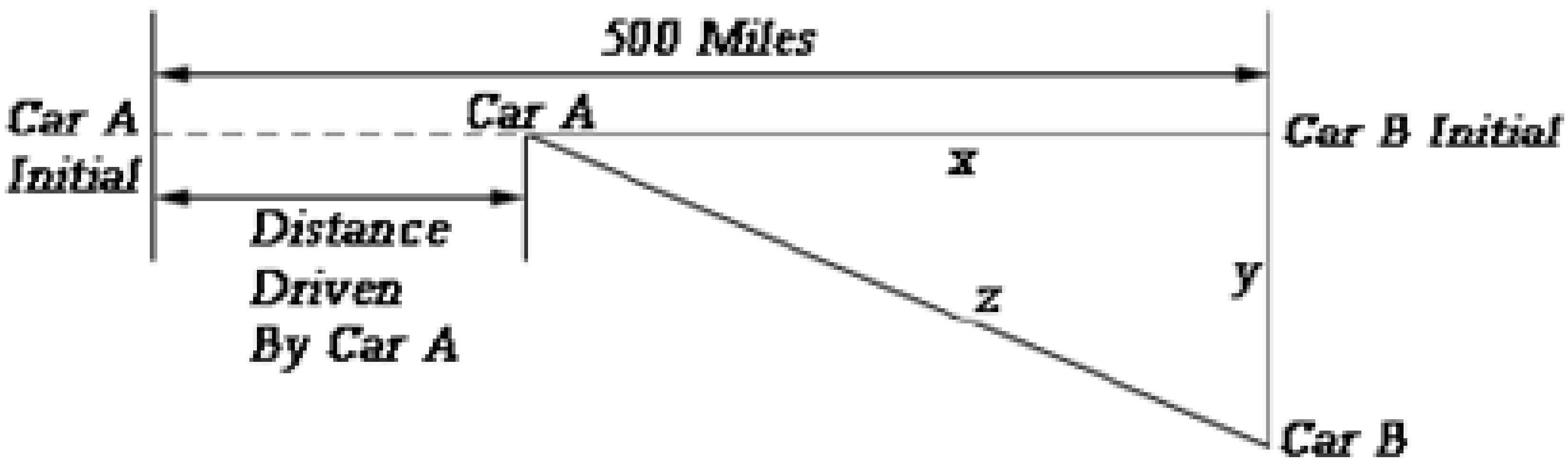
1. Air is being pumped into a spherical balloon at a rate of $5cm^3/min$. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is $20cm$.

2. Two people are 50 feet apart. One of them starts walking north at a rate so that the angle shown in the diagram below is changing at a constant rate of $0.01\ rad/min$. At what rate is the distance between the two people changing when $\theta = 0.5\ radians$?

3. Two cars start out 500 miles apart. Car A is to the west of Car B and starts driving to the east (i.e towards car B) at 35 mph and at the same time car B starts driving south at 50mph.

After 3 hours of driving at what rate is the distance between the two cars changing? Is it increasing or decreasing?

Lets sketch a figure to show the information:



Application of Derivatives

- More Application of Derivatives include:
  - Finding critical points of functions
  - Finding minimum and maximum values of a function
  - What the derivative tells us about the shape of the graph.
  - L'hopital's rule
  - Finding optimal solutions to problems

Critical points of a function

Definition

We say that $x = c$ is a critical point of the function $f(x)$ if $f(c)$ exists and if either of the following are true.

$$f'(c) = 0 \qquad \text{OR} \qquad f'(c) \text{ doesn't exist}$$

Critical points of a function

Examples:

Determine all the critical points for the functions below;

1. $$R(w) = \frac{w^2+1}{w^2-w-6}$$
2. $$h(t) = 10te^{(3-t^2)}$$
3. $$f(x) = x^2 \ln(3x) + 6$$

Minimum and Maximum values of a function

Definition

1. We say that $f(x)$ has an absolute (or global) maximum at $x = c$ if $f(x) \le f(c)$ for every x in the domain we are working on.
2. We say that $f(x)$ has a relative (or local) maximum at $x = c$ if $f(x) \le f(c)$ for every x in some open interval around $x = c$.
3. We say that $f(x)$ has an absolute (or global) minimum at $x = c$ if $f(x) \ge f(c)$ for every x in the domain we are working on.
4. We say that $f(x)$ has a relative (or local) minimum at $x = c$ if $f(x) \ge f(c)$ for every x in some open interval around $x = c$.

Minimum and Maximum values of a function

Meaning:

- We will have an absolute maximum/minimum at $x=c$ provided $f(c)$ is the largest (or smallest) value that the function will ever take on in the domain we are working with.

- For relative maximum or minimum, the point has to give the maximum or minimum value of the function in some interval of x's around $x=c$.

Minimum and Maximum values of a function

Note:

- “open interval around x=c” means some interval (a,b) not including endpoints. So c will not be either of the endpoints. Its just contained somewhere within the interval.
- The minimum and maximum points are to be called the extrema of the function.
- Relative extrema will refer to relative minimums and maximums.
- Absolute extrema will refer to absolute minimums and maximums.

Minimum and Maximum values of a function

- The relative extrema can never occur at endpoints.
- Absolute extrema occurs either at the endpoints of the domain or at relative extrema.

Minimum and Maximum values of a function

Extreme Value Theorem

Suppose that $f(x)$ is continuous on the interval $[a,b]$ then there are two numbers $a \le c, d \le b$ so that $f(c)$ is an absolute maximum for the function and $f(d)$ is an absolute minimum for the function.

Minimum and Maximum values of a function

Fermat's Theorem

If $f(x)$ has a relative extrema at $x = c$ and $f'(c)$ exists then $x = c$ is a critical point of $f(x)$. In fact, it will be a critical point such that $f'(c) = 0$.

Note that we can say that $f'(c) = 0$ because we are also assuming that $f'(c)$ exists.

Minimum and Maximum values of a function

- Note:
  - Since a relative extrema must be a critical point, the list of all critical points will give us the list of all possible relative extrema.
  - However, the theorem doesn’t say that a critical point will be a relative extrema.

Finding Absolute Extrema

Finding Absolute Extrema of $f(x)$ on $[a,b]$.

0. Verify that the function is continuous on the interval $[a,b]$.
1. Find all critical points of $f(x)$ that are in the interval $[a,b]$. This makes sense if you think about it. Since we are only interested in what the function is doing in this interval we don't care about critical points that fall outside the interval.
2. Evaluate the function at the critical points found in step 1 and the end points.
3. Identify the absolute extrema.

Finding Absolute Extrema

Examples:

1. Determine the absolute extrema for the following function in the following interval.

$$Q(y) = 3y(y+4)^{\frac{2}{3}} \text{ on } [-5,1]$$

2. Suppose the amount of money in a bank account after t years is given by ;

$$A(t) = 2000 - 10te^{5-\frac{t^2}{8}}$$

Determine the minimum and maximum amount of money in the account during the first 10 years that it is open.

The shape of the graph-part1

First Derivative Test

Suppose that $x = c$ is a critical point of $f(x)$ then,

1. If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$ then $x = c$ is a relative maximum.
2. If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$ then $x = c$ is a relative minimum.
3. If $f'(x)$ is the same sign on both sides of $x = c$ then $x = c$ is neither a relative maximum nor a relative minimum.

The shape of the graph-part1

Example:

Find and classify all the critical points of the following function.

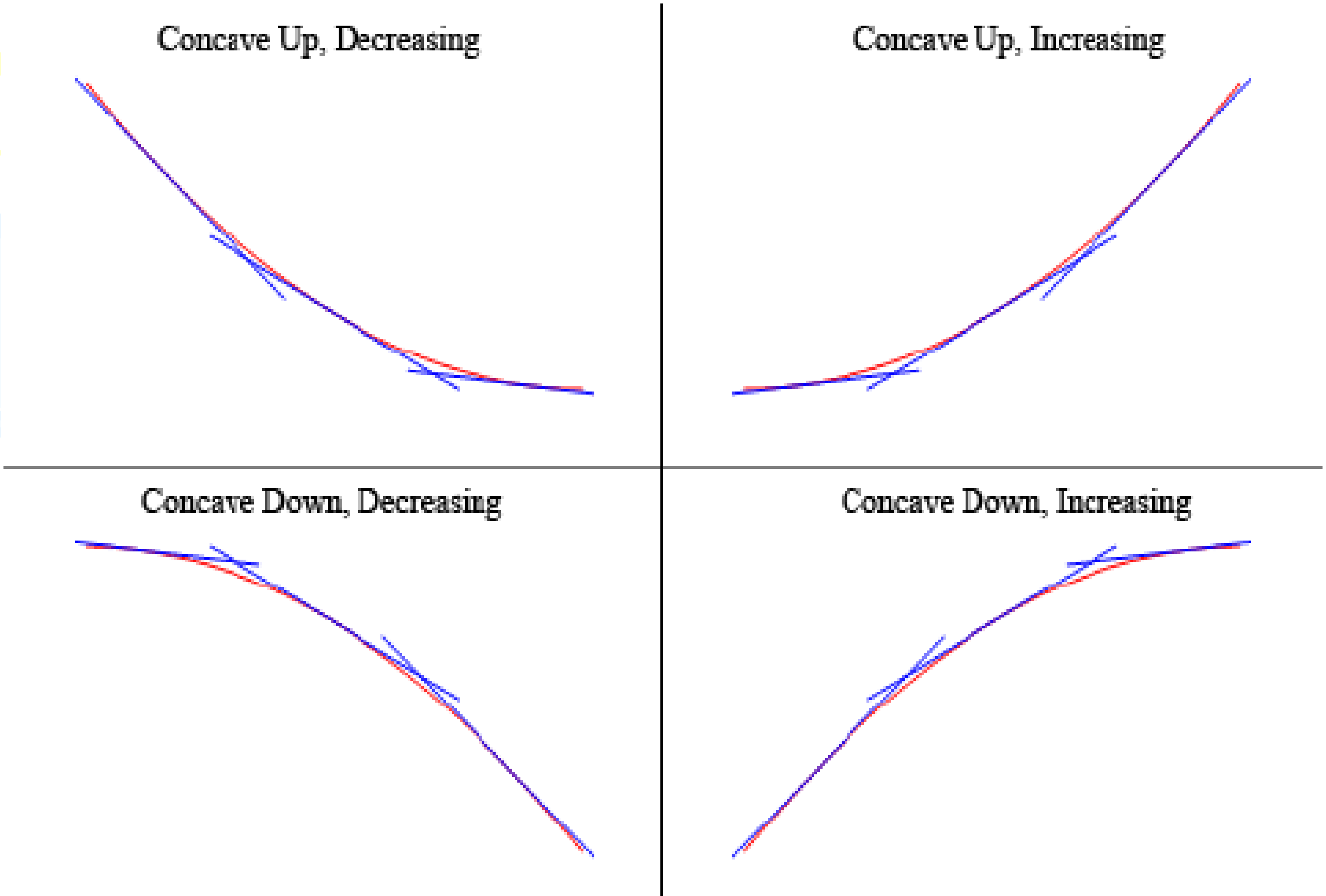
Give the intervals where the function is increasing or decreasing.

$$g(t) = t\left(\sqrt[3]{t^2 - 4}\right)$$

The shape of the graph-part2

- Here we look at what the second derivative tells us about the graph of a function.
- The main concept convered is the concavity of the graph.
- A function is said to be concave up if it “opens” up and a function is said to be concave down if it “opens” down.
- Concavity has nothing to do with increasing or decreasing.
- A function can be concave up/down and either increasing or decreasing.

The shape of the graph-part2



The shape of the graph-part2

Definition 1

Given the function $f(x)$ then

1. $f(x)$ is concave up on an interval I if all of the tangents to the curve on I are below the graph of $f(x)$.

2. $f(x)$ is concave down on an interval I if all of the tangents to the curve on I are above the graph of $f(x)$.

The shape of the graph-part2

Definition 2

A point $x = c$ is called an inflection point if the function is continuous at the point and the concavity of the graph changes at that point.

The shape of the graph-part2

Fact

Given the function $f(x)$ then,

1. If $f''(x) > 0$ for all x in some interval I then $f(x)$ is concave up on I.

2. If $f''(x) < 0$ for all x in some interval I then $f(x)$ is concave down on I.

The shape of the graph-part2

Example:

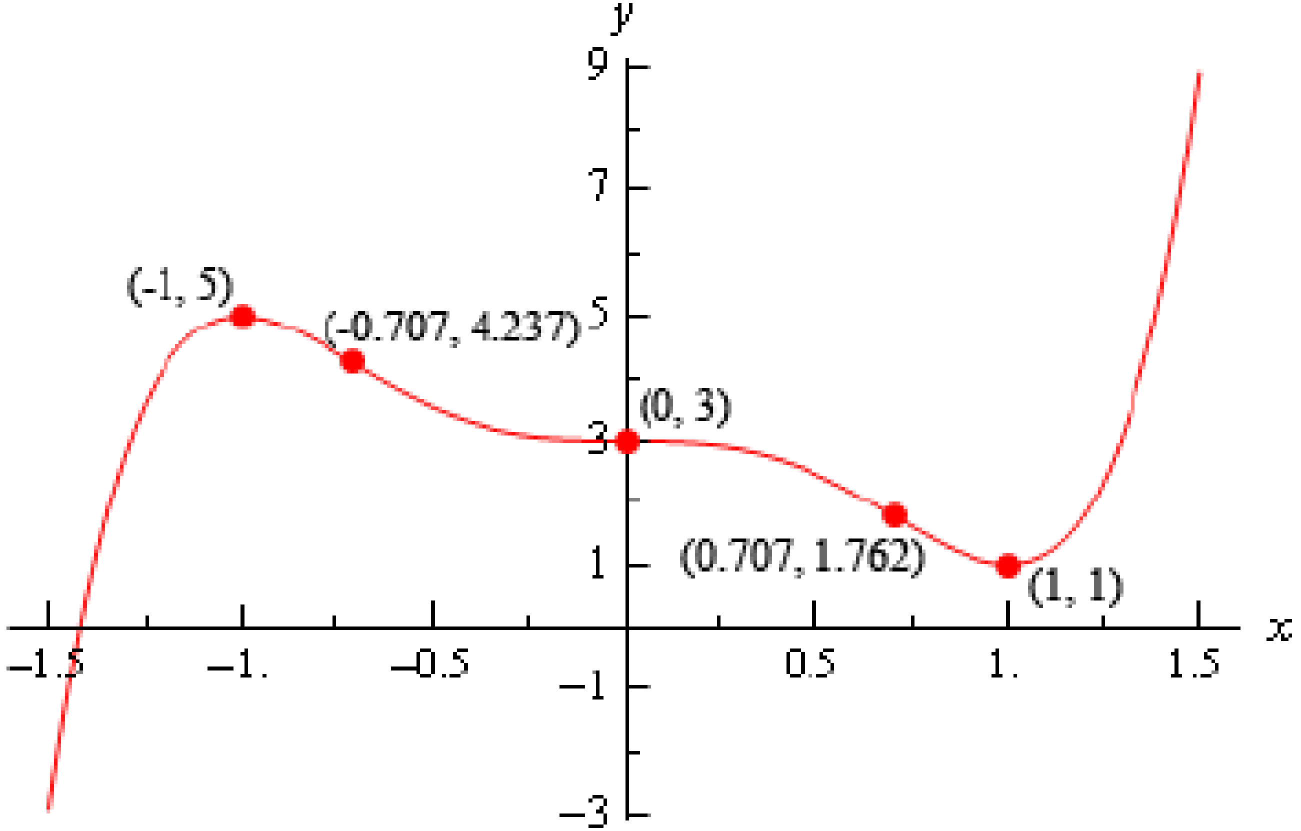
For the following function, $h(x) = 3x^5 - 5x^3 + 3$

a) Identify the intervals where the function is increasing and decreasing.

b) Identify the intervals where the function is concave up and concave down.

c) Use the Information to sketch the graph.

Using all this information to sketch the graph gives the following graph.



The shape of the graph-part2

Second Derivative Test

Suppose that $x = c$ is a critical point of $f'(c)$ such that $f'(c) = 0$ and that $f''(x)$ is continuous in a region around $x = c$. Then,

1. If $f''(c) < 0$ then $x = c$ is a relative maximum.
2. If $f''(c) > 0$ then $x = c$ is a relative minimum.
3. If $f''(c) = 0$ then $x = c$ can be a relative maximum, relative minimum or neither.

The shape of the graph-part2

TRY THIS OUT!!

1. Use the second derivative test to classify the critical points of the function below.

$$h(x) = 3x^5 - 5x^3 + 3$$

2. For the following function, find the inflection points and use the second derivative test, if possible, to classify the critical points. Also determine the intervals of increase/decrease and the intervals of concave up/concave down and sketch the graph of the function.

$$f(t) = t(6-t)^{\frac{2}{3}}$$

Indeterminate Forms and L'Hospital's Rule

- If we have limits as stated below, we can evaluate them as follows:

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4}(x + 4) = 8$$

$$\lim_{x \to \infty} \frac{4x^2 - 5x}{1 - 3x^2} = \lim_{x \to \infty} \frac{4 - \frac{5}{x}}{\frac{1}{x^2} - 3} = -\frac{4}{3}$$

- However, what about the following two limits.

$$\lim_{x \to 0} \frac{\sin x}{x} \qquad \lim_{x \to \infty} \frac{e^x}{x^2}$$

- This first is a 0/0 indeterminate form, but we can’t factor this one. The second is an ∞/∞ indeterminate form, but we can’t just factor an x^2 out of the numerator. So, nothing that we’ve got in our bag of tricks will work with these two limits.
- This is where the subject of **L'Hospital's Rule** comes into play.

Indeterminate Forms and L'Hospital's Rule

Suppose that we have one of the following cases,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad \text{OR} \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

- So, L'Hospital's Rule tells us that if we have an indeterminate form 0/0 or ∞/∞ all we need to do is differentiate the numerator and differentiate the denominator and then take the limit.

Indeterminate Forms and L'Hospital's Rule

Example 1 Example 1 Evaluate each of the following limits.

(a) $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

(b) $\lim\limits_{t \to 1} \dfrac{5t^4 - 4t^2 - 1}{10 - t - 9t^3}$

$$\lim_{t \to 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3} = \lim_{t \to 1} \frac{20t^3 - 8t}{-1 - 27t^2} = \frac{20 - 8}{-1 - 27} = -\frac{3}{7}$$

(c) $\lim\limits_{x \to \infty} \dfrac{\mathbf{e}^x}{x^2}$

$$\lim_{x \to \infty} \frac{\mathbf{e}^x}{x^2} = \lim_{x \to \infty} \frac{\mathbf{e}^x}{2x} = \lim_{x \to \infty} \frac{\mathbf{e}^x}{2} = \infty$$

- Sometimes we will need to apply L’Hospital’s Rule more than once.

Optimization Problem

- Here, we are looking for the largest value or the smallest value that a function can take subject to some kind of constraint.

- The constraint will be some condition (that can usually be described by some equation) that must absolutely, positively be true no matter what our solution is.

- This part is generally difficult mainly because:
  - a slight change of wording can completely change the problem.
  - Identifying the quantity that we'll be optimizing and the quantity that is the constraint and writing down equations for each is also a problem.

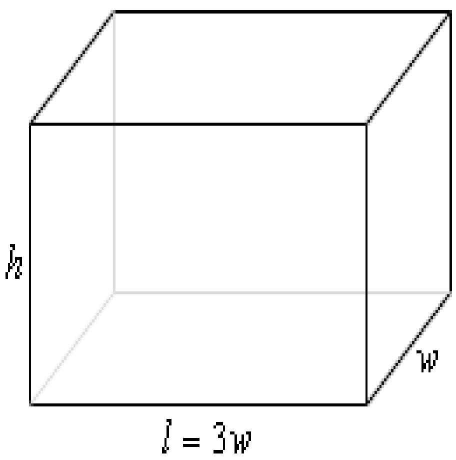
Optimization Problem

- The steps in this part involves:
  - Carefully reading the problem
  - Identify the quantity to be optimized and the constraint. Remember the constraint is the quantity that must be true regardless of the solution.

Optimization Problem

- ***Example 1*** We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $\$10/ft^2$ and the material used to build the sides cost $\$6/ft^2$. If the box must have a volume of $50ft^3$ determine the dimensions that will minimize the cost to build the box.

- Start with a sketch:



Differentials

- Given a function $y = f(x)$ we call dy and dx differentials and the relationship between them is given by,

$$dy = f'(x)\,dx$$

- Note that if we are just given $f(x)$ then the differentials are df and dx and we compute them in the same manner.

$$df = f'(x)\,dx$$

Example 1 Compute the differential for each of the following.

(a) $y = t^3 - 4t^2 + 7t$ $\qquad dy = \left(3t^2 - 8t + 7\right) dt$

(b) $w = x^2 \sin(2x)$ $\qquad dw = \left(2x\sin(2x) + 2x^2\cos(2x)\right) dx$

(c) $f(z) = \mathbf{e}^{3-z^4}$ $\qquad df = -4z^3\mathbf{e}^{3-z^4}\,dz$

Differentials

- If Δx is the change in x then $\Delta y = f(x+ \Delta x) - f(x)$ is the corresponding change in y. Therefore, if Δx is small, we assume $\Delta y \approx dy$.

Example:

A sphere was measured and its radius was found to be 45 inches with a possible error of no more than 0.01 inches. What is the maximum possible error in the volume if we use this value of the radius?

Differentials

Example 2 Compute dy and Δy if $y = \cos\left(x^2 + 1\right) - x$ as x changes from $x = 2$ to $x = 2.03$.

First let's compute actual the change in y, Δy.

$$\Delta y = \cos\left(\left(2.03\right)^2 + 1\right) - 2.03 - \left(\cos\left(2^2 + 1\right) - 2\right) = 0.083581127$$

Now let's get the formula for *dy*.

$$dy = \left(-2x\sin\left(x^2 + 1\right) - 1\right) dx$$

Next, the change in x from $x = 2$ to $x = 2.03$ is $\Delta x = 0.03$ and so we then assume that $dx \approx \Delta x = 0.03$. This gives an approximate change in y of,

$$dy = \left(-2\left(2\right)\sin\left(2^2 + 1\right) - 1\right)\left(0.03\right) = 0.085070913$$

We can see that in fact we do have that $\Delta y \approx dy$ provided we keep Δx small.

Self Study

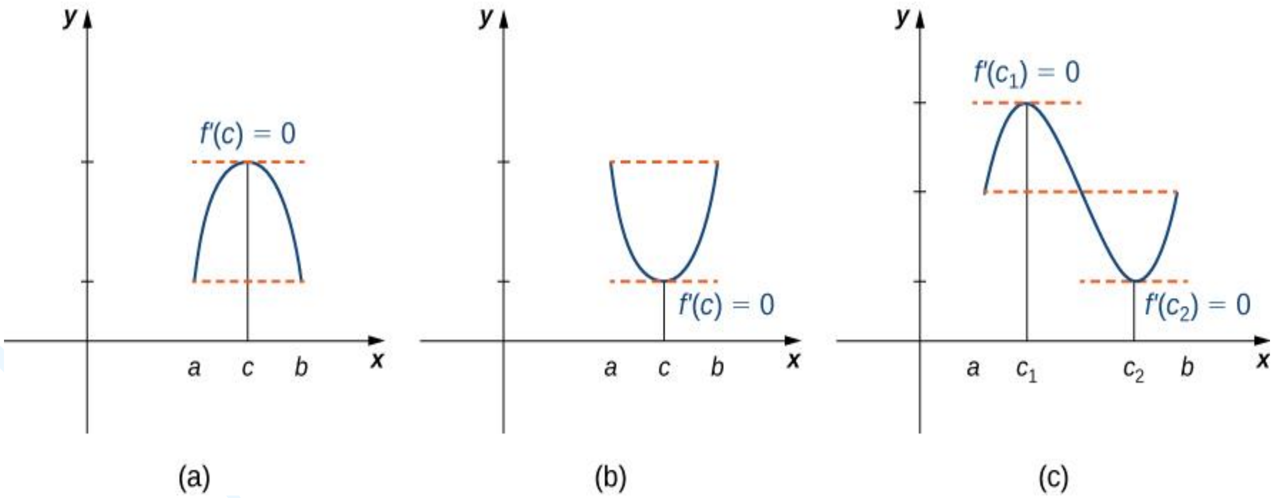
- For the remaining application of derivatives, use the textbook provided to read and understand them.
  - The mean value theorem
  - Linear Approximation
  - Newton's method
  - Business Application.

  NOTE:

  - Try to go through as many examples as you can.

Mean Value Theorem: Rolle's Theorem

- **Rolle's Theorem** is a special case of the Mean Value Theorem
- Informally, **Rolle's theorem** states that if the outputs of a differentiable function *f* are equal at the endpoints of an interval, then there must be an interior point *c* where $f'(c) = 0$.



- If a differentiable function *f* satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between *a* and *b*.

Rolle's Theorem

Definition

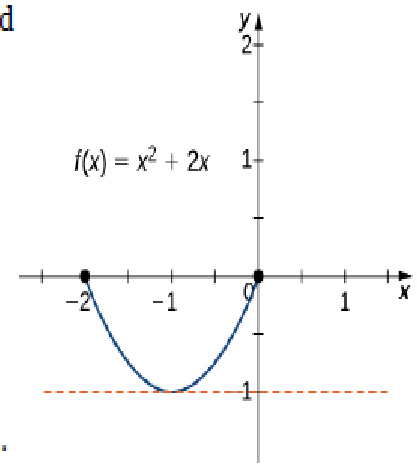
Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Using Rolle's Theorem

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0$.

a. $f(x) = x^2 + 2x$ over $[-2, 0]$

b. $f(x) = x^3 - 4x$ over $[-2, 2]$

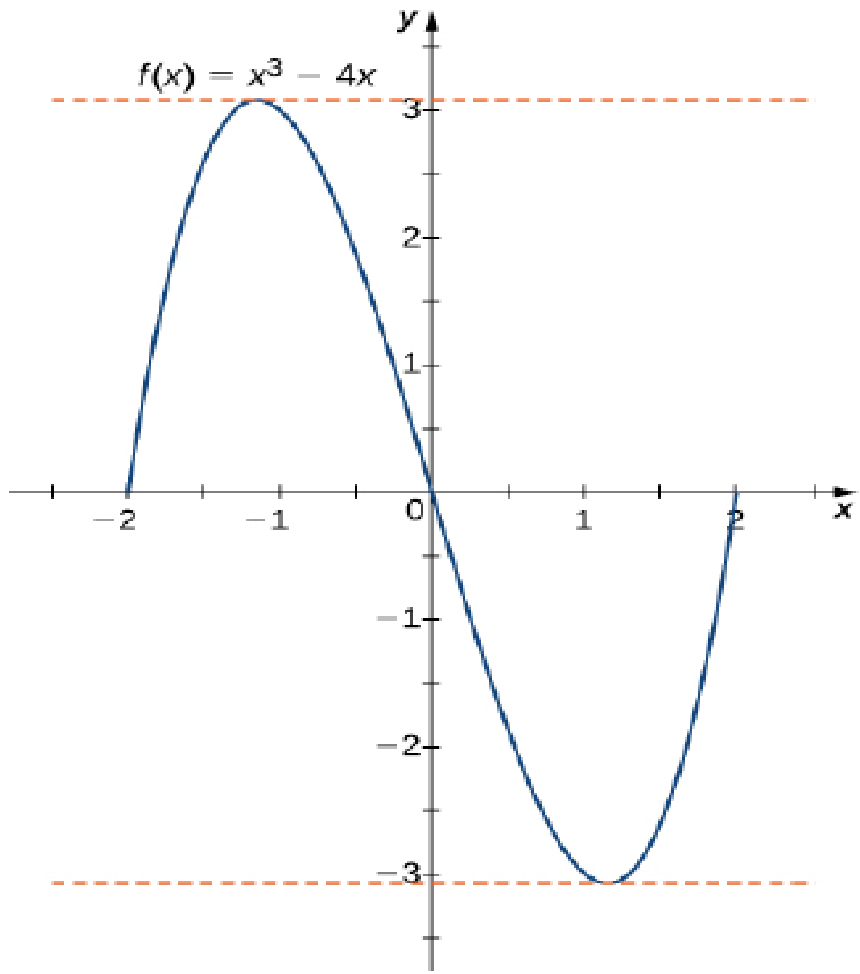


Solution

a. Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0)$. Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0$. Since $f'(x) = 2x + 2 = 2(x + 1)$, we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in the following graph.

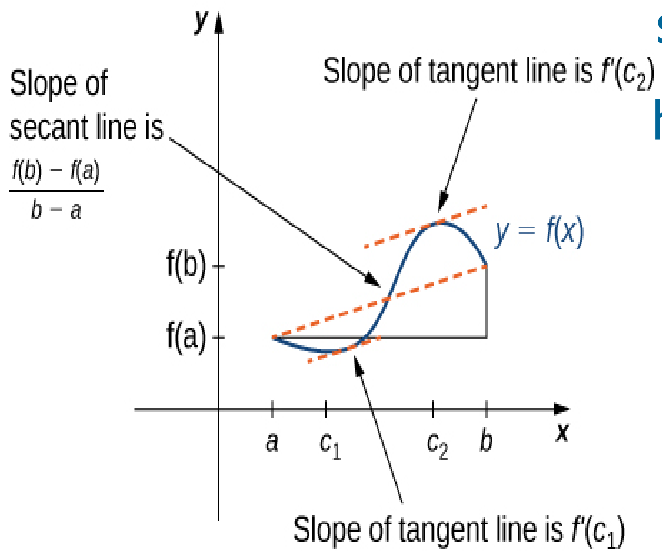
Rolle's Theorem

b. As in part a. f is a polynomial and therefore is continuous and differentiable everywhere. Also, $f(-2) = 0 = f(2)$. That said, f satisfies the criteria of Rolle's theorem. Differentiating, we find that $f'(x) = 3x^2 - 4$. Therefore, $f'(c) = 0$ when $x = \pm\frac{2}{\sqrt{3}}$. Both points are in the interval $[-2, 2]$, and, therefore, both points satisfy the conclusion of Rolle's theorem as shown in the following graph.



The Mean Value Theorem

- The Mean Value Theorem generalizes Rolle's theorem by considering functions that do not necessarily have equal value at the endpoints.
- The Mean Value Theorem states that if *f* is continuous over the closed interval [*a*, *b*] and differentiable over the open interval s a point $c \in (a, b)$ such that the h of *f* at *c* is parallel to the secant and *(b, f(b))*



The Mean Value Theorem

Definition

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 1 Determine all the numbers *c* which satisfy the conclusions of the Mean Value Theorem for the following function.

$$f(x) = x^3 + 2x^2 - x \quad \text{on} \quad [-1, 2]$$

Solution

First let's find the $f'(x) = 3x^2 + 4x - 1$

Now, to find the numbers that satisfy the conclusions of the Mean Value Theorem all we need to do is plug this into the formula given by the Mean Value Theorem.

The Mean Value Theorem

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$3c^2 + 4c - 1 = \frac{14 - 2}{3} = \frac{12}{3} = 4$$

Now, this is just a quadratic equation,

$$3c^2 + 4c - 1 = 4$$

$$3c^2 + 4c - 5 = 0$$

Using the quadratic formula on this we get,

$$c = \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{6} = \frac{-4 \pm \sqrt{76}}{6}$$

So, solving gives two values of c.

$$c = \frac{-4 + \sqrt{76}}{6} = 0.7863 \qquad c = \frac{-4 - \sqrt{76}}{6} = -2.1196$$

The number that we're after in this problem is,

$$c = 0.7863$$

The Mean Value Theorem

Example 2 Suppose that we know that $f(x)$ is continuous and differentiable on [6,15]. Let's also suppose that we know that $f(6) = -2$ and that we know that $f'(x) \leq 10$. What is the largest possible value for $f(15)$?

Solution

Let's start with the conclusion of the Mean Value Theorem.

$$f(15) - f(6) = f'(c)(15-6)$$

Plugging in for the known quantities and rewriting this a little gives,

$$f(15) = f(6) + f'(c)(15-6) = -2 + 9f'(c)$$

Now we know that $f'(x) \leq 10$ so in particular we know that $f'(c) \leq 10$. This gives us the following,

$$\begin{aligned} f(15) &= -2 + 9f'(c) \\ &\leq -2 + (9)10 \\ &= 88 \end{aligned}$$

All we did was replace $f'(c)$ with its largest possible value.

This means that the largest possible value for $f(15)$ is 88.

Exercise: Mean Value theorem

5. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7,0]$, that $f(-7)=-3$ and that $f'(x)\le 2$. What is the largest possible value for $f(0)$?

6. Show that $f(x)=x^3-7x^2+25x+8$ has exactly one real root.

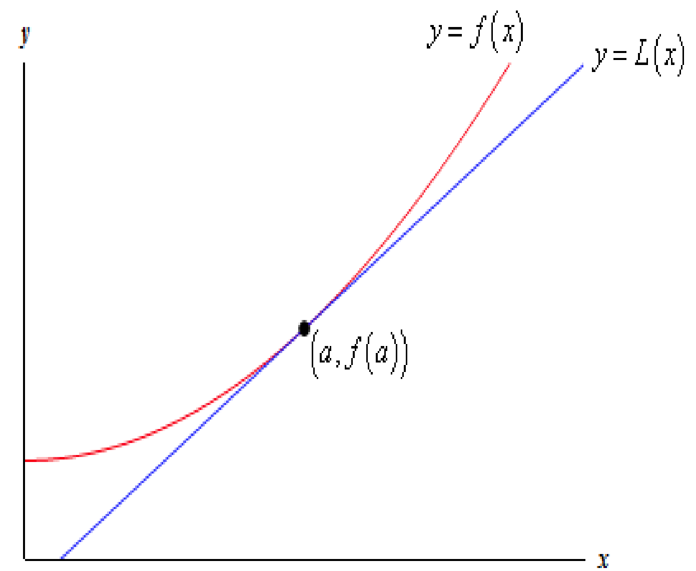
Linear Approximations

- In Linear approximation we take a look at an application not of derivatives but of the tangent line to a function. Of course, to get the tangent line we do need to take derivatives, so in some way this is an application of derivatives as well.
- Given a function, $f(x)$, we can find its tangent at $x = a$. The equation of the tangent line, which we'll call *L(x)* for this discussion, is,

$$L(x) = f(a) + f'(a)(x - a)$$

- Take a look at the following graph of a function and its tangent line.

From this graph we can see that near $x = a$ the tangent line and the function have nearly the same graph. On occasion we will use the tangent line, $L(x)$, as an approximation to the function, $f(x)$, near $x = a$. In these cases we call the tangent line the **linear approximation** to the function at $x = a$.



Linear Approximations

- ***Example 1*** Determine the linear approximation for $f(x)=\sqrt[3]{x}$ at $x=8$. Use the linear approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$.

Since this is just the tangent line there really isn't a whole lot to finding the linear approximation.

$$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{x^2}} \qquad f(8)=2 \quad f'(8)=\frac{1}{12}$$

The linear approximation is then,

$$L(x)=2+\frac{1}{12}(x-8)=\frac{1}{12}x+\frac{4}{3}$$

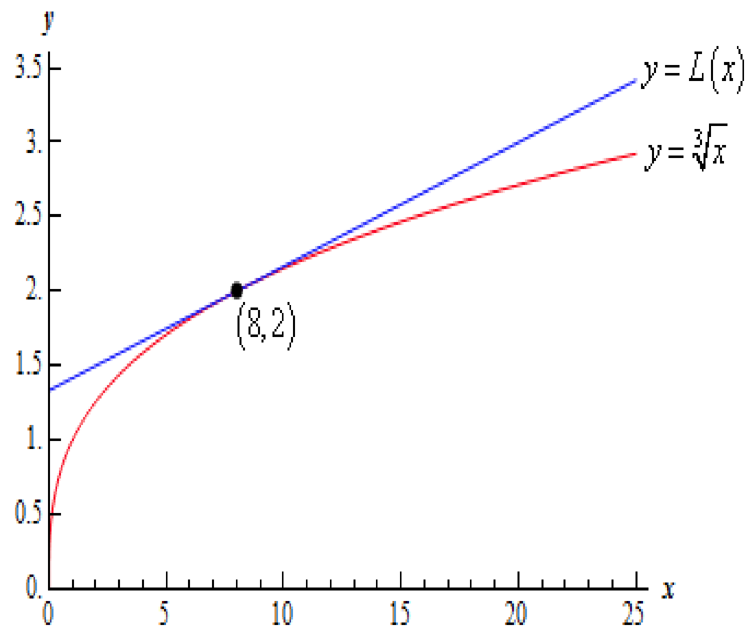
Now, the approximations are nothing more than plugging the given values of x into the linear approximation. For comparison purposes we'll also compute the exact values.

$$L(8.05)=2.00416667 \qquad \sqrt[3]{8.05}=2.00415802$$
$$L(25)=3.41666667 \qquad \sqrt[3]{25}=2.92401774$$

So, at $x=8.05$ this linear approximation does a very good job of approximating the actual value. However, at $x=25$ it doesn't do such a good job.

Linear Approximations

Here's a quick sketch of the function and its linear approximation at x=8



As noted above, the farther from $x = 8$ we get the more distance separates the function itself and its linear approximation.

Linear Approximations (Exercise)

- For problems 1 & 2 find a linear approximation to the function at the given point.

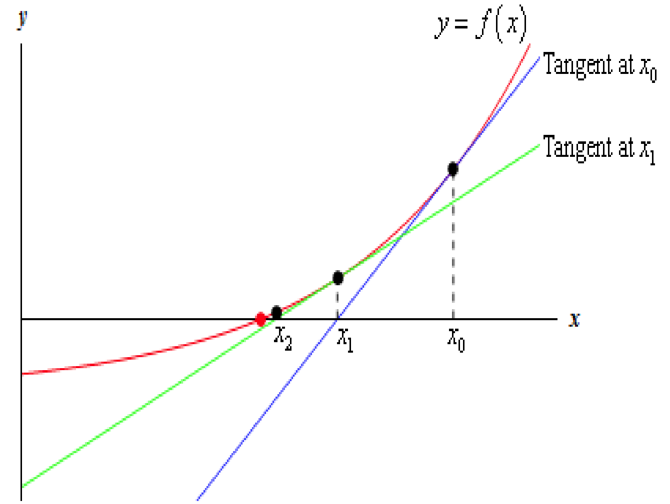
  1. $f(x) = 33x^{e^{2x-10}}$ at $x = 5$
  2. $h(t) = t^4 - 6t^3 + 3t - 7$ at $t = -3$
  3. Find the linear approximation to $g(z) = \sqrt[4]{z}$ at $z = 2$. Use the linear approximation to approximate the value of $\sqrt[4]{3}$ and $\sqrt[4]{10}$. Compare the approximated values to the exact values.
  4. Find the linear approximation to $f(t) = \cos(2t)$ at $t = 12$. Use the linear approximation to approximate the value of $\cos(2)$ and $\cos(18)$. Compare the approximated values to the exact values.

Newton's Method

- **Newton's Method** looks at a method for approximating solutions to equations.
- Let's suppose that we want to approximate the solution to $f(x) = 0$ and let's also suppose that we have somehow found an initial approximation to this solution say, x_0. This initial approximation is probably not all that good, in fact it may be nothing more than a quick guess we made, and so we'd like to find a better approximation. This is easy enough to do.
- First, we will get the tangent line to $f(x)$ at x_0.

$$y = f(x_0) + f'(x_0)(x - x_0)$$

- Now, take a look at the graph below.



Newton's Method

- The blue line is the tangent line at x_0. We can see that this line will cross the x-axis much closer to the actual solution to the equation than x_0 does. Let's call this point where the tangent at x_0 crosses the x-axis x_1 and we'll use this point as our new approximation to the solution.
- So, how do we find this point? Well we know it's coordinates, $(x_1,0)$ and we know that it's on the tangent line so plug this point into the tangent line and solve for x_1 as follows,

$$\begin{aligned} 0 &= f(x_0) + f'(x_0)(x_1 - x_0) \\ x_1 - x_0 &= -\frac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \end{aligned}$$

- So, we can find the new approximation provided the derivative isn't zero at the original approximation.
- Now we repeat the whole process to find an even better approximation. We form up the tangent line to $f(x)$ at x_1 and use its root, which we'll call x_1, as a new approximation to the actual solution. If we do this we will arrive at the following formula.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Newton's Method

If x_n is an approximation a solution of $f(x) = 0$ and if $f'(x_n) \neq 0$ the next approximation is given by,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example 1: Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval [1, 2]. Let $x_0 = 2$ and find $x_1, x_2, x_3, x_4,$ and x_5.

Solution

To find the next approximation, we use $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Since $f(x) = x^3 - 3x + 1$, the derivative is $f'(x) = 3x^2 - 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{9} \approx 1.666666667.$$

To find the next approximation, x_2,

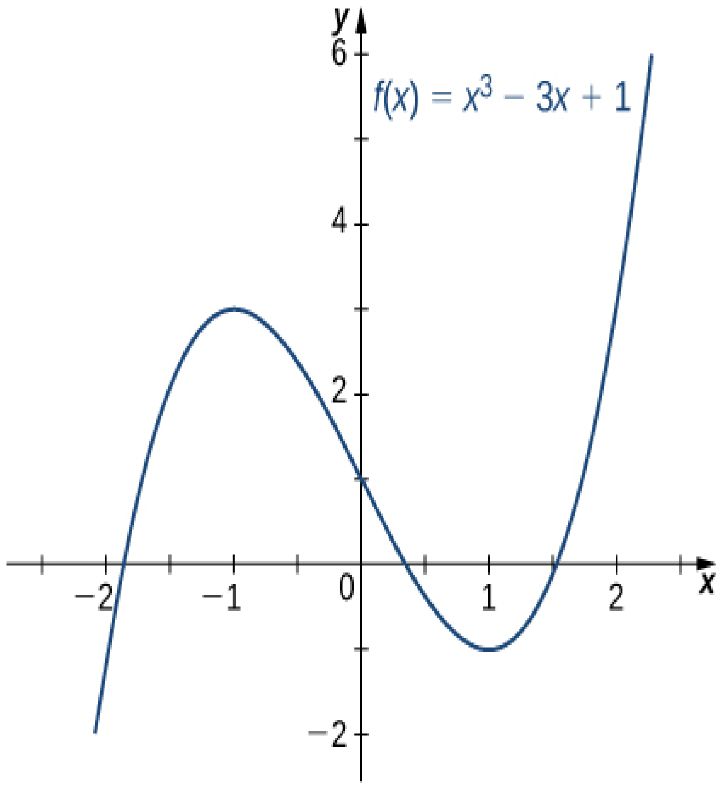
$$x_2 = x_1 = \frac{f(x_1)}{f'(x_1)} \approx 1.548611111.$$

Newton's Method

- Continuing in this way, we obtain the following results:

$$x_1 \approx 1.666666667$$
$$x_2 \approx 1.548611111$$
$$x_3 \approx 1.532390162$$
$$x_4 \approx 1.532088989$$
$$x_5 \approx 1.532088886$$
$$x_6 \approx 1.532088886.$$

- We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's method will most likely give the same value for x_n.



Newton's Method

- ***Example 1*** Use Newton's Method to determine an approximation to the solution to cosx=x that lies in the interval [0,2]. Find the approximation to six decimal places.
- Next, recall t ... ust have the function in the form f(x ... , we first rewrite the equation as,

$$\cos x - x = 0$$

$$x_{n+1} = x_n - \frac{\cos x - x}{-\sin x - 1}$$

- Let's nov ... roximation.

$$x_1 = 1 - \frac{\cos(1) - 1}{-\sin(1) - 1} = 0.7503638679$$

$$x_2 = 0.7503638679 - \frac{\cos(0.7503638679) - 0.7503638679}{-\sin(0.7503638679) - 1} = 0.7391128909$$

$$x_3 = 0.7390851334$$

$$x_4 = 0.7390851332$$

Exercise: Newton's Method

For problems 1 & 2 use Newton's Method to determine x_2 for the given function and given value of x_0.

1. $f(x) = x^3 - 7x^2 + 8x - 3$, $x_0 = 5$

2. $f(x) = x\cos(x) - x^2$, $x_0 = 1$

For problems 3 & 4 use Newton's Method to find the root of the given equation, accurate to six decimal places, that lies in the given interval.

3. $x^4 - 5x^3 + 9x + 3 = 0$ in $[4, 6]$

4. $2x^2 + 5 = e^x$ in $[3, 4]$

Business Applications

- Let's take a look at some applications of derivatives in the business world.

Example 1 An apartment complex has 250 apartments to rent. If they rent x apartments then their monthly profit, in dollars, is given by,

$$P(x) = -8x^2 + 3200x - 80,000$$

All that we're really being asked to do here is to maximize the profit subject to the constraint that x must be in the range $0 \le x \le 250$.

First, we'll need the derivative and the critical point(s) that fall in the range $0 \le x \le 250$.

$$P'(x) = -16x + 3200 \quad \Rightarrow \quad 3200 - 16x = 0 \quad , \Rightarrow \quad x = \frac{3200}{16} = 200$$

$$P(0) = -80,000 \qquad P(200) = 240,000 \qquad P(250) = 220,000$$

So, it looks like they will generate the most profit if they only rent out 200 of the apartments instead of all 250 of them.

Exercise: Business Applications

1. A company can produce a maximum of 1500 widgets in a year. If they sell x widgets during the year then their profit, in dollars, is given by,

$$P(x) = 30,000,000 - 360,000x + 750x^2 - \tfrac{1}{3}x^3$$

How many widgets should they try to sell in order to maximize their profit?

2. A management company is going to build a new apartment complex. They know that if the complex contains x apartments the maintenance costs for the building, landscaping etc. will be,

$$C(x) = 4000 + 14x - 0.04x^2$$

The land they have purchased can hold a complex of at most 500 apartments. How many apartments should the complex have in order to minimize the maintenance costs?

3. The production costs, in dollars, per day of producing x widgets is given by,

$$C(x) = 1750 + 6x - 0.04x^2 + 0.0003x^3$$

What is the marginal cost when $x = 175$ and $x = 300$? What do your answers tell you about the production costs?