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Written Homework 1.2

1. (6 pts) Suppose the coefficient matrix of a linear system of three equations in three variables has a pivot in each column. Explain why the system has a unique solution.

Since there is a pivot in each column, each variable is a basic variable, and thus there are no free variables and the solution is unique.

2. (6 pts) What would need to be known about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?

Every column but the right-most are pivot columns. This means each variable must be a basic variable, and thus the system has no free variables. This means that the linear system is consistent and has a unique solution.

3. (30 pts) Applications. For this problem it is not necessary to write all of the introductory material. For the Goal, just write the title of the application and then the question that is Asked.

(a) Newton's Law of Cooling says that there is a linear relationship between temperature difference ($\Delta T = T1 - T2$) and heat flux, q, i.e. the bigger the temperature difference between two points in space, the more heat will flow:

$$q = -K\Delta T.$$

The coefficient K is called the conductivity coefficient. If K is large then heat goes through the material quickly (think the bottom of a pot used to cook on a stove) and if K = 0, then no heat can transfer through the material (ideal insulation). Your boss tells you to measure the conductivity of a brand new material just invented by your company. Being very diligent, you measure the heat flux through the material at three different ΔT values. Then you decide to be extra sure. So you repeat the experiment twice more, for a total of 3 different experiments. Proud of the great job you have done, you plot the results, and you get the points below, where open circle, closed circle, and x's denote the three experiments.
You want to find K. You first recall that technically, when ΔT is zero, the heat flux should be zero, but then you also recall that there is always experimental error, so that there is a residual heat flux, qr, even when ΔT is zero, so your equation is now:

$$q = -k\Delta T + q_r$$

(i) (4 pts) List the unknowns.
k, q_r These are the unknowns since we do not know these values. The other values, ΔT and q are given in the experiment.
(ii) (4 pts) How many equations do you have? Explain.
There are 9 equations, since for each pair of temperature difference and heat flux(or each point on the graph) can be written into an equation in the form of $q = -k\Delta T + q_r$, where q is

substituted with the corresponding heat flux and ΔT is substituted with the corresponding temperature difference.

(iii) (6 pts) How many solutions are there likely to be? Explain how you obtained your answer *mathematically*.
It depends on how many heat flux residuals there are

If there is one uniform heat flux residual across all experiments:
There would be 2 unknowns and 9 equations, and it would be highly unlikely to find a solution as there is way to many requirements (the solution has to satisfy all 9 equations)

If there is one uniform heat flux residual for *each* experiment(group of three points):
There would be 4 unknowns (1 K and 3 heat flux residuals), and 9 equations. This would also make it extremely hard to get a solution to satisfy all 9 equations.

Finally, if there is a different heat flux residual for *each trial*:
There would be 10 unknowns(1K and 9 heat flux residuals), and 9 equations. This would probably have a solution as there are more unknowns than equations.

(b) CT Scan The radiation x-ray was discovered by a German physicist, Wilhelm Roentgen, who did not know what it was, so he simply called it “X-radiation”. A single x-ray passing through a body is absorbed at rates depending upon the material it goes through. Thus if an x-ray is passed through bone, a certain amount of intensity units is absorbed, while if it passes through soft material, a different amount of intensity units is absorbed. A CT scan is a scan in which an x-ray is passed through an object (part of a body) from multiple angles (on the order of 360), to produce a detailed, high quality three-dimensional image.
How does it work?
Consider the figure below, consisting of a toy two-dimensional figure, where each square represents a pixel. Each pixel can absorb 1 intensity unit (i.u.) from the X-ray beams, or none. Let’s assume the white boxes are bone and absorb 1 i.u., and the black pixels absorb no i.u.’s and represent e.g. soft tissue. Let $\mu 1$ be the attenuation coefficient of element 1, $\mu 2$ be the attenuation coefficient of element 2, ... $\mu 16$ be the attenuation coefficient of element 16, ... Ideally, each μ is either 0 or 1, indicating soft tissue or bone, respectively. A beam from the x-ray can be sent through the material at any location and from any direction, but must start on the outside of the material, and then the amount absorbed is measured where the x-ray exits. The goal is to use x-ray data to reconstruct this figure.
Consider the first row of the figure. The amount absorbed by an X-ray beam passing through the first row is given by
0 + 1 + 1 + 1 + 1 + 1... + 1 + 0 = 9.
However in real life, we do not know what is bone and what is soft tissue. So performing the x-ray gives
$\mu 1 + \mu 2 + \mu 3 + \mu 4 + \mu 5 + \mu 6 + \mu 7 + \mu 8 + \mu 9 + \mu 10 + \mu 11 = 9$.

(i) (4 pts) How many unknowns are in the entire figure? Explain how you obtained your answer. (Hint: It is NOT necessary to write down all of the equations).

There are 44 unknowns since it is a 4 by 11 pixel figure so there are 44 pixels each with their own attenuation coefficient.

Let us assume we make 4 horizontal x-ray measurements (one for each row) and 11 vertical measurements (one for each column).
(ii) (4 pts) How many equations do we have?
We have 15 equations, since we made 4 horizontal x-ray measurements and 11 vertical measurements, each yielding an equation. 11+4 = 15.

(iii) (8 pts) Will we be able to determine which pixels are black and which pixels are white with this systems of equations? Explain mathematically why or why not.
No, as for a consistent system, you need at least the same amount of variables as equations to determine all the variables. There are 44 variables but only 15 equations.