HW 5

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Linear Optimization - Dr. Tom Asaki

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1. Determine whether or not the following sets are polyhedra. Use clear reasoning and appeal to the definition of a polyhedron and/or relevant geometric theorems from Chapter 2 of the textbook.

(a) $\Omega_1 = \emptyset$

> **Theorem.** $\Omega_1 = \emptyset$ is a polyhedron.
>
> *Proof:*
> By *Definition 2.1* in our textbook, a polyhedron is a set that can be described in the form $\{x \in \mathbb{R}^n \mid Ax \geq b\}$ where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$.
>
> Take A as the zero matrix, and b as an all-ones matrix. This is an equivalent representation of the empty set, as no value of $x \in \mathbb{R}^n$ can satisfy the inequality $Ax \geq b$. Since the empty set can be described in the form from *Definition 2.1*, it is a polyhedron.

(b) $\Omega_2 = \{x \in \mathbb{R} \mid x \geq 5, x \leq 0\}$

> **Theorem.** $\Omega_2 = \{x \in \mathbb{R} \mid x \geq 5, x \leq 0\}$ is *NOT* a polyhedron.
>
> *Proof:*
> BWOC, assume that Ω_2 is a polyhedron.
> From *Theorem 2.1* in our textbook, every polyhedron must be convex, so Ω_2 must be convex.
> By *Definition 2.4* in our textbook, for $x, y \in \Omega_2$, $\lambda x + (1 - \lambda)y \in \Omega_2$ must be satisfied for Ω_2 to be convex.
> Take $x = 5, y = 0$, and $\lambda = 0.5$. Since $x, y \in \Omega_2$ and $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y$

should be in Ω_2.

However, $\lambda x + (1 - \lambda)y = 0.5 \cdot 5 + 0.5 \cdot 0 = 2.5 \notin \Omega_2$, a contradiction.

Thus, Ω_2 is *NOT* a polyhedron.

(c) $\Omega_3 = \{x \in \mathbb{R} \mid x^2 - 9 \leq 0\}$

Theorem. $\Omega_3 = \{x \in \mathbb{R} \mid x^2 - 9 \leq 0\}$ is a polyhedron.

Proof:

Take $Y_1 = \{x \in \mathbb{R} \mid -x \geq -3\}$ and $Y_2 = \{x \in \mathbb{R} \mid x \geq -3\}$.

$x^2 - 9 \leq 0$

$\leftrightarrow x \leq 3$ and $x \geq -3$

$\leftrightarrow -x \geq -3$ and $x \geq -3$

So $\Omega_3 = Y_1 \cap Y_2$.

By *Definition 2.1* in our textbook, Y_1 is a halfspace as it is in the form $\{x \in \mathbb{R} \mid Ax \geq b\}$. Similarly, Y_2 is a halfspace.

Since a polyhedron is an intersection of finite halfspaces and $\Omega_3 = Y_1 \cap Y_2$, Ω_3 is a polyhedra.

(d) $\Omega_4 = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$.

Theorem. $\Omega_4 = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ is *NOT* a polyhedron.

Proof:

We can write that $x = (x_1, x_2, \ldots, x_n)$, where $x \in \mathbb{R}^n$. This means that $||x|| = x_1^2 + x_2^2 + \ldots x_n^2$. Since $||x|| = x_1^2 + x_2^2 + \ldots x_n^2 \leq 1$ is nonlinear, it must require an intersection of infinite halfspaces.

Thus, Ω_4 is *not* a polyhedron.

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2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let c be some constant. Prove that
$S = \{x \in \mathbb{R}^n \mid f(x) \leq c\}$ is convex.

Theorem. $S = \{x \in \mathbb{R}^n \mid f(x) \leq c\}$ is convex.

Proof:

Let $a, b \in S$.

By *Definition 1.1* in our textbook, since f is convex,

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b)$$

for $\lambda \in [0, 1]$.

It follows that

$$f(\lambda a + (1 - \lambda)b) \leq c\lambda + (1 - \lambda)c \leq c$$

So,

$$\lambda a + (1 - \lambda)b \in S$$

and S is convex by *Definition 2.4*.