Continuity

Definitions

- ***Continuity at a point***: A function *f* is continuous at *c* if the following three conditions are met:
  1. $f(c)$ **is defined**
  2. $\lim_{x \to c} f(x)$ **exists**
  3. $\lim_{x \to c} f(x) = f(c)$

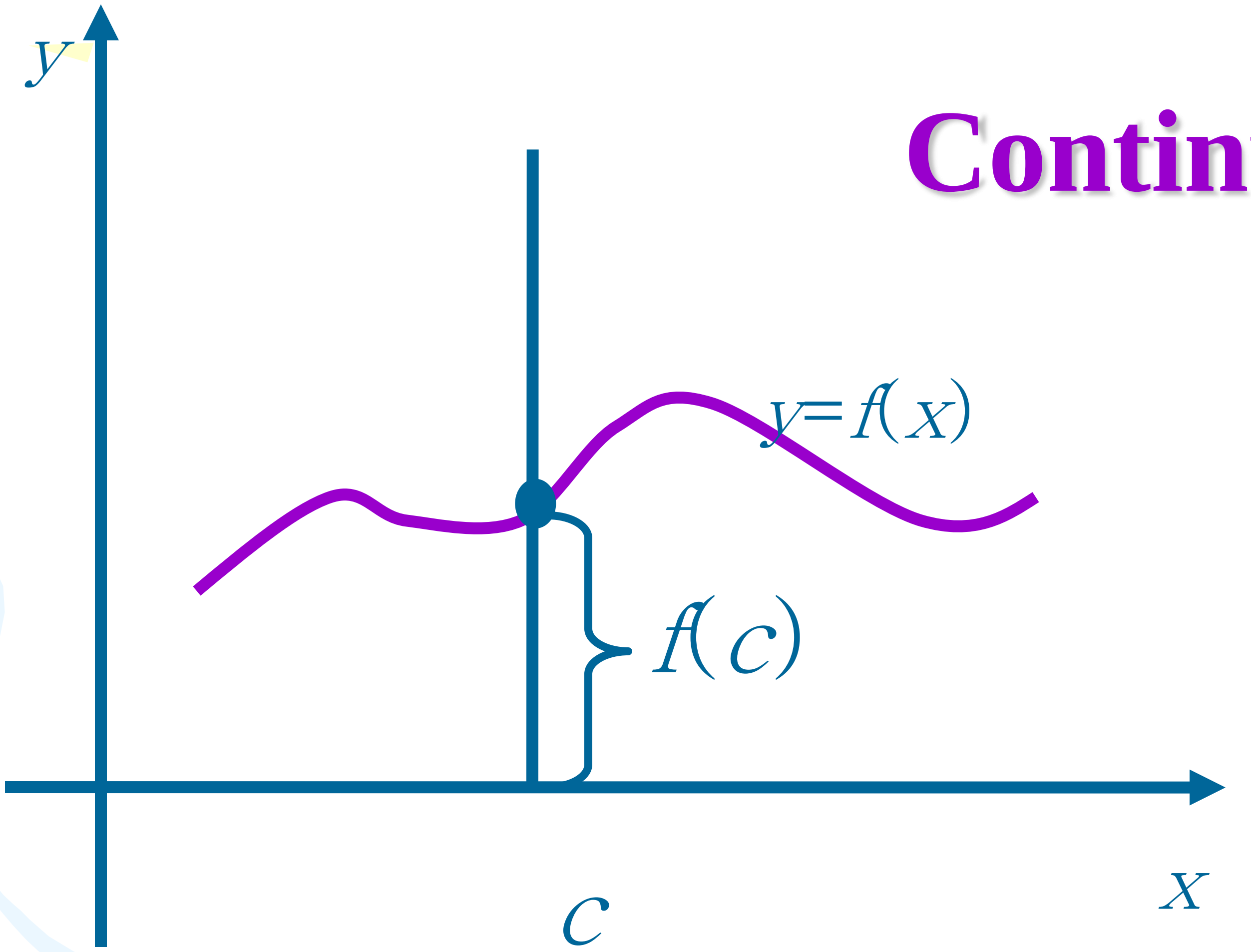
Definitions

- ***Continuity on an open interval:*** A function is continuous on an open interval (a,b) if it is continuous at each point in the interval.
- A function that is continuous on the entire real line $(-\infty,\infty)$ is everywhere continuous.

Definitions

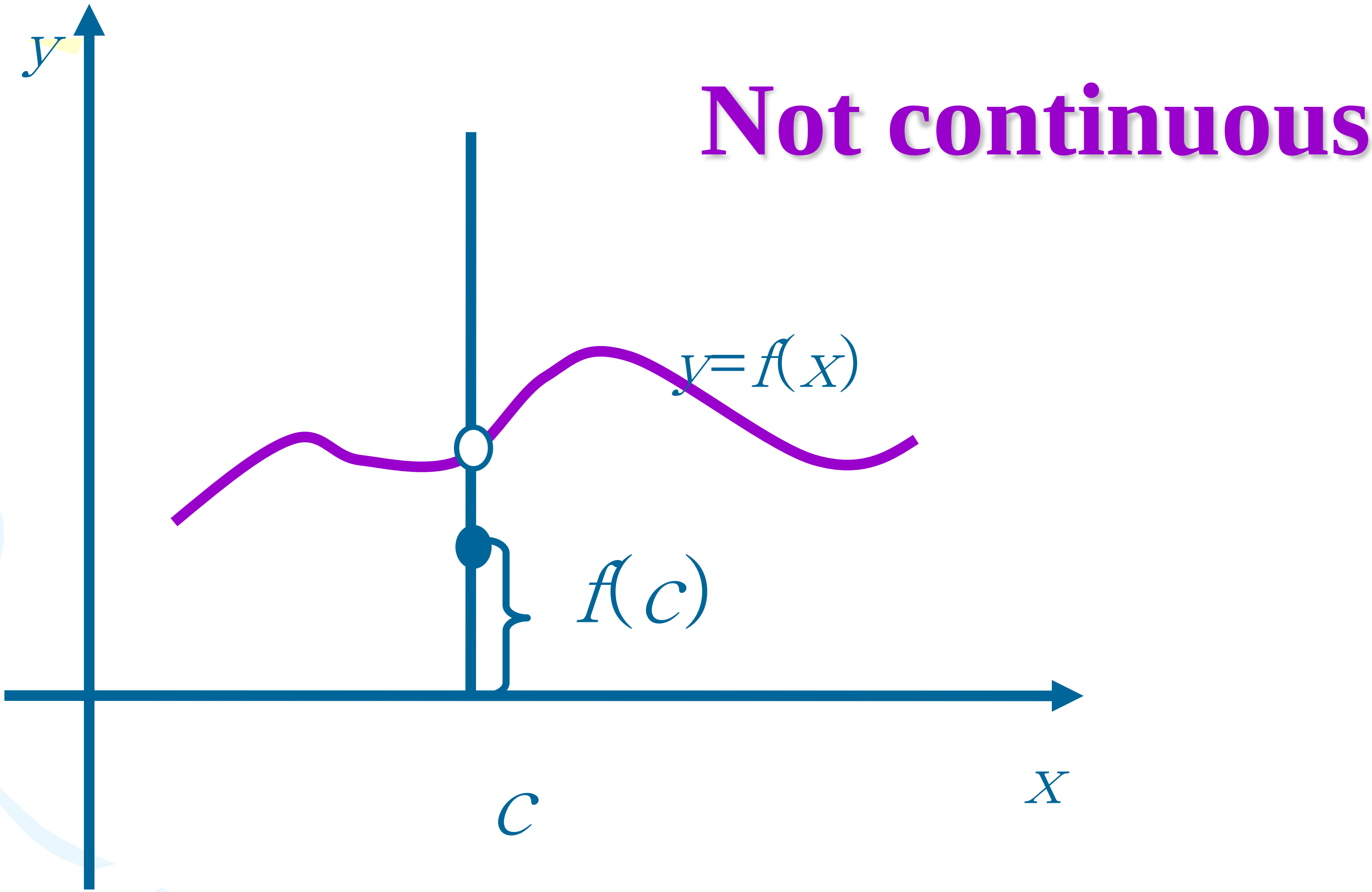
- A function is continuous on a closed interval $[a,b]$ if it is continuous on the open interval (a,b) and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$

Continuous



$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$, so the limit $\lim_{x \to c} f(x)$ exists

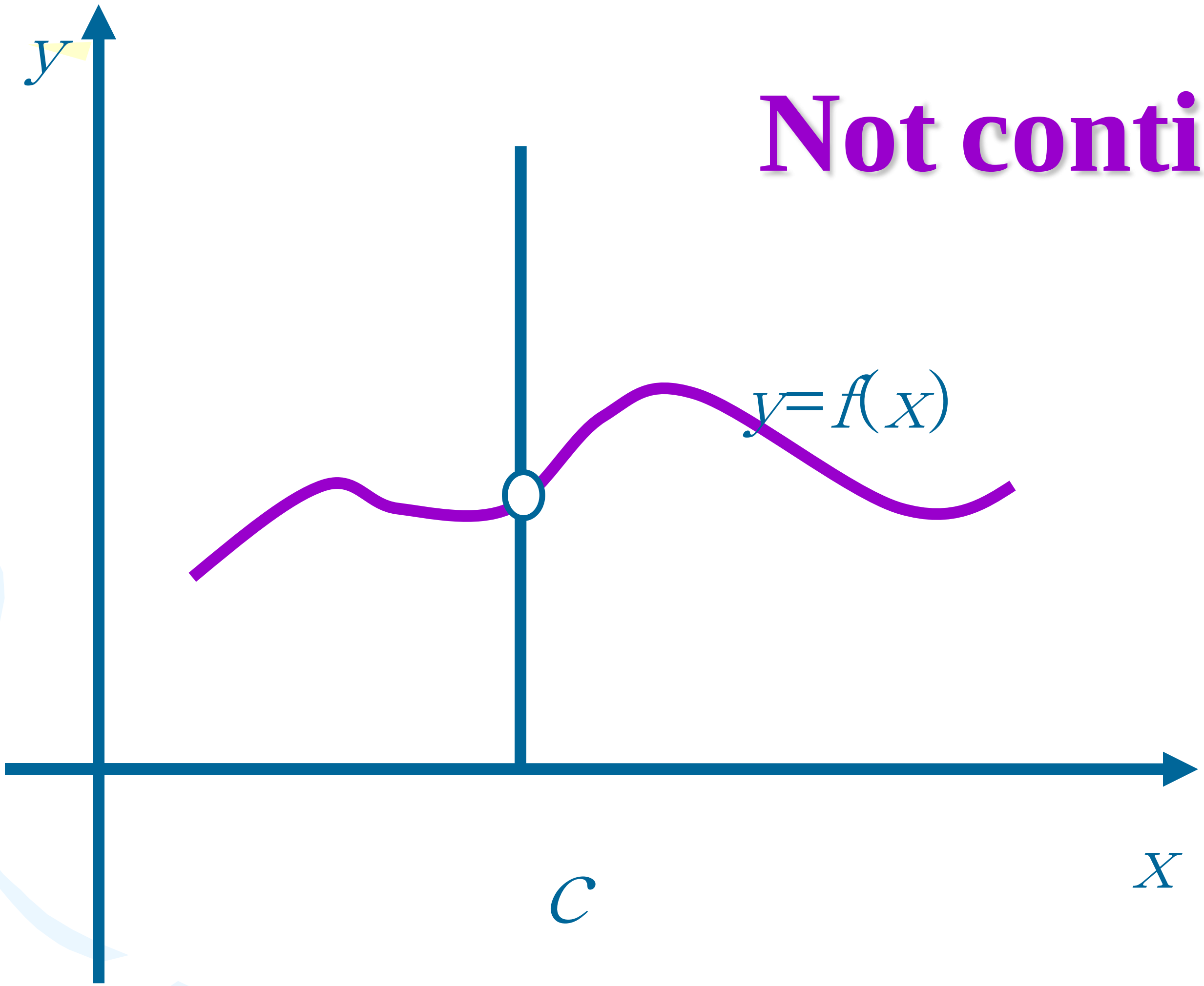
$\lim_{x \to c} f(x) = f(c)$



$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$, so the limit $\lim_{x \to c} f(x)$ exists;

$\lim_{x \to c} f(x) \neq f(c)$

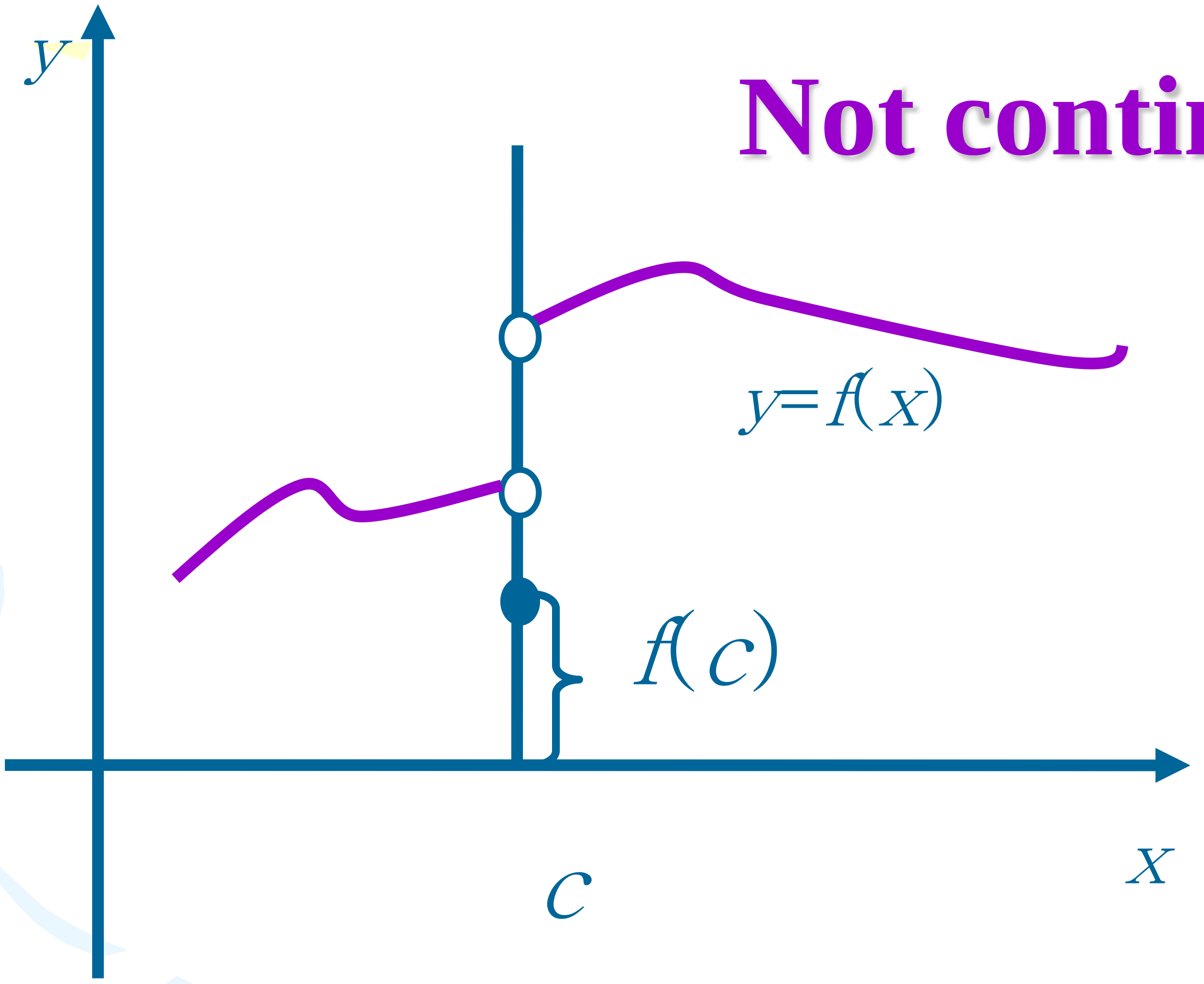
Not continuous



$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$, so the limit $\lim_{x \to c} f(x)$ exists;

$f(c)$ is not defined

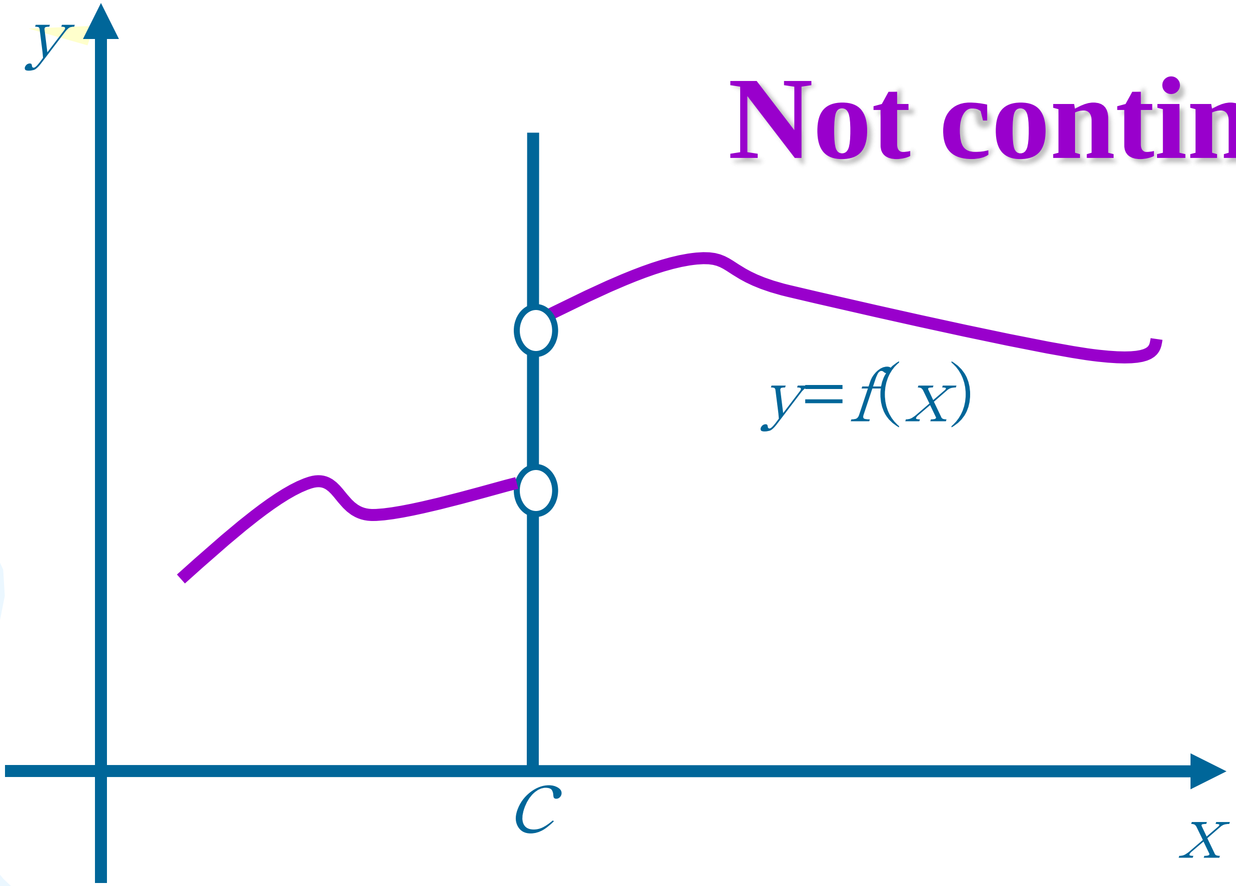
Not continuous



$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, so the

limit $\lim_{x \to c} f(x)$ does not exist ; $f(c)$ is defined

Not continuous



$\lim\limits_{x \to c^-} f(x) \neq \lim\limits_{x \to c^+} f(x)$, so the

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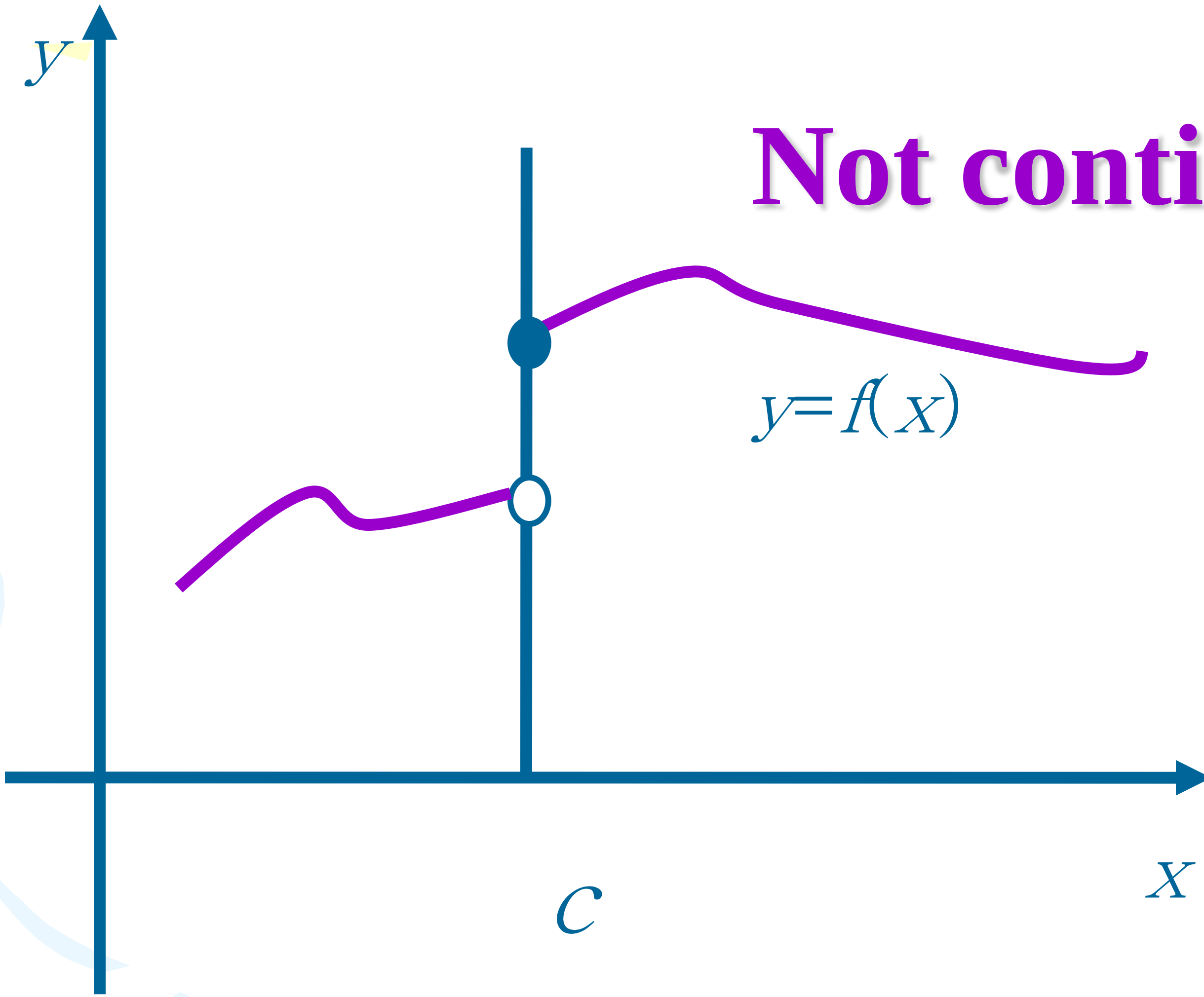


$\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x) = f(c)$, so the limit $\lim_{x\to c} f(x)$ does not exist ; $f(c)$ is defined

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that,

1. $a < c < b$
2. $f(c) = M$

The Intermediate Value Theorem means that a function, continuous on an interval, takes any value between any two values that it takes on that interval. A continuous function cannot grow from being negative to positive without taking the value 0.

One use of the theorem is in locating roots of equations.

Intermediate Value Theorem

Example 1

Show that there is a root of the equation

$4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$

We are looking for a solution of the given equation that is, a number c between 1 and 2 such that $f(c) = 0$.

Therefore, we take $a = 1$, $b = 2$, and $M = 0$ in the theorem.

We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Intermediate Value Theorem

Thus, $f(1) < 0 < f(2)$ that is, $M = 0$ is a number between $f(1)$ and $f(2)$.

Now, f is continuous since it is a polynomial.

So, the theorem states that there is a number c between 1 and 2 such that $f(c) = 0$.

In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root in the interval [1, 2].

Intermediate Value Theorem

Example 2 Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval [-1,2].

Definition

Discontinuity

If a function is not continuous at a point c, then c is called a point of discontinuity.

Types of Discontinuities

- Removable
- Non-removable
   - jump
   - oscillating
   - infinite

Types of Discontinuities

- **Removable Discontinuities** – <u>can</u> be "repaired"
  - ***Hole*** (factor can be "factored out" of the denominator)
- **Essential Discontinuities** – <u>cannot</u> be "repaired"
  - ***Jumps*** (usually found in piecewise functions)
  - ***Asymptotes*** (can't remove a factor/problem in the denominator) --- (like 1/x)
  - ***Wildly oscillating functions*** – (like graph 1/sin(x))

Removable Discontinuity

Example 1: Is $f(x)=\dfrac{x^2-9}{x-3}$. continuous at x=3?

Taking the limit of the function as *x* approaches 3.

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}x+3=6 \quad \text{but} \quad f(3)\neq 6$$

- Hence *f(x)* is **not continuous** or **<u>discontinuous</u>** at *x* = 3. This type of discontinuity is called **<u>removable discontinuity</u>** because we could remove the discontinuity by redefining *f* at the point of discontinuity. This could be done by redefining function *f* in this way:

$$f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & x<>3\\ 6, & x=3\end{cases}$$

Infinite Discontinuity

Example 2 Is $f(x)=\begin{cases}\dfrac{1}{x^3}, & x\neq 0\\ 1, & x=0\end{cases}$ continuous at x=0?

Taking the limit of the function as *x* approaches 0.

We can see that $\lim_{x\to 0} f(x)\neq f(0)$ so this function is discontinuous at *x* = 0.

This type of discontinuity is called **<u>infinite discontinuity.</u>**

Jump Discontinuity

Example 3: Is $g(x) = \begin{cases} 2x, & x < 2 \\ 2, & x \geq 2 \end{cases}$ ***continuous at x=2?***

Taking the limit of the function as x approaches 2.

$\lim_{x \to 2^+} f(x) = 2$ and $\lim_{x \to 2^-} f(x) = 4$

Therefore $\therefore \lim_{x \to 2} f(x)$ does not exist.

This is called a <u>**Jump discontinuity.**</u>

The THREE requirements for a function to be continuous at x=c .

1. C must be in the domain of the function - you can find *f(c)*,
2. The right-hand limit must equal the left-hand limit which means that there is a LIMIT at x=c, and
3. $\lim_{x \to c} f(x) = f(c)$

Properties of Continuity

If b is a real number and f and g are continuous at $x=c$, then the following functions are already continuous at c.

1. bf

Recall: $\lim\limits_{x\to c} bf(x) = b\lim\limits_{x\to c} f(x)$

So if $\lim\limits_{x\to c} f(x) = f(c)$

then $\lim\limits_{x\to c} bf(x) = b\lim\limits_{x\to c} f(x) = bf(c)$ i.e., bf is continuous

Properties of Continuity

If *b* is a real number and *f* and *g* are continuous at $x = c$, then the following functions are already continuous at *c*...

2. $f \pm g$

Recall: $\lim_{x \to c}\left[f(x) \pm g(x)\right] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

So if $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$

then $\lim_{x \to c}\left[f(x) \pm g(x)\right] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) = f(c) + g(c)$

i.e., $f \pm g$ is continuous

Properties of Continuity

If *b* is a real number and *f* and *g* are continuous at $x = c$, then the following functions are also continuous at *c*…

3. fg

Recall: $\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

So if $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$

then $\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = f(c)g(c)$

i.e., fg is continuous

Properties of Continuity

If b is a real number and *f* and *g* are continuous at $x = c$, then the following functions are also continuous at *c*…

4. $\dfrac{f}{g}$

Recall: $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$

So if $\lim\limits_{x\to c} f(x) = f(c)$ and $\lim\limits_{x\to c} g(x) = g(c)$

then $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)} = \dfrac{f(c)}{g(c)}$

i.e., $\dfrac{f}{g}$ is continuous

Properties of Continuity

If g is continuous at c and f is continuous at $g(c)$, then the composite function $(f \circ g)(x)$ is also continuous at c.

Recall: $\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x))$

So if $\lim_{x \to c} g(x) = g(c)$ and $\lim_{x \to g(c)} f(x) = f(g(c))$

then $\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x)) = f(g(c))$ i.e., $(f \circ g)(x)$ is continuous

Continuity

Example 1 Determine whether the function is continuous at (a) x = -1 (b) x = 2.

$$f(x) = \begin{cases} x^2 & x < -1 \\ 2x+3 & -1 \le x < 2 \\ -x+5 & x \ge 2 \end{cases}$$

Is the function continuous at -1?

$$f(x)=\begin{cases} x^2 & x<-1 \\ 2x+3 & -1\le x<2 \\ -x+5 & x\ge 2 \end{cases}$$

(REQ#1) $f(-1) = 2(-1) + 3 = 1$

(REQ#2)
$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^2 = 1$$
$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} 2x+3 = 1$$

$$\text{So, } \lim_{x\to -1} f(x) = 1$$

(REQ#3)
$$f(-1) = \lim_{x\to -1} f(x) = 1$$

f (x) is **CONTINUOUS** at x = -1.

Is f (x) continuous at x = 2?

$$f(x)=\begin{cases} x^2 & x<-1 \\ 2x+3 & -1\le x<2 \\ -x+5 & x\ge 2 \end{cases}$$

(REQ#3) $f(2) = -2 + 5 = 3$

(REQ#2) $$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x+3 = 7$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -x+5 = 3$$

There is **NO LIMIT**

(REQ#3) (no need to test – does not meet requirement #2!)

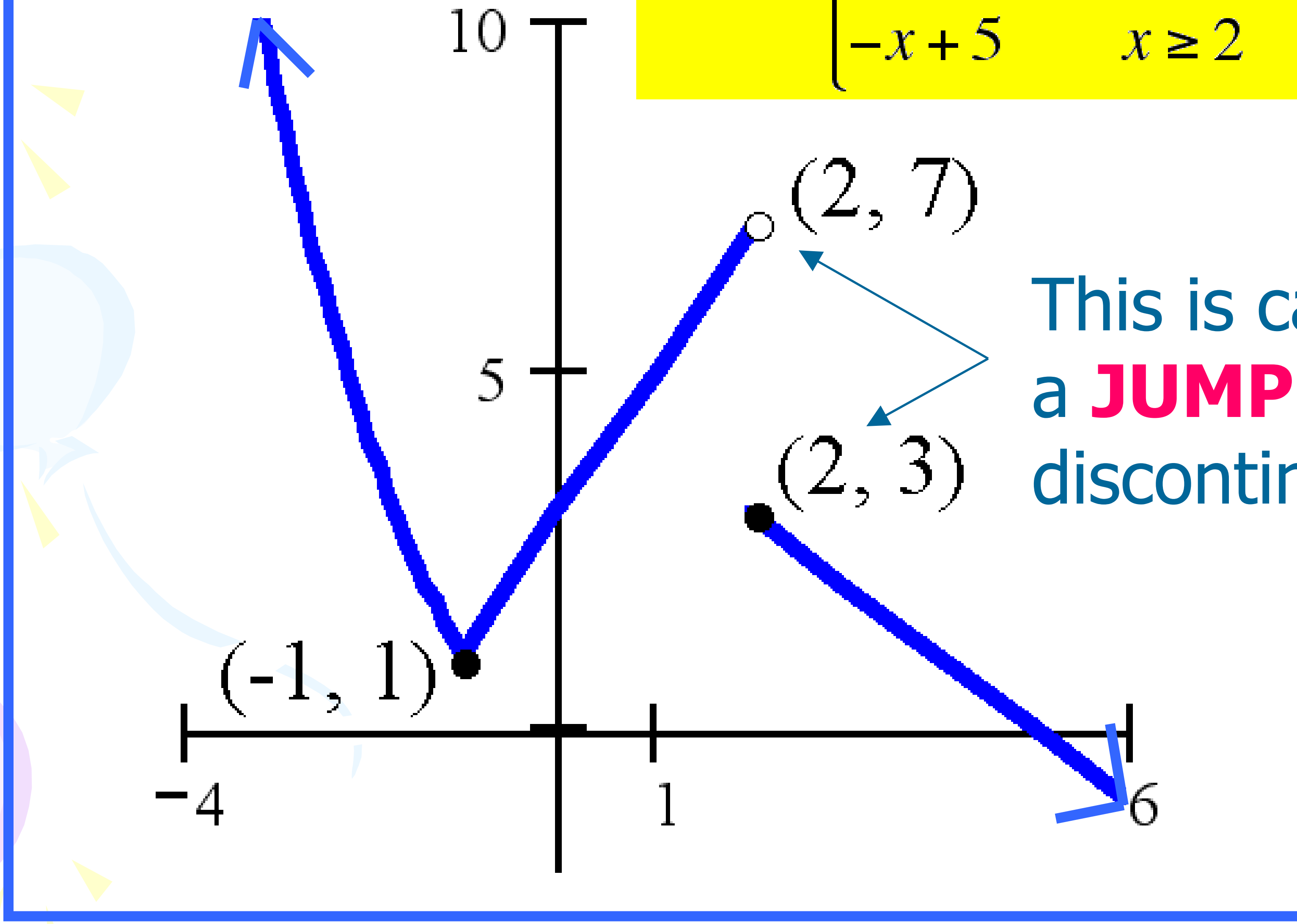
f (x) is **NOT CONTINUOUS** at $x = 2$.

- What does the Graph look like?

$$f(x)=\begin{cases} x^2 & x<-1 \\ 2x+3 & -1\le x<2 \\ -x+5 & x\ge 2 \end{cases}$$

- Graphing this piecewise function, it confirms our conclusions.

$$f(x)=\begin{cases} x^2 & x<-1 \\ 2x+3 & -1\le x<2 \\ -x+5 & x\ge 2 \end{cases}$$



This is called a **JUMP** discontinuity!

Polynomial Functions

Every polynomial function is continuous at every real number.

Functions that are continuous in their respective domains.

Rational Functions

$$R(x) = \frac{p(x)}{q(x)}$$ $\forall x$ for which $q(x) \neq 0$

Power Functions

$$f(x) = \sqrt[n]{x}$$

if n is odd: all real numbers

if n is even: $x > 0$

Functions that are continuous in their respective domains.

Logarithmic Functions

$x > 0$

Exponential Functions

All Real Numbers

Functions that are continuous in their respective domains.

Sin(x) or Cos(x)

All real numbers

Tan(x) or Sec(x)

$$\left\{\theta : \theta \neq \frac{\pi}{2} + n\pi,\ n \text{ any integer}\right\}$$

Cot(x) or Csc(x)

$$\left\{\theta : \theta \neq n\pi,\ n \text{ any integer}\right\}$$

Functions that are continuous in their respective domains…

Arccos(x) or Arcsin(x)

$-1 \leq x \leq 1$

Arctan(x) or arccot(x)

All real numbers

Arcsec(x) or Arccsc (x)

$|x| \geq 1$

Review Questions

Determine if the given function is continuous or discontinuous at the indicated points.

$$g(x)=\begin{cases} 1-3x & x<-6 \\ 7 & x=-6 \\ x^3 & -6<x<1 \\ 1 & x=1 \\ 2-x & x>1 \end{cases}$$

(a) $x=-6$, (b) $x=1$?

$$g(z)=\frac{6}{z^2-3z-10}$$

(a) $z=-2$, (b) $z=0$, (c) $z=5$?

Review Questions

Determine where the given function is discontinuous

$$h(t)=\frac{4t+10}{t^2-2t-15}$$

$$h(z)=\frac{1}{2-4\cos(3z)}$$

$$y(x)=\frac{x}{7-e^{2x+3}}$$