HW 4

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Graph Theory - Dr. Abigail Cortez
March 2nd, 2025

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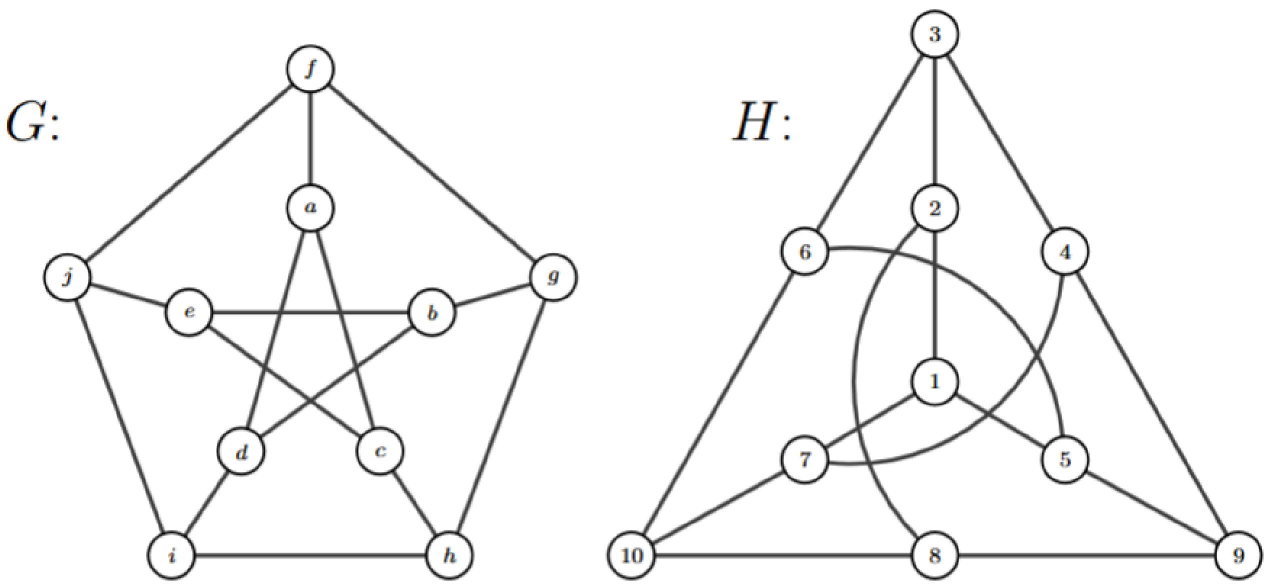
1. Determine whether the graphs below are isomorphic.
   If the graphs are isomorphic, give an appropriate one-to-one correspondence.
   Otherwise, prove that the graphs are not isomorphic.



Let $\phi : V(G) \to V(H)$.

We can take

$\phi(a) = 1$
$\phi(f) = 2$
$\phi(j) = 3$
$\phi(e) = 6$
$\phi(c) = 5$
$\phi(h) = 9$
$\phi(g) = 8$
$\phi(b) = 10$

$\phi(d) = 7$
$\phi(i) = 4$

Since ϕ is bijective and preserves edges, it is an isomorphism. Since an isomorphism exists between G and H, G and H are isomorphic.

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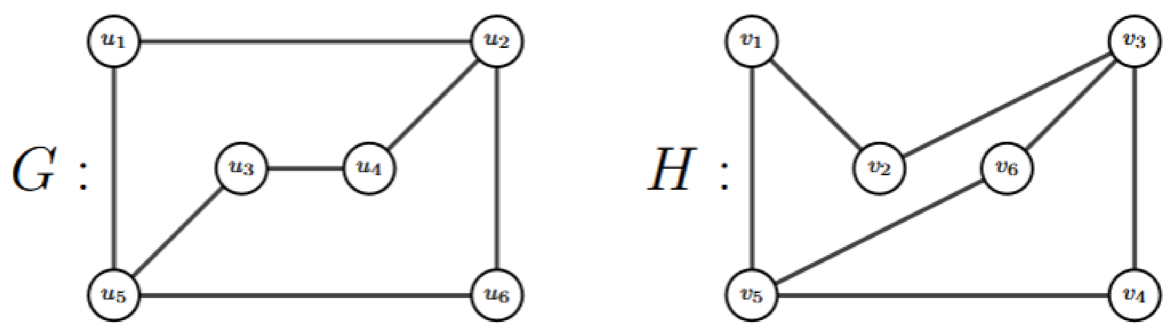
2. Determine whether the graphs below are isomorphic.
   If the graphs are isomorphic, give an appropriate one-to-one correspondence.
   Otherwise, prove that the graphs are not isomorphic.



We can find isomorphism $\phi : V(G) \to V(H)$, where
$\phi(u_3) = v_5$
$\phi(u_4) = v_4$
$\phi(u_2) = v_1$
$\phi(u_1) = v_2$
$\phi(u_5) = v_3$
$\phi(u_6) = v_6$

ϕ is bijective and preserves edges, so it is an isomorphism. Since an isomorphism exists between G and H, G and H are isomorphic.

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3. Determine whether the graphs below are isomorphic.
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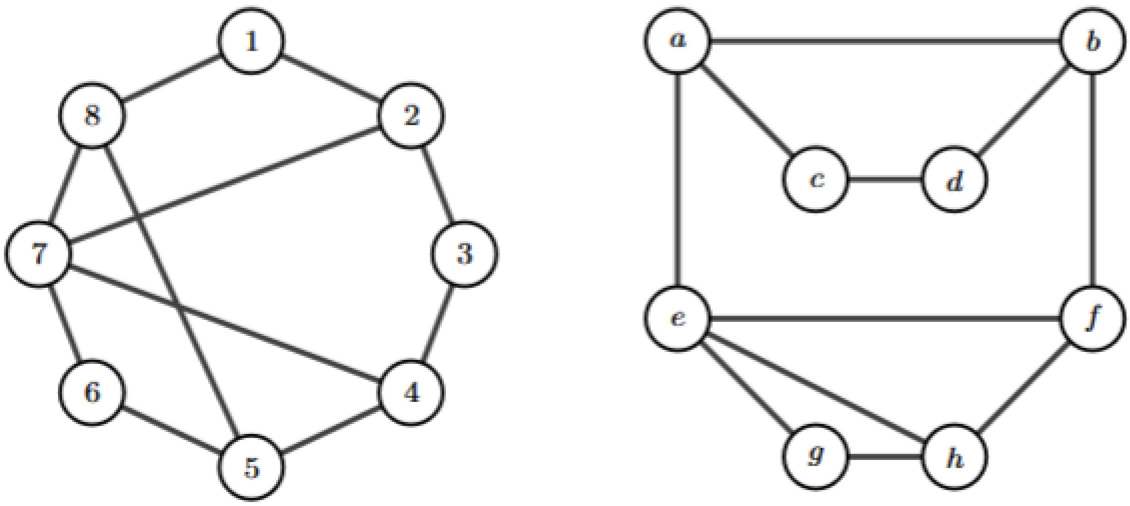

No, the graph on the right of the image has a three-cycle (e, g, h) but the graph on the left doesn't, so it can't preserve that structure.

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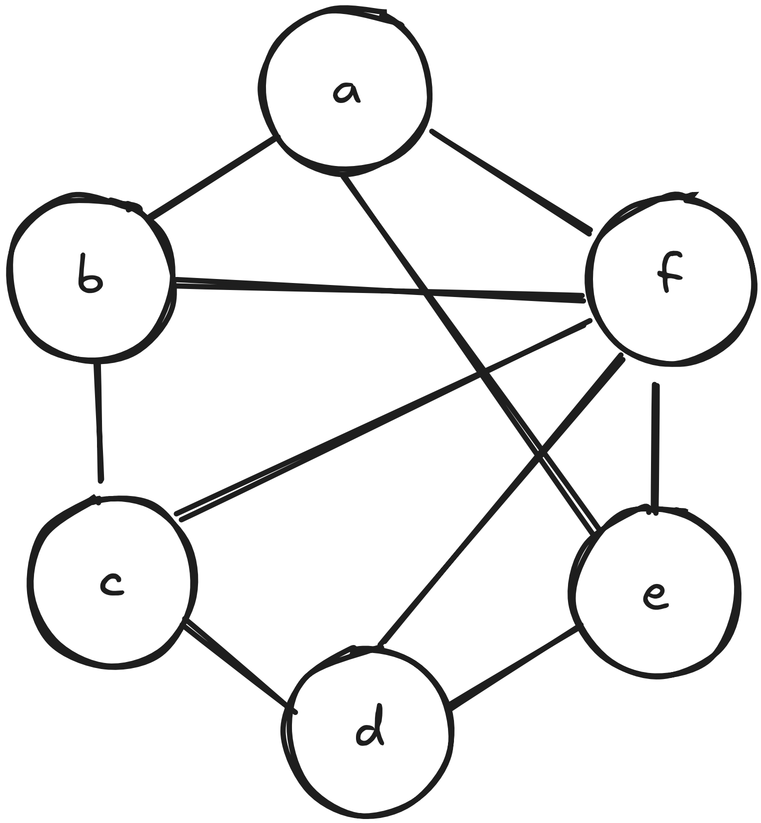
4. Consider the following incidence matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) Draw the graph G having B as its incidence matrix.



(b) Compute the adjacency matrix A and the degree matrix D for the graph B.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

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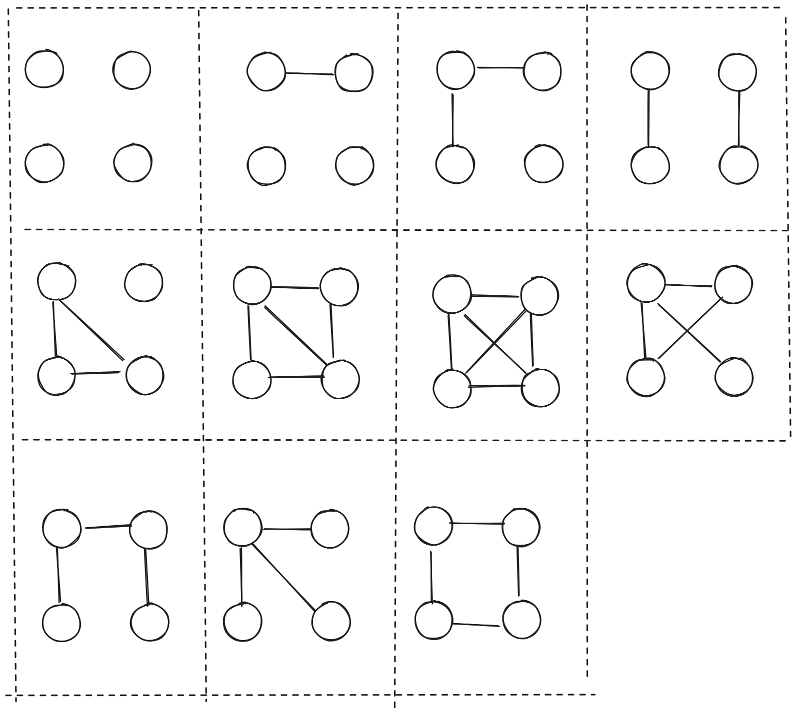
5. Draw all nonisomorphic graphs with 4 vertices.



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6. Show that the maximum size of a bipartite graph of order 10 is 25.

Theorem. The maximum size of a bipartite graph of order 10 is 25.

Proof:

Let G be a bipartite graph of order 10. Since G is bipartite, it can be written as $G = U \cup V$ where $U, V \neq \emptyset$ are disjoint subsets of G and $|U| + |V| = 10$.

Let the order of U be n. Then the order of V is $10 - n$.

All edges must connect one vertex in U to one in V. At maximum, each vertex in U can share an edge with every vertex in V for a total of $(10 - n)$ edges per each of the n vertices.

Thus, the size of G is less than or equal to $(10 - n)n$ where $1 \leq n \leq 10$ (there must be at least one vertex in both U and V).

To find the maximum of $f(n) = (10 - n)n = 10n - n^2$, we will take the derivative.
We solve that $f'(n) = -2n + 10$.
We get a critical point at $n = 5$.
We find that when $n > 5$, $f'(n) < 0$, and when $n < 5$, $f'(n) > 0$, so the point $(5, f(5))$ is an absolute maximum.

We can then solve that $f(5) = 50 - 25 = 25$, so the maximum size of G is 25.

7. Determine the maximum size of a bipartite graph of order $n \geq 2$. Justify your conclusion.

Theorem. The maximum size of a bipartite graph of order $n \geq 2$ is $\frac{n^2}{4}$ when n is even, and $n^2 - \frac{1}{2}n$ when n is odd.

Proof:

Let G be a bipartite graph of order $n \geq 2$. It can be written as $G = U \cup V$ where $U, V \neq \emptyset$ are disjoint subsets of G and $|U| + |V| = n$.

Let the order of U be m. Then the order of V is $n - m$.

All edges must connect one vertex in U to one in V. At maximum, each vertex in U can share an edge with every vertex in V for a total of $(10 - m)$ edges per each of the m vertices.

Thus, the size of G is less than or equal to $(n - m)m$ where $1 \leq m \leq 10$ (there must be at least one vertex in both U and V).

To find the maximum of $f(m) = (n - m)m = nm - m^2$, we will take the derivative with respect to m.
We solve that $f'(m) = -2m + n$.
We get a critical point at $m = \frac{n}{2}$.

We find that when $m > \frac{n}{2}$, $f'(m) < 0$, and when $m < \frac{n}{2}$, $f'(m) > 0$, so the point $(\frac{n}{2}, f(\frac{n}{2}))$ is an absolute maximum.

Case 1: n is even
We can then solve that $f(\frac{n}{2}) = \frac{n^2}{2} - \frac{n^2}{4} = \frac{n^2}{4}$, so the maximum size of G is $\frac{n^2}{4}$.

Case 2: n is odd
Since $m = \frac{n}{2}$ isn't an integer, we need to take the closest integer that still maximizes $f(m)$.
Since n is odd, we can write it as $n = 2k + 1$ where $k \in \mathbb{N}$.
The closest two integers are $\lfloor \frac{n}{2} \rfloor = 2k$ and $\lceil \frac{n}{2} \rceil = 2k + 2$
We can solve that

$$f'(\lfloor \frac{n}{2} \rfloor) = n\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor^2 = n(2k) - 4k^2$$

$$f'(\lceil \frac{n}{2} \rceil) = n\lceil \frac{n}{2} \rceil - \lceil \frac{n}{2} \rceil^2 = n(2k + 2) - 4(k + 2)^2$$

$= n(2k) + 2n - 4k^2 - 16k - 16$
$= n(2k) + 2n - 4k^2 - 8n - 8$
$= n(2k) - 4k^2 - 6n - 8$

So $f'(\lfloor \frac{n}{2} \rfloor) \geq f'(\lceil \frac{n}{2} \rceil)$.
Thus, for the odd case, the maximum size of G is
$f(\lfloor \frac{n}{2} \rfloor) = n\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n}{2} \rfloor(n-1) = n(\frac{n-1}{2}) = n^2 - \frac{1}{2}n$.

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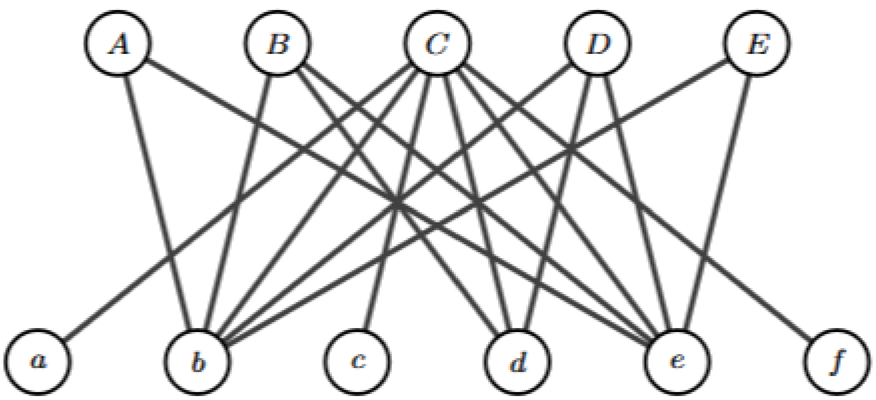
8. There are five women at a party and six men.
   The women: Anise (A), Betty (B), Celina (C), Daniela (D), and Evelyn (E).
   The men: Alejandro (a), Ben (b), Charles (c), Douglas (d), Eddie (e), and Frank (f).
   The compatible dancing partnerships are displayed by means of a graph below.



Is it possible to have all women dancing so that each dancing partnership is a compatible one? Justify your conclusion.

No.
Alejandro, Charles, and Frank are only compatible with Celina. This means after Celina goes with someone, the other two will be compatible partnerless. This leaves 3 more men (Ben, Douglas, and Eddie) with compatible partners and 4 women (Anise, Betty, Daniela, and Evelyn). Since there are less men than women, at least one women will go partnerless.

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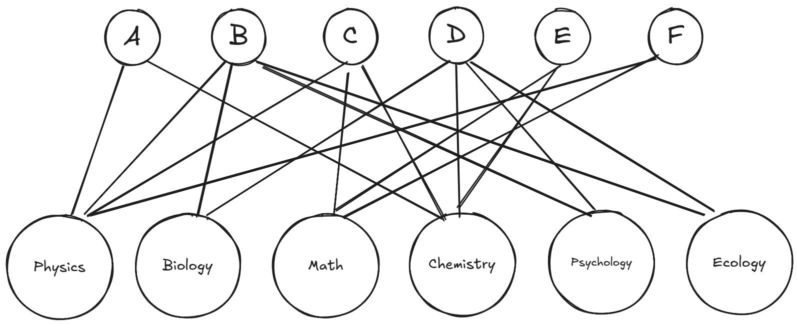
9. A high school has openings for six teachers, with one teacher needed for each of these areas:
   mathematics, chemistry, physics, biology, psychology, and ecology.
   In order to hire a teacher in a particular area, he or she must have either majored or

minored
in that subject. The school receives six applicants for these positions:

Mr. Arrowsmith. major: physics; minor: chemistry
Mr. Beckman. major: biology; minors: physics, psychology, ecology
Ms. Chase. major: chemistry; minors: mathematics, physics
Mrs. Deerfield. majors: chemistry, biology; minors: psychology, ecology
Mr. Evans. major: chemistry; minor: mathematics
Ms. Form. major: mathematics; minor: physics

(a) Draw a graph to represent this situation.



(b) What is the largest number of applicants the school can hire? Justify your answer.

The school can hire at maximum 5. Mr. Beckman and Mr. Deerfield are the only teachers for biology, psychology, and ecology, so one of those subjects will be guaranteed to lack a teacher. As a result, one teacher won't be hired as he would otherwise teach a repeat subject.