HW 8

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Exercise 3.17

Solve completely (i.e., both Phase I and Phase II) via the simplex method the following problem:

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 \\ \text{subject to} \quad & x_1 + 3x_2 + 0x_3 + 4x_4 + x_5 = 2 \\ & x_1 + 2x_2 + 0x_3 - 3x_4 + x_5 = 2 \\ & -x_1 - 4x_2 + 3x_3 + 0x_4 + 0x_5 = 1 \\ & x_1, \dots, x_5 \geq 0 \end{aligned}$$

Below is our tableau.

	x_1	x_2	x_3	x_4	x_5
0	2	3	3	1	-2
2	1	3	0	4	1
2	1	2	0	-3	1
1	-1	-4	3	0	0

There is no easy basic feasible solution.

Phase 1:

We have to construct a basis, so we add two slack variables (already have x_3 basis vector)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	0	0	0	0	1	1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	2	3	3	1	-2	0	0
2	1	3	0	4	1	1	0
2	1	2	0	-3	1	0	1
1	-1	-4	3	0	1	0	0

We want to make x_3, x_6, and x_7 columns basic.

Fix x_3 column by multiplying -1 to 5th row and adding to 2th row

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	0	0	0	0	1	1
-1	3	7	0	1	-2	0	0
2	1	3	0	4	1	1	0
2	1	2	0	-3	1	0	1
1	-1	-4	3	0	1	0	0

Fix x_6, x_7 columns by multiplying -1 to rows 3 and 4 and adding to row 1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-4	-2	-5	0	-1	-2	0	0
-1	3	7	0	1	-2	0	0
2	1	3	0	4	1	1	0
2	1	2	0	-3	1	0	1
1	-1	-4	3	0	1	0	0

We take the x_1 column since it has a reduced cost that's negative. Using ratio test, we take the 3rd row. We exit x_6 from the basis and let x_1 in the basis.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	1	0	7	0	2	0
-7	0	-2	0	-11	-5	-3	0
2	1	3	0	4	1	1	0

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	-1	0	-7	0	-1	1
3	0	-1	3	4	1	1	0

We gave x_1, x_3, and x_7 in our basis. We will now shift the basis to columns within x_1 to x_5.

Multiply 4th row by -1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	1	0	7	0	2	0
-7	0	-2	0	11	-5	-3	0
2	1	3	0	4	1	1	0
0	0	1	0	7	0	1	-1
3	0	-1	3	4	1	1	0

Enter x_2 in our basis, exit x_3.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	0	0	0	0	1	1
-7	0	0	0	3	-5	-1	-2
2	1	0	0	-17	1	-2	3
0	0	1	0	7	0	1	-1
3	0	0	3	11	1	2	-1

Phase 2:

Now we proceed with the simplex method.

	x_1	x_2	x_3	x_4	x_5
-7	0	0	0	3	-5
2	1	0	0	-17	1
0	0	1	0	7	0
3	0	0	3	11	1

Take x_5 column as it has negative reduced cost, do the ratio test.
Enter x_5 into basis, exit x_1.

	x_1	x_2	x_3	x_4	x_5
3	5	0	0	-82	0
2	1	0	0	-17	1
0	0	1	0	7	0
1	-1	0	3	28	0

Now take x_4 column which also has negative reduced cost. Do ratio test, enter x_4 into basis, exit x_2.

	x_1	x_2	x_3	x_4	x_5
3	5	$\frac{82}{7}$	0	0	0
2	1	$\frac{17}{7}$	0	0	1
0	0	$\frac{1}{7}$	0	1	0
1	-1	$-\frac{28}{7}$	3	0	0

All reduced costs are positive, so we are done and at an optimal solution.
The optimal vector $x = (0, 0, \frac{1}{3}, 0, 2)$, where the minimal value is -3.

Exercise 3.19

While solving a standard form problem, we arrive at the following tableau, with x_3, x_4, and x_5 being the basic variables:

-10	δ	-2	0	0	0
4	-1	η	1	0	0
1	α	-4	0	1	0
β	γ	3	0	0	1

The entries $\alpha, \beta, \gamma, \delta, \eta$ in the tableau are unknown parameters. For each one of the following statements, find some parameter values that will make the statement true.

a) The current solution is optimal and there are multiple optimal solutions

$\beta = 0, \delta = 0, \alpha > 0, \gamma = 0, \eta \in \mathbb{R},$

b) The optimal cost is $-\infty$

$\beta \geq 0, \delta < 0, \alpha \leq 0, \gamma \leq 0, \eta \in \mathbb{R}$

c) The current solution is feasible but not optimal.

Rewriting tableau:

$-10 + \delta$	0	$-2 + \eta\delta$	δ	0	0
4	-1	η	1	0	0
$1 + 4\alpha$	0	$-4 + \eta\alpha$	α	1	0
$\beta + 4\gamma$	0	$3 + \eta\gamma$	γ	0	1

$\beta \geq 0, \delta < 0, \alpha = 0, \gamma = 1, \eta = -3$