HW 1

Charles Liu
Elementary Modern Algebra - Dr. Ben Clarke
January 18th 2025

---

1. Given the sets below, perform the following operations:

$$A = \{0, 1, 2\},\ B = \{1, 2, 3\},\ C = \{0, 2, 4, 5\}$$

(a) $A \cap B \cap C$
By associativity,
$A \cap B \cap C = (A \cap B) \cap C = \{1, 2\} \cap \{0, 2, 4, 5\} = \{2\}$

(b) $(A \cup B) - (B \cup C)$
$(A \cup B) - (B \cup C) = \{0, 1, 2, 3\} - \{0, 1, 2, 3, 4, 5\} = \{0\}$

(c) $(B \cap A) \cup (A \cap C)$
$(B \cap A) \cup (A \cap C) = \{1, 2\} \cup \{0, 2\} = \{0, 1, 2\}$

(d) $B \times C$
$B \times C = \{(1, 0), (1, 2), (1, 4), (1, 5), (2, 0), (2, 2), (2, 4), (2, 5), (3, 0), (3, 2), (3, 4), (3$

---

2. Given the sets below, determine if the claims are true or false. Justify your response

$$X = \{0, 1, 2, 3, 4\}, Y = \{1, 3, 5, 7\}, Z = \{x \in \mathbb{R} \mid x > 0\}$$

(a) $(X \cup \emptyset) \subseteq Z$
False.

Since $0 \in X$ but $0 \notin Z$, $(X \cup \emptyset) = X \nsubseteq Z$.

(b) $X \cap Y \neq \emptyset$
True.
$X \cap Y = \{1, 3\} \neq \emptyset$

(c) $|X \cap Z| = 4$

True.

$X \cap Z = \{1, 2, 3, 4\}$, which means $|X \cap Z| = |\{1, 2, 3, 4\}| = 4$.

(d) $Y \cap Z \subseteq Z$

True.

In general, for an element $a \in Y \cap Z, a \in Z$. Thus, $Y. \cap Z \subseteq Z$.

(e) $3 \in X \cap Y \cap Z$

True.

Since $3 \in X, 3 \in Y$, and $3 \in Z, 3 \in X \cap Y \cap Z$.

---

3. Let n be a positive integer, and let A_i be an arbitrary subset of X for each $i \in Z$. Prove DeMorgan's laws:

(a) $X - \bigcap_{i=1}^{n} A_i = \bigcup_{i=1}^{n}(X - A_i)$

Theorem. $X - \bigcap_{i=1}^{n} A_i = \bigcup_{i=1}^{n}(X - A_i)$

Proof.

For any set $D \subseteq X$, let $D^c = X - D$.

Let n be a positive integer, and let A_i be an arbitrary subset of X for each $i \in Z$. It suffices to show $(\bigcap_{i=1}^{n} A_i)^c = \bigcup_{i=1}^{n}(A_i)^c$.

Lemma 1. $(\bigcap_{i=1}^{n} A_i)^c \subseteq \bigcup_{i=1}^{n}(A_i)^c$

Proof.

Let $y \in (\bigcap_{i=1}^{n} A_i)^c$. We want to show $y \in \bigcup_{i=1}^{n}(A_i)^c$. Since $y \in (\bigcap_{i=1}^{n} A_i)^c$, we can see that $y \notin \bigcap_{i=1}^{n} A_i$. Therefore, $y \notin A_1$ or $y \notin A_2$, or ... $y \notin A_n$. Thus, $y \in A_1{}^c$ or $y \in A_2{}^c$, or ... $y \notin A_n{}^c$. By definition of union, $y \in \bigcup_{i=1}^{n}(A_i)^c$. Thus, $(\bigcap_{i=1}^{n} A_i)^c \subseteq \bigcup_{i=1}^{n}(A_i)^c$.

Lemma 2. $\bigcup_{i=1}^{n}(A_i)^c \subseteq (\bigcap_{i=1}^{n} A_i)^c$

Proof.

Let $z \in \bigcup_{i=1}^{n}(A_i)^c$. We want to show $z \in (\bigcap_{i=1}^{n} A_i)^c$. Since $z \in \bigcup_{i=1}^{n}(A_i)^c$, $z \in A_1{}^c$ or $z \in A_2{}^c$, or ... $z \in A_n{}^c$. So $z \notin A_1$ or $z \notin A_2$ or ... $z \notin A_n$. So we can see that, $z \notin (\bigcap_{i=1}^{n} A_i)$ so $z \in (\bigcap_{i=1}^{n} A_i)^c$. Thus, $\bigcup_{i=1}^{n}(A_i)^c \subseteq (\bigcap_{i=1}^{n} A_i)^c$.

Because of Lemma 1 and Lemma 2,

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n}(A_i)^c$$

.

(b) $X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n}(X - A_i)$

Theorem. $X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n}(X - A_i)$

Proof.

For any $D \subseteq X$,let $D^c = X - D$. It suffices to show $\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$. By part (a), we know $(\bigcap_{i=1}^{n} B_i)^c = \bigcup_{i=1}^{n}(B_i)^c$ for any $B_i \subseteq X$ where $i = 1, 2, \ldots n$.
By taking the complement of both sides, we can find that $\bigcap_{i=1}^{n} B_i = (\bigcup_{i=1}^{n}(B_i)^c)^c$.
Set $B_i = (A_i)^c$. Then $(B_i)^c = (A_i^c)^c = A_i$ for $i = 1, 2, \ldots n$.

So the above equation becomes

$$\bigcap_{i=1}^{n} A_i^c = \left(\bigcup_{i=1}^{n} A_i\right)^c$$

---

4. Prove or provide a counter example to the following claims about set operations:

(a) The union of two sets is commutative.

Theorem. The union of two sets is commutative.

Proof.

Suppose we have two sets A and B. Suppose there is an element x which is in $A \cup B$. By definition of union, it must be in sets A or B. This means x is also in $B \cup A$, which means $(A \cup B) \subseteq (B \cup A)$
Likewise, suppose we have an element y in $B \cup A$. Since it is in B or A, it must be in $A \cup B$. Therefore, $B \cup A \subseteq A \cup B$
Since $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$, we can see that $A \cup B = B \cup A$, so the union of two sets is commutative.

(b) The union of two sets is associative.

Theorem. The union of two sets is associative.

Proof.

Suppose we have three sets A, B, C. Let x be an element in $(A \cup B) \cup C$. By definition of union, it must be in one ore more of the sets A, B and C. Therefore, it must also be in $A \cup (B \cup C)$, which means $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

Similarly, let y be any element in $A \cup (B \cup C)$. By definition of union, it must be in one or more of the sets A, B and C. Therefore, it must also be in $(A \cup B) \cup C$, which means $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.
Since $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ and $A \cup (B \cup C) \subseteq (A \cup B) \cup C$, $(A \cup B) \cup C = A \cup (B \cup C)$, so the union of two sets is associative.

(c) The intersection of two sets is commutative.

Theorem. The intersection of two sets is commutative.

Proof.

Suppose we have two sets A and B. Let x be an element in $A \cap B$. By the definition of intersection, x must be in both A and B. Thus, it must also be in $B \cap A$. Therefore, $A \cap B \subseteq B \cap A$.
Likewise, let y denote any element in $B \cap A$. By definition of intersection, $y \in B$ and $y \in A$, so it must also be in $A \cap B$. This means $B \cap A \subseteq A \cap B$.
Since $A \cap B \subseteq B \cap A$ and $B \cap A \subseteq A \cap B$, $A \cap B = B \cap A$. Thus, the intersection of two sets is commutative.

(d) The intersection of two sets is associative.

Theorem. The intersection of two sets is associative.

Proof.

Suppose we have three sets A, B, C. Let x be an element in $(A \cap B) \cap C$. By definition of intersection, x must be in A, B, and C. This means it is also in $A \cap (B \cap C)$, so $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.
Similarly, let y be an element in $A \cap (B \cap C)$. By definition of intersection, x must be in A, B, and C. This means it is also in $(A \cap B) \cap C$, so $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.
Since $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ and $(A \cap B) \cap C$, so $A \cap (B \cap C) \subseteq (A \cap B) \cap C$, $(A \cap B) \cap C = A \cap (B \cap C)$, so intersection is associative.

(e) The Cartesian product of two sets is commutative.

Theorem. The Cartesian product of two sets is *not* commutative.

Proof.

Suppose $A = \{0, 1\}$ and $B = \{2, 3\}$.
$A \times B = \{(0, 2), (0, 3), (1, 2), (1, 3)\} \neq B \times A = \{(2, 0), (2, 1), (3, 0), (3, 1)\}$
Thus, the Cartesian products between two sets are not commutative.

(f) The Cartesian product of two sets is associative.

Theorem. The Cartesian product of sets is *not* associative.

Proof.

Suppose there are three sets A, B, C, where $A = \{0, 1\}$, $B = \{2, 3\}$, and $C = \{4, 5\}$. Then

$$\begin{aligned}(A \times B) \times C &= \{(0,2),(0,3),(1,2),(1,3)\} \times \{4,5\} \\ &= \{(0,2,4),(0,3,4),(1,2,4),(1,3,4),(0,2,5),(0,3,5),(1,2,5),(1,3,\end{aligned}$$

In contrast, $A \times (B \times C) = \{0, 1\} \times (\{(2,4),(2,5),(3,4),(3,5)\}$

Since $(A \times B) \times C \neq A \times (B \times C)$, the Cartesian product of sets is *not* associative.
