Intuitive Derivation of Fourier Series and the Fourier Transform

Charles M. Roberts

June 21, 2023

1 Introduction

I have seen far too many articles, videos and lectures in which the properties of Fourier series and Fourier transforms are proved, yet are never derived. Often times, the speaker or writer will pull a magical integral or sum out of thin air and proclaim that it has some remarkable properties, and will then proceed to prove them. This is not merely an occurrence with this particular topic but with many, the Gamma function is a prime example of this. I will try to avoid such things as much as possible. My aim is to make every step to be clear, not as to why such a step is allowed but as to why one would wish to make it.

I will focus more on motivation than on rigour. This does not necessarily mean that each step will obviously be necessary for the final result (in this case the Fourier Transform) but it should at least be obvious as to why each thing I prove would be something of some interest to somebody, (and hence why anybody would have proved such a thing).

To make one final comment on rigour, I will not be focusing heavily on rigorous proof, I may extend certain finite ideas to infinite cases, or may switch sum and integral signs without justification. I do understand that such things are not always possible, but the focus of this article is on gaining an understanding rather than rigour.

2 Periodic Functions

A periodic function is one which repeats on regular intervals. A function $g(t)$ is L-periodic if it repeats every time t increases by L. In other words, if

$$g(t+L) = g(t)\ \forall\ t \in \mathbb{R}$$

Note that I have specified "for all *real* numbers t". This article will be focusing only on real valued periodic functions. The two most familiar periodic functions are likely to be the sine and cosine functions which each have a period of 2π (e.i., $\sin(t+2\pi) = \sin(t), \cos(t+2\pi) = \cos(t)\ \forall\ t \in \mathbb{R}$)

The big idea of Fourier series is to write any periodic functions in terms of sines cosines. As a sum of them. It seems clear that to write an L-periodic function as a sum of sines and cosines, those sines and cosines would also need to repeat every L.

To change the period of the cosine functions, we can scale it down horizontally by a factor of 2π, changing its period from 2π to 1. $\cos(2\pi t)$ is 1-periodic since

$$\cos(2\pi(t+1)) = \cos(2\pi t + 2\pi) = \cos(2\pi t)$$

In other words, if I let $f(t) = \cos(2\pi t)$ then $f(t+1) = f(t)\ \forall\ t \in \mathbb{R}$, so $f(t)$ is 1-periodic. Scaling the new function up horizontally by a factor of L will give us a n L-periodic function.

$$\cos\left(\frac{2\pi(t+L)}{L}\right) = \cos\left(\frac{2\pi t + 2\pi L}{L}\right) = \cos\left(\frac{2\pi t}{L} + \frac{2\pi L}{L}\right) = \cos\left(\frac{2\pi t}{L} + 2\pi\right) = \cos\left(\frac{2\pi t}{L}\right)$$

So if I let $f(t) = \cos\left(\frac{2\pi t}{L}\right)$ then $f(t+L) = f(t)$ so $f(t)$ is L-periodic. The same can be applied to sine, so $\sin\left(\frac{2\pi t}{L}\right)$ is L-periodic.

There is no need to stop here. Yes $\cos\left(\frac{2\pi t}{L}\right)$ is L-periodic, but so is $\cos\left(\frac{2\pi 2t}{L}\right)$. This function repeats every $2L$ instead of every L, but this still means that it repeats every L and so this function is also, in some sense L-periodic. $\cos\left(\frac{2\pi 3t}{L}\right)$ is $3L$-periodic and so is also technically L-periodic. In fact $\cos\left(\frac{2\pi nt}{L}\right)$ is nL-periodic and as long as $n \in \mathbb{N}$, this function is L-periodic. When $n = 0$ the input to the function is just zero which is constant, and so the function is also going to be a constant (in this case $\cos(0) = 1$) which is also technically L-periodic. The above can all also be said for the sine function, except $\sin(0) = 0$.

This means that for some L-periodic function $g(t)$ (assuming it is possible to write it as a sum of sines and cosines) will be written a sum of sines and cosines which also have periods of L, that is $\cos(\frac{2\pi nt}{L})$ and $\sin(\frac{2\pi nt}{L})$ where $n \in \mathbb{W}$. It is important to note that some of these terms will contribute more, some will contribute less, and some not at all. To use sigma notation, writing $g(t)$ as a sum of the above sines and cosines:

$$g(t) = \sum_{n=0}^{\infty} \left(a_n \cos\left(\frac{2\pi nt}{L}\right) + b_n \sin\left(\frac{2\pi nt}{L}\right) \right)$$

or

$$g(t) = \sum_{n \in \mathbb{W}} \left(a_n \cos\left(\frac{2\pi nt}{L}\right) + b_n \sin\left(\frac{2\pi nt}{L}\right) \right)$$

This is all assuming that we can write $g(t)$ as such a sum, if we cannot, none of the following will work. We do know however that $g(t)$ can sometimes be written as a sum of sines and cosines because of course, you could define $g(t)$ to be a sum of sines and cosines. I will not discuss when it is and is not possible to write a periodic function as a sum of trigonometric functions, but just know that you *almost* always can.

The above sum, is a Fourier series. At this stage however, it is not very useful as we have no way of working out what the values of a_n and b_n are. Before moving on to this, I will first write the above in a different, more conventional way. I will remove the $n = 0$ term from the sum and add it back to the beginning.

$$g(t) = (a_0 \cos(0) + b_0 sin(0)) + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nt}{L}\right) + b_n \sin\left(\frac{2\pi nt}{L}\right) \right)$$

$$g(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nt}{L}\right) + b_n \sin\left(\frac{2\pi nt}{L}\right) \right) \tag{1}$$

or

$$g(t) = a_0 + \sum_{n \in \mathbb{N}} \left(a_n \cos\left(\frac{2\pi nt}{L}\right) + b_n \sin\left(\frac{2\pi nt}{L}\right) \right)$$

This is still slightly different from what is conventional, but I will get to that later. The task now is to find the coefficients a_0, a_n and b_n

3 Finding the Fourier Coefficients

Equation (1) is useless with no way to find the values of the coefficients. That will be the purpose of this section. I will first break the sum into two parts to make it more manageable:

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{L}\right)$$

I would first like to find the value of a_0. To do this one might observe that sine is an odd function $(\sin(-x) = -\sin(x))$ and so $\displaystyle\int_{-a}^{a} \sin(x)dx = 0 \ \forall \ a \in \mathbb{R}$ so integrating both sides from, for example, -1 to 1 would eliminate the sine term.

$$\int_{-1}^{1} g(t)dt = \int_{-1}^{1} a_0 dt + \int_{-1}^{1} \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{L}\right) dt + \int_{-1}^{1} \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{L}\right) dt$$

$$\int_{-1}^{1} g(t)dt = \int_{-1}^{1} a_0 dt + \sum_{n=1}^{\infty} \int_{-1}^{1} a_n \cos\left(\frac{2\pi nt}{L}\right) dt + \sum_{n=1}^{\infty} \int_{-1}^{1} b_n \sin\left(\frac{2\pi nt}{L}\right) dt$$

I will not show why you are allowed to swap the integral and sum signs here, but in this case you can.

$$\int_{-1}^{1} g(t)dt = \int_{-1}^{1} a_0 dt + \sum_{n=1}^{\infty} a_n \int_{-1}^{1} \cos\left(\frac{2\pi nt}{L}\right) dt + \sum_{n=1}^{\infty} b_n \int_{-1}^{1} \sin\left(\frac{2\pi nt}{L}\right) dt$$

$$\int_{-1}^{1} g(t)dt = \int_{-1}^{1} a_0 dt + \sum_{n=1}^{\infty} a_n \left[\frac{L}{2\pi n} \sin\left(\frac{2\pi nt}{L}\right)\right]_{-1}^{1} + \sum_{n=1}^{\infty} 0$$

$$\int_{-1}^{1} g(t)dt = [a_0 t]_{-1}^{1} + \sum_{n=1}^{\infty} a_n \left[\frac{L}{2\pi n} \sin\left(\frac{2\pi nt}{L}\right)\right]_{-1}^{1} + \sum_{n=1}^{\infty} 0$$

$$\int_{-1}^{1} g(t)dt = 2a_0 + \sum_{n=1}^{\infty} a_n \frac{L}{2\pi n} \left[\sin\left(\frac{2\pi nt}{L}\right)\right]_{-1}^{1}$$

Here, the sine term is gone, and the cosine term has turned into a sine term (due to the integral). When we let $t = 1$ and let $t = -1$ and subtract we get $2\sin\left(\frac{2\pi n}{L}\right)$. If only the L weren't there, this this would be zero. If we integrated from $-L$ to L, then the L here would cancel and the sine function from before would still have disappeared as it is an odd function. We would end up with:

$$\int_{-L}^{L} g(t)dt = [a_0 t]_{-L}^{L} + \sum_{n=1}^{\infty} a_n \frac{L}{2\pi n} \left[\sin\left(\frac{2\pi nt}{L}\right)\right]_{-L}^{L}$$

$$\int_{-L}^{L} g(t)dt = 2a_0 L + \sum_{n=1}^{\infty} a_n \frac{L}{\pi n} \sin(2\pi n)$$

$$\int_{-L}^{L} g(t)dt = 2a_0 L$$

because $\sin(\pi n) = 0 \ \forall \ n \in \mathbb{Z}$.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(t)dt$$

We usually use $\frac{L}{2}$ as our bounds as this would also have canceled out the 2. This would give us:

$$a_0 = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} g(t)dt$$

If our function has a period of $2L$ instead of L we can replace L with $2L$, giving:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(t)dt$$

So make note of the fact that from here on out, $g(t)$ is $2L$-periodic, not L-periodic. This means that all occurrences of $\frac{2\pi nt}{L}$ will be replaced with $\frac{\pi nt}{L}$.

Conventionally, a_0 is defined: $a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} g(t)dt$ and $\frac{1}{2}a_0$ is used in the Fourier series from before. That is:

$$g(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi nt}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi nt}{L}\right)$$

I will talk more about this convention later but will instead use the my original definition and value for a_0 for now.

The above has revealed to us that that integrating from $-L$ to L may be a useful technique in finding the Fourier coefficients. Also that odd and even functions may play a role. For the next step I will use the fact that an odd function multiplied by an even function is an odd function. This is is not difficult to see as, if $f(x)$ is odd and $g(x)$ is even and $h(x) = f(x)g(x)$ then $h(-x) = f(-x)g(-x) = -f(x)g(x) = -h(x)$. Similarly an even function multiplied by an even function or an odd function multiplied by and odd function, both form even functions.

Using this fact in the above equation, I can get rid of the cosine term by multiplying it my a sine term (even function multiplied by odd function) hence creating an odd function. Integrating this between $-L$ and L will produce zero. When I do this, of course some other parts of the equation will be affected.

I multiply both sides by $\sin\left(\frac{\pi mt}{L}\right)$ and then integrate both sides from $-L$ to L. The reason for using and m instead of an n inside the sine function is because n is defined in terms of the sum. m is simply some constant positive integer.

$$\int_{-L}^{L} g(t)\sin\left(\frac{\pi mt}{L}\right)dt = a_0\int_{-L}^{L}\sin\left(\frac{\pi mt}{L}\right)dt+\sum_{n=1}^{\infty}a_n\int_{-L}^{L}\cos\left(\frac{\pi nt}{L}\right)\sin\left(\frac{\pi mt}{L}\right)dt$$
$$+\sum_{n=1}^{\infty}b_n\int_{-L}^{L}\sin\left(\frac{\pi nt}{L}\right)\sin\left(\frac{\pi mt}{L}\right)dt$$

Sine multiplied by cosine produces an odd function, and sine itself is an odd function, so those two integrals become zero.

$$\int_{-L}^{L} g(t)\sin\left(\frac{\pi mt}{L}\right)dt = \sum_{n=1}^{\infty}b_n\int_{-L}^{L}\sin\left(\frac{\pi nt}{L}\right)\sin\left(\frac{\pi mt}{L}\right)dt \tag{2}$$

The only Fourier coefficient left in equation (2) is now b_n. I must find a way to isolate it. I should also evaluate the integral. Let

$$I = \int_{-L}^{L}\sin\left(\frac{\pi nt}{L}\right)\sin\left(\frac{\pi mt}{L}\right)dt$$

The easiest way to do this is probably to use the identity:

$$\sin(A)\sin(B) = \frac{1}{2}\left(\cos(A-B)+\cos(A+B)\right)$$

Which can be easily derived from the addition formulae for $\cos(A+B)$ and $\cos(A-B)$.

$$\begin{aligned}
I &= \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{\pi nt}{L}-\frac{\pi mt}{L}\right)+\cos\left(\frac{\pi nt}{L}+\frac{\pi mt}{L}\right)dt \\
&= \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{\pi(n-m)t}{L}\right)+\cos\left(\frac{\pi(n+m)t}{L}\right)dt \\
&= \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{\pi(n-m)t}{L}\right)dt+\frac{1}{2}\int_{-L}^{L}\cos\left(\frac{\pi(n+m)t}{L}\right)dt
\end{aligned}$$

$$= \frac{1}{2}\left[\frac{L}{\pi(n-m)}\sin\left(\frac{\pi(n-m)t}{L}\right)\right]_{-L}^{L}+\frac{1}{2}\left[\frac{L}{\pi(n+m)}\sin\left(\frac{\pi(n+m)t}{L}\right)\right]_{-L}^{L}$$

$$= \frac{L}{2\pi(n-m)}\left[\sin\left(\frac{\pi(n-m)t}{L}\right)\right]_{-L}^{L}+\frac{L}{2\pi(n+m)}\left[\sin\left(\frac{\pi(n+m)t}{L}\right)\right]_{-L}^{L}$$

Here, we are dividing, and must be careful do not divide by zero. Since n and m are both natural numbers (they are both > 0) there is no risk that

$n + m = 0$. There is however the possibility that $n - m = 0$, that is if $n = m$. I will now assume that $n \neq m$ and proceed, but I will return to this later.

$$\begin{aligned} I = & \frac{L}{2\pi(n-m)} \left(\sin\left(\pi(n-m)\right) - \sin\left(-\pi(n-m)\right)\right) \\ & + \frac{L}{2\pi(n+m)} \left(\sin\left(\pi(n+m)\right) - \sin\left(-\pi(n+m)\right)\right) \\ = & \frac{L}{2\pi(n-m)} 2\sin\left(\pi(n-m)\right) + \frac{L}{2\pi(n+m)} 2\sin\left(\pi(n+m)\right) \\ = & \frac{L}{\pi(n-m)} \sin\left(\pi(n-m)\right) + \frac{L}{\pi(n+m)} \sin\left(\pi(n+m)\right) \end{aligned}$$

$m, n \in \mathbb{N}$ so $(n-m), (n+m) \in \mathbb{Z}$ and $\sin(k\pi) = 0 \ \forall \ k \in \mathbb{Z}$.

$$I = 0$$

This is only given that $m \neq n$. Given that m is a constant positive integer, there will be exactly one term in the series for which $n = m$. In this case:

$$\begin{aligned} I &= \int_{-L}^{L} \sin\left(\frac{\pi n t}{L}\right) \sin\left(\frac{\pi m t}{L}\right) dt \\ &= \int_{-L}^{L} \sin\left(\frac{\pi n t}{L}\right) \sin\left(\frac{\pi n t}{L}\right) dt \\ &= \int_{-L}^{L} \sin^2\left(\frac{\pi n t}{L}\right) \end{aligned}$$

Using $cos(2\theta) = 1 - 2sin^2(\theta) \rightarrow sin^2(\theta) = \frac{1}{2} - \frac{1}{2}cos(2\theta)$:

$$\begin{aligned} I &= \int_{-L}^{L} \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi n t}{L}\right) dt \\ &= \int_{-L}^{L} \frac{1}{2} dt - \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{2\pi n t}{L}\right) dt \\ &= \frac{1}{2}L - \frac{1}{2}(-L) - \frac{1}{2}\left(\frac{L}{2\pi n}\left[\sin\left(\frac{2\pi n t}{L}\right)\right]_{-L}^{L}\right) \\ &= L - \frac{L}{4\pi n}\left(\sin(2\pi n) - \sin(-2\pi n)\right) \\ &= L - \frac{L}{4\pi n}(0 - 0) \\ &= L \end{aligned}$$

So $\int_{-L}^{L} \sin\left(\frac{\pi nt}{L}\right) \sin\left(\frac{\pi mt}{L}\right) dt = 0$ when $n \neq m$ and $= L$ when $n = m$.

We can write this result using Kronecker delta notation. $\delta_{ab} = 0$ when $a \neq b$ and $\delta_{ab} = 1$ when $a = b$. Using this notation:

$$\int_{-L}^{L} \sin\left(\frac{2\pi nt}{L}\right) \sin\left(\frac{\pi mt}{L}\right) dt = \delta_{nm} L$$

Substitute this back into equation (2).

$$\int_{-L}^{L} g(t) \sin\left(\frac{\pi mt}{L}\right) dt = \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\left(\frac{\pi nt}{L}\right) \sin\left(\frac{\pi mt}{L}\right) dt$$

$$\int_{-L}^{L} g(t) \sin\left(\frac{\pi mt}{L}\right) dt = \sum_{n=1}^{\infty} b_n \delta_{nm} L$$

$$\int_{-L}^{L} g(t) \sin\left(\frac{\pi mt}{L}\right) dt = L \sum_{n=1}^{\infty} b_n \delta_{nm}$$

The key observation to make now is that the term $b_n \delta_{nm}$ will be zero for every term where $n \neq m$ by definition of δ_{nm}. The only non-zero term is the term for which $n = m$. This term is b_m. This means that the sum has only one non-zero term, that is b_m and so the sum totals to b_m.

$$\int_{-L}^{L} g(t) \sin\left(\frac{\pi mt}{L}\right) dt = L b_m$$

$$b_m = \frac{1}{L} \int_{-L}^{L} g(t) \sin\left(\frac{\pi mt}{L}\right) dt \tag{3}$$

Equation (3) can now be used to find the b_n coefficients.

Repeating the same steps as before but instead multiplying both sides by $\cos\left(\frac{\pi mt}{L}\right)$ and then integrating from $-L$ to L will allow us to find the a_n coefficients. For this I will need to know $I = \int_{-L}^{L} \cos\left(\frac{\pi nt}{L}\right) \cos\left(\frac{\pi mt}{L}\right) dt$.

When $n \neq m1$:

Using the identity $\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) - \cos(A-B)\right)$ (which can be easily derived from the addition formulae):

$$\begin{aligned} I &= \frac{1}{2} \int_{-L}^{L} \cos\left(\frac{\pi nt}{L} + \frac{\pi mt}{L}\right) - \cos\left(\frac{\pi nt}{L} - \frac{\pi mt}{L}\right) dt \\ &= \frac{1}{2} \int_{-L}^{L} \cos\left(\frac{\pi(n+m)t}{L}\right) - \cos\left(\frac{\pi(n-m)t}{L}\right) dt \end{aligned}$$

$$= \frac{1}{2}\left[\frac{L}{\pi(n+m)}\sin\left(\frac{\pi(n+m)t}{L}\right)\right]_{-L}^{L} - \frac{1}{2}\left[\frac{L}{\pi(n-m)}\sin\left(\frac{\pi(n-m)t}{L}\right)\right]_{-L}^{L}$$

$$= \frac{1}{2}(0-0) - \frac{1}{2}(0-0)$$

$$= 0$$

In the case where $n = m$:

$$I = \int_{-L}^{L} \cos^2\left(\frac{\pi nt}{L}\right) dt$$

Using $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$

$$= \int_{-L}^{L} \frac{1}{2} dt + \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{2\pi nt}{L}\right) dt$$

$$= L + \frac{1}{2}\frac{L}{2\pi nt}\left[\sin\left(\frac{2\pi nt}{L}\right)\right]_{-L}^{L}$$

$$= L + \frac{L}{4\pi nt}\left[\sin\left(\frac{2\pi nt}{L}\right)\right]_{-L}^{L}$$

$$= L + \frac{L}{4\pi nt} \times 0$$

$$= L$$

Using Kronecker delta notation:

$$\int_{-L}^{L} \cos\left(\frac{\pi nt}{L}\right)\cos\left(\frac{\pi mt}{L}\right) dt = \delta_{nm} L \tag{4}$$

Multiplying equation (1) by $\cos\left(\frac{\pi mt}{L}\right)$ and integrating from $-L$ to L:

$$g(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{\pi nt}{L}\right) + b_n \sin\left(\frac{\pi nt}{L}\right)\right)$$

$$\int_{-L}^{L} g(t)\cos\left(\frac{\pi mt}{L}\right) dt = \int_{-L}^{L} a_0 \cos\left(\frac{\pi mt}{L}\right) dt + \sum_{n=1}^{\infty}\int_{-L}^{L} a_n \cos\left(\frac{\pi nt}{L}\right)\cos\left(\frac{\pi mt}{L}\right) dt$$

$$+ \sum_{n=1}^{\infty}\int_{-L}^{L} b_n \sin\left(\frac{\pi nt}{L}\right)\cos\left(\frac{\pi mt}{L}\right) dt$$

Sine Multiplied by cosine is an odd function multiplied by an even function, which is an odd function. This means that that integral becomes zero.

$$\int_{-L}^{L} g(t)\cos\left(\frac{\pi mt}{L}\right) dt = a_0\int_{-L}^{L} \cos\left(\frac{\pi mt}{L}\right) dt + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{\pi nt}{L}\right)\cos\left(\frac{\pi mt}{L}\right) dt \tag{5}$$

$$\int_{-L}^{L} \cos\left(\frac{\pi m t}{L}\right) dt$$

$$= \frac{L}{\pi m}\left[\sin\left(\frac{\pi m t}{L}\right)\right]_{-L}^{L}$$

$$= 0$$

Substituting into (5):

$$\int_{-L}^{L} g(t)\cos\left(\frac{\pi m t}{L}\right) dt = \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{\pi n t}{L}\right)\cos\left(\frac{\pi m t}{L}\right) dt \tag{6}$$

Substituting (4) into (6):

$$\int_{-L}^{L} g(t)\cos\left(\frac{\pi m t}{L}\right) dt = \sum_{n=1}^{\infty} a_n \delta_{nm} L$$

Using the same argument as before:

$$\int_{-L}^{L} g(t)\cos\left(\frac{\pi m t}{L}\right) dt = a_m L$$

$$a_m = \frac{1}{L}\int_{-L}^{L} g(t)\cos\left(\frac{\pi m t}{L}\right) dt$$

We now know how to find each of the Fourier coefficients.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(t) dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} g(t)\cos\left(\frac{\pi n t}{L}\right) dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} g(t)\sin\left(\frac{\pi n t}{L}\right) dt$$

You may notice that the definition for a_0 is inconsistent with that of a_n when $n = 0$. It is for this reason that the convention from earlier is used.

$$a_0 = \frac{1}{L}\int_{-L}^{L} g(t) dt$$

and then we use $\frac{1}{2}a_0$ instead of a_0 in the Fourier series equation.

The reason why this inconsistency arises is that when we were integrating we said that there was no way for $(m+n)$ to equal zero. This is only correct as both m and n are natural numbers. If a_0 was left inside the sum then the sum would go from 0 to ∞ and we would need to define m to be a whole number as opposed to as a natural number as otherwise the a_m formula would not work for $m = 0$. These two facts mean that it would indeed be possible in this case for $(n+m)$ to be zero and this would need to be taken into account. However because we split up the sum so it went from 1 to ∞ this did not need to be accounted for.

4 Complex form of the Fourier Series

This section will work to write the Fourier series in a more compact form which happens to utilise complex exponentials. You may have already sensed this coming as we are dealing with sines and cosines which are very much related to complex exponentials via Euler's formula. To start, the Fourier series:

$$g(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{\pi n t}{L}\right) + b_n \sin\left(\frac{\pi n t}{L}\right)\right)$$

Where

$$a_n = \frac{1}{L}\int_{-L}^{L} g(t)\cos\left(\frac{\pi n t}{L}\right)dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} g(t)\sin\left(\frac{\pi n t}{L}\right)dt$$

What we will do is write sine and cosine in their exponential forms as this will allow us to write the sum as a sum of just one term instead of two.
$\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ and $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$. These can be easily derived from Euler's formula ($e^{i\theta} = \cos(\theta) + i\sin(\theta)$).

Substitute these into the Fourier series:

$$g(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\frac{e^{\frac{\pi i n t}{L}} + e^{\frac{-\pi i n t}{L}}}{2} + b_n\frac{e^{\frac{\pi i n t}{L}} - e^{\frac{-\pi i n t}{L}}}{2i}\right)$$

since $\frac{1}{i} = -i$:

$$g(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\frac{e^{\frac{\pi i n t}{L}} + e^{\frac{-\pi i n t}{L}}}{2} - ib_n\frac{e^{\frac{\pi i n t}{L}} - e^{\frac{-\pi i n t}{L}}}{2}\right)$$

$$g(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n\left(e^{\frac{\pi i n t}{L}} + e^{\frac{-\pi i n t}{L}}\right) - ib_n\left(e^{\frac{\pi i n t}{L}} - e^{\frac{-\pi i n t}{L}}\right)}{2}\right)$$

$$g(t) = \frac{1}{2}a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n e^{\frac{\pi i n t}{L}} + a_n e^{\frac{-\pi i n t}{L}} - ib_n e^{\frac{\pi i n t}{L}} + ib_n e^{\frac{-\pi i n t}{L}}\right)$$

$$g(t) = \frac{1}{2}a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\left(a_n - ib_n\right)e^{\frac{\pi i n t}{L}} + \left(a_n + ib_n\right)e^{\frac{-\pi i n t}{L}}\right)$$

$$g(t) = \frac{1}{2}a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n - ib_n\right)e^{\frac{\pi i n t}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n + ib_n\right)e^{\frac{-\pi i n t}{L}}$$

Here I will use a trick. I will replace all ns in the second sum with $-n$ and change the sum to go from $-\infty$ to -1. This means that both sums will now have a positive exponential.

$$g(t) = \frac{1}{2}a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)\,e^{\frac{\pi int}{L}} + \frac{1}{2}\sum_{n=-\infty}^{-1}(a_{-n} + ib_{-n})\,e^{\frac{\pi int}{L}}$$

To compactify this I will write:

$$g(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{\pi int}{L}} + \sum_{n=-\infty}^{-1} c_n e^{\frac{\pi int}{L}}$$

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{\pi int}{L}} \tag{7}$$

or

$$g(t) = \sum_{n\in\mathbb{Z}} c_n e^{\frac{\pi int}{L}}$$

Where c_n is defined differently depending on whether n is $<$, $>$ or $=$ zero.

When $n < 0$:

$$c_n = \frac{(a_{-n} + ib_{-n})}{2}$$

(Since $n < 0$, $-n > 0$ so a_{-n} and b_{-n} are defined.)

$$c_n = \frac{1}{2L}\int_{-L}^{L} g(t)\cos\left(\frac{-\pi nt}{L}\right)dt + i\frac{1}{2L}\int_{-L}^{L} g(t)\sin\left(\frac{-\pi nt}{2L}\right)dt$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} g(t)\left(\cos\left(\frac{-\pi nt}{L}\right) + i\sin\left(\frac{-\pi nt}{L}\right)\right)dt$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} g(t)e^{\frac{-\pi int}{L}}\,dt$$

When $n > 0$:

$$c_n = \frac{a_n - ib_n}{2}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} g(t)\cos\left(\frac{\pi nt}{L}\right)dt - i\frac{1}{2L}\int_{-L}^{L} g(t)\sin\left(\frac{\pi nt}{L}\right)dt$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} g(t)\left(\cos\left(\frac{\pi nt}{L}\right)dt - i\sin\left(\frac{\pi nt}{L}\right)\right)dt$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} g(t)e^{\frac{-\pi int}{L}}\,dt$$

When $n = 0$:

$$c_0 = \frac{1}{2}a_0$$

$$c_0 = \frac{1}{2L}\int_{-L}^{L} g(t)dt$$

$$c_0 = \frac{1}{2L}\int_{-L}^{L} g(t)e^0 dt$$

$$c_0 = \frac{1}{2L}\int_{-L}^{L} g(t)e^{\dfrac{-\pi i0t}{L}} dt$$

So for all $n \in \mathbb{Z}$:

$$c_n = \frac{1}{2L}\int_{-L}^{L} g(t)e^{\dfrac{-\pi int}{L}} dt \tag{8}$$

Substituting into equation (7):

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2L}\int_{-L}^{L} g(t)e^{\frac{-\pi int}{L}} dt\ e^{\frac{\pi int}{L}} \tag{9}$$

5 The Fourier Transform

The complex form for a Fourier series as we now know is:

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} g(t) e^{\frac{-\pi int}{L}} dt \ e^{\frac{\pi int}{L}}$$

The aim of this section, and of the Fourier transform is to write any function as a sum of sines and cosines. The Fourier series only works for functions which are periodic. To extend this to functions which are non-periodic, we can think of them in a new way. A non-periodic function can be thought of as having an infinite period. That is, it repeats every infinity, or essentially, it does not necessarily repeat on finite intervals. To find a way to do this we must find the limit of the above as $L \to \infty$.

To do this, I will attempt to turn the infinite sum into a Riemann sum so that it can be written as an integral. First I will use a substitution to make this expression easier to work with. An obvious substitution might be to let $\omega = \frac{\pi n}{L}$.

$$\sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} g(t) e^{-i\omega t} dt \ e^{i\omega t}$$

For the $n = 0$ term of the above series: $\omega = 0$, for the $n = 1$ term: $\omega = \frac{\pi}{L}$ for the $n = 2$ term: $\omega = \frac{2\pi}{L}$. The change in ω between any two terms, $\Delta\omega = \frac{\pi(n+1)}{L} - \frac{\pi n}{L} = \frac{\pi}{L}$. This implies $\frac{1}{2L} = \frac{\Delta\omega}{2\pi}$ Substituting this in:

$$\sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-L}^{L} g(t) e^{-i\omega t} dt \ e^{i\omega t}$$

Finally as $L \to \infty$, $\Delta\omega \to d\omega$ and the bounds should also be changed, so the above Riemann sum becomes the integral:

$$g(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt \ e^{i\omega t} d\omega$$

Which works for any function, not just periodic ones (if our assumption that a function can be written as a sum of sines and cosines is correct, which it is in most cases). This itself is not the Fourier transform, but rather the inner integral is the Fourier transform. Sometimes this includes the $\frac{1}{2\pi}$ on the outside, sometimes it does not. Annoyingly there are many different conventions for the Fourier transform which differ slightly, some are equivalent expressions, others are not. One such convention is

$$\mathcal{F}_t\{g\}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt$$

With this notation meaning, the Fourier transform of g from a function of t into a function of ω. This is sometimes just written as $F(\omega)$. Using this convention:

$$g(t) = \int_{-\infty}^{\infty} F(\omega)\ e^{i\omega t} d\omega$$

Other conventions may use $i\omega t$ instead of $-i\omega t$ (since you are integrating from $-\infty$ to ∞ this will not affect the overall value of the integral). Others may not include the $\frac{1}{2\pi}$ at the front. The important thing is not the specific detail but the overall idea and what the Fourier transform allows you to do. An additional note is that whichever convention you use, as long as you adjust the above accordingly, you will get two equations. One to find the Fourier transform of a given function, and another to find the original function given its Fourier transform (the inverse Fourier transform for your chosen convention).

6 Conclusion

Fourier series and the Fourier transform have applications in areas of physics, medicine and engineering. They can be used to decompose sound signals into pure sine and cosine waves which allows certain frequencies to be removed. This can be used to remove frequencies which the human ear cannot detect as a form of audio compression. They also have medical applications, for example in MRI machines. They also, most importantly of course, have many applications in pure mathematics such as with the Poisson summation formula.

My hope is that this article has made the derivation more clear, though I do encourage that you look into such things as when you can swap the sum and integral signs and other moments lacking in rigour if you are interested.