Trigonometric Functions

Trigonometric Functions

- A function which involves one or more of the trigonometric ratios (sine, cosine, tangent, cosecant, secant, or cotangent) is called a trigonometric function.
- For example,

$$f(x) = \sin 2x$$
$$g(x) = 3\cos^2 x - \cot x$$

are trigonometric functions.

Trigonometric Functions

- First let's start with the six trigonometric functions and how they relate to each other.

$$\cos(x)$$

$$\sin(x)$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

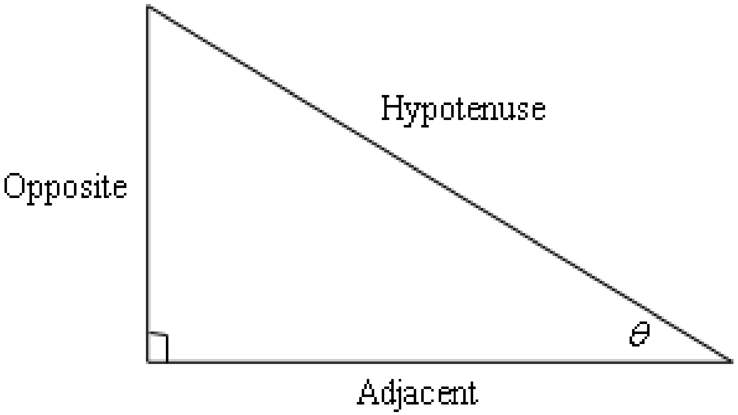
$$\cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

Trigonometric Functions

- All the trigonometric functions can be defined in terms of a right triangle



$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

The fundamental Identities

Reciprocal Identities

$$\csc x = \frac{1}{\sin x}$$

$$Secx = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

The beauty of the identities is that we can get all functions in terms of sine and cosine.

The Fundamental Identities

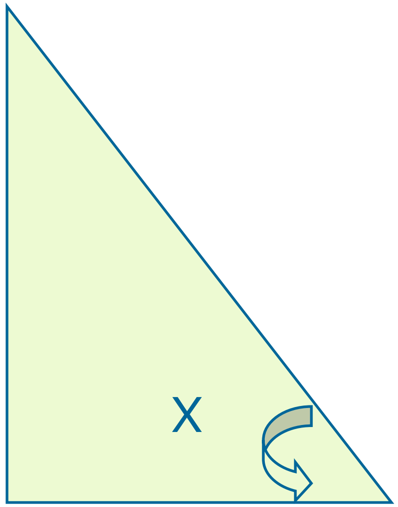
Identities for Negatives

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos(x)$$

$$\tan(-x) = -\tan x$$

The Fundamental Identities



Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

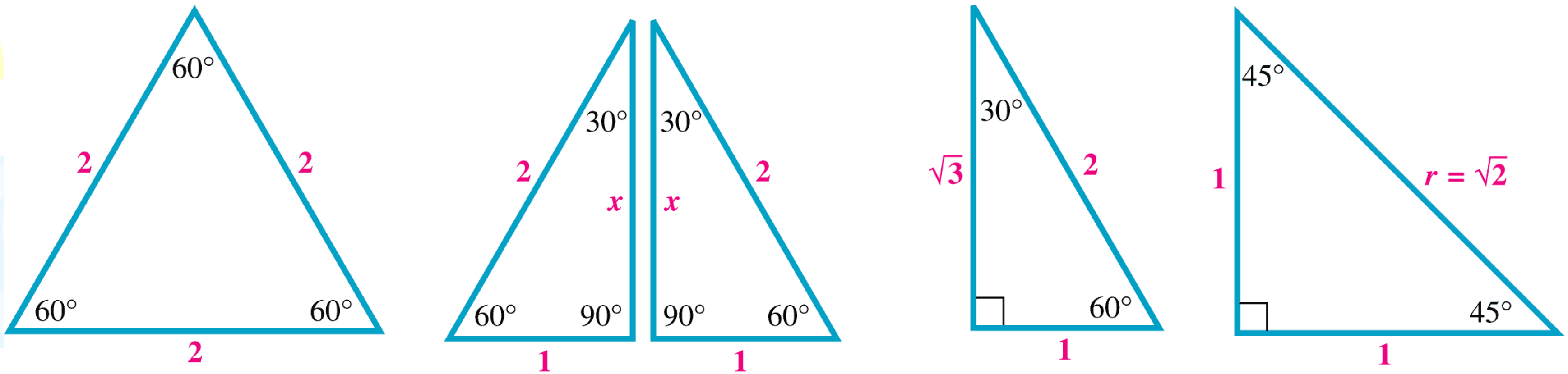
$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

The only unique Identity here is the top one, the other two can be obtained using the top identity.

Trigonometric Function Values of Special Angles

- Angles that deserve special study are 30°, 45°, and 60°.



Equilateral Triangle

30°– 60° Right Triangle

45°– 45° Right Triangle

Function Values for Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$30° = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$45° = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$60° = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

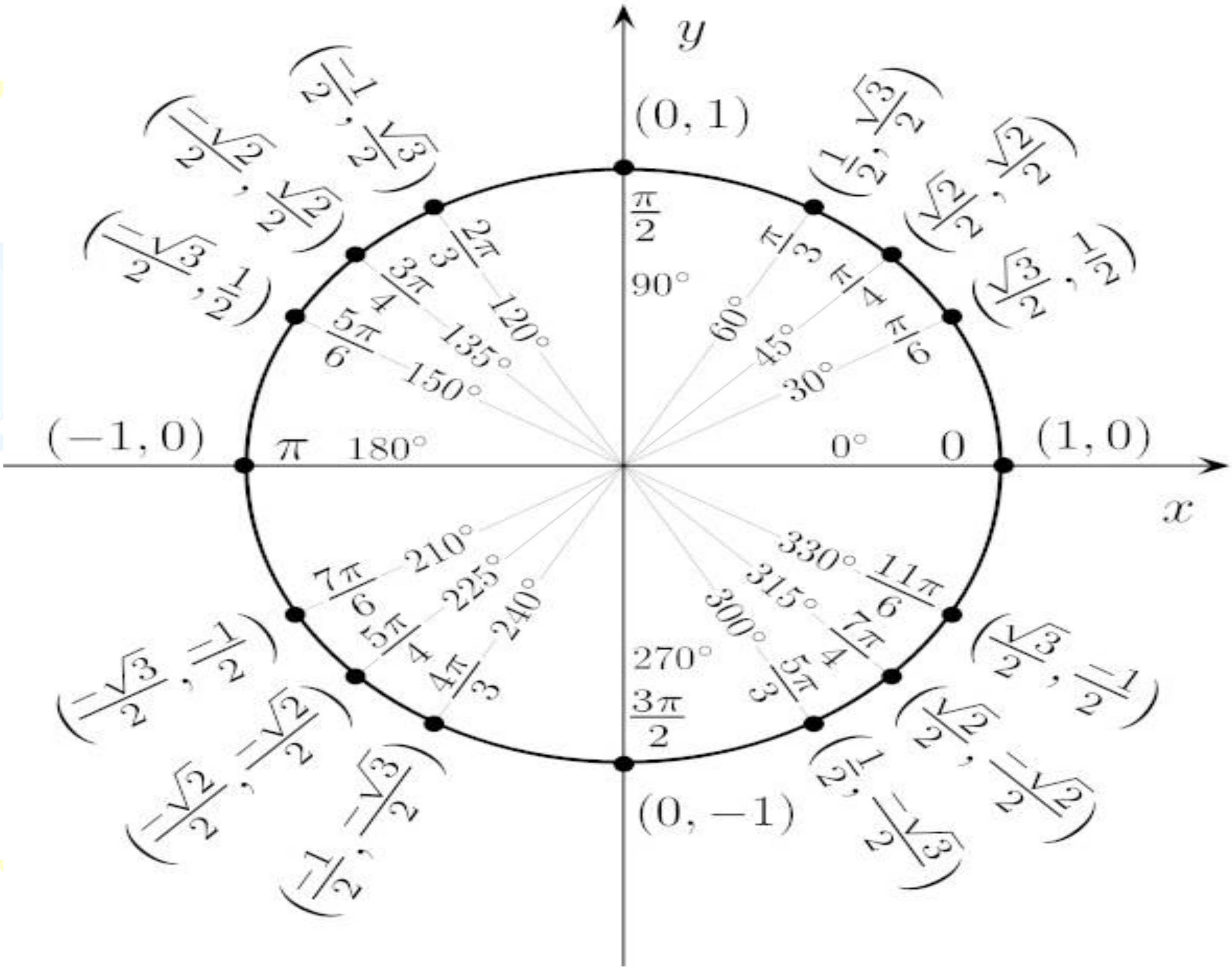
Using the figures above, we have the exact values of the special angles summarized in the table on the right.

Be forewarned, everything in most calculus classes will be done in radians!

Unit Circle

- The way the unit circle works is to draw a line from the center of the circle outwards corresponding to a given angle.
- Then look at the coordinates of the point where the line and the circle intersect.
- The first coordinate is the cosine of that angle and the second coordinate is the sine of that angle.
- We'll take a look at some *basic* angles along with the coordinates of their intersections on the unit circle.

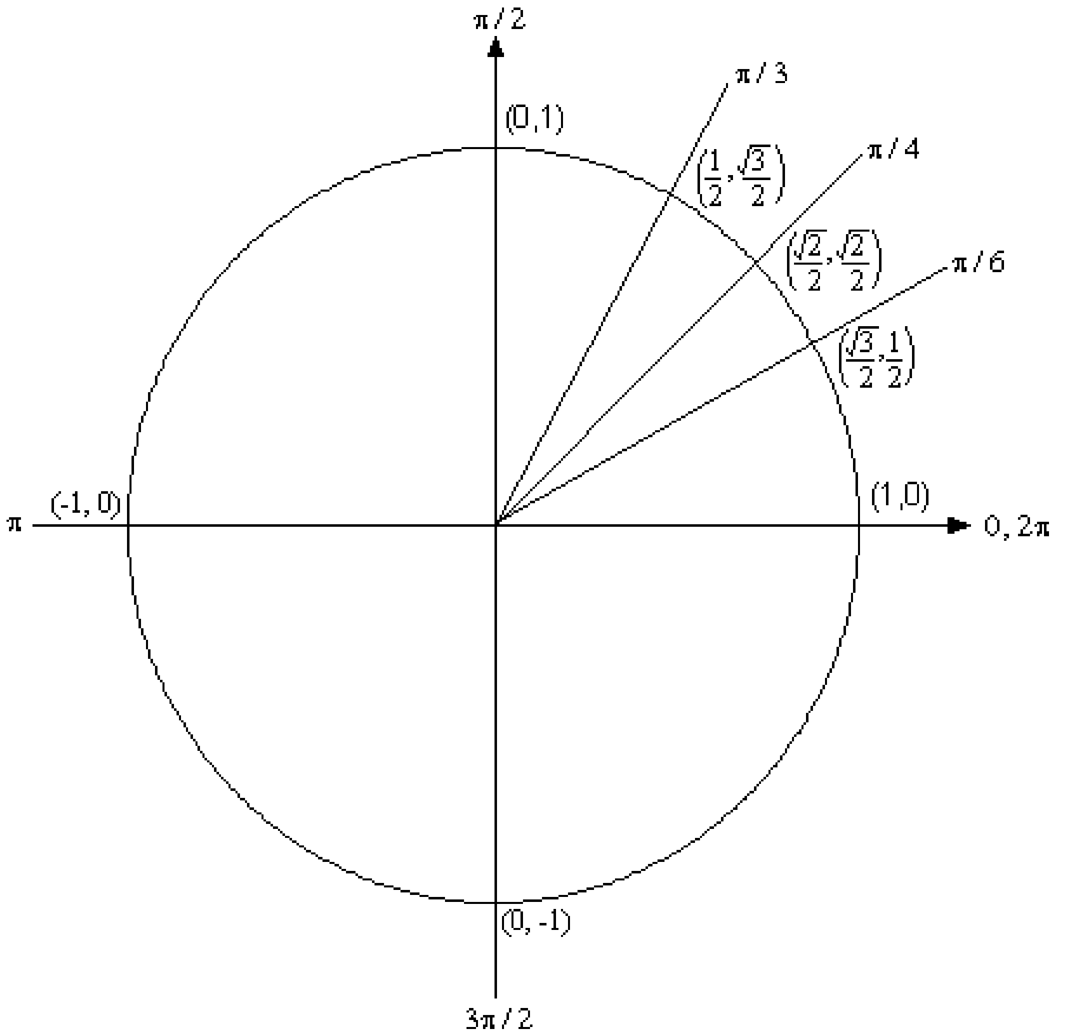
Points of Special Interest on the Unit Circle



Unit Circle

From the unit circle we can see that

$$\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2} \text{ and } \sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$



Trigonometric Functions

- One complete revolution is 2π, so the positive x-axis can correspond to either an angle of 0 or 2π (or 4π , or 6π ,or -2π , or -4π , *etc*. depending on the direction of rotation).
- Picking the angle $\pi/6$, it can also be any of the following angles:

Trigonometric Functions

$$\frac{\pi}{6}+2\pi=\frac{13\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate once around counter clockwise)}$$

$$\frac{\pi}{6}+4\pi=\frac{25\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate around twice counter clockwise)}$$

$$\frac{\pi}{6}-2\pi=-\frac{11\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate once around clockwise)}$$

$$\frac{\pi}{6}-4\pi=-\frac{23\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate around twice clockwise)}$$

In fact $\frac{\pi}{6}$ can be any of the following angles $\frac{\pi}{6}+2\pi n$, $n=0,\pm1,\pm2,\pm3,\ldots$ In this case n is the number of complete revolutions you make around the unit circle starting at $\frac{\pi}{6}$. Positive values of n correspond to counter clockwise rotations and negative values of n correspond to clockwise rotations.

Trigonometric Functions

- ***Example 1:*** Evaluate each of the following.

(a) $\sin\left(\frac{2\pi}{3}\right)$ and $\sin\left(-\frac{2\pi}{3}\right)$

(b) $\cos\left(\frac{7\pi}{6}\right)$ and $\cos\left(-\frac{7\pi}{6}\right)$

(c) $\tan\left(-\frac{\pi}{4}\right)$ and $\tan\left(\frac{7\pi}{4}\right)$

(d) $\sec\left(\frac{25\pi}{6}\right)$

Trigonometric Functions

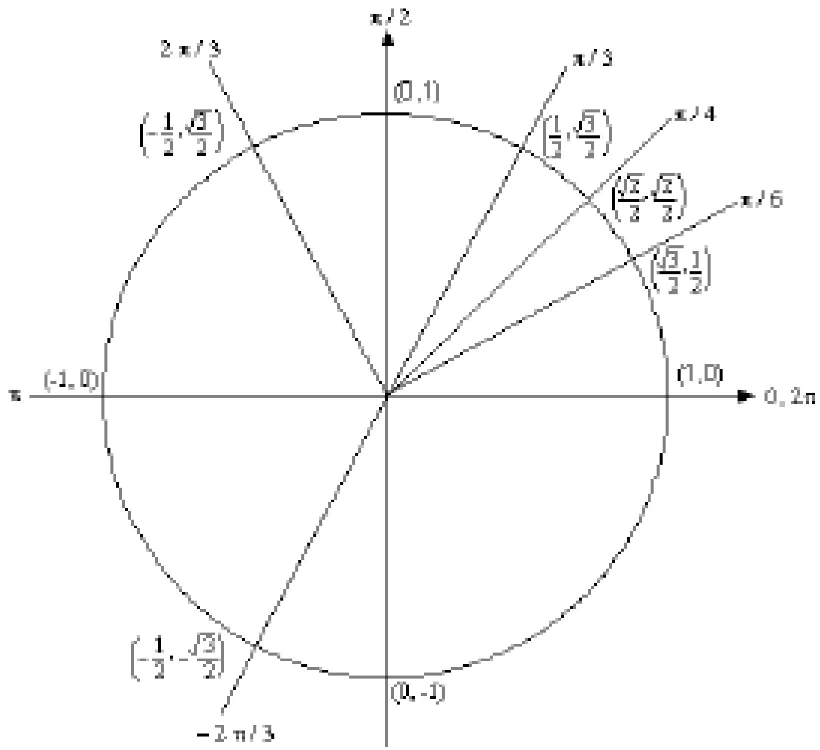
Solution

a) *Notice that* $\frac{2\pi}{3} = \pi - \frac{\pi}{3}$. So $\frac{2\pi}{3}$ is found by rotating up $\frac{\pi}{3}$ from the negative x-axis.

This means that the line for $\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the second quadrant. The coordinates for $\frac{2\pi}{3}$ will be the coordinates for $\frac{\pi}{3}$ except the x coordinate will be negative.

Likewise for $-\frac{2\pi}{3}$ we can notice that $-\frac{2\pi}{3} = -\pi + \frac{\pi}{3}$, so this angle can be found by rotating down $\frac{\pi}{3}$ from the negative x-axis. This means that the line for $-\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the third quadrant and the coordinates will be the same as the coordinates for $\frac{\pi}{3}$ except both will be negative.

Trigonometric Functions



From this unit circle we can see that $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

The sine function is called an **odd** function and so for ANY angle we have

$$\sin(-\theta) = -\sin(\theta)$$

Trigonometric Functions

Example 1 Solve $2\cos(t) = \sqrt{3}$ on $[-2\pi, 2\pi]$.

Dividing through by 2,

$$\cos(t) = \frac{\sqrt{3}}{2}$$

we are therefore looking for the values of t whose cosine value is √3/2. From the unit circle we shall have two values(1st quadrant & 4th quadrant) i.e $\pi/6$ and $11\pi/6$

So all the possible values will be

$$\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\frac{11\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Next we choose values of n and their solutions should lie within the given interval

Trigonometric Functions

- *For n=0 ,*

$$\frac{\pi}{6}+2\pi(0)=\frac{\pi}{6}<2\pi$$

$$\frac{11\pi}{6}+2\pi(0)=\frac{11\pi}{6}<2\pi$$

- we don't need to take any positive value of *n since* we will be adding on positive multiples of *2π* onto a positive quantity and this will take us past the upper bound of our interval.
- *For n=-1,*

$$\frac{\pi}{6}+2\pi(-1)=-\frac{11\pi}{6}>-2\pi$$

$$\frac{11\pi}{6}+2\pi(-1)=-\frac{\pi}{6}>-2\pi$$

- *n=-2 will give us solutions outside the interval. Therefore , all the possible solutions that lie within the interval* $[-2\pi, 2\pi]$. *are*

$$\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{\pi}{6}, -\frac{11\pi}{6}$$

Trigonometric Functions

Example 2 Solve $2\sin(5x) = -\sqrt{3}$ on $[-\pi, 2\pi]$

Example 3 Solve $\sin(2x) = -\cos(2x)$ on $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

Example 4 Solve $\cos(3x) = 2$.

Inverse Trigonometric Functions

- Functions that involve one or more of the inverses of the trigonometric ratios (arcsine, arccosine, arctangent, arccosecant, arcsecant, or arccotangent) are called inverse trigonometric functions.
- For example,

$$f(x) = \cot^{-1} x,\ g(x) = \cos^{-1} x + \tan^{-1} x$$

are inverse trigonometric functions.

Inverse Trigonometric Functions

- Trigonometric functions are periodic, hence not one-to-one in the their domains.
- If we restrict the trigonometric functions to intervals on which they are one-to-one, then we can define their inverses.

Inverses for the restricted trigonometric functions

$$y = \sin^{-1} x = \arcsin x$$

$$y = \cos^{-1} x = \arccos x$$

$$y = \tan^{-1} x = \arctan x$$

$$y = \cot^{-1} x = \operatorname{arc}\cot x$$

$$y = \sec^{-1} x = \operatorname{arc}\sec x$$

$$y = \csc^{-1} x = \operatorname{arc}\csc x$$

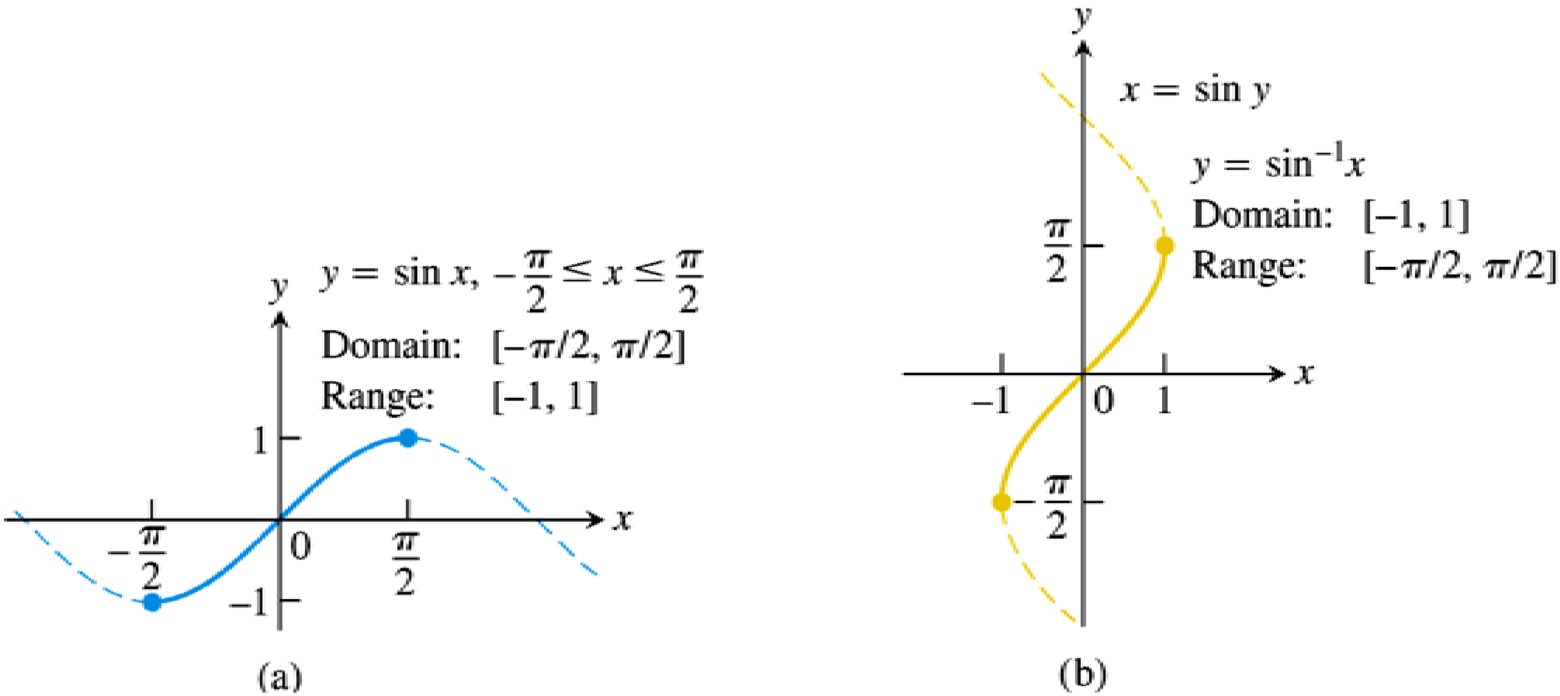
Inverse Trigonometric Functions

DEFINITION **Arcsine and Arccosine Functions**

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

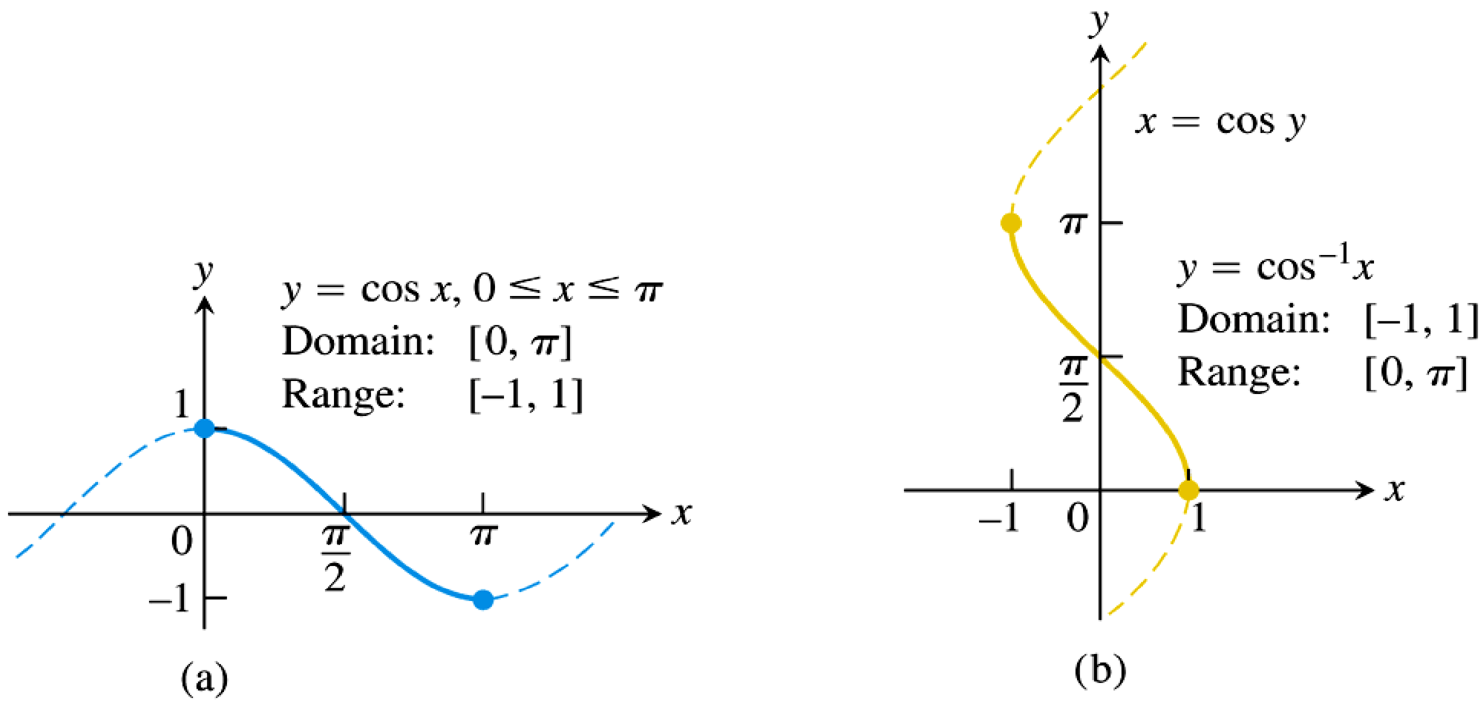
$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

Arcsine Function



The graphs of (a) $y = \sin x, -\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

Arccosine Function



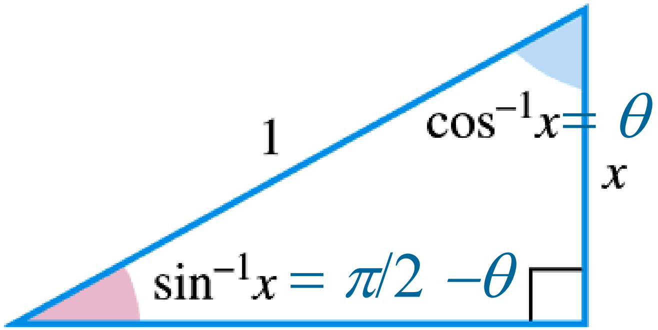
The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Some specific values of $\sin^{-1} x$ and $\cos^{-1} x$

x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$

x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$

The thing to remember is that for the trig function the input is the angle and the output is the ratio, but for the inverse trig function the input is the ratio and the output is the angle.



$\sin^{-1} x$ and $\cos^{-1} x$ are complementary angles (so their sum is $\pi/2$).

$$\cos^{-1} x = \theta; \sin^{-1} x = \left(\frac{\pi}{2} - \theta\right);$$

$$\cos^{-1} x + \sin^{-1} x + = \theta + \left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2}$$

Some Inverse Function Identities

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

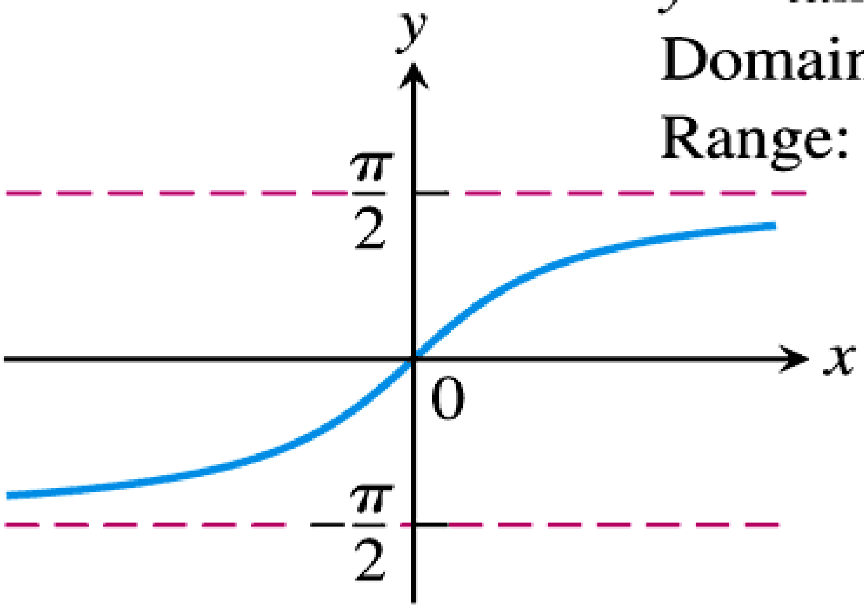
Inverse Trigonometric Functions

DEFINITION **Arctangent and Arccotangent Functions**

$y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

Arctangent Function



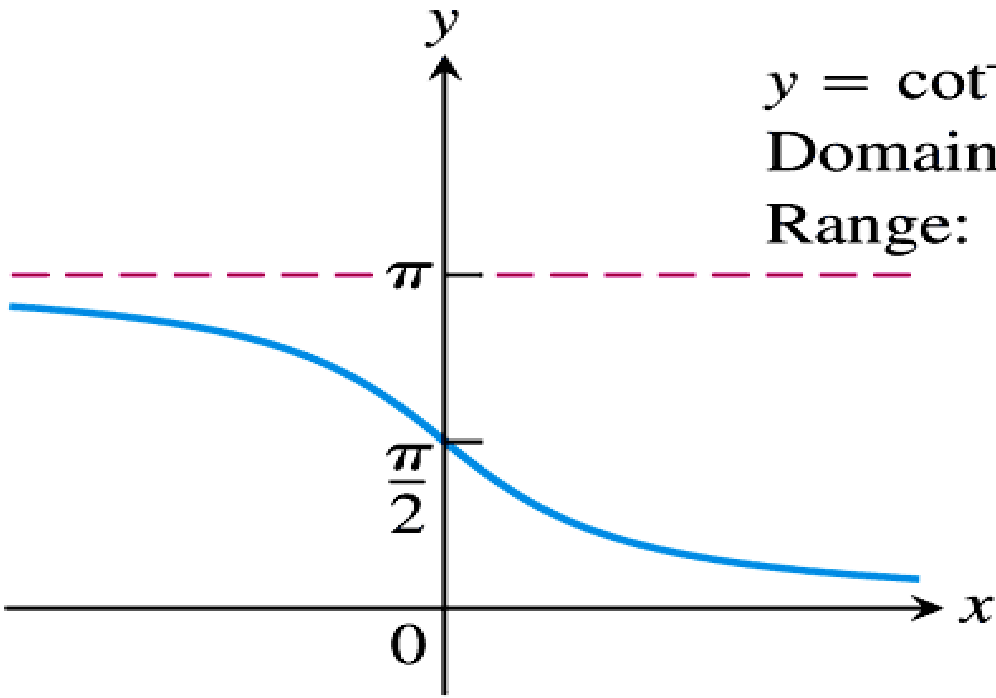
$y = \tan^{-1}x$
Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$

The graph of $y = \tan^{-1} x$.

Some specific values of $\tan^{-1} x$

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

Arccotangent Function

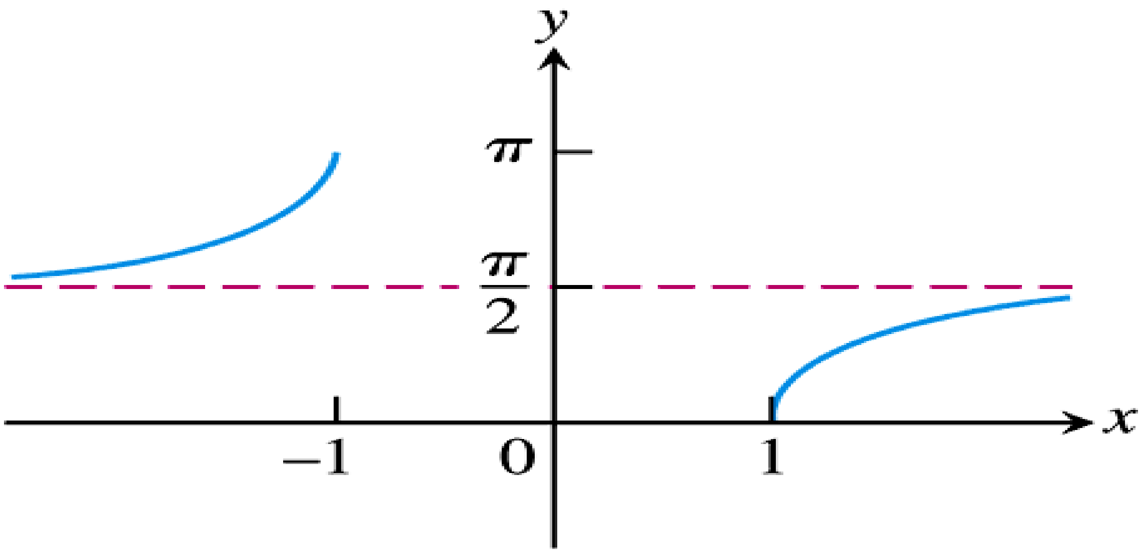


$y = \cot^{-1} x$
Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

The graph of $y = \cot^{-1} x$.

Arcsecant Function

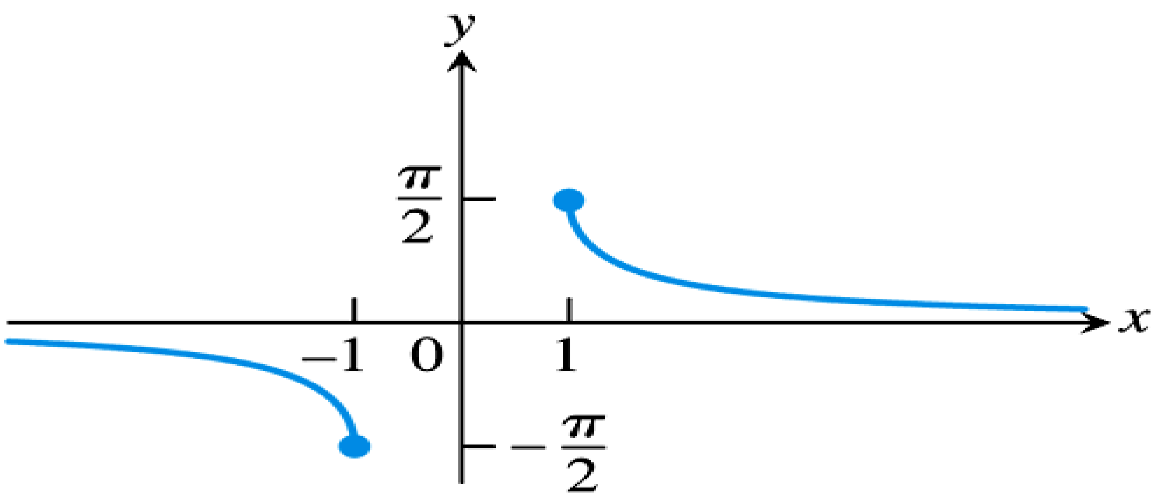
$y = \sec^{-1}x$
Domain: $|x| \geq 1$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$



The graph of $y = \sec^{-1} x$.

Arccosecant Function

$y = \csc^{-1} x$
Domain: $|x| \geq 1$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$



The graph of $y = \csc^{-1} x$.

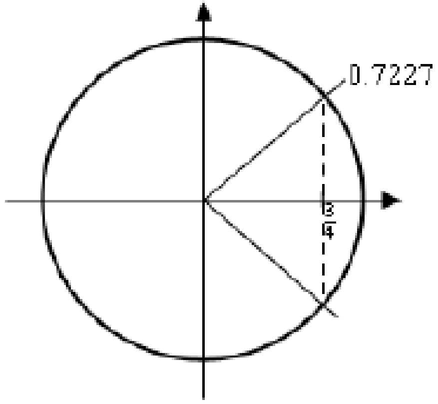
Inverse Trigonometric Functions

Example 1 Solve $4\cos(t)=3$ on[-8,10].

First we need to make the cosine the subject then get the first answer using the calculator.

$$\cos(t)=\frac{3}{4} \qquad \Rightarrow \qquad t=\cos^{-1}\left(\frac{3}{4}\right)=0.7227$$

From the unit circle, we can have two possibilities(1st and 4th quadrant)



This means that we can use either -0.7227 as the second angle or 2π - 0.7227 = 5.5605 .

So, all possible solutions are then

$$t=0.7227+2\pi n$$
$$t=5.5605+2\pi n \qquad n=0,\pm1,\pm2,\ldots$$

Inverse Trigonometric Functions

- Next we plug in values of *n* to determine the angles that are actually in the interval.

$$
\begin{array}{llll}
n=-2: & t=\cancel{-11.8437} & \text{and} & -7.0059 \\
n=-1: & t=-5.5605 & \text{and} & -0.7227 \\
n=0: & t=0.7227 & \text{and} & 5.5605 \\
n=1: & t=7.0059 & \text{and} & \cancel{11.8437}
\end{array}
$$

- *The solutions to this equation, in the given interval, are,*

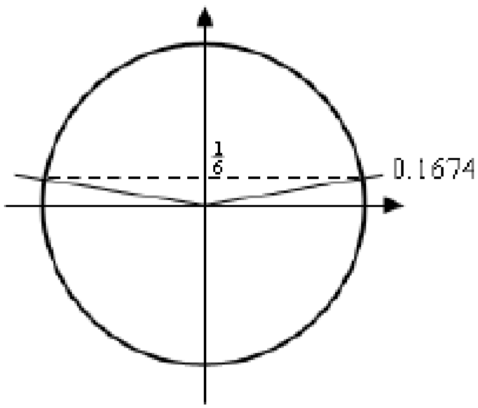
t = -7.0059, - 5.5605, - 0.7227, 0.7227, 5.5605, 7.0059

Inverse Trigonometric Functions

Example 2

Solve $6\sin\left(\frac{x}{2}\right)=1$ on [-20,30]

$$\sin\left(\frac{x}{2}\right)=\frac{1}{6} \quad \Rightarrow \quad \frac{x}{2}=\sin^{-1}\left(\frac{1}{6}\right)=0.1674$$



From the unit circle , there are two possibilities(1st and 2nd quadrant).
so the second angle is $\pi - 0.1674 = 2.9742$.
All the possible solutions will then be

$$\frac{x}{2}=0.1674+2\pi n \qquad \frac{x}{2}=2.9742+2\pi n \quad \Rightarrow \quad x=0.3348+4\pi n \qquad x=5.9484+4\pi n \qquad n=0,\pm1,\pm2,\ldots$$

Inverse Trigonometric Functions

- Plugging in values of n, we have

$n=-2$:	$x=\cancel{-24.7980}$	and	-19.1844
$n=-1$:	$x=-12.2316$	and	-6.6180
$n=0$:	$x=0.3348$	and	5.9484
$n=1$:	$x=12.9012$	and	18.5148
$n=2$:	$x=25.4676$	and	$\cancel{31.0812}$

- The solutions to this equation are then,
  x = -19.1844, -12.2316, - 6.6180, 0.3348, 5.9484, 12.9012, 18.5128, 25.4676

Inverse Trigonometric Functions Review questions

1. Solve $5\cos(2x-1)=-3$.
2. Solve $7\sec(3t)=-10$.
3. Solve $9\sin(2x)=-5\cos(2x)$ on[-10,0].
4. Solve $3\sin(5z)=-2$ on [0,1].
5. Solve $-10\cos(3t)=7$ on [-2,5].
6. Solve $2\cos(6y)+11\cos(6y)\sin(3y)=0$.
7. Solve $5x\tan(8x)=3x$.
8. Solve $8\cos^2(1-x)+13\cos(1-x)-5=0$.