2.1 Polyhedra and Convex Sets

Introduction to Linear Optimization
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Hyperplanes, Halfspaces, and Polyhedra

> **Definition 2.1** A *polyhedron* is a set that can be described in the form $\{x \in \mathbb{R}^n \mid Ax \geq b\}$, where A is an $m \times n$ matrix and b is a vector in $\mathbb{R}^m$.

The *feasible set* of any linear programming problem can be described by the inequality constraints of the form $Ax \geq b$, and is therefore a polyhedron.

A set of the form $\{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ is also a polyhedron and is called *standard form representation*.

> **Definition 2.2** A set $S \subset \mathbb{R}^n$ is *bounded* if there exists a constant K such that the absolute value of every component of every element of S is less than or equal to K.

The polyhedron can either "extend to infinity" or be confined.

> **Definition 2.3.** Let a be a nonzero vector in $\mathbb{R}^n$ and let b be a scalar.
> *(a)* The set $\{x \in \mathbb{R}^n \mid a^T x = b\}$ is called a *hyperplane*.
> *(b)* The set $\{x \in \mathbb{R}^n \mid a^T x \geq b\}$ is called a *halfspace*.

Notes:

- A hyperplane is the boundary of a corresponding halfspace.
- Vector a is perpendicular to the hyperplane itself.
  - $a^T x = a^T y$ where x, y in hyperplane
  - $\implies a^T(x - y) = 0$
  - $\implies a$ is orthogonal to the hyperplane.
- Any polyhedron is the intersection of finite number of halfspaces

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Convex Sets

> **Definition 2.4.** A set $S \subset \mathbb{R}^n$ is *convex* if for any $x, y \in S$, and any $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda) y \in S$

Notes:

The line segment connecting any two of a convex sets elements must be contained in the set.

> **Definition 2.5.** Let $x^1, \ldots x^k$ be vectors in $\mathbb{R}^n$ and let $\lambda_1, \ldots \lambda_k$ be nonnegative scalars whose sum is unity.
>
> *a)* The vector $\sum_{i=1}^{k} \lambda_1 x^1$ is said to be a *convex combination* of the vectors $x^1, \ldots, x^k$.
>
> *b)* The *convex hull* of the vectors $x^1, \ldots, x^k$ is the set of all convex combinations of these vectors.

Notes:

"Sum is unity" means they sum to 1.

> ### Theorem 2.1
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> *a)* The intersection of convex sets is also convex
>
> *b)* Every polyhedron is a convex set.
>
> *c)* A convex combination of a finite number of elements of a convex set also belongs to that set.
>
> *d)* The convex hull of a finite number of vectors is a convex set.

Notes:

The convex hull of a finite number of polyhedra is a polyhedra, so it is a convex set.
