Written Homework 1.3

For each of the exercises below, represent the vectors $\mathbf{u}$ and $\mathbf{v}$ as follows:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

1. (8 pts) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Show that for any scalar c, it is always the case that

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

Explain which computation requires the fewest arithmetic operations.

When calculating $c(u+v)$, the calculation looks like

$$c\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right)$$

$$= c\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} = \begin{bmatrix} c(u_1+v_1) \\ c(u_2+v_2) \\ \vdots \\ c(u_n+v_n) \end{bmatrix} = \begin{bmatrix} cu_1+cv_1 \\ cu_2+cv_2 \\ \vdots \\ cu_n+cv_n \end{bmatrix}$$

When calculating $cu+cv$, the calculation looks like

$$c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + c\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$

$$= \begin{bmatrix} cu_1+cv_1 \\ cu_2+cv_2 \\ \vdots \\ cu_n+cv_n \end{bmatrix}$$

Notice how the end product is the same. This means they are equal.

A scalar can always be distributed through added vectors. Since scalars are multiplied to each entry, it does not matter whether you add the entries then multiply or multiply the scalar to each of the vector's entries then add. This is the same principle of the distributive law of multiplication but enlarged to contain multiple entries.

This is called the **Distributive Law** of Scalar Multiplication.

The c(u+v) method requires n addition operations(one for each entry) to solve u+v, then n more multiplication operations to find c(u+v). This means the total amount of arithmetic operations required for c(u+v) is **2n**.
The cu + cv method requires n multiplication operations to calculate cu, another n to calculate cv, then n addition operations to solve for cu+cv. This means that there is a total required arithmetic operation amount of **3n**.
Therefore, since 2n<3n, the c(u+v) method requires less arithmetic operations.

2. (6 pts) Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$. Show that

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad u + (-u) = 0$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} -u_1 \\ -u_n \\ \vdots \\ -u_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1 - u_1 \\ u_2 - u_2 \\ \vdots \\ u_n - u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Vector addition adds each entry to the corresponding entry in the other vector to gain the result. If each entry is added by the negative of itself, that means each entry is subtracted by itself, which yields the result 0. If each entry is zero, then the result of u + (-u) is the zero vector.
This property is called the **Inverse Law** of vector addition.

3. (8 pts) Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$. Show that for any scalars c and d,

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

Explain which computation requires the fewest multiplications.

$$du = d\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} du_1 \\ du_2 \\ \vdots \\ du_n \end{bmatrix}$$

$$c(du) = c\begin{bmatrix} du_1 \\ du_2 \\ \vdots \\ du_n \end{bmatrix} = \begin{bmatrix} cdu_1 \\ cdu_2 \\ \vdots \\ cdu_n \end{bmatrix}$$

$$(cd) = c \times d$$

$$(cd)u = \begin{bmatrix} cdu_1 \\ cdu_2 \\ \vdots \\ cdu_n \end{bmatrix}$$

^^ Notice how the results of c(du) and (cd)u are the same.

Since scalars are multiplied by each entry of the vector u, and the commutative property of multiplication states the order of multiplying scalars does not matter, the end result will be the same.

This is called the **Associative Law** of scalar multiplication.

The (cd)u method requires less multiplications since you do one multiplication to find the scalar cd, then n multiplications to solve for each entry in the resulting vector. This means the total amount of arithmetic operations required is **n+1**. The c(du) method, however, requires n multiplications to solve for (du), and n more to find c(du), as you have to multiply a scalar by each entry of the vector every time. This means the total amount of arithmetic operations required is **2n**. Therefore, since n+1 < 2n for n>1, (cd)u requires fewer arithmetic operations.