

Finding correlation for acmPRS

November 9, 2016

Goals:

- Find $Var(\text{acmPRS})$
- Find $Cov(Y, \text{acmPRS})$

0.1 Finding $Var(\text{acmPRS})$:

Define

$$\hat{S}_i = \sum_{j=1}^{m_i} \hat{\beta} \mathbf{G}_j$$
$$\hat{S}_{acm} = \sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} \hat{S}_i$$

Theorem 1. $E[\hat{S}_{acm}] = 0$

Proof.

$$\begin{aligned} E[\hat{S}_{acm}] &= E\left[\sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} \hat{S}_i\right] \\ &= \sum_{i=1}^c E\left[n_i r_i \sqrt{\frac{h_i}{h_1}} \hat{S}_i\right] \\ &= \sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} E[\hat{S}_i] \end{aligned}$$

$$\because E[\hat{S}_i] = 0$$

$$\therefore E[\hat{S}_{acm}] = 0$$

□

Theorem 2. $Var(\hat{S}_{acm}) = \sum_{i=1}^c \frac{m_i n_i^2 r_i^2 h_i}{h_1} Var(\hat{\beta}_i) + \sum_{i \neq j} n_i n_j r_i r_j \sqrt{\frac{h_i h_j}{h_1^2}} Cov(\hat{S}_i, \hat{S}_j)$

Proof.

$$\begin{aligned}
Var\left(\sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} \hat{S}_i\right) &= \sum_{i=1}^c Var\left(n_i r_i \sqrt{\frac{h_i}{h_1}} \hat{S}_i\right) + \sum_{i \neq j} Cov\left(n_i r_i \sqrt{\frac{h_i}{h_1}} \hat{S}_i, n_j r_j \sqrt{\frac{h_j}{h_1}} \hat{S}_j\right) \\
&= \sum_{i=1}^c Var\left(n_i r_i \sqrt{\frac{h_i}{h_1}} \hat{S}_i\right) = \sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} Var(\hat{S}_i) \\
&= \sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} m_i Var(\hat{\beta}_i)
\end{aligned} \tag{1}$$

$$\begin{aligned}
\sum_{i \neq j} Cov\left(n_i r_i \sqrt{\frac{h_i}{h_1}} \hat{S}_i, n_j r_j \sqrt{\frac{h_j}{h_1}} \hat{S}_j\right) &= \sum_{i \neq j} E\left[\left(n_i r_i \sqrt{\frac{h_i}{h_1}} \hat{S}_i\right), \left(n_j r_j \sqrt{\frac{h_j}{h_1}} \hat{S}_j\right)\right] + E\left[\sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} \hat{S}_i\right] E\left[\sum_{j=1}^c n_j r_j \sqrt{\frac{h_j}{h_1}} \hat{S}_j\right] \\
&= \sum_{i \neq j} n_i n_j r_i r_j \sqrt{\frac{h_i h_j}{h_1^2}} E[\hat{S}_i \hat{S}_j] \\
&= \sum_{i \neq j} n_i n_j r_i r_j \sqrt{\frac{h_i h_j}{h_1^2}} Cov(\hat{S}_i, \hat{S}_j)
\end{aligned} \tag{2}$$

The above is because

$$\begin{aligned}
Cov(\hat{S}_i, \hat{S}_j) &= E[\hat{S}_i \hat{S}_j] + E[\hat{S}_i] E[\hat{S}_j] - E[\hat{S}_i] E[\hat{S}_j] \\
&= E[\hat{S}_i \hat{S}_j]
\end{aligned}$$

Combining (1) and (2):

$$Var(\hat{S}_{acm}) = \sum_{i=1}^c \frac{m_i n_i^2 r_i^2 h_i}{h_1} Var(\hat{\beta}_i) + \sum_{i \neq j} n_i n_j r_i r_j \sqrt{\frac{h_i h_j}{h_1^2}} Cov(\hat{S}_i, \hat{S}_j)$$

□

Theorem 3. $Cov(Y, acmPRS) = \frac{1}{c} \sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} m_i Cov(\hat{\beta}_i, \beta_Y)$

Proof.

$$\begin{aligned} Cov(Y, acmPRS) &= E[Y acmPRS] + \cancel{E[Y]} + \cancel{E[acmPRS]} \xrightarrow{0} 0 \\ &= E \left[\left(\sum_{j=1}^m \beta_{jY} G_j \right) \left(\sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} \sum_{j=1}^m \hat{\beta}_{ji} G_j \right) \right] \end{aligned}$$

I'm really not sure if I'm allowed to do this.

$$E \left[\left(\sum_{j=1}^m \beta_{jY} G_j \right) \left(\sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} \sum_{j=1}^m \hat{\beta}_{ji} G_j \right) \right] = E \left[\sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} \left[\sum_{j=1}^{m_i} \beta_j G_j \right] \frac{1}{c} \left[\sum_{j=1}^{m_Y} \beta_{jY} G_j \right] \right]$$

Here I've brought the Y score inside the summation with a $\frac{1}{c}$ coefficient.

$$= \frac{1}{c} \sum_{i=1}^c \left[n_i r_i \sqrt{\frac{h_i}{h_1}} E \left[\sum_{j=1}^{m_i} \beta_j G_j \sum_{j=1}^{m_Y} \beta_{jY} G_j \right] \right]$$

And since by equation (7) in power and predicitive accuracy: $E \left[\sum_{j=1}^{m_i} \beta_j G_j \sum_{j=1}^{m_Y} \beta_{jY} G_j \right] = m Cov(\hat{\beta}_i, \beta_Y)$

Then

$$Cov(Y, acmPRS) = \frac{1}{c} \sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} m_i Cov(\hat{\beta}_i, \beta_Y)$$

□

Theorem 4. $R_{Y,acmPRS}^2 = \frac{\frac{1}{c^2} \left(\sum_{i=1}^c n_i r_i \sqrt{\frac{h_i}{h_1}} m_i \text{Cov}(\hat{\beta}_i, \beta_Y) \right)^2}{\left(\sum_{i=1}^c \frac{m_i n_i^2 r_i^2 h_i}{h_1} \text{Var}(\hat{\beta}_i) + \sum_{i \neq j}^c n_i n_j r_i r_j \sqrt{\frac{h_i h_j}{h_1^2}} \text{Cov}(\hat{S}_i, \hat{S}_j) \right) (\sigma_g^2 + \sigma_e^2)}$

Proof.

$$R_{X,Y}^2 = \frac{\text{Cov}(X, Y)^2}{\text{Var}(X) \text{Var}(Y)}$$

Sub in equations in theorems (2) and (3).

□