Written Homework 1.4

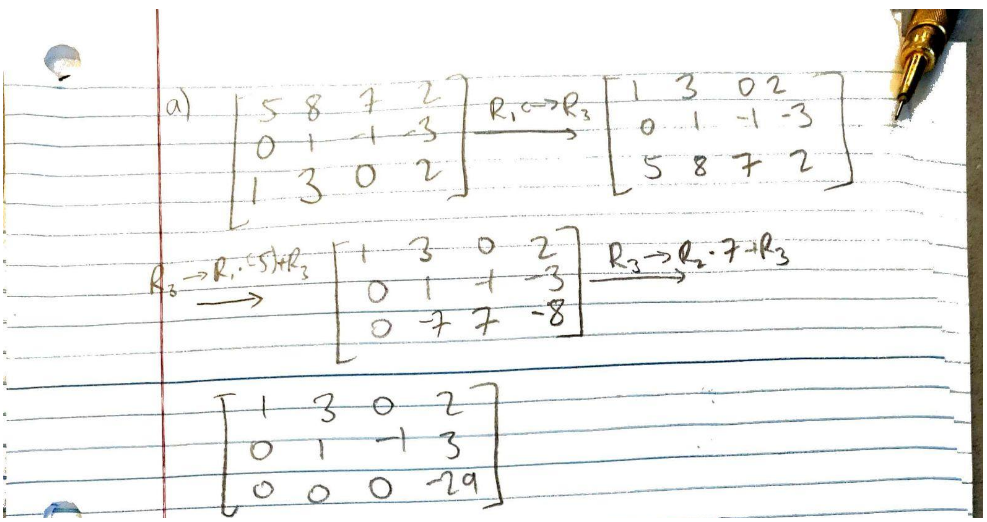
1. (15 pts) Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

   (a) (10 pts) Is $\mathbf{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of $\mathbf{A}$? Why or why not?

   (b) (5 pts) If it is, find the linear combination of the columns that give $\mathbf{u}$. If it is not, find a vector, $\mathbf{v}$, that is in the span of the columns of $\mathbf{A}$.

a) In order to determine whether u is in the subset spanned by the columns of A, we need to find whether
[A] [x] = [u]



a) $\begin{bmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{bmatrix}$

$\xrightarrow{R_3 \to R_1 \cdot (-5) + R_3} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{bmatrix} \xrightarrow{R_3 \to R_2 \cdot 7 + R_3}$

$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -29 \end{bmatrix}$

Which is equivalent to the augmented matrix [A | u] having infinite solutions. However, the echelon form of the augmented matrix has a row [0 0 0 -29], which means it is inconsistent, and cannot have infinite solutions.
This means u **is not in the subset spanned by the column vectors of A**.

b) Find a vector v, that spans column vectors of A
A vector in the Span of the column vectors must be a linear combination of the column vectors

b) Vector in Span$\left\{ \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} \right\}$

$$1 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$$

is in the span stated above.

So $\Rightarrow \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \in \text{Span}\left\{ \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} \right\}$

2. (8 pts) Construct a 3×3 matrix, not in echelon form, whose columns span $\mathbb{R}^3$. Show that the constructed matrix has the desired property.

2. For a 3×3 matrix to span $\mathbb{R}^3$, each column must be a pivot column.

The matrix below works

pivot in each column

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(reduced) echelon form

3. (6 pts) Let A be a 3×4 matrix, let $\mathbf{y}_1$ and $\mathbf{y}_2$ be vectors in $\mathbb{R}^3$, and let $\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2$. Suppose $\mathbf{y}_1 = A\mathbf{x}_1$ and $\mathbf{y}_2 = A\mathbf{x}_2$ for some vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^4$. What property of matrix multiplication allows you to conclude that the system $A\mathbf{x} = \mathbf{w}$ is consistent?

3. Distributive form
Property of Matrix Multipication

Since $\overset{(1)}{y_1 = Ax_1}$ and $\overset{(2)}{y_2 = Ax_2}$ have some solution $(x_1$ and $x_2) \in \mathbb{R}^4$,

$\underbrace{Ax_1 + Ax_2 = y_1 + y_2}_{(1)+(2)}$ (Recall $w = y_1 + y_2$)

$\Rightarrow A(x_1 + x_2) = w$

Since $x_1, x_2 \in \mathbb{R}^4$, $x_1 + x_2$ must also exist and be in $\mathbb{R}^4$, so there must be a solution to

$Ax = w$, the solution being

$x = x_1 + x_2$.

4. (6 pts) Let A be a 5×3 matrix, let $\mathbf{y}$ be a vector in $\mathbb{R}^3$, and let $\mathbf{z}$ be a vector in $\mathbb{R}^5$. Suppose $A\mathbf{y} = \mathbf{z}$. What property of matrix multiplications allows you to conclude that the system $A\mathbf{x} = 4\mathbf{z}$ is consistent?

4. The Linear Property of Matrix Multiplication

$A(cu) = c(Au)$

Since $Ay = z$,

$\Rightarrow 4Ay = 4z$

$\Rightarrow A(4y) = (4z)$ (Linear Property)

Since y exists in $\mathbb{R}^3$, that means $4y$ must also exist in $\mathbb{R}^3$. That means there must be a solution to $Ax = 4z$, where $x = 4y$.