Differentiation

Derivative

The first interpretation of a derivative is rate of change. If $f(x)$ represents a quantity at any x then the derivative $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x = a$.

Definition

The derivative of $f(x)$ with respect to x is the function $f'(x)$ and is defined as,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

Example 1 Find the derivative of the following function using the definition of the derivative.

$$g(t) = \frac{t}{t+1}$$

Solution

First, we plug the function into the definition of the derivative

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{t+h}{t+h+1} - \frac{t}{t+1}\right)$$

we can't just plug in $h = 0$. So we will need to simplify things a little. In this case we will need to combine the two terms in the numerator into a single rational expression as follows.

Derivatives

$$g'(t) = \lim_{h \to 0} \frac{1}{h}\left(\frac{(t+h)(t+1) - t(t+h+1)}{(t+h+1)(t+1)}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{t^2 + t + th + h - (t^2 + th + t)}{(t+h+1)(t+1)}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{h}{(t+h+1)(t+1)}\right)$$

So, upon canceling the *h* we can evaluate the limit and get the derivative.

$$g'(t) = \lim_{h \to 0} \frac{1}{(t+h+1)(t+1)}$$

$$= \frac{1}{(t+1)(t+1)}$$

$$= \frac{1}{(t+1)^2}$$

Derivatives

Example 2 Find the derivative of the following function using the definition of a derivative.

$$R(z) = \sqrt{5z-8}$$

Example 3 Determine $f\,'(0)$ for $f(x) = |x|$.

Continuity & Differentiability

Definition

A function $f(x)$ is called differentiable at $x = a$ if $f'(x)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval.

Theorem

If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$.

Differentiation

Properties

1) $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$ **OR** $\dfrac{d}{dx}\left(f(x) \pm g(x)\right) = \dfrac{df}{dx} \pm \dfrac{dg}{dx}$

2) $\left(cf(x)\right)' = cf'(x)$ **OR** $\dfrac{d}{dx}\left(cf(x)\right) = c\dfrac{df}{dx}$, c is any number

Note that the derivative of products or quotients of two functions is not the product or quotient of the derivatives of the individual pieces.

Differentiation

Formulas

1) If $f(x)=c$ then $f'(x)=0$ **OR** $\dfrac{d}{dx}(c)=0$

2) If $f(x)=x^n$ then $f'(x)=nx^{n-1}$ **OR** $\dfrac{d}{dx}(x^n)=nx^{n-1}$, n is any number.

Review Questions

Differentiate each of the following functions.

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

(b) $g(t) = 2t^6 + 7t^{-6}$

(c) $y = 8z^3 - \dfrac{1}{3z^5} + z - 23$

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \dfrac{2}{\sqrt[5]{x^2}}$

(e) $h(x) = x^{\pi} - x^{\sqrt{2}}$

Differentiation

Product Rule

If the two functions *f(x)* and *g(x)* are differentiable (*i.e.* the derivative exist) then the product is differentiable and,

$$(f\,g)' = f'\,g + f\,g'$$

Quotient Rule

If the two functions *f(x)* and *g(x)* are differentiable (*i.e.* the derivative exist) then the quotient is differentiable and,

$$\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$$

Chain Rule

Suppose that we have two functions *f(x)* and *g(x)* and they are both differentiable.

1. If we define $F(x) = (f \circ g)(x)$ then the derivative of *F(x)* is,

$$F'(x) = f'(g(x))\; g'(x)$$

2. If we have $y = f(u)$ and $u = g(x)$ then the derivative of *y* is,

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Review Questions

Differentiate each of the following functions.

(a) $W(z) = \dfrac{3z+9}{2-z}$

(b) $f(x) = (6x^3 - x)(10 - 20x)$

(c) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

(d) $y = \dfrac{w^6}{5}$

Product rule

- Finally the product rule can be extended to more than two functions as shown in the following formulas.

$$(f\,g\,h)' = f'\,g\,h + f\,g'\,h + f\,g\,h'$$

$$(f\,g\,h\,w)' = f'\,g\,h\,w + f\,g'\,h\,w + f\,g\,h'\,w + f\,g\,h\,w'$$

Derivatives of Trig Functions

Derivatives of the six trig functions

$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x) \qquad \frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$$

$$\frac{d}{dx}\left(\tan(x)\right) = \sec^2(x) \qquad \frac{d}{dx}\left(\cot(x)\right) = -\csc^2(x)$$

$$\frac{d}{dx}\left(\sec(x)\right) = \sec(x)\tan(x) \qquad \frac{d}{dx}\left(\csc(x)\right) = -\csc(x)\cot(x)$$

Review Questions

Differentiate each of the following functions.

(a) $g(x) = 3\sec(x) - 10\cot(x)$

(b) $h(w) = 3w^{-4} - w^2\tan(w)$

(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$

(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

Derivatives of Inverse Functions

If $f(x)$ and $g(x)$ are inverse functions then,

$$g'(x) = \frac{1}{f'(g(x))}$$

This is also referred to as the inverse function rule.

Recall that two functions are inverses if $f(g(x)) = x$ and $g(f(x)) = x$.

Derivative of Inverse Sine

Inverse Sine

We start with definition of the inverse sine.

$$y = \sin^{-1} x \qquad \Leftrightarrow \qquad \sin y = x \qquad \text{for} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

We have the following relationship between the inverse sine function and the sine function.

$$\sin\left(\sin^{-1} x\right) = x \qquad\qquad \sin^{-1}\left(\sin x\right) = x$$

This means that we can use the fact above to find the derivative of inverse sine. Let's start with,

$$f\left(x\right) = \sin x \qquad\qquad g\left(x\right) = \sin^{-1} x$$

Derivative of Inverse sine

Then,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos\left(\sin^{-1} x\right)}$$

Let's try to make this a better formula. Let's start by recalling the definition of the inverse sine function.

$$y = \sin^{-1}(x) \qquad \Rightarrow \qquad x = \sin(y)$$

Using the first part of this definition the denominator in the derivative becomes,

$$\cos\left(\sin^{-1} x\right) = \cos(y)$$

Using the identity,

$$\cos^2 y + \sin^2 y = 1 \qquad \Rightarrow \qquad \cos y = \sqrt{1 - \sin^2 y}$$

Derivative of Inverse sine

Therefore,

$$\cos\left(\sin^{-1} x\right) = \cos\left(y\right) = \sqrt{1 - \sin^2 y}$$

Now, using the second part of the definition of the inverse sine function. The denominator is then,

$$\cos\left(\sin^{-1} x\right) = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Putting everything together, implies that

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$$

Derivatives of Inverse Trig Functions

Derivatives of the inverses of the six trig functions

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1}x\right)=-\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1}x\right)=-\frac{1}{|x|\sqrt{x^2-1}}$$

Derivatives of Inverse Trig Functions

- Examples:

(a) $f(t) = 4\cos^{-1}(t) - 10\tan^{-1}(t)$

(b) $y = \sqrt{z}\sin^{-1}(z)$

Derivatives of Exponential & Logarithm Functions

$$\frac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x \qquad \frac{d}{dx}\left(a^x\right) = a^x \ln a$$

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x} \qquad \frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}$$

Let's consider the derivative of the exponential function. Going back to our limit definition of the derivative:

$$\frac{d}{dx}\left[f(x)\right]=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}\left(e^{x}\right)=\lim_{h\to 0}\frac{e^{x+h}-e^{x}}{h}$$

First rewrite the exponential using exponent rules.

$$=\lim_{h\to 0}\frac{e^{x}e^{h}-e^{x}}{h}$$

Next, factor out e^x.

$$=\lim_{h\to 0}\frac{e^{x}\left(e^{h}-1\right)}{h}$$

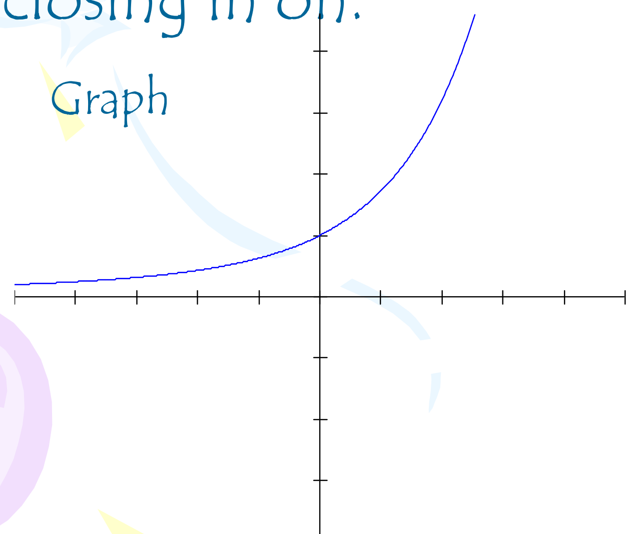
Since e^x does not contain h, we can move it outside the limit.

$$=e^{x}\lim_{h\to 0}\frac{e^{h}-1}{h}$$

Substituting h=0 in the limit expression results in the indeterminate form $\frac{0}{0}$, thus we will need to determine it.

We can look at the graph of $f(x) = \dfrac{e^x - 1}{x}$ and observe what happens as x gets close to 0. We can also create a table of values close to either side of 0 and see what number we are closing in on.

Graph



Table

x	-.1	-.01	-.001	.001	.01	.1
y	.95	.995	.999	1.0005	1.005	1.05

At x = 0, f(0) appears to be 1. As x approaches 0, y approaches 1.

We can safely say that from the last slide that $\lim_{h\to 0}\frac{e^h-1}{h}=1$

Thus $\frac{d}{dx}\left(e^x\right)=e^x\lim_{h\to 0}\frac{\left(e^h-1\right)}{h}=e^x\cdot 1=e^x$

Rule 1: Derivative of the Exponential Function

$$\frac{d}{dx}\left(e^x\right)=e^x$$

The derivative of the exponential function is the exponential function.

Example 1: Find the derivative of $f(x) = x^2e^x$.

Solution: Do you remember the product rule? You will need it here.

Product Rule:
(1^{st})(derivative of 2^{nd}) + (2^{nd})(derivative of 1^{st})

$$f(x) = x^2e^x$$

$$f'(x) = x^2e^x + e^x 2x$$

Factor out the common factor xe^x.

$$f'(x) = xe^x(x+2)$$

Example 2: Find the derivative of $f(t) = \left(e^{t}+2\right)^{\frac{3}{2}}$

Solution: We will need the chain rule for this one.

Chain Rule:
(derivative of the outside)(derivative of the inside)

$$f(t) = \left(e^{t}+2\right)^{\frac{3}{2}}$$

$$f'(t) = \frac{3}{2}\left(e^{t}+2\right)^{\frac{1}{2}} e^{t}$$

What if the exponent on e is a function of x and not just x?

Rule 2: If f(x) is a differentiable function then

$$\frac{d}{dx}\left(e^{f(x)}\right)=e^{f(x)}\cdot f'(x)$$

In words: the derivative of e to the f(x) is an exact copy of e to the f(x) times the derivative of f(x).

Example 3: Find the derivative of $f(x) = e^{3x}$

Solution: We will have to use Rule 2.
The exponent, 3x is a function of x whose derivative is 3.

$$f(x) = e^{3x}$$

$$f'(x) = e^{3x} \cdot 3$$

An exact copy of
the exponential function

Times the derivative of
the exponent

Example 4: Find the derivative of $f(x) = e^{2x^2+1}$

Solution:

$$f(x) = e^{2x^2+1}$$

$$f'(x) = e^{2x^2+1}(4x)$$

Again, we used Rule 2. So the derivative is the exponential function times the derivative of the exponent.

Or rewritten:

$$f'(x) = 4xe^{2x^2+1}$$

Example 5: Differentiate the function $f(t) = \dfrac{e^t}{e^t + e^{-t}}$

Solution: Using the quotient rule

$$f'(t) = \frac{\left(e^t + e^{-t}\right)e^t - e^t\left(e^t - e^{-t}\right)}{\left(e^t + e^{-t}\right)^2}$$

Keep in mind that the derivative of e^{-t} is $e^{-t}(-1)$
or $-e^{-t}$

Distribute e^t into the ( )'s.

$$f'(t) = \frac{e^{2t} + e^0 - e^{2t} + e^0}{\left(e^t + e^{-t}\right)^2}$$

Recall that $e^0 = 1$.

$$f'(t) = \frac{2}{\left(e^t + e^{-t}\right)^2}$$

Derivatives of Exponential & Logarithm Functions

- Examples:

(a) $R(w) = 4^w - 5\log_9 w$

(b) $f(x) = 3e^x + 10x^3 \ln x$

(c) $y = \dfrac{5e^x}{3e^x + 1}$

Derivatives of Hyperbolic Functions

The six hyperbolic functions are defined as follows,

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

We know that,

$$\frac{d}{dx}\left(e^{-x}\right) = -e^{-x}$$

Therefore,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^{x} - e^{-x}}{2}\right) = \frac{e^{x} - \left(-e^{-x}\right)}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

Derivatives of Hyperbolic Functions

For the rest we can either use the definition of the hyperbolic function and/or the quotient rule.
Here are all the six derivatives.

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Review Questions

Example 1 Differentiate each of the following functions.

(a) $f(x) = 2x^5 \cosh x$

(b) $h(t) = \dfrac{\sinh t}{t+1}$