HW 3

Charles Liu
Astronomy 138 - Dr. Matthew Duez
January 31st 2025

Recall, an object with mass m in a circular orbit with period P and radius d has orbital angular momentum $J_{\text{orb}} = \dfrac{2\pi m d^2}{P}$. If the object is in Keplerian orbit about an object of much higher mass M, its orbital period is $P = \sqrt{\dfrac{4\pi^2 d^3}{GM}}$.

A spherical object with uniform density of mass M and radius R spinning with period P has spin angular momentum $J_{\text{sp}} = \dfrac{4\pi}{5}\dfrac{MR^2}{P}$.

A) Compute the angular momentum of the Earth's spin, of the moon's spin, and of the moon's orbit about the Earth. Which of the three do you think it is safest to ignore?

HINTS: Be careful to use correct units when performing your calculations and in expressing your answers. Angular momentum has units kg $\mathrm{m^2 s^{-1}}$. For the spin angular momentum, your M, R , and P will be different for the Earth and the moon. In figuring out one of these numbers, it will be helpful to remember that the moon's spin is tidally locked to its orbit.

Earth's Spin's Angular Momentum

$$J_{\text{sp}} = \frac{4\pi}{5}\frac{MR^2}{P} = \frac{4\pi}{5}\frac{5.9722 \times 10^{24} \times (6371 \times 10^3)^2}{23.9345 \times 3600} = 7.07 \times 10^{33} \text{kg m}^2/\text{s}$$

Moon's Spin's Angular Momentum

$$P = \sqrt{\frac{4\pi^2 d^3}{GM}} = \sqrt{\frac{4\pi^2 \times (0.3844 \times 10^6 \times 10^3)^3}{6.67 \times 10^{-11} \times 5.9724 \times 10^{24}}} = 21,392,747.14\text{s} = 2372562\text{s}$$

$$J_{\text{sp}} = \frac{4\pi}{5}\frac{MR^2}{P} = \frac{4\pi}{5}\frac{0.0736 \times 10^2 4 \times (1737.4 \times 10^3)^2}{2372562} = 2.349 \times 10^{29} \text{kg m}^2/\text{s}$$

Moon's Orbit's Angular Momentum

$$J_{\text{orb}} = \frac{2\pi m d^2}{P} = \frac{2\pi \times 0.07346 \times 10^{24} \times (0.3844 \times 10^6 \times 10^3)^2}{2372562}$$

$$= 2.875 \times 10^{34} \text{kg m}^2/\text{s}.$$

B) We know that the moon has been slowly moving farther away from the Earth, so in the past it must have been closer. Suppose in the distant past, the moon started out basically touching the Earth (so that the center of the moon was one lunar radius from the surface of the Earth). If the moon were in Keplerian orbit at this time, what would have been its orbital period? Express this as a fraction of a day.

HINTS: For the distance between the Earth and the moon, use the distance between the center of the Earth and the center of the moon.

$$P = \sqrt{\frac{4\pi^2 d^3}{GM}} = \sqrt{\frac{4\pi^2 \times ((1737.4 + 6371) \times 10^3)^3}{6.67 \times 10^{-11} \times 5.9724 \times 10^{24}}} = 7268.5\text{s}$$

C) Continuing with the scenario in part B, what would have been moon's orbital angular momentum? Would the moon's orbital angular momentum have been higher then or now?

$$J_{\text{orb}} = \frac{2\pi m d^2}{7268.5} = \frac{2\pi \times 0.07346 \times 10^{24} \times ((1737.4 + 6371) \times 10^3)^2}{7268.5}$$

$$= 4.175 \times 10^{33} \text{kg m}^2/\text{s}$$

It's *higher now*.

D) Assume that the moon then had negligible spin and had the orbital angular momentum you found in part C. Work out how much more angular momentum the moon has now compared to how much it had then $\Delta J_{moon} = J_{\text{moon}} - J_{\text{moon then}}$. Because the total angular momentum of the whole Earth-moon system is conserved (ignoring small tidal torques from the sun), the Earth must have had this much *more* angular momentum then. What then would have been the spin angular momentum of the Earth?

$$\Delta J_{moon} = 2.875 \times 10^{34} - 4.175 \times 10^{33} = 2.4575 \times 10^{34} \text{kg m}^2/\text{s}$$

Thus, the moon has $2.46 \times 10^{34} \text{kg m}^2/\text{s}$ more angular momentum compared to the past.

All angular momentum lost by Earth is transferred to the moon (assuming negligible moon spin).

Thus,
$J_{\text{sp}} = 2.46 \times 10^{34} + 7.07 \times 10^{33} = 3.167 \times 10^{34} \text{kg m}^2/\text{s}$

E) In your own words, explain why the moon has been moving farther from the Earth, and the Earth's spin has been slowing down, for the past few billion years.

Tidal bulges on Earth are caused by the Moon's pull on Earth's oceans. Since Earth rotates, the bulges are a little bit ahead of the moon, causing energy to transfer from Earth's spin to the Moon's orbit. This tug pushes the Moon further out.

Citations

"Earth Fact Sheet." NASA. Accessed January 31, 2025. https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html.
"Moon Fact Sheet." NASA. Accessed January 31, 2025. https://nssdc.gsfc.nasa.gov/planetary/factsheet/moonfact.html.
