HW 6

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1. Consider the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$, and assume that the rows of matrix A are linearly independent. Prove that if two different bases correspond to the same basic solution, then this basic solution is degenerate.

Theorem. For the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ where A has linearly independent rows, if two different bases correspond to the same basic solution, then that basic solution is degenerate.

Proof:
We want to show that the basic solution is degenerate, which is equivalent to showing that at least one variable in x is zero.
Assume that we have two distinct bases B_1 and B_2 that produce the same basic solution, x.
BWOC, assume that x has all (m) nonzero variables. This means all m variables in B_1 and B_2 are nonzero.
As B_1 and B_2 are distinct, there's at least one variable in B_1 that's not in B_2.
However, this means there is $m + 1$ nonzero variables in x, a contradiction, so x must be degenerate.

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2. Consider nonempty polyhedron $P \subset \mathbb{R}^n$ and suppose that for each variable x_i we have either the constraint $x_i \geq 0$ or the constraint $x_i \leq 0$. Prove that P has at least one basic feasible solution.

Theorem. For the nonempty polyhedron $P \subset \mathbb{R}^n$, if we have that each variable x_i either satisfies the constraint $x_i \geq 0$ or $x_i \leq 0$, then P has at least one basic feasible solution.

Proof:
Since P is nonempty, we can take an element x where $x \in P$ and x is feasible. Consider a

line $S = \{y \in \mathbb{R}^n \,|\, y = x + \lambda d\}$, where $\lambda \in \mathbb{R}$, $d \neq 0$ and $d \in \mathbb{R}^n$.
Take some constraint in the polyhedra, where $i = j$. Also consider another element in d which is non-zero, d_j.

Case 1: $x_j \geq 0$ and $d_j > 0$, then $x_j + \lambda d_j \notin P$ when $\lambda < -\frac{x_j}{d_j}$.
Case 2: $x_j \geq 0$ and $d_j < 0$, then $x_j + \lambda d_j \notin P$ when $\lambda > -\frac{x_j}{d_j}$.
Case 3: $x_j \leq 0$ and $d_j > 0$, then $x_j + \lambda d_j \notin P$ when $\lambda < -\frac{x_j}{d_j}$.
Case 4: $x_j \leq 0$ and $d_j < 0$, then $x_j + \lambda d_j \notin P$ when $\lambda > -\frac{x_j}{d_j}$.

Since there can be no line in P, by **Theorem 2.6**, P has an extreme point which by **Theorem 2.3** is equivalent to P having a basic feasible solution.