HW 8

Charles Liu
Elementary Modern Algebra - Dr. Ben Clark
April 8th, 2025

1. Is the mapping from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{10}$ given by $x \to 2x$ a ring homomorphism?

Theorem. $\mathbb{Z}_{10}$ to $\mathbb{Z}_{10}$ given by $x \to 2x$ is not a ring homomorphism.
Proof:

$\phi : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$
$\phi(x) = 2x$

Consider $3, 4 \in \mathbb{Z}_{10}$
$\phi(3 \cdot 4) = \phi(12) = 2 \neq \phi(3)\phi(4) = 6 \cdot 8 \text{ mod } 10 = 48 \text{ mod } 10 = 8$

2. Let

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} \text{ and } H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$$

Show that $\mathbb{Z}[\sqrt{2}]$ and H are isomorphic as rings.

Theorem. $\mathbb{Z}[\sqrt{2}]$ and H are isomorphic as rings.
Proof:
Consider the mapping $\phi : \mathbb{Z}[\sqrt{2}] \to H$ defined as $\phi(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ where $a, b \in \mathbb{Z}$.

Ring Homomorphism

Consider $x, y \in \mathbb{Z}[\sqrt{2}]$, where $x = a_1 + b_1\sqrt{2}$ and $y = a_2 + b_2\sqrt{2}$.

$$\phi(x+y) = \begin{bmatrix} a_1+a_2 & 2(b_1+b_2) \\ b_1+b_2 & a_1+a_2 \end{bmatrix} = \begin{bmatrix} a_1 & 2b_1 \\ b_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & 2b_2 \\ b_2 & a_2 \end{bmatrix} = \phi(x)+\phi(y)$$

$$\phi(x \times y) = \phi((a_1+b_1\sqrt{2}) \times (a_2+b_2\sqrt{2})) = \phi(a_1a_2+b_1b_2+(a_1b_2+a_2b_1)\sqrt{2})$$

$$= \begin{bmatrix} a_1a_2+b_1b_2 & 2(a_1b_2+a_2b_1) \\ a_1b_2+a_2b_1 & a_1a_2+b_1b_2 \end{bmatrix} = \begin{bmatrix} a_1 & 2b_1 \\ b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & 2b_2 \\ b_2 & a_2 \end{bmatrix} = \phi(x)\phi(y)$$

Bijective

Assume $\phi(x) = \phi(y)$. This means $\begin{bmatrix} a_1 & 2b_1 \\ b_1 & a_1 \end{bmatrix} = \begin{bmatrix} a_2 & 2b_2 \\ b_2 & a_2 \end{bmatrix}$, so $a_1 = a_2$ and $b_1 = b_2$. This also implies that $a_1 + b_1\sqrt{2} = a_2 + b_2\sqrt{2}$, so $x = y$ and ϕ is one-to-one.

Consider $h = \begin{bmatrix} a_h & 2b_h \\ b_h & a_h \end{bmatrix} \in H$. We can find $x \in \mathbb{Z}[\sqrt{2}]$ where $\phi(x) = y$ is $a_h + b_h\sqrt{2}$, so h is onto.

Since there exists a mapping ϕ between $\mathbb{Z}[\sqrt{2}]$ and H that is both bijective and ring homomorphic, and thus isomorphic, the two rings $\mathbb{Z}[\sqrt{2}]$ and H are isomorphic as rings.

---

3. Determine all the ring isomorphisms from $\mathbb{Z}_n$ to itself.

Theorem. $\phi(x) = x$ represents all ring isomorphisms from $\mathbb{Z}_n$ to itself.
Proof:
We first find all the generators of the group $\mathbb{Z}_n$. 1 is the standard generator.
Any number a where $\gcd(a, n) = 1$ is also a generator.
Any ring isomorphism on cyclic groups has the property that generators map to generators.
So, $\phi(1) = a$.
We then produce a mapping, $\phi(x) = ax$.

To make an isomorphism, we need to prove ring homomorphism and bijectiveness.
Ring Homomorphism:
$\phi(x+y) = a(x+y) = ax+ay = \phi(x)+\phi(y)$
$\phi(xy) = a(xy) = \phi(x)\phi(y) = axay = a^2xy$
From preserving homomorphism with multiplication, we see that we need $a^2 \bmod n = a \bmod n$. The only element that works for this is $a = 1$
Bijectiveness

1-1

Assume $\phi(x) = \phi(y)$. This means $ax = ay$, and $x = y$. So ϕ is one-to-one.

Onto

Take $y \in \mathbb{Z}_n$. Since the generator a produces all elements in $\mathbb{Z}_n$, we can always find some $k \in \mathbb{Z}_n$ where $ka = y$. So ϕ is onto.

So, $\phi(x) = x$ represents all ring isomorphisms of $\mathbb{Z}_n$ to itself.

---

4. Let n be a positive integer. Show that there is a ring isomorphism from $\mathbb{Z}_2$ to a subring of $\mathbb{Z}_{2n}$ if and only if n is odd.

Theorem. There is a ring isomorphism from $\mathbb{Z}_2$ to a subring of $\mathbb{Z}_{2n}$ if and only if n is odd.

Proof:

"$\Rightarrow$"

Assume n is odd.

We can take the subring $\mathbb{Z}_{\mathbb{2}} \to \mathbb{Z}_{\mathbb{2}}$

Take $\phi : \mathbb{Z}_2 \to \mathbb{Z}_2$, where $\phi(x) = x$.

ϕ is a *ring homomorphism*.

Consider $x, y \in \mathbb{Z}_2$

$\phi(x + y) = x + y = \phi(x) + \phi(y)$

$\phi(xy) = xy = \phi(x)\phi(y)$

ϕ is *bijective*.

Assume $\phi(x) = \phi(y)$. This means $x = y$, so ϕ is one-to-one.

Take $y \in \mathbb{Z}_2$. We can always find $x = y \in \mathbb{Z}_2$ where $\phi(x) = y$. So ϕ is onto.

Thus, ϕ is a *ring isomorphism* when n is odd, and there doest exist a ring isomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}_{2n}$.

"$\Leftarrow$"

Assume there exists an isomorphism from $\mathbb{Z}_2$ to a subring of $\mathbb{Z}_{2n}$

By way of contradiction, assume n is even. Lets denote $n = 2k$.

Then we would need a ring isomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}_{4k}$.

However, we can't take a subring from $\mathbb{Z}_{4k}$ of order 2, so we can never establish a ring isomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}_{4k}$ as it would never be bijective.

Thus, n must be odd.

---

5. Give that f is a polynomial of degree n in P_n (vector space of polynomials of degree at most n with real coefficients), show that $\{f, f', f'', \ldots, f^{(n)}\}$ is a basis for P_n.

Theorem. $\{f, f', f'', \ldots, f^{(n)}\}$ is a basis for P_n.

Proof:

Consider the polynomial f which is order n.

By definition f' is an order $n-1$ polynomial, f'' is order $n-2$, and $f^{(k)}$ is an order $n-k$ polynomial.

Consider any polynomial $g \in P^n = a_0 + a_1x + a_2x^2 + \ldots a_nx^n$.

We can write $g = c_1f + c_2f' + c_3f'' \ldots c_nf^{(n)}$, as each component ($k$th derivative of f) has a unique order spanning 0 to n, so the polynomial can always be generated.

---

6. Prove that for a vector space V over a field that does not have characteristic 2, the hypothesis that V is commutative under addition is redundant (we can prove it from the other properties).

Theorem. For a vector space V, commutativity under addition is redundant if

Proof:

Consider $v, w \in V$. We know that

$v + w + v + w = 2 \times (v + w) = 2v + 2w = v + v + w + w$, as $2(v + w) \neq 0$, $2v \neq 0$, and $2w \neq 0$ (their is no characteristic of 2).

So, $v + w + v + w = v + v + w + w$ and $w + v = v + w$.

Thus, commutative is redundant.
