HW 7

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1. Let R be the ring of real-valued continuous functions on $[-1, 1]$. Show that R has zero-divisors. Note that in this ring $f(x) = 0$ is the additive identity.

Theorem. For a ring R of the real-valued continuous functions on $[-1, 1]$, R has zero-divisors.

Proof:

Take two functions, $g(x), h(x)$.

Let $g(x) = \begin{cases} 0 & \text{if } x \in [-1, 0) \\ x & \text{if } x \in [0, 1] \end{cases}$

and $h(x) = \begin{cases} x & \text{if } x \in [-1, 0] \\ 0 & \text{if } x \in [0, 1] \end{cases}$

Since $\lim_{x \to 0^-} g(x) = 0 = \lim_{x \to 0^+} g(x)$ and $g(x)$ from $[-1, 0)$ and $[0, 1]$ are both continuous, $g(x)$ on $[-1, 1]$ is continuous.

Similarly, $h(x)$ is continuous on $[-1, 1]$.

Thus, $g(x) \in R$ and $h(x) \in R$.

Although $g(x) \neq 0$ and $h(x) \neq 0$, $g(x) \cdot h(x) = 0$, So, $g(x)$ and $h(x)$ are both zero-divisors and R has zero divisors.

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2. Show that the intersection of two ideals is an ideal.

Theorem. The intersection of two ideals is an ideal.

Proof:

Let S_1, S_2 be two ideals of ring R.
Since $S_1 \subset R$ and $S_2 \subset R$, $S_1 \cap S_2 \subset R$.

For any given element $s \in S_1 \cap S_2$, it must lie in both S_1 and S_2.

Since $s \in S_1$, it must satisfy the definition of an ideal, that being $rs \in S_1$ and $sr \in S_1$ for any $r \in R$.

Similarly, since $s \in S_2$, it must satisfy $rs \in S_2$ and $sr \in S_1$.

Since rs and sr are both in S_1 and S_2, $rs \in S_1 \cap S_2$ and $sr \in S_1 \cap S_2$.

Thus, $S_1 \cap S_2$ is an ideal.

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3. Show, by example, that the intersection of two prime ideals need not be a prime ideal.

Theorem. The intersection of two prime ideals need not be a prime ideal.
Proof:
Take prime ideals $3\mathbb{Z}$ and $5\mathbb{Z}$ of $\mathbb{Z}$.

$3\mathbb{Z} \cap 5\mathbb{Z} = 15\mathbb{Z}$.

We can solve that $3 \cdot 5 = 15 \in 15\mathbb{Z}$, but $3 \notin 15\mathbb{Z}$ and $5 \notin 15\mathbb{Z}$.

From this example, we can see that the intersection of prime ideals is not necessarily a prime ideal.

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4. Let $I = \langle 2 \rangle$. Prove that $I[x]$ is not a maximal ideal of $\mathbb{Z}[x]$ even though I is a maximal ideal of $\mathbb{Z}$.

Theorem. $I[x]$ where $I = \langle 2 \rangle$ is not a maximal ideal of $\mathbb{Z}[x]$
Proof:
Consider $\langle 2, x \rangle$. This is the set of all polynomials with integer coefficients where the constant term is even. We know that $\langle 2, x \rangle \subset \mathbb{Z}[x]$.

Given any $a = a_0 + a_1x + ... + a_kx^k \in \langle 2, x \rangle$ and
$r = r_0 + r_1x_1 + ... + r_mx^m \in \mathbb{Z}[x]$ we know that $a_0 \in 2\mathbb{Z}$,
$a_0, a_1, ... a_k, r_0, r_1, ... r_k \in \mathbb{Z}$ and $k, m \in \mathbb{N}$.

We find that ra and ar must be in $\langle 2, x\rangle$ as $r_0 \cdot a_0$ must be even and all other coefficients are linear combinations of integers and must be integers.
Thus, $\langle 2, x\rangle$ is an ideal of $\mathbb{Z}[x]$.

We also know that $I[x] \subset \langle 2, x\rangle$ where $I = \langle 2\rangle$. Given $a \in \langle 2, x\rangle$ as defined earlier, we can take $a_1, a_2, \ldots a_k = 0$ and produce all elements in $I[x]$.

Thus, $I[x]$ cannot be a maximal ideal of $\mathbb{Z}[x]$.

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5. Find the characteristic of $\mathbb{Z}[i]/\langle 2+i\rangle$.

Theorem. The characteristic of $\mathbb{Z}[i]/\langle 2+i\rangle$ is 5.
Proof:
We know that $\mathbb{Z}[i]/\langle 2+i\rangle$ has unity 1.
We want to find $n \in \mathbb{N}$ where when $a + bi \in \mathbb{Z}[i]$, $(a+bi)(2+i) = n$.
This means $(2a - b + (a+2b)i) = n$.
Since n has no complex component as it is an integer, $a + 2b = 0$ so $a = -2b$. Using substitution, we get that $-5b = n$. Taking $b = -1$, we get $n = 5$, the minimal possible value.
Thus, the characteristic of $\mathbb{Z}[i]/\langle 2+i\rangle$ is 5.

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6. Let $S = \{a + bi \mid a, b \in \mathbb{Z}, b \text{ is even}\}$. Show that S is a subring of $\mathbb{Z}[i]$, but not an ideal of $\mathbb{Z}[i]$.

Theorem. S is a subring of $\mathbb{Z}[i]$.
Proof:
Take $a = 0, b = 0$. Since $0 \in S$, S is nonempty.
Since $2\mathbb{Z} \in \mathbb{Z}$ (the evens are a subset of all integers), $S \subset \mathbb{Z}[i]$.

Take $x = a_1 + b_1 i$ and $y = a_2 + b_2 i$ where $a_1, a_2 \in \mathbb{Z}$ and $b_1, b_2 \in 2\mathbb{Z}$. We know that $x, y \in S$.

We can solve that

$$\begin{aligned} xy &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1a_2 + a_2b_1 i + a_1b_2 i - b_1b_2 \\ &= (a_1a_2 - b_1b_2) + (a_2b_1 + a_1b_2)i \end{aligned}$$

Note that since both b_1 and b_2 are even, $a_2b_1 + a_1b_2$ is also even.
Since $a_1a_2 - b_1b_2 \in \mathbb{Z}$ and $a_2b_1 + a_1b_2 \in 2\mathbb{Z}$, $xy \in S$.

We can also solve that
$x + y = a_1 + b_1i + a_2 + b_2i = (a_1 + a_2) + (b_1 + b_2)i$

Note that since both b_1 and b_2 are even, $b_1 + b_2$ is even.
Since $a_1 + a_2 \in \mathbb{Z}$ and $b_1 + b_2 \in 2\mathbb{Z}$, $x + y \in S$.

Thus, S is a subring of $\mathbb{Z}[i]$.

Theorem. S is not an ideal of $\mathbb{Z}[i]$.
Proof:
If S is an ideal of $\mathbb{Z}[i]$, then for arbitrary elements $s \in S$ and $z \in \mathbb{Z}[i]$, $sz \in S$.
By definition, $s = 1 + 2i$ and $z = 1 + i$.
$sz = (1 + 2i)(1 + i) = 1 + 2i + i - 2 = -1 + 3i \notin S$.
Thus, S is not an ideal of $\mathbb{Z}[i]$.