Written Homework 1.9

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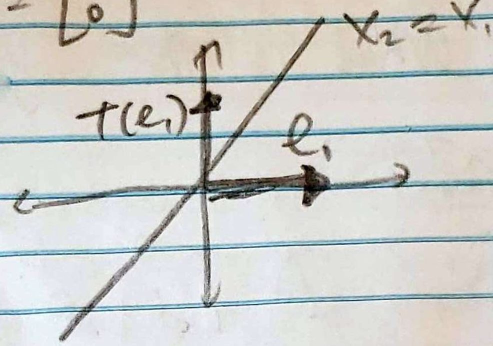
1. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$.

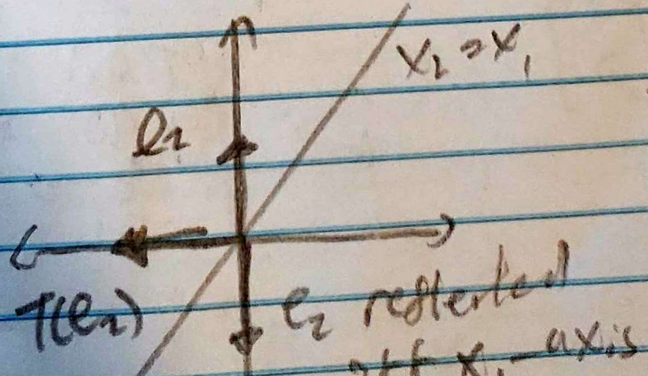
a) What is the standard matrix of T?

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$T(e_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$A = [T(e_1)\ \ T(e_2)] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

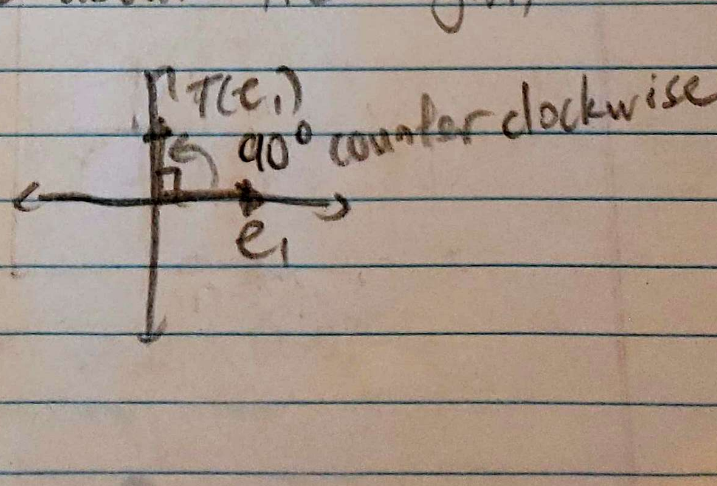
standard matrix of T

b) Show that the transformation, T, is merely a rotation about the origin. What is the angle of that rotation?

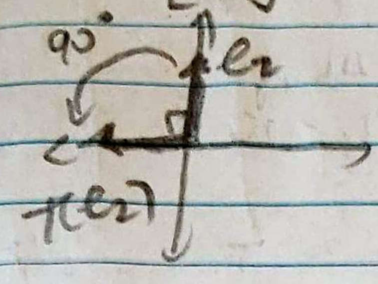
$T(x) = Ax \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

As proved before, $T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is a 90° (or $\frac{\pi}{2}$) rotation counterclockwise about the origin, of

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$T(e_2) = \begin{bmatrix}-1\\0\end{bmatrix}$ is also a $90°$ (or $\frac{\pi}{2}$) rotation counter-clockwise of $e_1 = \begin{bmatrix}0\\1\end{bmatrix}$.



Since every linear transformation of $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ $= T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)$ can be written as

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = T\left(x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= T\left(x_1 e_1 + x_2 e_2\right)$$

$$= \underline{x_1 T(e_1) + x_2 T(e_2)}$$

and thus x_1 and x_2 are both rotated by the same $90°$ counter-clockwise and T is merely a rotation about the origin.

The angle of that rotation is
90° counter-clockwise

Rotational Matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$-\sin\theta = -1 \quad \theta = 90°$ ✓

$\sin\theta = 1$

$\cos\theta = 0$

2. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

a) If T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, decide which of m and n is greater than or equal to the other. Justify your answer.

n is greater than or equal to m.

If the transformation T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then its standard matrix must have a pivot in every row, and be a $m \times n$ matrix. If $n < m$, there will not be enough columns for the standard matrix to have a pivot in every row, so $n \geq m$.

b) If T is one-to-one, decide which of m or n is greater than or equal to the other. Justify your answer

m is greater than or equal to n. If the transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one, the standard matrix of the transformation is an $m \times n$ matrix and has a pivot in each column.

If there is not enough rows for there to be a pivot in every column ($m<n$), then T is not one-to-one.

So $m \geq n$.

c) If T is one-to-one and onto, what can you say about the relationship between m and n?

m and n must be equal, since each row and column of the $m \times n$ matrix of the transformation T must have pivots. This means the amount of pivots = amount of pivot = amount of pivots columns = amount of rows = amount of columns = $m=n$