Quasi-Isometries, Groups, and Metric Spaces

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Group as a Metric Space

A fundamental idea in geometric group theory is seeing a group as a metric space!

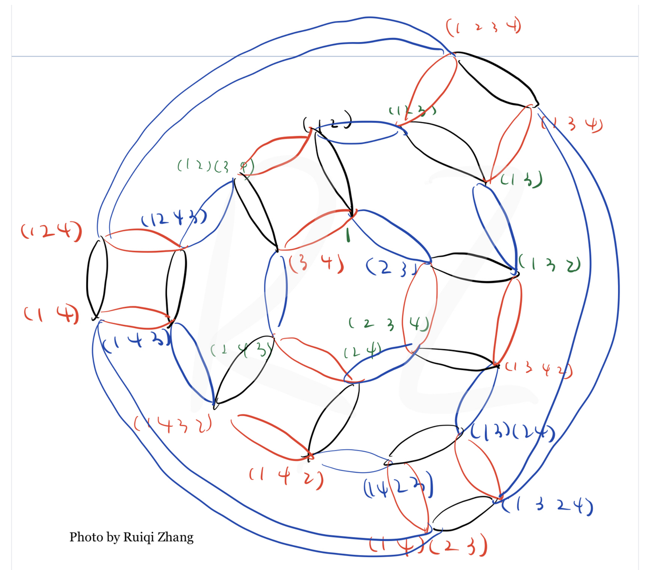
Let S be a generating set of a group, and S^{-1} is the set of inverses of elements of S. A **word** is a finite string of elements of $S \cup S^{-1}$.

The **word length** of $g \in G$ with respect to S is the length of the shortest word in $S \cup S^{-1}$ that is equal to $g \in G$. For example, the word length of the identity in G is 0.

We can now define a **word metric** on a group G. Let S be a generating set. We let the distance between $g, h \in G$ to be the word length of $g^{-1}h$ with respect to the generating set S.

Cayley Graphs

Let G be a group with generating set S. The **Cayley graph** of G with respect to S is a directed, labeled graph $\Gamma(G, S)$, where the vertices are the elements of G and there is a directed edge from g to gs for all $g \in G$ and $s \in S$, and this edge is labeled by the element s.



Cayley Graph of S_4

On the image above you can see that each vertex is an element of group S_4 and the black, blue and red edges correspond to generators $(1\,2)$, $(2\,3)$, and $(3\,4)$ respectively.

Another Cayley Graph of S_4

This is the same graph as the one on the right but a three dimensional version if you follow the paths along the square and hexagon faces of this solid!

Cayley Graph as a Metric Space

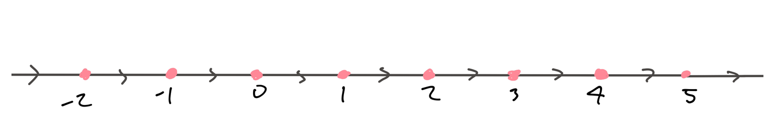
The **path metric** on a Cayley graph of a group G is defined as the length of the shortest edge path between the corresponding vertices.

We can actually find the word metric on a group G by using the path metric on its Cayley graph when considering the same generators! In the Cayley graph of S_4 above we see that the path distance between $(1\,2\,4)$ and $(1\,4)$ is 1 and this is the same as the word metric between these two elements.

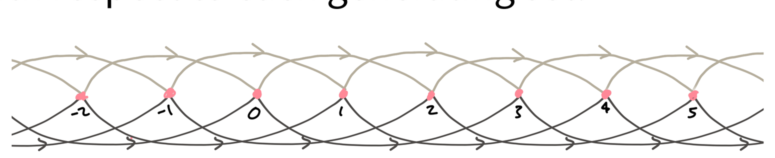
Two "Different" Cayley Graphs

It is important to note that the same group can have many different generators, and this means that we could have two different Cayley graphs, and so gives two different metrics.

An example of this is the group $\mathbb{Z}$. We can consider some different generation sets, $S_1 = \{1\}$ and $S_2 = \{2, 3\}$. The following images are the Cayley graphs with respect to each generating set:



(a) Generated with S_1, each edge is labeled with a 1



(b) Generated with S_2, the edges above correspond with the generator 2 and the edges below correspond to the generator 3

These in fact also give different metrics. Under the path metric, consider distance with respect to S_1 from -2 to 5 this would be 7, and the distance with respect to S_2 would be 3.

Even though they seem different, there is a notion where they are "equivalent."

Bi-Lipschitz Equivalence

We would like to develop a language to describe exactly how different (or similar) are our two word metrics on $\mathbb{Z}$.

Let (X, d_X) and (Y, d_Y) be metric spaces.

A function $f : X \to Y$ is called an **isometric embedding** if f preserves distances, that is, for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

An isometric embedding f is called an **isometry** if it is also surjective.

We're now going to weaken the notion of isometry by allowing distances to be stretched and compressed by bounded amounts.

A function $f : X \to Y$ is called a **bi-Lipschitz embedding** if there is some constant $K \geq 1$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

A bi-Lipschitz embedding f is a **bi-Lipschitz equivalence** if it is also surjective.

Theorem 7.2. *Let G be a finitely generated group and let S and S' be two finite generating sets for G. Then the identity map $f : G \to G$ is a bi-Lipschitz equivalence from the metric space (G, d_S) to the metric space $(G, d_{S'})$.*

That is, changing the generating set can stretch or compress distances between elements of G, but only by a bounded amount.

Quasi-Isometric Equivalence

Let (X, d_X) and (Y, d_Y) be metric spaces.

A function $f : X \to Y$ is called a **quasi-isometric embedding** if there are constants $K \geq 1$ and $C \geq 0$ so that

$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$$

for all $x_1, x_2 \in X$.

A quasi-isometric embedding $f : X \to Y$ is called a **quasi-isometric equivalence** or just **quasi-isometry** if there is a constant $D > 0$ so that for every point $y \in Y$, there is an $x \in X$ so that

$$d_Y(f(x), y) \leq D.$$

This last property can be seen as a relaxation of surjectivity.

We say two metric spaces (X, d_X), $(Y, d_Y)$are **quasi-isometric** if there is a quasi-isometry $f : X \to Y$.

This may seem really "loose" but there are many interesting quasi-isometries!

One example is the floor function function, $f : \mathbb{R} \to \mathbb{R}$ that takes $f(x) = n$ where n is the unique integer such that $n \leq x < n + 1$. It is a quasi-isometric embedding when you take constants $K = 1$ and $C = 1$ and taking $D = 1$ for it to be a quasi-isometry.

Theorem 7.3. *Let G be a finitely generated group and let S and S be two finite generating sets for G. Then the Cayley graph $\Gamma(G, S)$ is quasi-isometric to the Cayley graph $\Gamma(G, S')$.*

Proof : First, there is a quasi-isometry from any graph to its set of vertices (with the path metric) obtained by sending every point on an edge to a nearest vertex (there are sometimes two choices). Theorem 7.2 tells us that the identity map $G \to G$ induces a bi-Lipschitz map—hence a quasi-isometry—between the vertex sets of $\Gamma(G, S)$ and $\Gamma(G, S')$. Since the composition of two quasi-isometries is a quasi-isometry, and we also have a notion of a quasi-inverse, we obtain a quasi-isometry $\Gamma(G, S) \to \Gamma(G, S')$ as a composition of three quasi-isometries.

With this theorem we see that we do not need to worry about our generating sets of groups when we see them as a metric space, since they would be the same up to quasi-isometry.

Proper and Geodesic Metric Spaces

Let (X, d) be a metric space.

X is **proper** if for all $x \in X$ and for all $r > 0$ the closed ball centered at r, is a compact subset of $\mathbb{R}$.

A **geodesic segment** is an isometric embedding $\gamma : [a, b] \to X$ where $a, b \in \mathbb{R}$ and $a \leq b$.

A **geodesic** is an isometric embedding $\gamma : \mathbb{R} \to X$. This means that geodesic lines are just isometrically embedded copies of $\mathbb{R}$ in X!

This lets us define a **geodesic metric space**. That is, for any $x_1, x_2 \in X$ there exists a geodesic segment $\gamma : [a, b] \to X$ such that $\gamma(a) = x_1$ and $\gamma(b) = x_2$. In other words, any two points in the metric space are connected by a geodesic segment!

Groups Acting on a Metric Space

A **group action by isometries** is a map $G \times X \to X$ where the image of (g, x) is written as $g \cdot x$ and also where the following is satisfied:

- $1 \cdot x = x$ for all $x \in X$
- $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$
- $d(g \cdot x, g \cdot y) = d(x, y)$ for all $g \in G$ and all $x, y \in X$, this is really what makes it an action by *isometries*

An example of this would be that every group acts by isometries on itself with the word metric by left multiplication.

Acting Geometrically

In order to go into the Milnor-Schwarz Lemma it is important to talk about when a group acts "nicely", in other words, when it acts *geometrically*.

A group action of a group G on X is **properly discontinuous** if for each compact set $K \subset X$, the set $\{g \in G \,|\, gK \cap K \neq \emptyset\}$ is finite.

A group action of a group G on X is **cocompact** if for any basepoint $x_0 \in X$, there is an $R > 0$ such that for any $x \in X$ there is some $g \in G$ so that the closed ball centered at $g \cdot x_0$ with radius R contains x. In other words, the closed R-neighbourhood of the G-orbit of x_0 is all of X.

Intuitively properly discontinuous and cocompact are properties that limit the amount of separation and stretching of a space from applying the group action.

Now we can define what a **geometric** group action of G on X is! It is a properly discontinuous and cocompact action by isometries.

Milnor-Schwarz Lemma

Let G be a group and (X, d) be a proper geodesic metric space. Suppose that G acts geometrically on X. Then G is finitely generated and G is quasi-isometric to X.

Small Application of Milnor-Schwarz Lemma

We can use the Milnow-Schwarz lemma to prove that $\mathbb{Z}^2$ is quasi-isometric to $\mathbb{R}^2$.

We can consider $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ by translation: if $(a, b) \in \mathbb{Z}^2$ then $(a, b) \cdot (x, y) = (x + a, y + b)$. This is an action by isometries. If you consider a closed unit square on the plane, it only has 8 neighboring squares, and so only 8 distinct translations that do not move it completely off of itself. This gives the intuition and is likewise for any compact set (which in $\mathbb{R}^2$ is equivalent to closed and bounded), and so is a properly discontinuous group action. It is also cocompact if we take the radius $R = 2$. So $\mathbb{Z}^2$ acts geometrically on $\mathbb{R}^2$ and so they are quasi-isometric!

References

[1] Matt Clay and Dan Margalit.
Office hours with a geometric group theorist.
Princeton University Press, 2017.