HW 7

Exercise 3.3

Let x be an element of the standard form polyhedron $P \in \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$. Prove that a vector $d \in \mathbb{R}^n$ is a feasible direction at x if only if $Ad = 0$ and $d_i \geq 0$ for every i such that $x_i = 0$.

Proof:

We will first prove the forward direction.

A feasible direction only exists when there is a positive scalar θ where $x + \theta d \in P$.

Assume $d \in \mathbb{R}^n$ is a feasible direction. We can solve that

$$A(x + \theta d) = Ax + \theta Ad = b + \theta Ad$$

However, since $x + \theta d \in P$,

$$A(x + \theta d) = b$$

So,
$b = b + \theta Ad$
and $\theta Ad = 0$, which means $Ad = 0$.
Since $x + \theta d \geq 0$, $x_i + \theta d_i \geq 0$. When $x_i = 0$, we can see that $\theta d_i \geq 0$.

Thus, the forward direction is proved.

We will now prove the converse.
Assume $Ad = 0$ and $d_i \geq 0$ for every i such that $x_i = 0$.

We can solve that

$$A(x + \theta d) = Ax + \theta Ad = Ax = b$$

So $A(x + \theta d) = b$.

Since $x \geq 0$ and $d \geq 0$, we can find $\theta \geq 0$ so that $x + \theta d \geq 0$.
So, $x + \theta d \in P$, and d is feasible.

Exercise 3.12

Consider the problem

$$\begin{aligned} \text{minimize } 2x_1 - x_2 & \\ x_1 - x_2 &\leq 2 \\ x_1 + x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

a) Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.

$$\begin{aligned} \min_x z = \ & 2x_1 - x_2 \\ & x_1 - x_2 + x_3 = 2 \\ & x_1 + x_2 + x_4 = 6 \\ & x \geq 0, x \in \mathbb{R}^4 \end{aligned}$$

b) Carry out the full tableau implementation of the simplex method, starting with the basic feasible solution of part (a).

Basic feasible solution: $(x_1, x_2) = (0, 0)$

	x_1	x_2	x_3	x_4
$-z = 0$	-2	-1	0	0
$x_3 = 2$	1	-1	1	0
$x_4 = 6$	1	1	0	1

	x_1	x_2	x_3	x_4
$-z = 4$	0	-3	2	0
$x_3 = 2$	1	-1	1	0
$x_4 = 6$	1	1	0	1

	x_1	x_2	x_3	x_4
$-z = 4$	0	-3	2	0
$x_3 = 2$	1	-1	1	0
$x_4 = 4$	0	2	-1	1

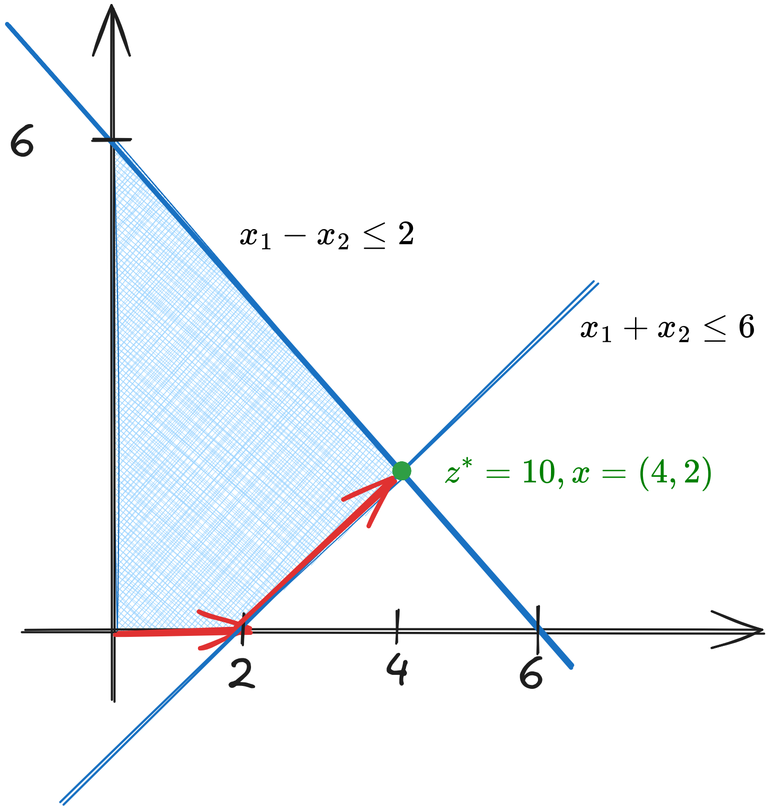
	x_1	x_2	x_3	x_4
$-z = 4$	0	-3	2	0
$x_1 = 2$	1	-1	1	0
$x_4 = 4$	0	1	-0.5	0.5

	x_1	x_2	x_3	x_4
$-z = 4$	0	-3	2	0
$x_1 = 2$	1	-1	1	0
$x_4 = 2$	0	1	-0.5	0.5

	x_1	x_2	x_3	x_4
$-z = 10$	0	0	0.5	1.5
$x_1 = 2$	1	-1	0.5	0
$x_4 = 2$	0	1	-0.5	0.5

	x_1	x_2	x_3	x_4
$-z = 10$	0	0	0.5	1.5
$x_1 = 4$	1	0	0.5	0.5
$x_4 = 2$	0	1	-0.5	0.5

c) Draw a graphical representation of the problem in the terms of the original variables x_1, x_2, and indicate the path taken by the simplex algorithm.



6
$x_1 - x_2 \leq 2$
$x_1 + x_2 \leq 6$
$z^* = 10, x = (4, 2)$
2
4
6