Limits of Functions

Introduction

- Informal definition of limits of functions
- One sided limits
- Techniques and theorems of evaluating limits

Limits

- Definition: **The limit of *f(x)* as *x* approaches *a* is L** and write this as

$$\lim_{x \to a} f(x) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a*, from both sides, without actually letting *x* be *a*.

- Limits are **not concerned with** what is going on at $x = a$.
- *Limits are only concerned with what is going on* ***around x = a.***

One Sided Limits

- **Definitions**

Right-handed limit

We say $\lim_{x \to a^+} f(x) = L$ provided we can make f(x) as close to L as we want for all x sufficiently close to **a** and **x>a** without actually letting x be a.

Left-handed limit

We say $\lim_{x \to a^-} f(x) = L$ provided we can make f(x) as close to L as we want for all x sufficiently close to **a** and **x<a** without actually letting x be a.

One Sided Limits

Example 1 Estimate the value of the following limits.

$$\lim_{t \to 0^+} H(t) \quad \text{and} \quad \lim_{t \to 0^-} H(t) \qquad \text{where, } H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

- ***The right-handed limit (i.e. $t > 0$) will be***

$$\lim_{t \to 0^+} H(t) = 1$$

- ***The left-handed limit (i.e. $t < 0$) will be***

$$\lim_{t \to 0^-} H(t) = 0$$

Relationship between one-sided limits & normal limits

- Given a function *f(x)* if,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

then the normal limit will exist and

$$\lim_{x \to a} f(x) = L$$

- Likewise, if

$$\lim_{x \to a} f(x) = L$$

then,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Properties of Limits

- First we will assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and that c is any constant.

1.Sum&Difference rule: $\lim_{x\to a}[f(x)\pm g(x)]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$

2.Product rule: $\lim_{x\to a}[f(x)g(x)]=\lim_{x\to a} f(x)\lim_{x\to a} g(x)$

3:Quotient rule: $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, ***provided*** $\lim_{x\to a} g(x)\neq 0$

4.Constant Multiple rule: $\lim_{x\to a}[cf(x)]=c\lim_{x\to a} f(x)$

Properties of Limits

5.Power rule: $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n,$ ***where n is any real number .***

6.Root rule: $\lim_{x\to a}[\sqrt[n]{f(x)}] = \sqrt[n]{\lim_{x\to a} f(x)}$

Other properties

7. $\lim_{x\to a} c = c,$ ***where c is a real number***

8. $\lim_{x\to a} x = a$

9.

$$\lim_{x\to a} x^n = a^n$$

Limits

Example 1: Compute the value of the following limit. $\lim_{x \to -2} (3x^2 + 5x - 9)$

solution

We will make great use of the properties in order to find the limit.

$$\lim_{x \to -2} (3x^2 + 5x - 9) = \lim_{x \to -2} 3x^2 + \lim_{x \to -2} 5x - \lim_{x \to -2} 9$$

$$= 3 \lim_{x \to -2} x^2 + 5 \lim_{x \to -2} x - \lim_{x \to -2} 9$$

$$= 3(-2)^2 + 5(-2) - 9$$

$$= -7$$

Limits

- ***Letting p (x) =*** $\lim_{x\to -2}(3x^2+5x-9)$

$$\Rightarrow \lim_{x\to -2} p(x) = p(-2)$$

- ***This leads to a fact that***

If p(x) is a polynomial then,

$$\lim_{x\to a} p(x) = p(a)$$

Limits

Example 2: Evaluate the following limit.

$$\lim_{z \to 1} \frac{6-3z+10z^2}{-2z^4+7z^3+1} = \frac{13}{6}$$

- ***This leads to a fact that provided f(x) is nice enough then***

$$\lim_{x \to a} f(x) = f(a) \qquad \lim_{x \to a^-} f(x) = f(a) \qquad \lim_{x \to a^+} f(x) = f(a)$$

Computing Limits

Example 1: Evaluate the following limit.

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}$$

Solution

First let's notice that if we try to plug in x = 2 we get,

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\frac{0}{0}$$

This implies that we need to factor both the numerator and Denominator.

Computing Limits

- ***On factoring, we have***

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}$$

- ***This now gives us***

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\lim_{x\to 2}\frac{(x+6)}{x}$$

- ***And at this point we can comfortably plug in x=2. Therefore,***

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\lim_{x\to 2}\frac{(2+6)}{2}=4$$

Computing Limits

- Evaluate the following limits.
- ***Example 2:*** $$\lim_{h \to 0} \frac{2(-3+h)^2 - 18}{h}$$
- ***Example 3:*** $$\lim_{t \to 4} \frac{t - \sqrt{3t+4}}{4-t}$$

Computing Limits

Example 4: Given the function,

$$g(y)=\begin{cases} y^2+5, if y<-2 \\ 1-3y, if y\geq -2 \end{cases}$$

Compute the following limits.

a) $\lim_{y\to 6} g(y)$

b) $\lim_{y\to -2} g(y)$

Computing Limits

- ***Solution***

a) $$\lim_{y \to 6} g(y) = \lim_{y \to 6}(1 - 3y) = -17$$

b) $$\lim_{y \to -2} g(y)$$

- In this case the point that we want to take the limit for is the cutoff point for the two intervals. In other words we can't just plug *y* = -2 into the second portion because this interval does not contain values of *y* to the left of *y* = -2 and we need to know what is happening on both sides of the point.
- So we need to test and see if two one sided limits exist and have the same value which implies that the normal limit will also exist with the same value.

Computing Limits

- *So doing the two one-sided limits we get,*

$$\lim_{y \to -2^-} g(y) = \lim_{y \to -2^-} y^2 + 5 = 9$$

$$\lim_{y \to -2^+} g(y) = \lim_{y \to -2^+} 1 - 3y = 7$$

- *And we can clearly not that,*

$$\lim_{y \to -2^-} g(y) \neq \lim_{y \to -2^+} g(y)$$

- *This implies that* $\lim_{y \to -2} g(y)$ *does not exist.*

Computing Limits

- ***Example 5***:Evaluate the following limit.

$$\lim_{y \to -2} g(y) \quad \text{where} \quad g(y) = \begin{cases} y^2 + 5, if y < -2 \\ 3 - 3y, if y \geq -2 \end{cases}$$

Squeeze Theorem

Suppose that for all x on $[a, b]$ (except possibly at $x = c$) we have,

$$f(x) \le h(x) \le g(x)$$

Also suppose that,

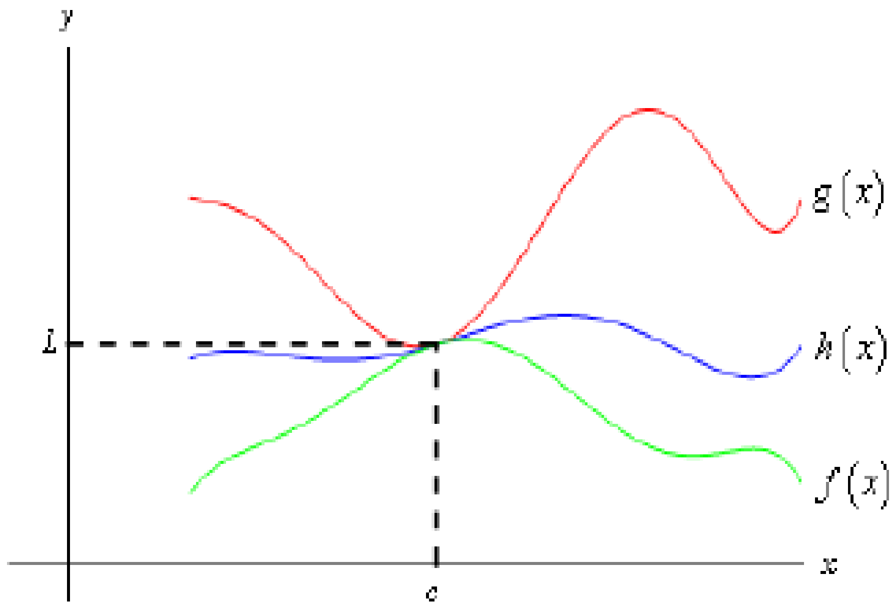
$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$$

for some $a \le c \le b$. Then,

$$\lim_{x \to c} h(x) = L$$

Squeeze Theorem

The figure below illustrates what happens in this theorem.



From the figure we can see that if the limits of f(x) and g(x) are equal at x = c then the function values must also be equal at x = c. However, because h(x) is "squeezed" between f(x) and g(x) at this point then h(x) must have the same value. Therefore, the limit of h(x) at this point must also be the same.

The Squeeze theorem is also known as the Sandwich Theorem and the Pinching Theorem.

Squeeze Theorem

- ***Example 6 :*** Evaluate the following limit.

$$\lim_{x \to 0} x^2 \cos(\frac{1}{x})$$

Solution

we know the following fact about cosine.

$$-1 \leq \cos(x) \leq 1$$

This leads us to say that,

$$-1 \leq \cos(\frac{1}{x}) \leq 1$$ ***provided we avoid x=0.***

We can multiply everything by an x^2 and get the following.

$$-x^2 \leq x^2 \cos(\frac{1}{x}) \leq x^2$$

Squeeze Theorem

- We've managed to squeeze the function that we were interested in between two other functions that are very easy to deal with. So, the limits of the two outer functions are.

$$\lim_{x\to 0}(-x^2) = 0 \quad \text{and} \quad \lim_{x\to 0}(x^2) = 0$$

- These are the same and so by the Squeeze theorem we must also have,

$$\lim_{x\to 0} x^2 \cos(\frac{1}{x}) = 0$$

Squeeze Theorem

Evaluate the following

Example 6: $$\lim_{x\to 0}\frac{\sin x}{x}$$

Example 7: $$\lim_{x\to 0}\frac{1-\cos x}{x}$$

Infinite Limits

- ***we will take a look at limits whose value is infinity or minus infinity.***

Definition

We say

$$\lim_{x \to a} f(x) = \infty$$

if we can make *f(x)* arbitrarily large for all *x* sufficiently close to *x*=*a*, from both sides, without actually letting $x = a$.

We say

$$\lim_{x \to a} f(x) = -\infty$$

if we can make *f(x)* arbitrarily large and negative for all *x* sufficiently close to *x*=*a*, from both sides, without actually letting $x = a$.

These definitions can be appropriately modified for the one-sided limits as well.

Infinite Limits

Example 1 Evaluate each of the following limits.

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right), \quad \lim_{x \to 0^-} \left(\frac{1}{x}\right), \quad \lim_{x \to 0} \left(\frac{1}{x}\right)$$

Solution

- ***So we're going to be taking a look at a couple of one-sided limits as well as the normal limit here. In all three cases notice that we can't just plug in x = 0 .***
- ***Lets plug in some points and see what value the function is approaching.***

Infinite Limits

- *Below is a table of values of x's from both the left and the right.*

x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10000	0.0001	10000

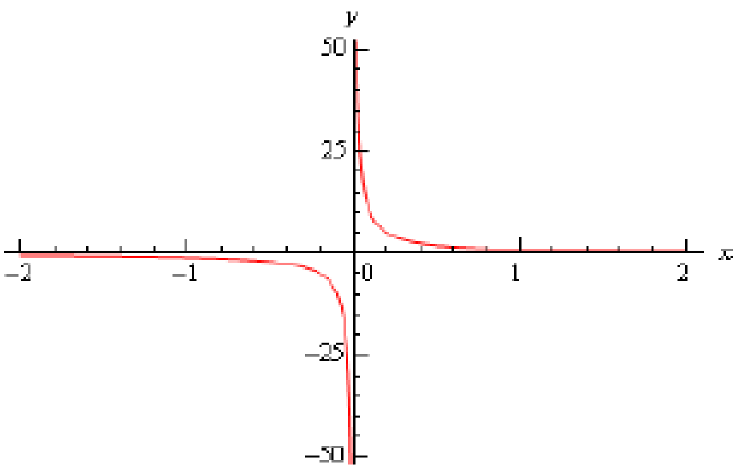
- ***From this table we can see that as we make x smaller and smaller the function 1/x gets larger and larger and will retain the same sign that x originally had.***
- ***We can make the function as large and positive as we want for all x's sufficiently close to zero while staying positive (i.e. on the right).***
- ***Likewise, we can make the function as large and negative as we want for all x's sufficiently close to zero while staying negative (i.e. on the left).***

Infinite Limits

- ***So, from our definition above , we should have the following values for the two one sided limits.***

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right)=\infty \qquad \lim_{x\to 0^-}\left(\frac{1}{x}\right)=-\infty$$

- ***Sketch of a graph of the functions.***



The normal limit, in this case, will not exist since the two one-sided have different values.

The values of the three limits for this example are,

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right)=\infty,\quad \lim_{x\to 0^-}\left(\frac{1}{x}\right)=-\infty,\quad \lim_{x\to 0}\left(\frac{1}{x}\right)\ \textit{doesn't exist}$$

Infinite Limits

- ***Evaluate each of the following limits.***

$$\lim_{x\to 0^+}\left(\frac{6}{x^2}\right),\quad \lim_{x\to 0^-}\left(\frac{6}{x^2}\right),\quad \lim_{x\to 0}\left(\frac{6}{x^2}\right)$$

$$\lim_{x\to -2^+}\left(\frac{-4}{x+2}\right),\quad \lim_{x\to -2^-}\left(\frac{-4}{x+2}\right),\quad \lim_{x\to -2}\left(\frac{-4}{x+2}\right)$$

$$\lim_{x\to 4^+}\frac{3}{(4-x)^3},\quad \lim_{x\to 4^-}\frac{3}{(4-x)^3},\quad \lim_{x\to 4}\frac{3}{(4-x)^3}$$

Infinite Limits

Facts

Given the functions $f(x)$ and $g(x)$ suppose we have,

$$\lim_{x \to c} f(x) = \infty \qquad\qquad \lim_{x \to c} g(x) = L$$

for some real numbers c and L. Then,

1. $\lim_{x \to c}\left[f(x) \pm g(x)\right] = \infty$

2. If $L > 0$ then $\lim_{x \to c}\left[f(x)\,g(x)\right] = \infty$

3. If $L < 0$ then $\lim_{x \to c}\left[f(x)\,g(x)\right] = -\infty$

5. $\lim_{x \to c} \dfrac{g(x)}{f(x)} = 0$

Note as well that the above set of facts also holds for one-sided limits. They will also hold if $\lim_{x \to c} f(x) = -\infty$, with a change of sign on the infinities in the first three parts.

Limits At Infinity

- By limits at infinity we mean one of the following two limits.

$$\lim_{x\to\infty} f(x) \quad \text{or} \quad \lim_{x\to-\infty} f(x)$$

Facts

1. If r is a positive rational number and c is any real number then,

$$\lim_{x\to\infty} \frac{c}{x^r} = 0$$

2. If r is a positive rational number, c is any real number and x^r is defined for $x < 0$ then,

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0$$

Limits At Infinity

Example 1 Evaluate each of the following limits.

(a) $\lim\limits_{x \to \infty}\left(2x^4 - x^2 - 8x\right)$

(b) $\lim\limits_{t \to -\infty}\left(\frac{1}{3}t^5 + 2t^3 - t^2 + 8\right)$

Limits At Infinity

Solution

(a) $\lim_{x\to\infty}\left(2x^4 - x^2 - 8x\right)$

- Plugging infinity into the polynomial and evaluating each term to determine the value of the limit would give us.

$$\lim_{x\to\infty}\left(2x^4 - x^2 - 8x\right) = \infty - \infty - \infty$$

- We are probably tempted to say that the answer is zero or -∞. However, in both cases we'd be wrong.
- In order to solve this, we need to factor the largest power of *x* out of the whole polynomial as follows,

Limits At Infinity

$$\lim_{x\to\infty}\left(2x^4 - x^2 - 8x\right) = \lim_{x\to\infty}\left[x^4\left(2 - \frac{1}{x^2} - \frac{8}{x^3}\right)\right]$$

- Now for each of the terms we have,

$$\lim_{x\to\infty} x^4 = \infty \qquad \lim_{x\to\infty}\left(2 - \frac{1}{x^2} - \frac{8}{x^3}\right) = 2$$

- Therefore using Fact 2 of infinite limits, we will have the following value for the limit.

$$\lim_{x\to\infty}\left(2x^4 - x^2 - 8x\right) = \infty$$

Limits At Infinity

- we can give a simple fact about polynomials in general.

If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n (*i.e.* $a_n \neq 0$) then,

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} a_n x^n \qquad \lim_{x \to -\infty} p(x) = \lim_{x \to -\infty} a_n x^n$$

Limits At Infinity

- ***Example 2*** Evaluate both of the following limits.

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7} \qquad \lim_{x\to-\infty}\frac{2x^4-x^2+8x}{-5x^4+7}$$

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x} \qquad \lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x}$$

Limits At Infinity

Lets review some of the basics of the exponential function that will help us solve for limits.

$$\lim_{x\to\infty} e^{x} = \infty \qquad \lim_{x\to-\infty} e^{x} = 0 \qquad \lim_{x\to\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = \infty$$

Example 3 Evaluate each of the following limits.

(a) $\lim\limits_{x\to\infty} e^{2-4x-8x^2}$

(b) $\lim\limits_{t\to-\infty} e^{t^4-5t^2+1}$

(c) $\lim\limits_{z\to 0^+} e^{\frac{1}{z}}$

Limits At Infinity

Example 4 Evaluate each of the following limits.

$$\lim_{x \to 0^+} \ln x \qquad \lim_{x \to \infty} \ln x$$

Solution

From the basics of Logarithms, we shall have

$$\lim_{x \to 0^+} \ln x = -\infty \qquad \lim_{x \to \infty} \ln x = \infty$$

Limits At Infinity

- ***Example 6*** Evaluate each of the following limits.

(a) $\lim_{x\to\infty} \ln\left(7x^3 - x^2 + 1\right)$

(b) $\lim_{t\to-\infty} \ln\left(\dfrac{1}{t^2 - 5t}\right)$