Homework 2.2

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1. Use row operations, Gaussian-Jordan method, to find the inverse of the following two matrices if it exists. If the inverse exists, be sure to check that your result works. Namely, if A is invertible and you find the inverse A^{-1}, you need to check that $AA^{-1}=I$

a)

$$A=\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -2 & 4 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_1 \cdot (-2)]{R_2 \to R_1 \cdot 3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -2 & 8 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_2 \cdot 2 + R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 4 & 4 & 2 & 1 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{4} \cdot R_3}$$

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{4} \end{array}\right] \xrightarrow[R_1 \to R_3 \cdot 2 + R_1]{R_2 \to R_3 \cdot 2 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & 5 & 2 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{4} \end{array}\right] \quad A^{-1} = \begin{bmatrix} 3 & 1 & \frac{1}{2} \\ 5 & 2 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$[I \mid A^{-1}]$

$$AA^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 & \frac{1}{2} \\ 5 & 2 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot(1) - 2\cdot(1) & 1\cdot(1) - 2(\frac{1}{2}) & \frac{1}{2}(1) - 2(\frac{1}{4}) \\ -3(3) + 1(5) + (4\cdot 1) & -3(1) + 1(2) + 4(\frac{1}{2}) & -3(\frac{1}{2}) + 1(\frac{1}{2}) + 4(\frac{1}{4}) \\ 2(3) - 2(5) + 4(1) & 2(1) - 2(2) + 4(\frac{1}{2}) & 2(\frac{1}{2}) + (-2)(\frac{1}{2}) + 4(\frac{1}{4}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

1. b)

$$B = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ 5 & -9 & 4 \end{bmatrix}$$

$$[B \mid I] = \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ 5 & -9 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3(-5)+R_1]{R_2 \to R_1(-4)+R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 1 & -1 & -5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_2(-1)+R_3]{R_1 \to R_2(2)+R_1} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & -1 & -7 & 2 & 0 \\ 0 & \boxed{1} & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

Inverse does not exist, REF of B does not have 3 pivots although it is a 3x3 matrix.

2. We have a theorem that says if A and B are invertible, then AB is also invertible, and $(AB)^{-1} = B^{-1}A^{-1}$. Use this result to show that if A, B, and C are $n \times n$ matrices that are invertible, then ABC is also invertible by producing a matrix D and showing that $ABC(D) = I$ and $D(ABC) = I$. Caution: The order you multiply the matrices is important.

$$(ABC)^{-1} = ((AB)(C))^{-1} = C^{-1}(AB)^{-1}$$

(by $(AB)^{-1} = B^{-1}A^{-1}$ where C is A and AB is B)

$$= C^{-1}\underbrace{B^{-1}A^{-1}}_{(AB)^{-1} = B^{-1}A^{-1}}$$

So $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

2. cont.

$(ABC)D = (ABC)(C^{-1}B^{-1}A^{-1})$

$D = C^{-1}B^{-1}A^{-1}$

$= AB(CC^{-1})B^{-1}A^{-1}$

$= ABIB^{-1}A^{-1} = A(BI)B^{-1}A^{-1}$

$= ABB^{-1}A^{-1} = A(BB^{-1})A^{-1}$

$= AIA^{-1} = (AI)A^{-1} = AA^{-1}$

$= I$

So $(ABC)D = I$ where $D = C^{-1}B^{-1}A^{-1}$

$D(ABC) = C^{-1}B^{-1}A^{-1}(ABC)$

$= C^{-1}B^{-1}(A^{-1}A)BC$

$= C^{-1}B^{-1}IBC = C^{-1}(B^{-1}I)BC$

$= C^{-1}B^{-1}BC = C^{-1}(B^{-1}B)C$

$= C^{-1}IC = C^{-1}(IC) = C^{-1}C = I$

So $D(ABC) = I$ where $D = C^{-1}B^{-1}A^{-1}$

6.

3. Let A be $n \times n$ matrix such that $Ax=b$ has a solution for each $b \in \mathbb{R}^n$. Explain why A must be invertible.

If $Ax=b$ has a solution for each $b \in \mathbb{R}^n$, then A must have a pivot in every row, or otherwise there would be some b such that $0 =$ some non-zero value. If A has a pivot in every row, then A has a pivot for each of the n rows, or n pivots, and thus one for each column as well. This means the REF of A is the identity matrix I_n. This means the Gauss-Jordan method will yield a result for $[A|I]$, Reducing it to $[I|A^{-1}]$, and A is invertible.