HW 10

Charles Liu
Linear Optimization - Dr. Tom Asaki
March 28th, 2025

Example 5.1

$$\begin{aligned} \text{minimize } & -5x_1 - x_2 + 12x_3 \\ \text{subject to } & 3x_1 + 2x_2 + x_3 = 10 \\ & 5x_1 + 3x_2 + x_4 = 16 \\ & x_1, \ldots, x_4 \geq 0 \end{aligned}$$

Optimal solution: $x = (2, 2, 0, 0)$
Corresponding simplex tableau:

	x_1	x_2	x_3	x_4
12	0	0	2	7
x_1 = 2	1	0	-3	2
x_2 = 2	0	1	5	-3

Exercise 5.4

Consider the problem in Example 5.1, with a_{11} changed from 3 to $3 + \delta$. Let us keep x_1 and x_2 as the basic variables and let $B(\delta)$ be the corresponding basis matrix, as a function of δ.

a) Compute $B(\delta)^{-1}b$. For which values of δ is $B(\delta)$ a feasible basis?

$$B(\delta) = \begin{bmatrix} 3+\delta & 2 \\ 5 & 3 \end{bmatrix}$$

$$B(\delta)^{-1} = \frac{1}{3(3+\delta) - 2 \cdot 5} \begin{bmatrix} 3 & -2 \\ -5 & 3+\delta \end{bmatrix} = \frac{1}{-1+3\delta} \begin{bmatrix} 3 & -2 \\ -5 & 3+\delta \end{bmatrix}$$

$$B(\delta)^{-1}b = \frac{1}{-1+3\delta}\begin{bmatrix} 3 & -2 \\ -5 & 3+\delta \end{bmatrix}\begin{bmatrix} 10 \\ 16 \end{bmatrix} = \frac{1}{-1+3\delta}\begin{bmatrix} -2 \\ -2+16\delta \end{bmatrix}$$

$B(\delta)$ is feasible when $\frac{1}{8} \leq \delta < \frac{1}{3}$, as this is where $x_B = B(\delta)^{-1}b \geq 0$.

(You can verify this with casework of signs, matching signs of numerator and denominator)

b) Compute $c_B^T B(\delta)^{-1}$. For which values of δ is $B(\delta)$ an optimal basis?

$c_B^T = \begin{bmatrix} -5 & -1 \end{bmatrix}$

$c_B^T \cdot B(\delta)^{-1} = \frac{1}{-1+3\delta}\begin{bmatrix} -10 & 7-\delta \end{bmatrix}$

$$c_N^T - c_B^T B(\delta)^{-1} A = \begin{bmatrix} 12 & 0 \end{bmatrix} - \frac{1}{-1+3\delta}\begin{bmatrix} -10 & 7-\delta \end{bmatrix}\begin{bmatrix} 3+\delta & 2 \\ 5 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 12 & 0 \end{bmatrix} - \frac{1}{-1+3\delta}\begin{bmatrix} -2+39\delta & 1-3\delta \end{bmatrix} = \begin{bmatrix} -2+39\delta & 1 \end{bmatrix} \leq 0$$

$B(\delta)$ is optimal when $\frac{1}{18} \leq \delta < \frac{1}{3}$, as this is where $c_B^T \cdot B(\delta)^{-1} A \geq 0$.

c) Determine the optimal cost, as a function of δ, when δ is restricted to those values for which $B(\delta)$ is an optimal basis matrix.

$$c_B^T(\delta)B^{-1}b = \frac{1}{-1+3\delta}\begin{bmatrix} -5 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ -2+16\delta \end{bmatrix} = \frac{12-16\delta}{-1+3\delta}$$