HW 5

Charles Liu
Elementary Modern Algebra - Dr. Ben Clark
March 5th, 2025

---

1. Let $|a| = 30$. How many left cosets of $\langle a^4 \rangle$ in $\langle a \rangle$ are there? List them.

Theorem. There are two left cosets of $\langle a^4 \rangle$ in $\langle a \rangle$ where $|a| = 30$.

Proof:

By Lagrange's Theorem, we know that the number of left cosets of $\langle a^4 \rangle$ in $\langle a \rangle$ is $\frac{|\langle a^4 \rangle|}{|\langle a \rangle}|$

We know that $|\langle a \rangle| = 30$.
We can solve that $|\langle a^4 \rangle| = \frac{30}{\gcd(30,4)} = 15$

Thus, we can find that there are $\frac{30}{15} = 2$ left cosets.

The two cosets are:
$\langle a^4 \rangle$
$a\langle a^4 \rangle$

---

2. What is the order of the element $14 + \langle 8 \rangle$ in the factor group $\mathbb{Z}_{24} / \langle 8 \rangle$?

Theorem. The order of the element $14 + \langle 8 \rangle$ in the factor group $\mathbb{Z}_{24} / \langle 8 \rangle$ is 4

Proof:

In the factor group, $14 + \langle 8 \rangle \equiv 6 + \langle 8 \rangle$.

We want to solve the smallest number $n \in \mathbb{N}$ where $n(6 + \langle 8 \rangle) = 0 + \langle 8 \rangle$

We find this number is $n = \frac{\operatorname{lcm}(6,8)}{6} = \frac{24}{6} = 4$.

Thus, the order is 4.

---

3. Let G be a group with $|G| = pq$, where p and q are prime. Prove that every proper subgroup of G is cyclic.

Lemma 1. Every group of prime order is cyclic.
Let G be a group with prime order p.
By Lagrange's Theorem, for $g \in G$, $|g| \mid p$ or $|g| \mid 1$.
Take $a \in G$ such that $a \neq e$. We know that $|\langle a \rangle| = p$ since a is not the identity element.
Since $\langle a \rangle \in G$ and $|\langle a \rangle| = |G| = p$, $|\langle a \rangle| = G$, so G must be cyclic.

Theorem. For a group G with $|G| = pq$ where p and q are prime, every proper subgroup of G is cyclic.
Proof:
By Lagrange's Theorem, for $H \leq G$, $|H| \mid pq$. However, $H \neq G$ as H is a proper subgroup so $|H| \neq pq$. This means either $|H| = 1$, $|H| = p$, or $|H| = q$.

Case 1: $|H| = 1$
If $|H| = 1$, $H = \{e\}$ and is thus cyclic.

Case 2: $|H| = p$ or $|H| = q$
If $|H| = p$ or $|H| = q$, is cyclic by *Lemma 1*.

Thus, all proper subgroups of G are cyclic.

---

4. The group $(\mathbb{Z}_4 + \mathbb{Z}_{12})/\langle (2,2) \rangle$ is isomorphic to one of $\mathbb{Z}_8$, $\mathbb{Z}_4 + \mathbb{Z}_2$ or $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$. Determine which one by eliminating the other two.

Theorem. $(\mathbb{Z}_4 + \mathbb{Z}_{12})/\langle (2,2) \rangle \cong \mathbb{Z}_4 + \mathbb{Z}_2$
Proof:
The order of quotient group $(\mathbb{Z}_4 + \mathbb{Z}_{12})/\langle (2,2) \rangle$ is $\frac{|\mathbb{Z}_4 + \mathbb{Z}_{12}|}{|\langle (2,2) \rangle|} = \frac{48}{6} = 8$, so by *Fundamental Theorem of Finite Commutative Groups*, $(\mathbb{Z}_4 + \mathbb{Z}_{12})/\langle (2,2) \rangle$ must be isomorphic to to one of $\mathbb{Z}_8$, $\mathbb{Z}_4 + \mathbb{Z}_2$ or $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$.

Eliminate: $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$
We notice that the maximum order of elements in $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$ is $\operatorname{lcm}(2,2,2) = 2$.
However, we can take $(0,3) \in \mathbb{Z}_4 + \mathbb{Z}_{12}$.

Let $H = \langle (2,2) \rangle = \{(0,0), (2,2)(0,4), (2,6), (0,8), (2,10)\}$.
We solve that the order of $(0,3) + H$ is 4 by doing the following:
$(0,3) + H \neq H$

$(0,6)+H \neq H$
$(0,9)+H \neq H$
$(0,0)+H = H$

Since an element of order 4 exists in $(\mathbb{Z}_4+\mathbb{Z}_{12})/\langle(2,2)\rangle$ but there isn't one in $\mathbb{Z}_2+\mathbb{Z}_2+\mathbb{Z}_2$, the two can't be isomorphic.

Eliminate: $\mathbb{Z}_8$
We want to show that in $\mathbb{Z}_8$ there are only 2 elements of order 4.

An element of order 4 in $\mathbb{Z}_8$ is $\langle \frac{8}{4} \rangle = \langle 2 \rangle$

We want to find any other $m \in \mathbb{Z}_8$ of order 4. In other words, we want to find m where $\langle m \rangle = \langle 2 \rangle$.

The only elements that work are elements where $\gcd(m,8)=2$, which can also be expressed as $\gcd(\frac{m}{2},4)=1$.

The only numbers coprime with 4 and less than it are 1 and 3, so $\frac{m}{2}=1,3$ and $m=2,6$.

Thus, we find that there are two elements in $\mathbb{Z}_8$ with order 4.

However, in $(\mathbb{Z}_4+\mathbb{Z}_{12})/\langle(2,2)\rangle$, the following all have order 4:
$(3,0)+H$
$(0,3)+H$
$(1,0)+H$
$(0,1)+H$

So, there must be at least 4 elements of order 4 in $(\mathbb{Z}_4+\mathbb{Z}_{12})/\langle(2,2)\rangle$, but there is only 2 in $\mathbb{Z}_8$, so the two can't be isomorphic.

Conclusion:

Thus, $(\mathbb{Z}_4+\mathbb{Z}_{12})/\langle(2,2)\rangle \cong \mathbb{Z}_4+\mathbb{Z}_2$.

---

5. Prove that a quotient group of a commutative group is commutative.

Theorem. The quotient group of a commutative group is commutative.
Proof:
Let G be a commutative group and $H \leq G$.

For $a, b \in G$, the group operation of quotient group is

$$(aH)(bH) = (ab)H$$

Since G is commutative, we can see that

$$(aH)(bH) = (ab)H = (ba)H = (bH)(aH)$$

so the quotient group G/H is commutative.