An explanation and history of Mathematical discoveries and which mathematical figures and discoveries have been the most influential and important.

Table of Contents:

Section 0: introduction

The goal of this project is to answer the above question. Which mathematical figures and discoveries have been the most influential and important?

How could I answer this question, how could I decide with mathematicians and discoveries have been the most influential and important if I don't even understand concepts being discussed? Such a task would be impossible. The first step therefore will be to uncover, to rediscover all the ancient and recent mathematics being discussed. This is the primary reason behind the bulk of this project. It is not essential that you the reader are able to understand and comment on the technicalities, however I do believe it to be essential that I can. I also would not recommend that you skip over the mathematical detail because it is (in my opinion) the most interesting part of this project!

Once I have done this, I can be certain that I fully understand the concepts being talked about (and hopefully you, the reader, will learn a great deal too) I will research who discovered what, when the discoveries were made, as well as any controversies and misnomers. This information, accompanied by appropriate sources, will be sprinkled throughout the document in what I have deemed to be the most appropriate locations. (They should be easily identifiable as each source contains a link which is highlighted in blue and is underlined. The sources are also written in the Georgia Font)

Once I have researched which people have done what, I will have some idea as to the conclusion of this project as certain names are likely to have cropped up more frequently than others. I will use data, which I have collected from laymen, as well as professionals to answer the question.

Once my final judgement has been made it will be detailed in the conclusion along with a summary of what has been done.

Section 1: Basics and geometry

This section will go over the foundations of mathematics starting with definitions and axioms and later proving more complex geometric theorems and identities.

Axioms and Definitions

Ancient Greek Philosopher, Aristotle was the first to coin the term “axiom”.

“The common notions are evidently the same as what were termed “axioms” by Aristotle” - Britannica, The Editors of Encyclopaedia. (1998) "axiom". *Encyclopedia Britannica*. Available at: https://www.britannica.com/topic/axiom

In this chapter I will define the terms “axiom” and “definition” in terms of mathematics.

A definition is something which mathematicians declare, for example the definitions of e, π, and i, given earlier. These do not require proof as they are defined, not proven.

An axiom is a mathematical statement, without a formal proof, as it is fundamental and cannot be proven true. Axioms are important as all complex mathematical ideas are built upon them, including that of Euler’s formula and identity and Isaac Newton’s approximation of π.

What are e, π and i, and what is a limit?

These are all examples of definitions as they have each been defined in a certain way.

I will begin with i, the most difficult to wrap your head around at first, but the easiest to define.

$$i = \sqrt{-1}$$

Many people first encounter this number, and ask what does that mean? Where on the number line does that fall? What is that supposed to even represent? These questions are all very difficult to answer, but the best solution I find is to simply think of it as any other number, with one key property. If you multiply it by itself the result is -1. That’s it, that’s really all there is to it.

Complex numbers were first invented (and thought to be useless) by Italian mathematician Gerolamo Cardano in 1545.

“They obtained these new numbers by extending the arithmetic operation of square root to whatever numbers appeared in solving in solving quadratic equations by... Cardan, in Chapter 37 of *Ars Magna* (1545), sets up and solves the problem of... He then states, “So progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless.”” Kline, M. (1990) "Mathematical Thought From Ancient to Modern Times". Volume 1. p.253. Oxford: *Oxford University Press*. Available at: https://books.google.co.uk/books/about/Mathematical_Thought_From_Ancient_to_Mod/lbPN34fg760C?hl=en

Though Swiss Mathematician Leonhard Euler was the first to represent the imaginary unit $(\sqrt{-1})$ using the symbol i. He also popularised the use of the symbol π to represent the famous constant, which was first used in that way by William Jones in 1706.

“We shall from now on write i for $\sqrt{-1}$. This notation was first introduced by the Swiss mathematician Leonhard Euler (1707-1783). Much of our modern notation is due to him including e and π. Euler was a giant in 18th century mathematics and the most prolific mathematician ever.” – Earl, R. (2003) "Week 4 – Complex Numbers". p.2. Oxford: *Mathematical Institute*. Available at: https://www.maths.ox.ac.uk/system/files/attachments/complex.pdf

Secondly π, the one which you are most likely to have heard of, its definition is relatively simple. Draw a shape with each point equidistant from a given point, this shape is a circle, that point is the centre, and the distance is the radius. Multiply this radius (r) by 2, this is the diameter (d). Measure the distance all the way around the circle, its perimeter, this is the circumference (c).

$$\pi = \frac{c}{d}$$

Of course, it is important to note that pi should not be calculated this way, due to imprecision in the result.

The discovery of pi is often accredited to Greek philosopher and mathematician Archimedes of Syracuse, although there is evidence of pi being known about long before he was alive (Born 287 BC), such as a clay tablet found from ancient Babylon (1900-600 BC).

“A clay tablet unearthed in 1936 from the Old Babylonian period, approx. 1900-1600 BC, states that the circumference of an hexagon [sic] is 0;57,36 (in base 60) = 96/100 = 24/25 times the circumference of the circumscribed circle [72, p. 18]. From $u_{hexagon} = 3 \cdot d = 24/25 \cdot u_{circle} = 24/25 \cdot \pi \cdot d$ we get what is perhaps the oldest approximation to π, $\pi_{Babylon} = 3\frac{1}{8} = 3.125$” - Arndt, J. and Haenel, C. (2001) "Pi - Unleashed". Second Edition. p.167. Translated by Lischka, C. and Lischka, D. Germany: *Springer Science and Publish Media*. Available at: https://books.google.co.uk/books?id=QwwcmweJCDQC&redir_esc=y

Finally, e. Whilst not named by him, the concept was first used by Jacob Bernoulli who wanted to answer a question about compound interest. The question was this:

He knew that if he put £1 in a bank which offered 100% interest per year, he would have £2 at the end of the year.

(It would have been in whatever currency was used in Switzerland during the seventeenth century, but that is not important).

He wanted to know what would happen if instead of 100% once a year, there was an interest rate of 50% twice a year.

This was not difficult to compute.

$$1 * 1.5 = 1.5$$

$$1.5 * 1.5 = 2.25$$

This thought process continued. What about an interest of 1/3 three times?

$$1 * \left(1 + \frac{1}{3}\right) = 1 + \frac{1}{3}$$

$$\left(1 + \frac{1}{3}\right) * \left(1 + \frac{1}{3}\right) = \frac{16}{9}$$

$$\frac{16}{9} * \left(1 + \frac{1}{3}\right) = \frac{64}{27} \approx 2.37$$

This could be more cleanly written as:

$$\left(1 + \frac{1}{3}\right)^3$$

He noticed that each time he did this, the end value increased.

A general expression for this might be:

$$\left(1 + \frac{1}{n}\right)^n$$

Where the larger the value of n, the larger the total.

He kept increasing the value of n, to see how high the total could get, but he found that it seemed to approach a finite value of approximately 2.718 which we today call "e" thanks to Euler who later named it. Despite what you may be thinking, he (at least supposedly) did not name the number after himself.

"In a letter written in 1731 the number e appeared again in connection with a certain differential equation… Why did he choose the letter e? There is no general consensus. According to one view, Euler chose it because it is the first letter of the word *exponential*. More likely, the choice came to him naturally as the first "unused" letter of the alphabet, since the letters a, b, c and d frequently appear elsewhere in mathematics. It seems unlikely that Euler chose the letter because it is the initial of his own name, as has occasionally been suggested: he was an extremely modest man and often delayed publication of his own work so that a colleague or student of his would get due credit. In any event, his choice of the symbol e, like so many other symbols of hs, became universally accepted." – Eli, M. (1994) "e: The Story Of A Number". p156 Princeton: *Princeton University Press*. Available at: https://archive.org/details/estoryofnumber0000maor_x8v0/page/156/mode/2up?view=theater

e can be more easily calculated by using calculus, which will be done later in this project.

As alluded to earlier, the larger the value of n, the closer the total is to the true value of e. This can be written in a more mathematical way using the concept of limits first developed by Archimedes but wouldn't gain standard notation until much later. The notation for the above expression would look like this:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

This limit notation was first used by English mathematician John Gaston Leathem in his Book titled "Volume and Surface Integrals Used in Physics" in 1905.

"Our present day expression $\lim_{x \to c}$ seems to have originated with the English mathematician John Gaston Leathem in his 1905 book *Volume and Surface Integrals Used in Physics." - Miller, J. (2017)* "Earliest Uses of Symbols of Calculus". Scotland: *University of St Andrews*. Available at: https://mathshistory.st-andrews.ac.uk/Miller/mathsym/calculus/#:~:text=The%20arrow%20notation%20for%20limits.&text=Our%20present%20day%20expression%20lim,Surface%20Integrals%20Used%20in%20Physics

This means that the closer the value of n to infinity (the larger n is) the closer the expression will be to the exact value of e. We use limits because sometimes simply writing ∞ or 0 makes the expression undefined, for example with the above function, the expression would reach the form 1^{∞} which is undefined. Similarly, if we had to divide by a very small number, for example in differentiation we may use the limit: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ In this case setting h = 0 would result in division by zero which is also an undefined form. Limits allow us to avoid this. We will review the above limit in much more detail later.

The discovery (or perhaps invention) of the constant "e" is often accredited to Euler, after whom it was named, though in reality it is thought that the first to discover the constant was Swiss mathematician Jacon Bernoulli.

"In 1683 Jacob Bernoulli looked at the problem of compound interest and, in examining continuous compound interest, he tried to find the limit of $\left(1 + \frac{1}{n}\right)^n$ as n tends to infinity. He used the binomial theorem to show that the limit had to lie between 2 and 3 so we could consider this to be the first approximation found to e. Also if we accept this as a definition of e, it is the first time that a number was defined by a limiting process" - O'Connor, J.J. and Robertson, E.F. (2001) "The number e". Scotland: *University of St Andrews. Available at:* *https://mathshistory.st-andrews.ac.uk/HistTopics/e/*

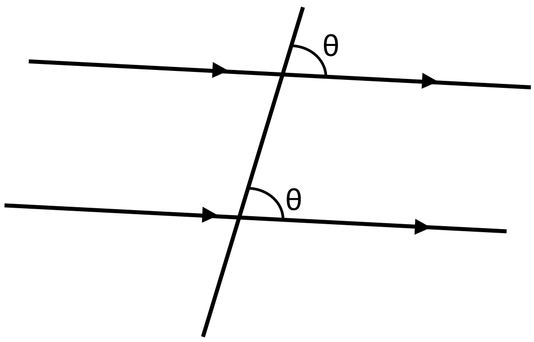
These definitions all come together in an equation often said to be “the most beautiful”

$$e^{i\pi} + 1 = 0$$

I will let you be the judge.

This equation is one which we will prove much later, for now we must continue with the basics.

The first axiom I will cover is the corresponding angles axiom. This axiom states that if two parallel lines each intersect a different straight line, the angles created will be equal.

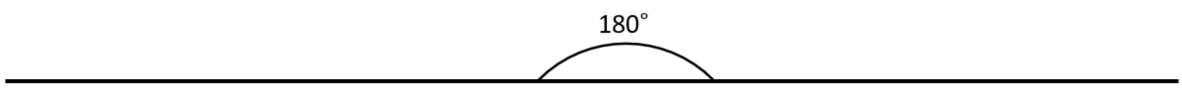


In this instance, the two angles labelled θ are equal because they are corresponding angles. This requires no proof as it is an intuitive enough observation. You could slide one of the intersecting lines up and down, without changing the slope of any of the lines. This means that the angle between them would not change. You could slide the top line down into the position of the bottom line and the angle would not have changed. Therefore, corresponding angles are equal.

That was an example of an axiom, a statement which is fundamental and usually intuitive and cannot be or doesn’t need to be proven.

The degree (a unit of measuring angles) is another example of a definition, and is defined as follows:

Let 1° (one degree) be a measurement of angle equal $\frac{1}{360}$ of a full rotation about a point. This requires no proof as it is a definition. We can use this definition to state facts about the



thing we have defined, e.g., a full rotation about a point = 360°, half a rotation about a point (a straight line) has an angle of 180° as shown here. A quarter rotation (a right-angle) is 90°.

These two ideas are the underlying pillars beneath all of mathematics.

Using axioms and definitions to discover new mathematics

In this chapter I will review how these basic ideas and assumptions can be used to reach far more complex and unexpected conclusions.

I will now prove that co-interior angles add to $180°$ using the corresponding angles axiom and the degree definition.

In the above diagram, I have labelled three angles, a, b and c as shown.

$a = b$ because they are corresponding angles.

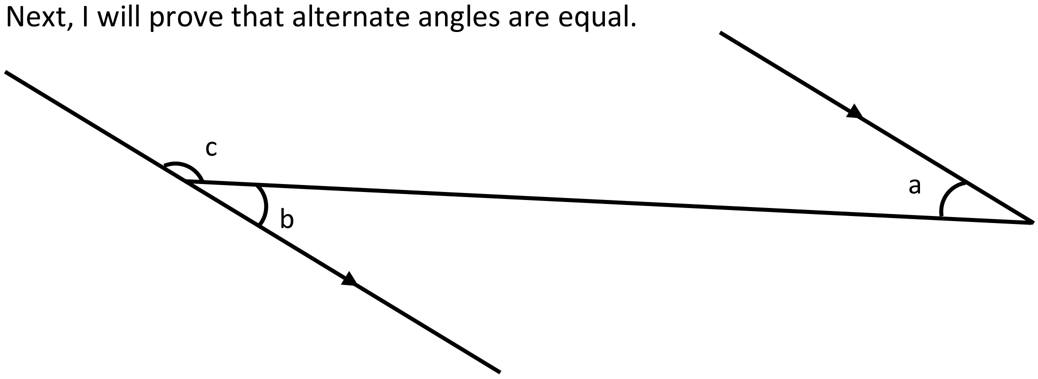
$c + a = 180$ because they are angles on a straight line.

Substitute a for b.

$c + b = 180$

$therefore\ co - interior\ angles\ add\ to\ 180°$

Next, I will prove that alternate angles are equal.



a and b are alternate angles. $a + c = 180$ because co-interior angles add to 180°

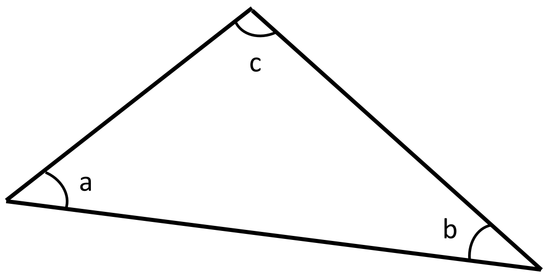
$c + b = 180$ because angles on a straight line add to 180° therefore

$180 = c + b = a + c$

so $a = b$ therefore alternate angles are equal.

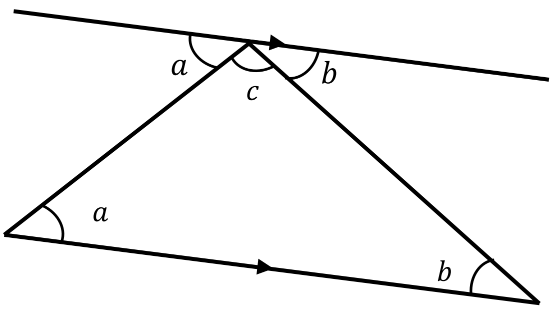
We can use the alternate angles theorem, and the degree definition to prove that angles in a triangle add to $180°$.

Here is a triangle with angles a, b and c. a, b and c are used to generalise to prove that everything said applies to all cases, as opposed to just one.



$$a + b + c = total\ number\ of\ degrees\ in\ a\ triangle$$

We can draw a line parallel to one of the triangles sides, which touches the vertex opposite.



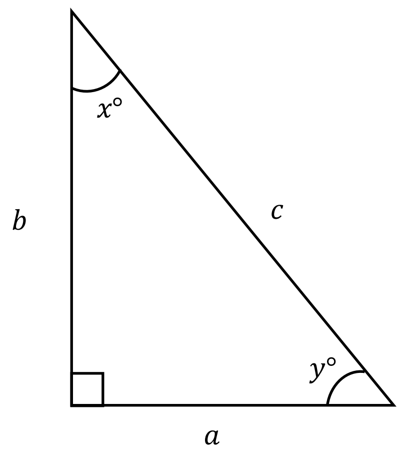
Because alternate angles are equal, we can label the two new angles a and b. We now have a straight line with angle of $180°$, comprised of a, b and c. Therefore $a + b + c = 180$. We also know that $the\ sum\ of\ angles\ in\ a\ triangle = a + b + c$.

Therefore $the\ sum\ of\ angles\ in\ a\ triangle = 180°$

The Pythagorean Theorem

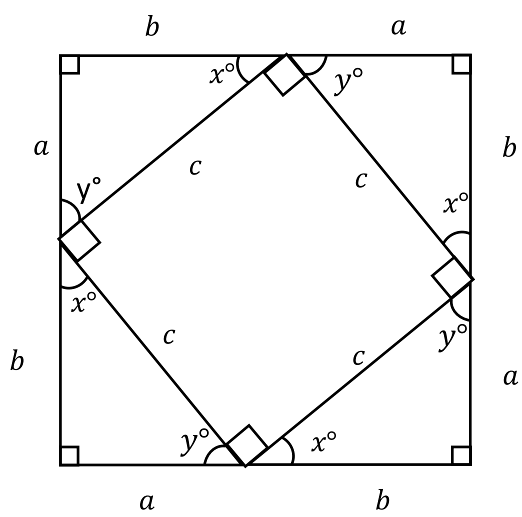
In this chapter I will prove arguably the most well-known theorem in all of mathematics, the Pythagorean theorem.

A right-angled triangle is a triangle with a $90°$angle in it. We will label its three sides a, b and c, with c being the hypotenuse (the side opposite the right angle), and angles $90°$, $x°$ and $y°$.



Note that $x + y + 90 = 180$ because angles in a triangle add to $180°$. This will be important in a moment.

In the diagram to the right, I have taken the above shape and copied it 4 times into the pattern shown forming a large square with side lengths $a + b$. We know it is a square as all 4 sides are the same length and each corner is $90°$. We also know that each side of the smaller shape has the same length, c.



$Let\ the\ angle\ of\ each\ corner\ of\ the\ smaller\ shape = z$

$x + y + z = 180°$ because they are angles on a straight line.

recall $x + y + 90° = 180°$

$x + y = 90°$

Substitute $x + y = 90°$ into the first equation.

$90° + z = 180°$

$z = 90°$

We now know that each corner of the smaller shape is ninety degrees. This means that this shape is a square and therefore its $area\ = c^2$. The area can also be calculated in a different way, by first calculating the area of the large square, then subtracting that of the small triangles.

$The\ area\ of\ the\ large\ square = (a+b)^2 = a^2 + 2ab + b^2$

The $area\ of\ each\ small\ triangle = \frac{ab}{2}$

There are 4 small triangles, so their $total\ area = \frac{4ab}{2} = 2ab$

So the $area\ of\ the\ small\ square = a^2 + 2ab + b^2 - 2ab = a^2 + b^2$

The $area\ of\ the\ small\ square = c^2$

Therefore $a^2 + b^2 = c^2$

Despite the name, the theorem was not actually discovered by Pythagoras himself. This is why in certain places, such as China, different names are used. In China, the theorem is referred to as the "Gougu Rule". There is evidence that the theorem had already been discovered even a thousand years before Pythagoras was ever alive.

"The above example of the determination of the diagonal of the square from its side is sufficient proof that the "Pythagorean" theorem was known more than a thousand years before Pythagoras. This is confirmed by many other examples of the use of this theorem... it was well known during the whole duration of Babylonian mathematics that the sum of the squares of the lengths of the sides of a right triangle equals the square of the hypotenuse." – Neugebauer, O. (1969) "The Exact Sciences in Antiquity". Second Edition. p.36. New York: *DOVER PUBLICATIONS, INC.* Available at: https://books.google.co.uk/books/about/The_Exact_Sciences_in_Antiquity/JVhTtVA2zr8C?hl=en

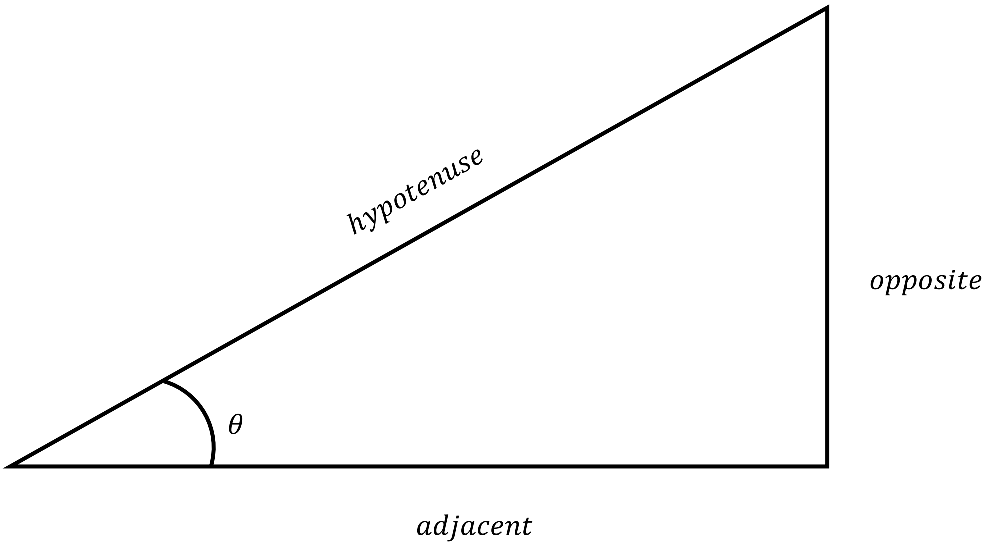
Trigonometric Functions: sin(ϑ), cos(ϑ) and tan(ϑ)

In this chapter I will introduce the trigonometric functions and will provide their special cases.

Modern trigonometry, including the sin cos and tan functions began in Ancient Greece. The first to have created a table of results to a trigonometric function was Ancient Greek Astronomer Hipparchus.

"Trigonometry in the modern sense began with the Greeks. Hipparchus (*c.* 190–120 BCE) was the first to construct a table of values for a trigonometric function." – Maor, E. (1998) "trigonometry". *Encyclopedia Britannica*. Available at: https://www.britannica.com/science/trigonometry#ref225327

The axiom of similarity states that if two triangles have the same set of angles, then they must be similar. This means that the ratio between two given side lengths will be equal for both triangles.



Given that the angles in a triangle add to $180°$, if you know that two triangles are right angled, and you know that they each have a common angle, then those triangles must be similar as the third angle must also be the same. This means that the ratio between any two given side lengths will be the same for each of the two triangles. This is important because it means that if you know that a triangle is right angled and you know a second angle from that triangle, you can determine the ratio between two of its sides. This is what the sin, cos and tan functions seek to do. Currently there is no way for us to calculate the sin, cos and tan of any given angle, however we can work out a few special cases, only two of which (ninety and zero) are necessary for proving Euler's identity, though for completeness I will cover the others.

First, create a right-angled triangle, with an angle θ (theta). The side opposite the right angled should be labelled hypotenuse (h), the side opposite theta, should be labelled opposite (o) and the side connecting theta to the right-angle should be labelled adjacent (a). Then let $sin(\theta) = \frac{o}{h}$, $cos(\theta) = \frac{a}{h}$ and $tan(\theta) = \frac{o}{a}$.

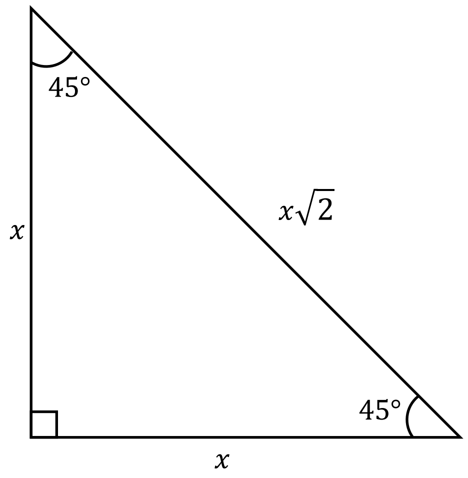
In order to calculate sin, cos and tan of a few basic angles I will construct right-angled triangles with the following angles in them: $0°$, $30°$, $45°$, $60°$, $90°$

I will start with $45°$, by drawing a right-angled isosceles triangle, with width and height, x. I am using x to demonstrate that the ratio between side lengths is maintained regardless of the specific side lengths used. We can calculate the length of the hypotenuse using the Pythagorean theorem.

$$a^2 + b^2 = c^2$$

$$2x^2 = c^2$$

$$c = \sqrt{2x^2} = x\sqrt{2}$$



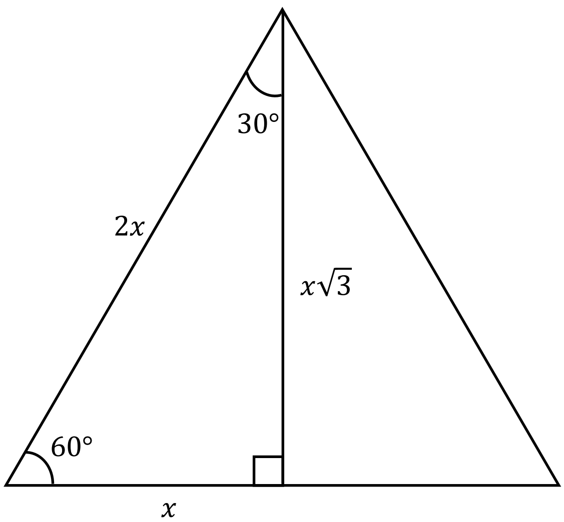
The sin, cos and tan of 45° can now be calculated:

$$\sin(45) = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos(45) = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan(45) = \frac{x}{x} = 1$$

For 30° and 60°, I will create an equilateral triangle with sidelengths $2x$, and then cut it in half. We can calculate the length of the third side using the pythagorean theorem.



$$a^2 + b^2 = c^2$$

$$(2x)^2 = x^2 + b^2$$

$$4x^2 = x^2 + b^2$$

$$3x^2 = b^2$$

$$b = \sqrt{3x^2} = x\sqrt{3}$$

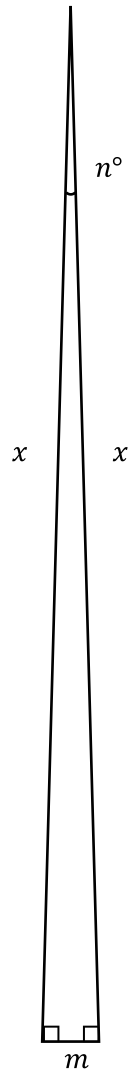
The sin, cos and tan of 30° and 60° can now be calculated

$\sin(30) = \frac{x}{2x} = \frac{1}{2}$ $\cos(30) = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$ $\tan(30) = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin(60) = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$ $\cos(60) = \frac{x}{2x} = \frac{1}{2}$ $\tan(60) = \frac{x\sqrt{3}}{x} = \sqrt{3}$

Finally, I will calculate the sin, cos and tan of 0° and 90°. Bear in mind that the following diagram shows the limiting case as an angle and side length cannot be zero. The angle and side length both approach zero, but aren't equal to each other which is why I have represented them with separate limits, both approaching the same value, but not necessarily at the same rate.

This diagram can be used to calculate the sin, cos and tan of 0° and 90°



Findingthe values of opposite, adjacent and hypotenuse is mostly as straightforward as with the others, though a few values may seem confusing as to what they are. Here are the explanations of those:

The hypotenuse is x because the side length opposite each right-angle $= x$

The adjacent of 0° is x because the distance from 0° to each right angle $= x$

The adjacent of 90° is 0 because the distance between the two right angles = 0

$\lim_{n\to 0}$

$\lim_{m\to 0}$

$$\lim_{n\to 0} \sin(n) = \sin(0) = \lim_{m\to 0} \frac{m}{x} = 0$$

$$\lim_{n\to 0} \cos(n) = cos(0) = \frac{x}{x} = 1$$

$$\lim_{n\to 0} \tan(n) = \tan(0) = \lim_{m\to 0} \frac{m}{x} = 0$$

$$\sin(90) = \frac{x}{x} = 1$$

$$\cos(90) = \frac{0}{x} = 0$$

$$\tan(90) = \frac{x}{0}\ which\ is\ undefined$$

Trigonometric Identities

In this chapter I will outline the relationship that the trigonometric functions have with each other via a set of identities.

$sin(\theta)$, $cos(\theta)$ and $tan(\theta)$ have a special relationship with each other, which can be expressed in the form of trigonometric identities. Equations which always hold true, regardless of the value of θ.

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{o}{h} \div \frac{a}{h} = \frac{o}{h} \times \frac{h}{a} = \frac{oh}{ha} = \frac{o}{a} = \tan(\theta)$$

$$\frac{\sin(\theta)}{\cos(\theta)} \equiv \tan(\theta)$$

$$sin^2(\theta) + cos^2(\theta) = \left(\frac{o}{h}\right)^2 + \left(\frac{a}{h}\right)^2 = \frac{o^2}{h^2} + \frac{a^2}{h^2} = \frac{o^2 + a^2}{h^2} = \frac{h^2}{h^2} = 1$$

$$sin^2(\theta) + cos^2(\theta) \equiv 1$$

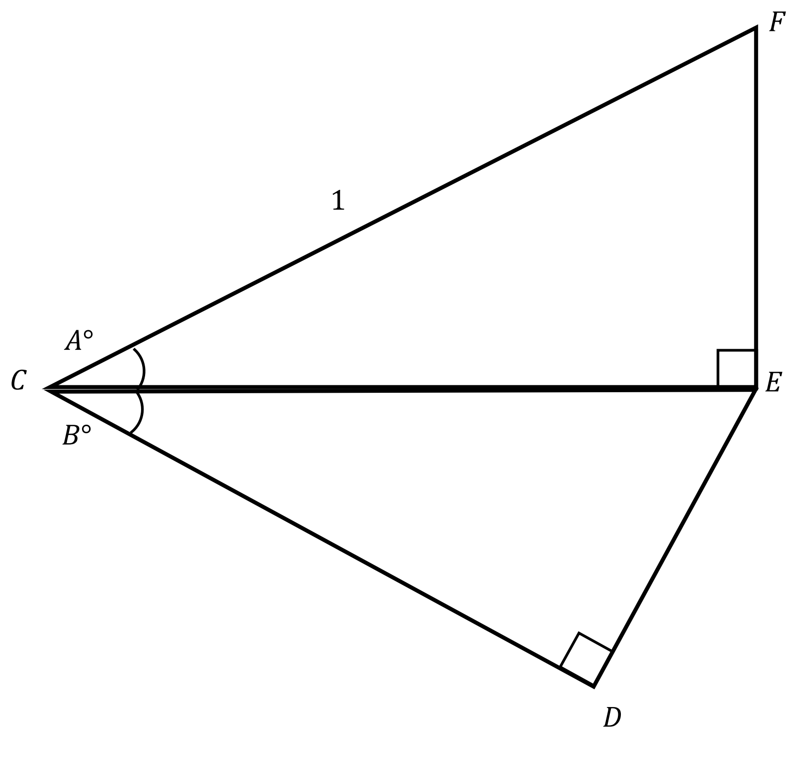
$o^2 + a^2 = h^2$
Pythagorean theorem

Sin(A+B) & Cos(A+B)

In this chapter I will prove two more, more complex identities.

Now that we have a list of known trigonometric function outputs, it will be necessary to work out new ones by adding the previous one together. For example, we know $sin(45)$ and $sin(30)$, and it might be useful to work out what $sin(45 + 30)$ equates to.

I will draw I diagram of the following: A right-triangle with hypotenuse 1 and angle A, -followed by a second right-triangle, whose hypotenuse is equal to the bottom side of the first triangle, with an angle B.



$$FC = 1$$

$$\sin(A) = \frac{FE}{FC}$$

$$FE = \sin(A)$$

$$\cos(A) = \frac{EC}{FC}$$

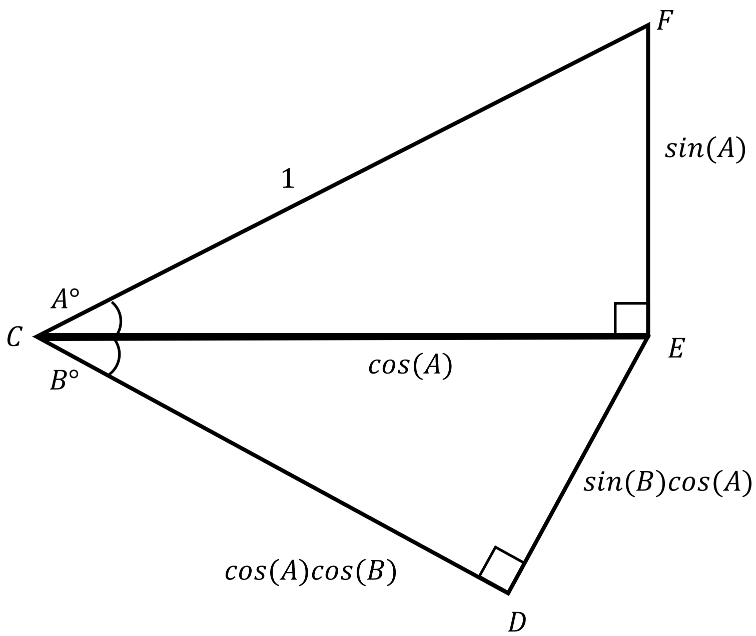
$$EC = \cos(A)$$

$$\sin(B) = \frac{ED}{EC} = \frac{ED}{\cos(A)}$$

$$ED = \sin(B)\cos(A)$$

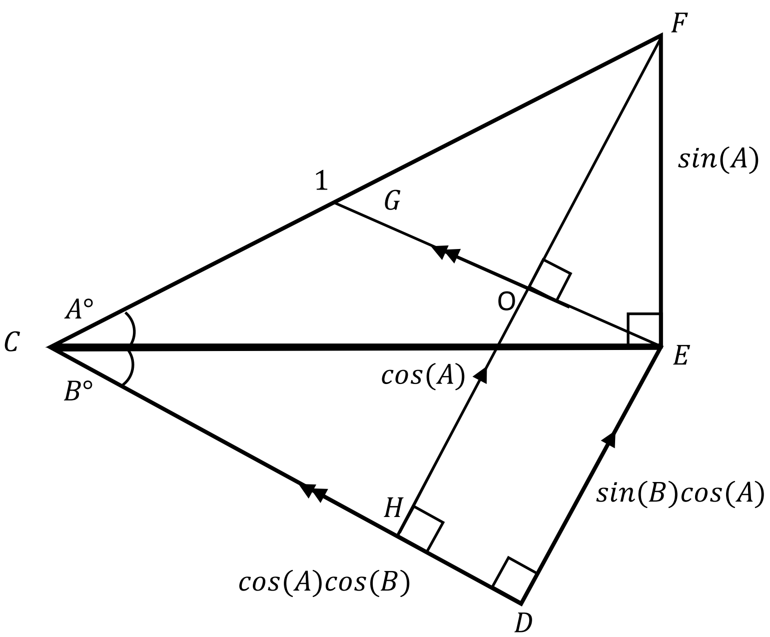
$$\cos(B) = \frac{DC}{EC} = \frac{DC}{\cos(A)}$$

$$DC = \cos(A)\cos(B)$$



Next, create a line passing through the point F, and perpendicular to side CD

Then create a line passing through point E, perpendicular to this new line.



$\widehat{CEG} = B$ because alternate angles are equal

$\widehat{FEG} = 90 - B$

$\widehat{HFE} = B$ angles in a triangle add to 180°

$\cos(B) = \frac{FO}{\sin(A)}$ $\quad FO = \sin(A)\cos(B)$

$HO = \sin(B)\cos(A)$

$HF = FO + HO$

$= \sin(A)\cos(B) + \sin(B)\cos(A)$

$\sin(B) = \frac{OE}{\sin(A)}$ $\quad OE = \sin(A)\sin(B)$

$$HD = OE = \sin(A)\sin(B)$$

$$CH = CD - HD$$

$$= \cos(A)\cos(B) - \sin(A)\sin(B)$$

Now focus on triangle CHF.

This is a right triangle where $angle\ \widehat{FCH} = A + B$

Angle $\widehat{FCH}$ has an opposite $HF = sin(A)cos(B) + sin(B)cos(A)$

Angle $\widehat{FCH}$ has an adjacent $CH = cos(A)cos(B) - sin(A)sin(B)$

The $hypotenuse\ of\ this\ right\ triangle = 1$

Therefore:

$$\sin(A + B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

These are known as the “addition formulae”.

The former of these two identities was first discovered by Hindu Mathematician Bhāskara II in the 12th century AD.

“He also gives the (now) well known results for sin(a + b) and sin(a - b)” – Pearce, I.G. (no date) "Indian Mathematics – Redressing the balance". p.13.

It is also thought that he had already discovered differential calculus hundreds of years before Sir Isaac Newton and Gottfried Wilhelm Leibniz, though this achievement often goes unappreciated.

“Evidence suggests Bhaskara was fully acquainted with the principle of differential calculus, and that his researches were in no way inferior to Newton's, asides the fact that it seems he did not understand the utility of his researches, and thus historians of mathematics generally neglect his outstanding achievement, which is extremely regrettable.” – Pearce, I.G. (no date) "Indian Mathematics – Redressing the balance". p.13. Scotland: *University of St Andrews*. Available at: https://mathshistory.st-andrews.ac.uk/Projects/Pearce/

In order to be able to calculate more specific angles, we will need to use a Taylor (or more specifically Maclaurin) series introduced by Brook Taylor and Colin Maclaurin in 1715. Before we can do this however, we will need to first reinvent differentiation.

Section 2: Calculus

This is a section all about differential calculus. The power rule for differentiating polynomials will be proven, as well as the derivatives for the functions: $\sin(x)$, $\cos(x)$ and e^x.

Differentiation

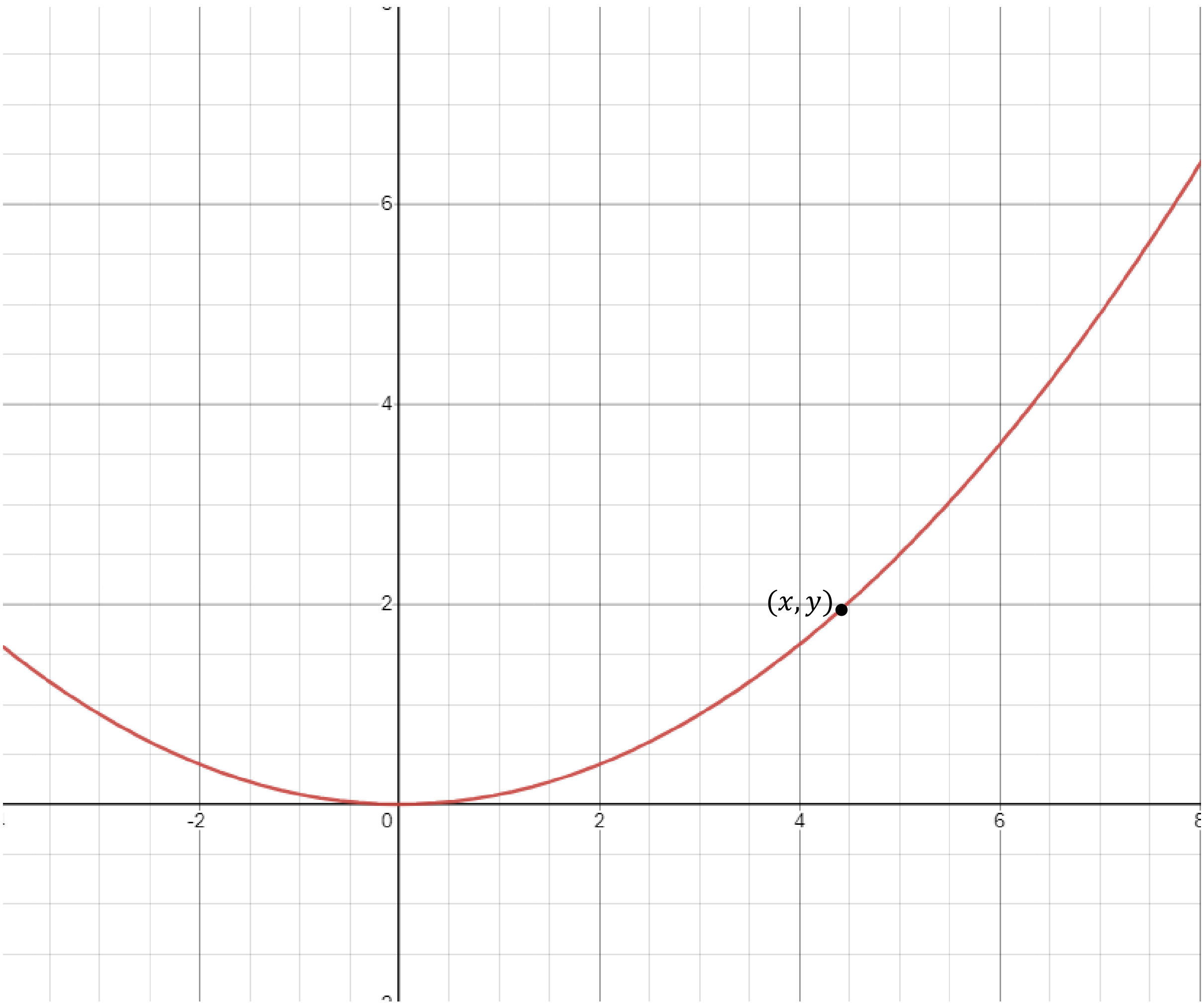
In this chapter I will show the difference quotient which can be used to differentiate a function. This can later be used to differentiate certain functions.

Differentiation is the process by which the gradient function can be derived from some function $f(x)$. The gradient function for $f(x)$ will return the gradient of the $f(x)$ function for any given x coordinate.

How is a gradient calculated?

A gradient is calculated by dividing the change in y by the change in x. On a curved graph, this means drawing a tangent to the curve and measuring its gradient to find the instantaneous rate of change. The infinitesimally small change in y (dy) divided by the infinitesimally small change in x (dx), hence the gradient is referred to as $\frac{dy}{dx}$ when referring to a curved graph.

Picture a graph $y = f(x)$

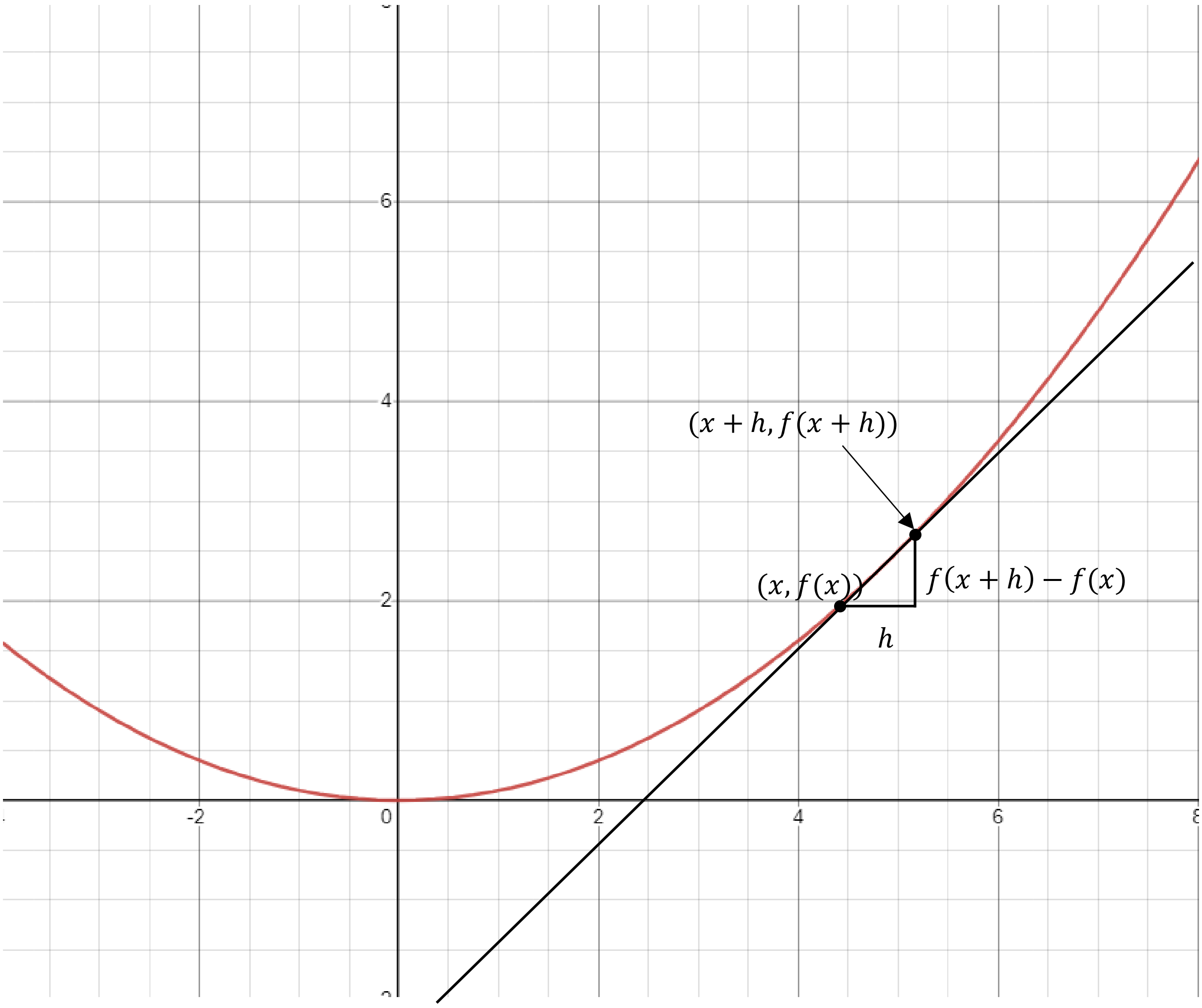


I want to calculate the gradient at coordinates (x, y). Because $y = f(x)$, this can be written as $(x, f(x))$. In order to calculate the gradient, I would first choose some point on the graph close to it, with an x-coordinate of $x + h$ where h is a very small number. As mentioned earlier this would be written as:

$$\lim_{h \to 0}(x + h)$$

This means the coordinates of the second point would be:

$$\lim_{h \to 0}((x + h), (f(x + h)))$$



The calculation of the gradient would be $\frac{y_2 - y_1}{x_2 - x_1}$

$$= \lim_{h \to 0}\frac{f(x + h) - f(x)}{h}$$

This limit is known as the difference quotient and can be used to differentiate functions such as $sin(x)$, $cos(x)$, e^x and polynomials, all of which we will later look at in greater detail.

Calculus was discovered by Sir Isaac Newton, who named it “the science of fluxions”, and Gottfried Wilhelm Leibniz who names it “calculus”. Both discovered calculus independently and each accused the other of plagiarism. Newton was able to use his position as Lucasian Professor of Mathematics at The University of Cambridge and President of the Royal Society to convince his colleagues that he was the first to discover calculus.

“Eventually, in 1711, Leibniz appealed to the Royal Society, of which he was a member, to adjudicate in the dispute. Newton, as President of the Society, set up a committee which barely needed to meet because Newton was already busy writing its report. Not surprisingly, it found in Newton’s favour. And, also not surprisingly, that was not the end of the matter: the dispute rumbled on until after Leibniz’ death in 1716. The dispute explains why om 1809 the English schoolboy George Peat in Cumbria learned a subject called ‘fluxions’ rather than a subject called calculus.” – Stedall, J. (2012) "The History of Mathematics: A Very Short Introduction". p.100. Oxford; New York: *Oxford University Press*. Available at: https://archive.org/details/historyofmathema0000sted/page/100/mode/2up

“the charges of plagiarism brought against Leibniz by partisans of Newton, and indeed by Newton himself in the *Recensio* published in the *Philosophical Transactions*, were unfounded. I considered that the charges in the *Recensio* were perhaps the hardest to be answered, since they were not only direct charges, backed with circumstantial evidence, but they were also set forth very cleverly” – Child, J.M. (1920) "The Early Mathematical Manuscripts of Leibniz". p.228. Chicago: *Open Court Publishing Company*. Available at: https://archive.org/details/in.ernet.dli.2015.161147/page/n223/mode/2up

Derivative of a monomial

In this chapter I will show how the difference quotient can be used to differentiate a monomial.

I will begin with an example, $f(x) = x^2$

$f'(x)$ is a notation which notates the gradient function of $f(x)$

The difference quotient can be used to find $f'(x)$

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0}\frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0}(2x + h)$$

$$= 2x$$

$$f'(x) = 2x$$

This is an example of how the difference quotient can be used to find the derivative of a monomial with just one term. I will in a moment discuss polynomials with multiple terms, but for now will focus on just the one. In order to understand what is happening, we must understand the expansion of $(x + h)^n$

I will do so by using $n = 3$ as an example.

$(x + h)^3 = (x + h)(x + h)(x + h)$

$$(x + h)(x + h) = x(x + h) + h(x + h)$$

$$= x^2 + xh$$

$$+xh + h^2$$

$$= x^2 + 2xh + h^2$$

$$(x + h)(x^2 + 2xh + h^2) = x(x^2 + 2xh + h^2) + h(x^2 + 2xh + h^2)$$

$$= x^3 + 2x^2h + xh^2$$

$$+x^2h + 2xh^2 + h^3$$

$$= x^3 + 3x^2h + 3xh^2 + h^3$$

The purpose of writing it out like this is to demonstrate that each time we multiply by

$(x + h)$, a copy of the previous polynomial is multiplied by x, shifting each power of x one to the right, another copy is multiplied by h, and then these two copies are added together by collecting like terms. This is effectively the same as adding the coefficient of one term to the coefficient of the term on its right, which is the same way in which the numbers of Pascal's triangle are arranged. This is why the binomial coefficients can be found on Pascal's triangle. The powers of x are also seen to ascend as the powers of h descend. This demonstrates $(x + h)^n$ expands in the way it does. The key thing to notice is that the first term will always equal x^n, the second term will always equal $nx^{n-1}h$ and that each subsequent term has a power of h, greater than 1.

Reviewing the difference quotient for a function: $f(x) = x^n$

$$\lim_{h \to 0} \frac{(x + h)^n - x^n}{h}$$

$$\lim_{h \to 0} \frac{x^n + nhx^{n-1} + \cdots + h^n - x^n}{h}$$

$$= \lim_{n \to 0} \frac{nhx^{n-1} + \cdots + h^n}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + \cdots h^{n-1}$$

$$= nx^{n-1}$$

This means that for a function $f(x) = x^n$

$$f'(x) = nx^{n-1}$$

If the original monomial term was multiplied by a constant, m, then all terms in the expansion will be multiplied by m, including the first and second terms and the $-x^n$, meaning for a function $f(x) = mx^n$

$$f'(x) = mnx^{n-1}$$

This is known as the power rule.

I will next discuss how to differentiate a function with multiple terms.

Differentiating multiple terms

In this chapter I will show how a sum of differentiable terms may be differentiated as a whole.

Two functions are defined:

$$f(x)$$

$$g(x)$$

I could differentiate each of these functions individually, but how would I differentiate their sum?

This can be done by defining a new function:

$$j(x) = f(x) + g(x)$$

And using the difference quotient to differentiate it.

$$j'(x) = \lim_{h \to 0} \frac{j(x+h) - j(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$j'(x) = f'(x) + g'(x)$$

This means that the derivative of the sum of two functions is equal to the sum of the derivatives of each of those functions. This means that a polynomial can be differentiated, by using to power rule on each term individually.

Differentiating sin(ϑ) and cos(ϑ)

In this chapter I will show how the trigonometric functions $sin(\theta)$ and $\cos(\theta)$ can be differentiated.

The difference quotient can be used to differentiate $sin(\theta)$ and $cos(\theta)$.

$$sin'(\theta) = \lim_{h \to 0} \frac{\sin(\theta + h) - \sin(\theta)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(\theta)\cos(h) + \sin(h)\cos(\theta) - \sin(\theta)}{h}$$

$$= \lim_{h \to} \frac{\sin(\theta)\,(\cos(\mathrm{h}) - 1) + \sin(\mathrm{h})\cos(\theta)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(\theta)(\cos(\mathrm{h}) - 1)}{h} + \lim_{h \to 0} \frac{\sin(h)\cos(\theta)}{h}$$

$$= \sin(\theta) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(\theta) \lim_{h \to 0} \frac{\sin(h)}{h}$$

The two limits now created cannot be solved algebraically. They are known as the fundamental trigonometric limits.

To solve the following limit: $\lim_{h \to 0} \frac{\sin(h)}{h}$ we must use radians, instead of degrees.

$$\pi\, rad = 180°$$

Remember the definition of pi was: $\pi = \frac{c}{d}$

$$c = \pi d$$

$$Let\ r = 1$$

$$d = 2r = 2$$

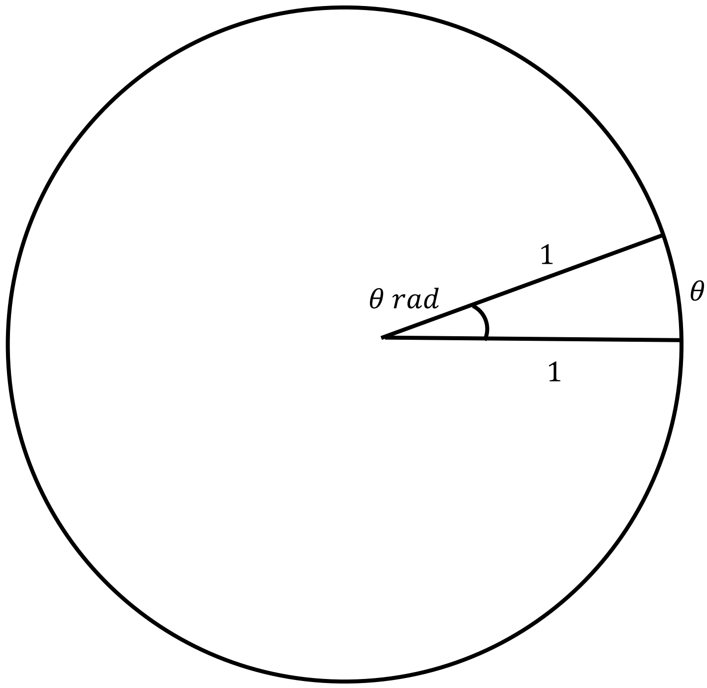
$$c = 2\pi$$

A full rotation around a unit circle would be a rotation of 360° or 2π rad.

A half rotation would cover half this distance and with half the angle, so the arc length would = π, and the angle would = π rad.

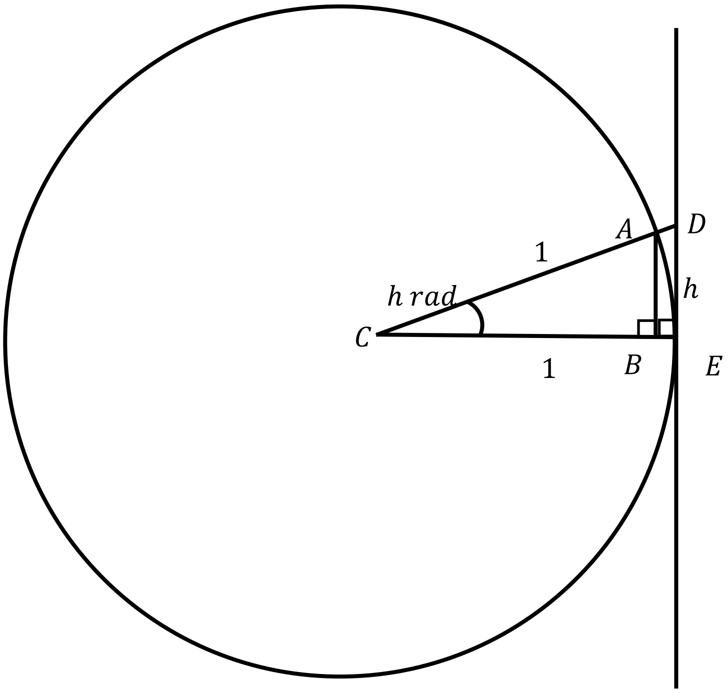
A quarter circle, radius 1 would have an arc of $\frac{c}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$ and an angle of $90° = \frac{\pi}{2}\ rad.$

So for a circle, radius 1, $the\ angle\ of\ a\ sector\ in\ radians = the\ length\ of\ the\ arc$ as shown in this diagram.



From this moment on, any mention of an angle will be measured in radians, unless stated otherwise.

Note that for the following diagram $h = \lim_{h\to 0} h$



There are now two right-triangles, one with the hypotenuse being a radius of the circle, the other with a hypotenuse extending until it touches the tangent.

$h = legnth\ of\ the\ arc\ AE = \widehat{ACB} = \widehat{DCE}.$

From the smaller triangle, ABC:

$$\sin(h) = \frac{AB}{1} = AB$$

From the larger triangle, DEC:

$$\tan(h) = \frac{DE}{1} = DE$$

This is where the $tan()$ function, short for tangent, gets its name, as
$the\ length\ of\ this\ section\ of\ the\ tangent = tan(h)$

h refers to the length of the arc, AE. As h approaches 0, the arc approaches a straight line which is longer than $AB = sin(h)$, and is shorter than $DE = \tan(h)$.

$$\lim_{h\to 0} \sin(h) < h < \tan(h)$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\lim_{h\to 0} \sin(h) < h < \frac{\sin(h)}{\cos(h)}$$

$$\lim_{h\to 0} \sin(h) < h$$

$$\lim_{h\to 0} \frac{\sin(h)}{h} < 1$$

$$\lim_{h\to 0} h < \frac{\sin(h)}{\cos(h)}$$

$$\lim_{h\to 0} \cos(h) < \frac{\sin(h)}{h}$$

$$\lim_{h\to 0} \cos(h) < \frac{\sin(h)}{h} < 1$$

$$\cos(0) < \lim_{h\to 0} \tfrac{\sin(h)}{h} < 1$$

$$1 < \lim_{h\to 0} \frac{\sin(h)}{h} < 1$$

It may seem impossible that $\lim_{h\to 0} \frac{\sin(h)}{h}$ can be less than and greater than 1. This is because the 1 on the left side of the inequality is actually the limit of $\lim_{h\to 0} \cos(h)$, which approaches 1. This means that the 1 on the left side of the inequality, should be seen as a number smaller than, but infinitesimally close to 1. This means that:

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1$$

This is called the squeeze theorem.

The first fundamental trigonometric limit has now been solved.

Next, I must solve the second:

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h}$$

$$= \lim_{h} \frac{(\cos(h) - 1)(\cos(h) + 1)}{h(\cos(h) + 1)}$$

$$= \lim_{h\to 0} \frac{cos^2(h) - 1}{h(\cos(h) + 1)}$$

$$sin^2(\theta) + \cos^2(\theta) = 1$$

$$cos^2(\theta) = 1 - \sin^2(\theta)$$

$$= \lim_{h\to 0} \frac{1 - sin^2(h) - 1}{h(\cos(h) + 1)}$$

$$= \lim_{h\to 0} \frac{-sin^2(h)}{h(\cos(h) + 1)}$$

$$= \lim_{h\to 0} \frac{sin(h)}{h} \times \lim_{h\to 0} \frac{-\sin(h)}{\cos(h) + 1}$$

$$= 1 \times \lim_{h \to 0} \frac{-\sin(h)}{\cos(h) + 1}$$

$$= \frac{-\sin(0)}{\cos(0) + 1}$$

$$= \frac{0}{1 + 1}$$

$$= 0$$

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$$

Both limits have now been solved, meaning $sin'(\theta)$ can now be evaluated.

$$sin'(\theta) = \sin(\theta) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(\theta) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= 0 \sin(\theta) + 1\cos(\theta)$$

$$= \cos(\theta)$$

$$sin'(\theta) = \cos(\theta)$$

The derivative of $sin(\theta)$ is $cos(\theta)$, but only when using radians, not degrees.

Next the derivative of $cos(\theta)$:

$$cos'(\theta) = \lim_{h \to 0} \frac{\cos(\theta + h) - \cos(\theta)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(\theta)\cos(h) - \sin(\theta)\sin(h) - \cos(\theta)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(\theta)\,(\cos(h) - 1) - \sin(\theta)\sin(h)}{h}$$

$$= \cos(\theta) \lim_{h \to 0} \frac{\cos(h) - 1}{h} - \sin(\theta) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= 0\cos(\theta) - 1(\sin\theta)$$

$$= -\sin(\theta)$$

$$cos'(\theta) = -\sin(\theta)$$

$$-\sin'(\theta) = \lim_{h \to 0} \frac{-\sin(\theta + h) + \sin(\theta)}{h}$$

$$= \lim_{h\to 0} \frac{-\sin(\theta)\cos(h) - \sin(h)\cos(\theta) + \sin(\theta)}{h}$$

$$= \lim_{h\to 0} \frac{-\sin(\theta)\,(\cos(\theta) - 1) - \sin(\mathrm{h})\cos(\theta)}{h}$$

$$= -\sin(\theta) \lim_{h\to 0} \frac{\cos(\theta) - 1}{h} - \cos(\theta) \lim_{h\to 0} \frac{\sin(h)}{h}$$

$$= -0\sin(\theta) - 1\cos(\theta)$$

$$= -\cos(\theta)$$

$$-sin'(\theta) = -\cos(\theta)$$

$$-\cos'(\theta) = \lim_{h\to 0} \frac{-\cos(\theta + \mathrm{h}) + \cos(\theta)}{h}$$

$$\lim_{h\to 0} \frac{-\cos(\theta)\cos(h) + \sin(\theta)\sin(\mathrm{h}) + \cos(\theta)}{h}$$

$$= \lim_{h\to 0} \frac{-\cos(\theta)\,(\cos(h) - 1) + \sin(\theta)(\sin(h))}{h}$$

$$= -\cos(\theta) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \sin(\theta) \lim_{h\to 0} \frac{\sin(h)}{h}$$

$$= -0\cos(\theta) + 1\sin(\theta)$$

$$= \sin(\theta)$$

$$-cos'(\theta) = \sin(\theta)$$

$$sin'(\theta) = cos(\theta)$$

$$cos'(\theta) = -sin(\theta)$$

$$-sin'(\theta) = -cos(\theta)$$

$$-cos'(\theta) = sin(\theta)$$

Roger Cotes was the first to differentiate the *sine* function, though Thomas Simpson found the same result independently, only do find that Cotes had already made the same discovery 15 years prior, via the same method. It was also found that a colleague of Leonhard Euler’s had reached the same conclusion via the same method also.

“Thomas Simpson… he proves geometrically the result that “the Fluxion of any circular arch is to the Fluxion of its Sine, as Radius to the Cosine”… The cited proof was not, however, original to Simpson. It had appeared some 15 years earlier when the manuscript of Roger Cotes were published 6 years after his untimely death at the age of 34... Cotes uses essentially the same diagram as Simpson did later and gives the same proof. (The same result with the same proof is also found in a paper of Euler's colleague at St. Petersburg, F. C. Maier [1727].)" – Katz, V.J. (1987) "The Calculus of the Trigonometric Functions". pp.314-315. Washington DC: *Department of Mathematics, University of the District of Columbia*. Available at: https://www.semanticscholar.org/paper/The-calculus-of-the-trigonometric-functions-Katz/443d004ca386215fc4aa3b984f9085f55326b530

The method they used is different from the method shown here.

Self-differentiating function

In this chapter I will find a function, which will differentiate to itself. I will then find a general expression to express all functions which satisfy this requirement.

Now that a function can be differentiated, the question may be posed: are there any functions which differentiate to themselves. In more mathematical terms, is there a function $f(x)$ which satisfies the following equation:

$$f(x) \equiv \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Once such function would be $f(x) = 0$ as it has a constant y-coordinate of 0 and a constant gradient of 0.

Notice the use of the ≡ symbol. This is because the equation must hold for all values of x.

$$\lim_{h\to 0} f(x+h) - f(x) \equiv \lim_{h\to 0} hf(x)$$

$$\lim_{h\to 0} f(x+h) \equiv \lim_{h\to 0} hf(x) + f(x)$$

$$\lim_{h\to 0} f(x+h) \equiv \lim_{h\to 0} f(x)(h+1)$$

We can define a single value of the function e.g., $f(0) = 0$. Depending on what we define it as, we may find a different result which satisfies the initial equation. Here is an example.

$$Let\ f(0) = 0$$

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0)(h+1)$$

$$f(h) = 0$$

$$\lim_{h\to 0} f(h+h) = \lim_{h\to 0} f(h)(h+1)$$

$$f(2h) = 0$$

$$\lim_{h\to 0} f(2h+h) = \lim_{h\to 0} f(2h)(h+1)$$

$$f(3h) = 0$$

.

.

.

$$f(nh) = 0 \; for\ all\ values\ of\ n$$

$$Let\ n = \frac{x}{h}$$

$$f\left(\frac{x}{h} \times h\right) = 0$$

$$f(x) = 0$$

We have now proven that, as mentioned before, $f(x) = 0$ does satisfy this equation.

Now I will find another function satisfying this equation.

$$Let\ f(0) = 1$$

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0)(h+1)$$

$$\lim_{h\to 0} f(h) = \lim_{h\to 0}(h+1)$$

$$\lim_{h\to 0} f(h+h) = \lim_{h\to 0} f(h)(h+1)$$

$$\lim_{h\to 0} f(2h) = \lim_{h\to 0}(h+1)(h+1)$$

$$\lim_{h\to 0} f(2h) = \lim_{h\to 0}(h+1)^2$$

$$\lim_{h\to 0} f(2h+h) = \lim_{h\to 0} f(2h)(h+1)$$

$$\lim_{h\to 0} f(3h) = \lim_{h\to 0}(h+1)^2(h+1)$$

$$\lim_{h\to 0} f(3h) = \lim_{h\to 0}(h+1)^3$$

.

.

.

$$\lim_{h\to 0} f(nh) = \lim_{h\to 0}(h+1)^n \ for\ all\ values\ of\ n$$

$$Let\ n = \lim_{h\to 0}\frac{x}{h}$$

$$f(x) = \lim_{h\to 0}(1+h)^{\frac{x}{h}}$$

This looks very similar to the definition of e from earlier.

If $\lim\limits_{h\to 0}$ becomes $\lim\limits_{n\to\infty}$ and h becomes $\frac{1}{n}$, the value of the expression remains the same but in a different form.

$$f(x) = \lim_{n\to\infty}(1+\frac{1}{n})^{xn}$$

$$= \lim_{n\to\infty}((1+\frac{1}{n})^n)^x = e^x$$

Therefore e^x is its own derivative!

As I mentioned before, there are many different functions (an infinite number) which satisfy this function. I will $Let\ f(0) = m$ where m can be any number, to find all possible functions, satisfying the equation.

$$Let\ f(0) = m$$

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0)(h+1)$$

$$\lim_{h\to 0} f(h) = \lim_{h\to 0} m(h+1)$$

$$\lim_{h\to 0} f(h+h) = \lim_{h\to 0} f(h)(h+1)$$

$$\lim_{h\to 0} f(2h) = \lim_{h\to 0} m(h+1)(h+1)$$

$$\lim_{h\to 0} f(2h) = \lim_{h\to 0} m(h+1)^2$$

$$\lim_{h\to 0} f(2h+h) = \lim_{h\to 0} f(2h)(h+1)$$

$$\lim_{h\to 0} f(3h) = \lim_{h\to 0} m(h+1)^2(h+1)$$

$$\lim_{h\to 0} f(3h) = \lim_{h\to 0} m(h+1)^3$$

.

.

.

$$\lim_{h \to 0} f(nh) = \lim_{h \to 0} m(h+1)^n \text{ for all values of } n$$

$$Let\ n = \lim_{h \to 0} \frac{x}{h}$$

$$f(x) = \lim_{h \to 0} m(1+h)^{\frac{x}{h}}$$

$$f(x) = \lim_{n \to \infty} m(1+\frac{1}{n})^{xn}$$

$$= \lim_{n \to \infty} m(1+\frac{1}{n})^{n^x} = me^x$$

m could be any number, e.g. 0 or 1, resulting in $f(x) = 0$ and $f(x) = e^x$ respectively. But m could represent another number, for example 3, so $3e^x$ is also its own derivative.

Creating a Taylor Series of a differentiable function

In this chapter I will show a method to turn a function which can be repeatedly differentiated into a polynomial using Taylor and Maclaurin Series.

Taylor series were first invented by English Mathematician Brook Taylor in 1715, in his "Methodus Incrementorum Directa et Inversa".

"x becomes, $x + \dot{x}\frac{v}{1\dot{z}} + \ddot{x}\frac{v^2}{1.2\dot{z}^2} + \dddot{x}\frac{v^3}{1.2.3\dot{z}^3} + \&c.$ or by changing the sign of v by which z decreases to $z - v$, x decreases to become : $x - \dot{x}\frac{v}{1\dot{z}} + \ddot{x}\frac{v^2}{1.2\dot{z}^2} - \dddot{x}\frac{v^3}{1.2.3\dot{z}^3}$+&c." – Taylor, B. (1715) "Methodus Incrementorum Directa et Inversa". p.23. London: St John's College, *University of Cambridge*. Available at: https://archive.org/details/UFIE003454_TO0324_PNI-2529_000000/page/34/mode/2up

Translated by Bruce, I. (2007). Available at: https://www.17centurymaths.com/contents/brooktaylor/methodofincrementsparttwo.pdf

The notation used by Taylor is rather outdated. In this notation:

x represents some function f

$\dot{x}$ is its derivative, $\ddot{x}$ is the second derivative, $\dddot{x}$ is the third derivative and so on.

$\&c$ means "etcetera".

To write what he wrote in a more modern notation:

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

Now that $sin(\theta)$, $cos(\theta)$ and e^x are all differentiable, we can create a Taylor series for them. This will allow us to represent the function as a polynomial. The more terms in the polynomial the more accurate the result will be. I will begin with e^x as it is the easiest example.

$$e^x = a + bx + cx^2 + dx^3 + fx^4 + gx^5 + \cdots$$

Note that the more polynomial terms you add, the closer the result will be. First ensure that both functions have the same y-coordinate where $x = 0$. This is done by setting $x = 0$ and solving for a. The gradient at $x = 0$ must also be the same for both functions, which is done by differentiating both sides, settings $x = 0$ on both sides and solving for b. The concavity (the second derivative) must also be the same at $x = 0$. This is done by differentiating both sides yet again, setting $x = 0$ on both sides and solving for c. This process continues until either an appropriate number of terms have been found, or a pattern emerges. A Taylor Series which is centred around $x = 0$ is also known as a Maclaurin Series.

Also note that $e^0 = 1$, because earlier I said, $Let\ f(0) = 1$ and also because $x^0 = 1$ for all values of x.

$$f(0) = e^0 = a + b(0) + c(0)^2 + d(0)^3 + f(0)^4 + g(0)^5 + \cdots$$

$$a = 1$$

$$f'(x) = e^x = b + 2cx + 3dx^2 + 4fx^3 + 5gx^4 + \cdots$$

$$f'(0) = e^0 = b + 2c(0) + 3d(0)^2 + 4f(0)^3 + 5g(0)^4 + \cdots$$

$$b = 1$$

$$f''(x) = e^x = 2c + 2 \times 3dx + 3 \times 4fx^2 + 4 \times 5gx^3 + \cdots$$

$$f''(0) = e^0 = 2c + 2 \times 3d(0) + 3 \times 4f(0)^2 + 4 \times 5g(0)^3 + \cdots$$

$$2c = 1$$

$$c = \frac{1}{2}$$

$$f'''(x) = e^x = 2 \times 3d + 2 \times 3 \times 4fx + 3 \times 4 \times 5gx^2 + \cdots$$

$$f'''(0) = e^0 = 2 \times 3d + 2 \times 3 \times 4f(0) + 3 \times 4 \times 5g(0)^2 + \cdots$$

$$2 \times 3d = 1$$

$$d = \frac{1}{2 \times 3} = \frac{1}{3!}$$

$f^{(n)}(x) = the\ n^{th}\ derivative\ of\ f(x)$

$$f^{(4)}(x) = e^x = 2 \times 3 \times 4f + 2 \times 3 \times 4 \times 5gx + \cdots$$

$$f^{(4)}(0) = e^0 = 2 \times 3 \times 4f + 2 \times 3 \times 4 \times 5g(0) + \cdots$$

$$2 \times 3 \times 4f = 1$$

$$f = \frac{1}{2 \times 3 \times 4} = \frac{1}{4!}$$

$$f^{(5)}(x) = e^x = 2 \times 3 \times 4 \times 5g + \cdots$$

$$f^{(5)}(0) = e^0 = 2 \times 3 \times 4 \times 5g + \cdots$$

$$2 \times 3 \times 4 \times 5g = 1$$

$$g = \frac{1}{2 \times 3 \times 4 \times 5} = \frac{1}{5!}$$

Substitute these values into the original polynomial:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

This can be written in the form:

$$\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

The powers of x increase by one each term as that is how the polynomial was written, the denominator of each term $=\ n!$, where n is the term's position in the polynomial, starting at $n = 0$ for the first position, due to the nature of the power rule, which was seen in the above repeated differentiation.

The more terms in the series, the closer to the actual value it will be. The above can be written in the form of an infinite series using sigma notation:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Leonhard Euler created this series expansion for e^x in 1748 in his "*Introduction to the Analysis of the infinite*".

"Let $x = \log a$, so that $a = e^x$. Then

$a = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$" – Euler, L. (1748/1998) "Introduction to analysis of the infinite". Book II. p.338. Translated by Blanton, J.D. New York: *Springer-Verlag*. Available at: https://archive.org/details/introductiontoan0000eule

substitute $x = 1$ to calculate e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Leonhard Euler was the first to create this series expansion to approximate the value of e.

"Euler used his definition of the exponential function (equation 1) to develop it in an infinite power series. As we saw in Chapter 4, for $x = 1$ equation 1 gives the numerical series

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$"

– Eli, M. (1994) "e: The Story Of A Number". pp.156-157. Princeton: *Princeton University Press*. Available at: https://archive.org/details/estoryofnumber0000maor_x8v0/page/156/mode/2up

Now repeat the process for $sin(\theta)$:

$$f(\theta) = sin(\theta) = a + b\theta + c\theta^2 + d\theta^3 + f\theta^4 + g\theta^5 + h\theta^6 + \cdots$$

$$f(0) = \sin(0) = a + b(0) + c(0)^2 + d(0)^3 + f(0)^4 + g(0)^5 + h(0)^6 + \cdots$$

$$a = 0$$

$$f'(\theta) = cos(\theta) = b + 2c\theta + 3d\theta^2 + 4f\theta^3 + 5g\theta^4 + 6h\theta^5 + \cdots$$

$$f'(0) = cos(0) = b + 2c(0) + 3d(0)^2 + 4f(0)^3 + 5g(0)^4 + 6h(0)^5 + \cdots$$

$$b = 1$$

$$f''(\theta) = -sin(\theta) = 2c + 2 \times 3d\theta + 3 \times 4f\theta^2 + 4 \times 5g\theta^3 + 5 \times 6h\theta^4 + \cdots$$

$$f''(0) = -sin(0) = 2c + 2 \times 3d(0) + 3 \times 4f(0)^2 + 4 \times 5g(0)^3 + 5 \times 6h(0)^4 + \cdots$$

$$2c = 0$$

$$c = 0$$

$$f'''(\theta) = -cos(\theta) = 2 \times 3d + 2 \times 3 \times 4f\theta + 3 \times 4 \times 5g\theta^2 + 4 \times 5 \times 6h\theta^3 + \cdots$$

$$f'''(0) = -cos(0) = 2 \times 3d + 2 \times 3 \times 4f(0) + 3 \times 4 \times 5g(0)^2 + 4 \times 5 \times 6h(0)^3 + \cdots$$

$$2 \times 3d = -1$$

$$d = \frac{-1}{2 \times 3} = -\frac{1}{3!}$$

$$f^{(4)}(\theta) = sin(\theta) = 2 \times 3 \times 4f + 2 \times 3 \times 4 \times 5g\theta + 3 \times 4 \times 5 \times 6h\theta^2 + \cdots$$

$$f^{(4)}(0) = sin(0) = 2 \times 3 \times 4f + 2 \times 3 \times 4 \times 5g(0) + 3 \times 4 \times 5 \times 6h(0)^2 + \cdots$$

$$2 \times 3 \times 4f = 0$$

$$f = 0$$

$$f^{(5)}(\theta) = cos(\theta) = 2 \times 3 \times 4 \times 5g + 2 \times 3 \times 4 \times 5 \times 6h\theta + \cdots$$

$$f^{(5)}(0) = cos(0) = 2 \times 3 \times 4 \times 5g + 2 \times 3 \times 4 \times 5 \times 6h(0) + \cdots$$

$$2 \times 3 \times 4 \times 5g = 1$$

$$g = \frac{1}{2 \times 3 \times 4 \times 5} = \frac{1}{5!}$$

$$f^{(6)}(\theta) = -sin(\theta) = 2 \times 3 \times 4 \times 5 \times 6h + \cdots$$

$$f^{(6)}(0) = -sin(0) = 2 \times 3 \times 4 \times 5 \times 6h + \cdots$$

$$2 \times 3 \times 4 \times 5 \times 6h = 0$$

$$h = 0$$

Substitute these values into the original polynomial:

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

Repeat the process for cos(θ)

$$f(\theta) = \cos(\theta) = a + b\theta + c\theta^2 + d\theta^3 + f\theta^4 + g\theta^5 + h\theta^6 + \cdots$$

$$f(0) = cos(0) = a + b(0) + c(0)^2 + d(0)^3 + f(0)^4 + g(0)^5 + h(0)^6 + \cdots$$

$$a = 1$$

$$f'(\theta) = -sin(\theta) = b + 2c\theta + 3d\theta^2 + 4f\theta^3 + 5g\theta^4 + 6h\theta^5 + \cdots$$

$$f'(0) = -sin(0) = b + 2c(0) + 3d(0)^2 + 4f(0)^3 + 5g(0)^4 + 6h(0)^5 + \cdots$$

$$b = 0$$

$$f''(\theta) = -\cos(\theta) = 2c + 2 \times 3d\theta + 3 \times 4f\theta^2 + 4 \times 5g\theta^3 + 5 \times 6h\theta^4 + \cdots$$

$$f''(0) = -cos(0) = 2c + 2 \times 3d(0) + 3 \times 4f(0)^2 + 4 \times 5g(0)^3 + 5 \times 6h(0)^4 + \cdots$$

$$2c = -1$$

$$c = -\frac{1}{2}$$

$$f'''(\theta) = sin(\theta) = 2 \times 3d + 2 \times 3 \times 4f\theta + 3 \times 4 \times 5g\theta^2 + 4 \times 5 \times 6h\theta^3 + \cdots$$

$$f'''(0) = sin(0) = 2 \times 3d + 2 \times 3 \times 4f(0) + 3 \times 4 \times 5g(0)^2 + 4 \times 5 \times 6h(0)^3 + \cdots$$

$$2 \times 3d = 0$$

$$d = 0$$

$$f^{(4)}(\theta) = cos(\theta) = 2 \times 3 \times 4f + 2 \times 3 \times 4 \times 5g\theta + 3 \times 4 \times 5 \times 6h\theta^2 + \cdots$$

$$f^{(4)}(0) = cos(0) = 2 \times 3 \times 4f + 2 \times 3 \times 4 \times 5g(0) + 3 \times 4 \times 5 \times 6h(0)^2 + \cdots$$

$$2 \times 3 \times 4f = 1$$

$$f = \frac{1}{2 \times 3 \times 4} = \frac{1}{4!}$$

$$f^{(5)}(\theta) = -sin(\theta) = 2 \times 3 \times 4 \times 5g + 2 \times 3 \times 4 \times 5 \times 6h\theta + \cdots$$

$$f^{(5)}(0) = -sin(0) = 2 \times 3 \times 4 \times 5g + 2 \times 3 \times 4 \times 5 \times 6h(0) + \cdots$$

$$2 \times 3 \times 4 \times 5g = 0$$

$$g = 0$$

$$f^{(6)}(\theta) = -\cos(\theta) = 2 \times 3 \times 4 \times 5 \times 6h + \cdots$$

$$f^{(6)}(0) = -\cos(0) = 2 \times 3 \times 4 \times 5 \times 6h + \cdots$$

$$2 \times 3 \times 4 \times 5 \times 6h = -1$$

$$h = \frac{-1}{2 \times 3 \times 4 \times 5 \times 6} = -\frac{1}{6!}$$

Substitute these values into the original polynomial:

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

Yet again, Leonhard Euler was the first to discover the series expansions for the $sine$ and $cosine$ functions in 1748.

"Let the arc z be infinitely small, then $\sin z = z$ and $\cos z = 1$. If n is an infinitely large number, so that nz is a finite number, say $nz = v$, then, since

$\sin z = z = \frac{v}{n}$, we have

$$\cos v = 1 - \frac{v^2}{1 \cdot 2} + \frac{v^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{v^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \cdots and$$

$\sin v = v - \frac{v^3}{1 \cdot 2 \cdot 3} + \frac{v^5}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} - \frac{v^7}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \cdots$. It follows that if v is a given arc, by means of these series, the sine and cosine can be found." – Euler, L. (1748/1998) "Introduction to analysis of the infinite". Book I. p.107. Translated by Blanton, J.D. New York: *Springer-Verlag*. Available at: https://books.google.co.uk/books/about/Introduction_to_Analysis_of_the_Infinite/LkJ5ngEACAAJ?hl=en

Euler's Formula

We know that $e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$

x could be any number, including complex ones (those containing the imaginary unit i). e^{ix} can be calculated, where x is some real number.

Also $i = \sqrt{-1}$ meaning:

$$i^0 = 1$$

$$i^1 = i^0 \times i = i$$

$$i^2 = i^1 \times i = -1$$

$$i^3 = i^2 \times i = -i$$

$$i^4 = i^2 \times i^2 = 1$$

$$i^5 = i^4 \times i = i$$

This pattern repeats forever.

$$Let\ x = ix$$

$$e^{ix} = \frac{(ix)^0}{0!} + \frac{(ix)^1}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$

$$= \frac{x^0}{0!} + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

$$Factor\ out\ i$$

$$= \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

We know that $\cos(x) = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

And $\sin(x) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Therefore $e^{ix} = \cos(x) + isin(x)$

I will write θ instead of x as I am using the $sin()$ and $cos()$ functions..

$$e^{i\theta} = \cos(\theta) + \mathrm{isin}(\theta)$$

Euler discovered this formula in his 1748 publication, "*Introduction to analysis of the infinite"* Though he did it slightly differently. He used the series for e^{ix} and e^{-ix} separately. He then proved his formula from there, though he used z instead of x and v instead of θ.

He showed:

$$\cos v = \frac{e^{iv} + e^{-iv}}{2}$$

$$and$$

$$\sin v = \frac{e^{iv} - e^{-iv}}{2i}$$

$$so$$

$$\cos v + i \sin v = \frac{e^{iv} + e^{-iv}}{2} + \frac{e^{iv} - e^{-iv}}{2}$$

$$= \frac{e^{iv} + e^{-iv} + e^{iv} - e^{-iv}}{2}$$

$$= \frac{2e^{iv}}{2} = e^{iv}$$

$$so\ e^{iv} = \cos v + i \sin v$$

"$\left(1 + \frac{z}{j}\right) = e^z$ where e is the base of the natural logarithms. When we let $z = iv$ and then $z = iv$ we obtain $\cos v = \frac{e^{iv}+e^{-iv}}{2}$ and $\sin v = \frac{e^{iv}-e^{-iv}}{2i}$. From these equations we understand how complex exponentials can be expressed by real sines and cosines, since $e^{iv} = \cos v + i \sin(v)$ and $e^{-iv} = \cos v - i \sin v$." – Euler, L. (1748/1998) "Introduction to analysis of the infinite". Book I. p.112. Translated by Blanton, J.D. New York: *Springer-Verlag*. Available at: https://books.google.co.uk/books/about/Introduction_to_Analysis_of_the_Infinite/LkJ5ngEACAAJ?hl=en

There is evidence to support that this formula was discovered (though in a different form) as far back as 1714 in the form:

$$ix = \ln(\cos(x) + isin(x))$$

The key difference that this formula is technically incorrect as the natural log of a complex value is multivaried. For example, $\ln(i) = \frac{\pi}{2} + 2n\pi$ where n is any integer. The reason for this is that a complex number can be represented on an "argand diagram" in cartesian form (in the form (x, y)) or polar form (in the form (r, θ)). Using trigonometry, the complex number $a + bi$ could be written as $r(\cos(\theta) + i \sin(\theta)) = re^{i\theta}$. The natural logarithm of this complex number therefore $= \ln(\mathrm{r}) + \ln\left(e^{i\theta}\right) = \ln(r) + i\theta$. The interesting part is that $\cos(2n\pi + \theta) = \cos(\theta)$ and $\sin(2n\pi + \theta) = \sin(\theta)$ for all integer values of n. This means that the natural logarithm of a complex number is $\ln(r) + i(\theta + 2n\pi) = \ln(r) + i\theta + 2in\pi$.

This means that, whilst $\ln(\cos(x) + i\,\sin(x))$ does equal ix, it also equals $ix + 2\pi$, $ix + 4\pi$, $ix + 6\pi$ etc., which are not equal to each other. Euler's version of the formula is, for this reason, technically more correct as it is not multivaried (meaning it only produces one result for a given input).

Cotes' findings were not published until 1722 after his death in 1716.

"In examining the relation between exponentials and trigonometry, Roger Cotes (1682-1716) came to the formula *ix* = log(cos *x* + *i* sin *x*). This appeared in his *Logometria* of 1714 (printed in the *Philosophical Transactions of the Royal Society*, then a widely read publication) and reprinted in his posthumous 1722 work *Harmonia Mensurarum*." – Shell-Gellasch, A. (2010) "Napier's e – Roger Cotes" Beloit: *Mathematical Association of America*. Available at: https://www.maa.org/publications/periodicals/convergence/napiers-e-roger-cotes#:~:text=In%20examining%20the%20relation%20between,x%20%2B%20i%20sin%20x

Euler's Identity

Using this formula, Euler's Formula, we can set $\theta = \pi$ to calculate $e^{i\pi}$

$$e^{i\pi} = \cos(\pi) + isin(\pi)$$

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + isin\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= \cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) + i\left(\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)\right)$$

$$= 0^2 - 1^2 + i(1 \times 0 + 0 \times 1)$$

$$= -1 + 0i$$

$$= -1$$

$$Therefore:$$

$$e^{i\pi} = -1$$

Section 3: Calculating π

Throughout this project, e has been mentioned many times and a series with with to efficiently calculate its digits has been found. So far, we have no such thing for π which I hope to rectify in this final section.

The Binomial Expansion

The Binomial expansion has already been introduced in the "Derivative of a monomial" chapter. In this chapter will go into further detail and look at a new notation.

As seen in the aforementioned chapter the expression $(A+B)^n$ expands into a series with descending powers of A, starting at A^n, ascending powers of B, starting at B^0, and with each term having a coefficient from the n^{th} row of Pascal's triangle, starting at $n=0$ for the first row and the k^{th} column of that row (again starting at $k=0$), where k is the term of the series. The third term of the expansion of $(A+B)^4$, for example, would have a coefficient from the 3$^{\text{rd}}$ column of the 4$^{\text{th}}$ row of Pascal's triangle. This is denoted by the notation:

$$\binom{4}{3}$$

With the general notation of the number in the n$^{\text{th}}$ row and k$^{\text{th}}$ column of Pascal's triangle being:

$$\binom{n}{k}$$

So $(A+B)^n = A^n + nA^{n-1}B + \binom{n}{2}A^{n-2}B^2 + \binom{n}{3}A^{n-3}B^3 + \binom{n}{4}A^{n-4}B^4 + \binom{n}{5}A^{n-5}B^5$

$$= \sum_{k=0}^{n} \binom{n}{k} A^{n-k}B^k$$

This new notation makes it easier to reference numbers in Pascal's triangle but does not make it any easier to find out what they are. The next chapter will be focusing on how one can do so. I will later return to the Binomial Expansion to discuss a few special cases considered by Sir Isaac Newton.

Combinatorics

Combinatorics is a field of Mathematics involved in calculating the number of possible arrangements of a set. Two definitions which are necessary to understand are permutation and combination. The main difference is that in a permutation order matters, in a combination it does not. For example, the sets $[1,2,3]$ and $[2,3,4]$ are both different combinations, and different permutatons. The sets $[1,2,3]$ and $[3,2,1]$ are different permutations (because they have a different order of items) but are the same combination (because order does not matter in a combination). It may not seem like combinatorics have anything to do with pi or with Pascal's triangle, but this will all be relevant later.

It may seem like calculating the number of possible combinations in a certain scenario would be simpler as you don't need to worry about the order of items in a set, however it is easier to calculate the number of permutations than the number of combinations. I will show how to do this now.

Imagine there are 7 distinct playing cards in front of you, if you lay them out randomly on a table how many possible ways could they be arranged, or in other words how many permutations are there?

Position 1 could have any of the 7 cards. Position 2 could have ay of the 6 remaining. Position 3 could have of the 5 remaining and so on. The total number of permutations is therefore $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$$= 7!$$

In general, k items in a set can have $k!$ different permutations.

Now imagine that there are 10 cards in front of you and at random take 7 cards and lay them on the table in a random order. How many possible permutations are there now?

Position 1 could have any of the 10 cards, position 2 could have any of the 9 remaining, position 3 could have ay of the remaining 8 and so on, with position 7 having any of the remaining 4 cards. The total number of permutations is therefore:

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$$

This may seem trickier to write down in a mathematical way. Any easy way of doing so would be to multiply and then divide by $3 \times 2 \times 1$ to have the numerator in the form of a factorial.

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$$

$$= \frac{10!}{3!}$$

Where 10 is the number of cards initially presented, and 3 is the number of cards remaining after having taken 7.

$$= \frac{10!}{(10-7)!}$$

In general, if presented with n items and you randomly insert k of these items into a set, the number of possible permutations of the set is:

$$\frac{n!}{(n-k)!}$$

This is denoted by:

$$P_k^n$$

If I have a set of k items, the total number of permutations is $k!$ as mentioned before. Each of these permutations are however the same single combination as the contents of the set do not change, only their order. This means that for each combination, there are $k!$ permutations. In other words, $P_k^n = k! \times C_k^n$ for a given set. Where C_k^n is the total number of combinations. This can be rearranged:

$$C_k^n = \frac{P_k^n}{k!}$$

Since $P_k^n = \frac{n!}{(n-k)!}$

$$C_k^n = \frac{n!}{k!\,(n-k)!}$$

This may seem irrelevant to Pascal's triangle at first, but mathematicians noticed that the expression $\frac{n!}{k!(n-k)!}$ seems to give the value of the number in Pascal's triangle with row n and column k. This link will be proven in the next chapter.

The field of Combinatorics is thought to have started with famous mathematicians Pascal and Fermat. In the 1600s.

"In the West, combinatorics may be considered to begin in the 17th century with Blaise Pascal and Pierre de Fermat, both of France, who discovered many classical combinatorial results in connection with the development of the theory of probability." – Grünbaum, B. (1999) "combinatorics". *Encyclopedia Britannica*. Available at: https://www.britannica.com/science/combinatorics

Linking the combination formula to Pascal's triangle

In this chapter I will prove the link between the numbers in Pascal's triangle and the combination formula from the previous chapter.

In order to prove the link, we need to first define a link between the numbers in Pascal's triangle. In order to write Pascal's triangle, you first write a 1 with a diagonal line of 1s going down the left and one to the right. Then, starting from the top, sum two numbers next to each other to find the number underneath, as shown.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
. \\
. \\
.
\end{array}
$$

Pascal's triangle extends down forever. The important thing to notice is that a number is equal to the sum of the number above and to its right and the number above and to its left. Using the new notation from before, this can be written:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

(This is except for the numbers on the left (where $k = 0$) and right (where $k = n$) of each row which are all 1s).

If $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ Then we should be able to prove that

$$\frac{n!}{k!\,(n-k)!} = \frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!} + \frac{(n-1)!}{k!\,((n-1)-k)!}$$

$$\frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!} + \frac{(n-1)!}{k!\,((n-1)-k)!}$$

$$= \frac{(n-1)!}{(k-1)!\,(n-1-k+1)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$

$$= \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-k-1)!}$$

$$Note\ that\ x(x-1)! = x(x-1)(x-2)(x-3)(x-4)\ldots = x!$$

$$= \frac{k(n-1)!}{k(k-1)!\,(n-k)!} + \frac{(n-k)(n-1)!}{(n-k)k!\,(n-k-1)!}$$

$$= \frac{k(n-1)!}{k!\,(n-k)!} + \frac{(n-k)(n-1)!}{k!\,(n-k)!}$$

$$= \frac{k(n-1)! + (n-k)(n-1)!}{k!\,(n-k)!}$$

$$= \frac{(n-1)!\,(k+n-k)}{k!\,(n-k)!}$$

$$= \frac{(n-1)!\,(n)}{k!\,(n-k)!}$$

$$\frac{n!}{k!\,(n-k)!}$$

$$Therefore \frac{n!}{k!\,(n-k)!} = \frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!} + \frac{(n-1)!}{k!\,((n-1)-k)!}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

When $k = 0$:

$$\binom{n}{k} = \binom{n}{0} = \frac{n!}{0!\,(n-0)!} = \frac{n!}{n!} = 1$$

When $k = n$:

$$\binom{n}{k} = \binom{n}{n} = \frac{n!}{n!\,(n-n)!} = \frac{n!}{n!\,0!} = \frac{n!}{n!} = 1$$

Meaning numbers in Pascal’s triangle can be found by using the combination formula.

$$C_k^n = \binom{n}{k}$$

Pascal’s triangle is named after French mathematician Blaise Pascal though, as seems to be a common theme, despite being named after him, he was not the first to come up with it.

“It is named for the 17th-century French mathematician Blaise Pascal, but it is far older. Chinese mathematician Jia Xian devised a triangular representation for the coefficients in the 11th century.” – Hosch, W.L. (2009) "Pascal’s Triangle". *Encyclopedia Britannica*. Available at: https://www.britannica.com/science/Pascals-triangle

The Binomial Expansion Continued

In this chapter I will look at a special case of the binomial theorem where $A = 1$ and $B = x$. I will then show how the binomial theorem can be used for negative and fractional values of n.

We already know that:

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k$$

We also now know that:

$$\binom{n}{k} = C_k^n = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)(n-3)(n-4)\dots}{k!\,(n-k)(n-k-1)(n-k-2)(n-k-3)\dots}$$

We can substitute this in:

$$(A+B)^n = \sum_{k=0}^{n} \frac{A^{n-k}B^k n(n-1)(n-2)(n-3)(n-4)\dots}{k!\,(n-k)(n-k-1)(n-k-2)(n-k-3)\dots}$$

$$= \frac{A^{n-0}B^0 n(n-1)(n-2)(n-3)(n-4)\dots}{0!\,(n-0)(n-0-1)(n-0-2)(n-0-3)\dots}$$

$$+ \frac{A^{n-1}B^1 n(n-1)(n-2)(n-3)(n-4)\dots}{1!\,(n-1)(n-1-1)(n-1-2)(n-1-3)\dots}$$

$$+ \frac{A^{n-2}B^2 n(n-1)(n-2)(n-3)(n-4)\dots}{2!\,(n-2)(n-2-1)(n-2-2)(n-2-3)\dots}$$

$$+ \frac{A^{n-3}B^3 n(n-1)(n-2)(n-3)(n-4)\dots}{3!\,(n-3)(n-3-1)(n-3-2)(n-3-3)\dots}$$

$$+ \frac{A^{n-4}B^4 n(n-1)(n-2)(n-3)(n-4)\dots}{4!\,(n-4)(n-4-1)(n-4-2)(n-4-3)\dots}$$

$$+ \frac{A^{n-5}B^5 n(n-1)(n-2)(n-3)(n-4)\dots}{5!\,(n-5)(n-5-1)(n-5-2)(n-5-3)\dots}$$

$$+$$

$$.$$

$$.$$

$$.$$

notice that when the term where $k - n$ is reached, the quotient will have a division by zero. This can only happen when n is a positive integer, meaning the series is invalid for positive integer values of n.

$$= \frac{A^{n-0}B^0 n(n-1)(n-2)(n-3)(n-4)\ldots}{(n)(n-1)(n-2)(n-3)\ldots}$$

$$+ \frac{A^{n-1}B^1 n(n-1)(n-2)(n-3)(n-4)\ldots}{(n-1)(n-2)(n-3)(n-4)\ldots}$$

$$+ \frac{A^{n-2}B^2 n(n-1)(n-2)(n-3)(n-4)\ldots}{2!\,(n-2)(n-3)(n-4)(n-5)\ldots}$$

$$+ \frac{A^{n-3}B^3 n(n-1)(n-2)(n-3)(n-4)\ldots}{3!\,(n-3)(n-4)(n-5)(n-6)\ldots}$$

$$+ \frac{A^{n-4}B^4 n(n-1)(n-2)(n-3)(n-4)\ldots}{4!\,(n-4)(n-5)(n-6)(n-7)\ldots}$$

$$+ \frac{A^{n-5}B^5 n(n-1)(n-2)(n-3)(n-4)\ldots}{5!\,(n-5)(n-6)(n-7)(n-8)\ldots}$$

$$+$$

$$.$$

$$.$$

$$.$$

Cancel terms on the top and bottom

$$= \frac{A^{n-0}B^0}{0!}$$

$$+ \frac{A^{n-1}B^1 n}{1!}$$

$$+ \frac{A^{n-2}B^2 n(n-1)}{2!}$$

$$+ \frac{A^{n-3}B^3 n(n-1)(n-2)}{3!}$$

$$+ \frac{A^{n-4}B^4 n(n-1)(n-2)(n-3)}{4!}$$

$$+ \frac{A^{n-5}B^5 n(n-1)(n-2)(n-3)(n-4)}{5!}$$

$$+$$

$$.$$

$$.$$

$$.$$

This is known as the Binomial Theorem. Sir Isaac Newton was particularly interested in the expression $(1+x)^n$

Using the Binomial Theorem:

$$Let\ A = 1$$

$$Let\ B = x$$

$$Let\ n = n$$

$$(1+x)^n$$

$$= \frac{1^n x^0}{0!}$$

$$+ \frac{1^{n-1} x^1 n}{1!}$$

$$+ \frac{1^{n-2} x^2 n(n-1)}{2!}$$

$$+ \frac{1^{n-3} x^3 n(n-1)(n-2)}{3!}$$

$$+ \frac{1^{n-4} x^4 n(n-1)(n-2)(n-3)}{4!}$$

$$+ \frac{1^{n-5} x^5 n(n-1)(n-2)(n-3)(n-4)}{5!}$$

$$+$$

$$.$$

$$.$$

$$.$$

$$= 1$$

$$+nx$$

$$+ \frac{n(n-1)x^2}{2!}$$

$$+ \frac{n(n-1)(n-2)x^3}{3!}$$

$$+ \frac{n(n-1)(n-2)(n-3)x^4}{4!}$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4)x^5}{5!}$$

$$+$$

$$.$$

$$.$$

$$.$$

Sir Isaac Newton wanted to find out if the theorem would break if n was a negative number.

$$Let\; n = -1$$

$$(1+x)^{-1}$$

$$= 1$$

$$-x$$

$$+\frac{x^2(-1)(-2)}{2!}$$

$$+\frac{x^3(-1)(-2)(-3)}{3!}$$

$$+\frac{x^4(-1)(-2)(-3)(-4)}{4!}$$

$$+\frac{x^5(-1)(-2)(-3)(-4)(-5)}{5!}$$

$$+$$

$$.$$

$$.$$

$$.$$

$$= 1 - x + \frac{x^2(2!)}{2!} + \frac{x^3(-3!)}{3!} + \frac{x^4(4!)}{4!} + \frac{x^5(-5!)}{5!} + \cdots$$

To explain the previous step more clearly,

$$(-1)(-2)(-3)(-4)(-5)(-6)\ldots(-n) = n!\; when\; n\; is\; even$$

and

$$(-1)(-2)(-3)(-4)(-5)(-6)\ldots(-n) = -n!\; when\; n\; is\; odd$$

This is the case because multiplying an even number of negative numbers together results in a positive number and multiplying an odd number of negative numbers together results in a negative number.

$$1 - x + \frac{x^2(2!)}{2!} + \frac{x^3(-3!)}{3!} + \frac{x^4(4!)}{4!} + \frac{x^5(-5!)}{5!} + \cdots$$
$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

Meaning if the Binomial Theorem works for negative values of n, then:

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$
$$1 = (1+x)(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots)$$

We can prove that this is the case:

$$(1+x)(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots)$$
$$= (1 - x + x^2 - x^3 + x^4 - x^5 + \cdots) + x(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots)$$
$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$
$$+x - x^2 + x^3 - x^4 + x^5$$
$$= 1$$

So $(1+x)(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots) = 1$

Meaning

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

Meaning the Binomial Theorem does not only work for positive integer values of n.

He next tried fractional values of n.

For example, if $n = \frac{1}{2}$ this the square root of a number could be calculated.

$$\sqrt{3} = \sqrt{4-1} = \sqrt{4\left(1 - \frac{1}{4}\right)} = \sqrt{4}\sqrt{1 - \frac{1}{4}} = 2\left(1 - \frac{1}{4}\right)^{\frac{1}{2}}$$

$\left(1-\frac{1}{4}\right)^{\frac{1}{2}}$ could be calculated using the Binomial Theorem where $x=-\frac{1}{4}$ and $n=\frac{1}{2}$, multiplying the result by 2 would give $\sqrt{3}$. Also, the result given by this calculating would only be the positive root, not the negative one. This will be important later.

Newton himself did not prove the Binomial theorem, meaning calling it such was rather a misnomer, it would not be proven until 1826 when it was proved by Niels Henrik Abel.

"the young Norweigan Niels Henrik Abel. Who attacked the formula of the binomial using rigorous criteria" – Delon, M. (2002) "Encyclopedia of the Enlightenment". p.562. USA: *Fitzroy Dearborn Publishers*. Available at: https://books.google.co.uk/books?id=rElJAgAAQBAJ&redir_esc=y

The reason why this seemingly more convoluted expansion was calculated instead of $(1+2)^{\frac{1}{2}}$ is because this sum only converges when $-1<x<1$. This is because otherwise each x term would be larger for a given term than for the previous term, by which I mean $<\cdots>x^3>x^2>x>1$, but if $-1<x<1$, then $1>x>x^2>x^3>\cdots$. These powers of x are known as the sequences *general term.* Because the general term only converges when $-1<x<1$, the series only converges when $-1<x<1$. Also, the smaller the value of x the more quickly the series converges, because each term has increasing powers of x, each term will be much smaller than the last, if x is as small as possible. The fact that the binomial expansion converges faster for smaller values of x will be important later. Newton noticed that this method of calculating $\sqrt{3}$ did indeed converge towards the familiar value of approximately 1.732. Newton now knew that The Binomial theorem would work for both negative and fractional values of n. This will be important later.

Integration

In this chapter I will review another key part of calculus, integral calculus. This will later be necessary to integrate the equation of a circle to find its area in the form of a decimal. We will be able to use this to form an equation in terms of π which can then be solved for π.

We have already seen the first half of calculus, differentiation, but now we meet integration. This chapter will discuss what it is and how and why it works. First, we must look at speed, or more specifically velocity, the difference being that velocity is a vector quantity, meaning has a direction (so can be negative), whereas speed is a scalar quantity, meaning it has no direction, so can only be positive.

$Velocity=\frac{Displacement}{Time}$

or

$$v=\frac{d}{t}$$

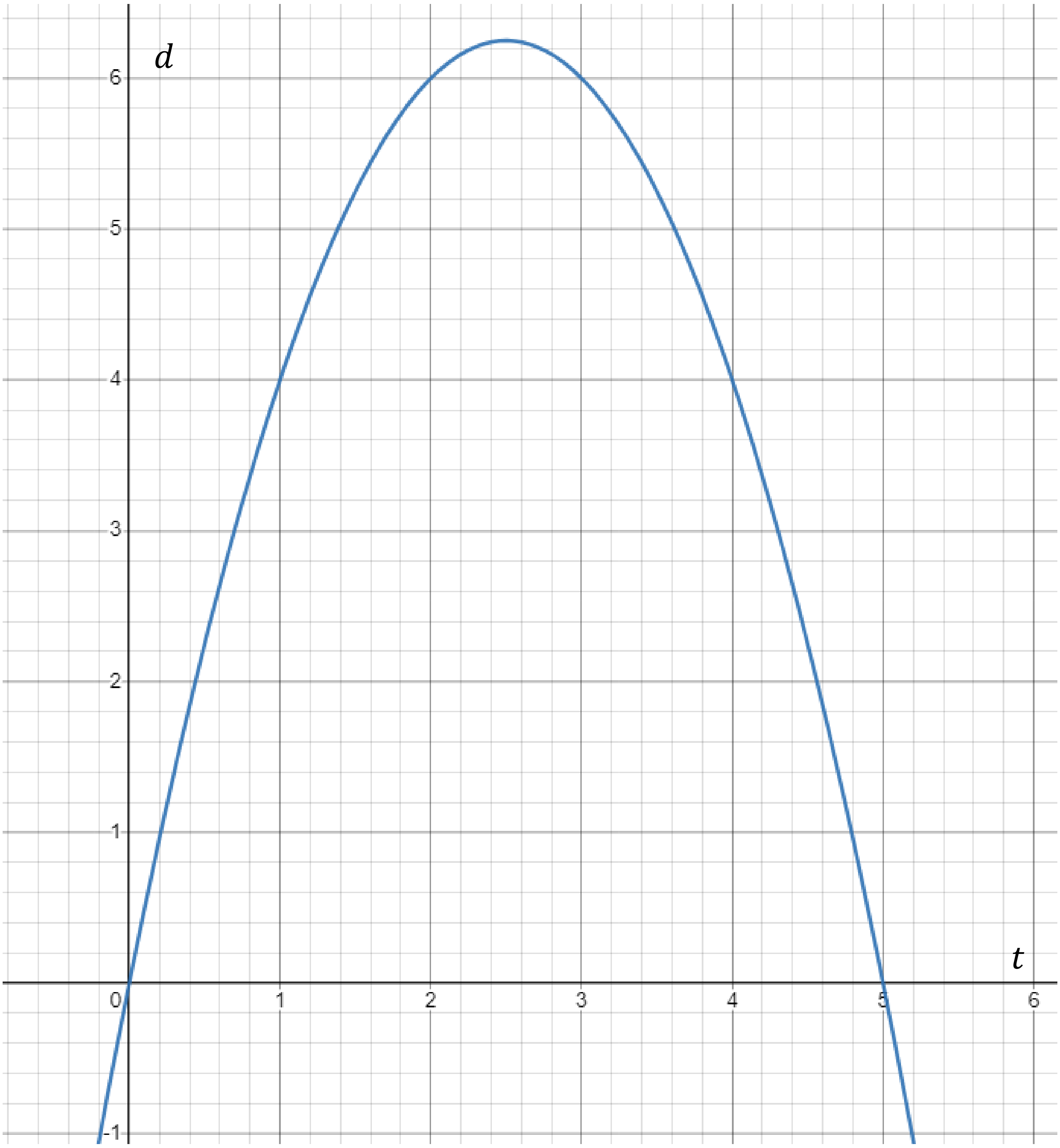
Displacement is like distance, but is a vector quantity, whereas distance is a scalar quantity.

This can be rearranged:

$$vt = d$$

We could now plot a graph, $d = f(t)$, where t is the horizontal-axis and d is the vertical axis.

In this example, I have $f(t) = -t^2 + 5t$

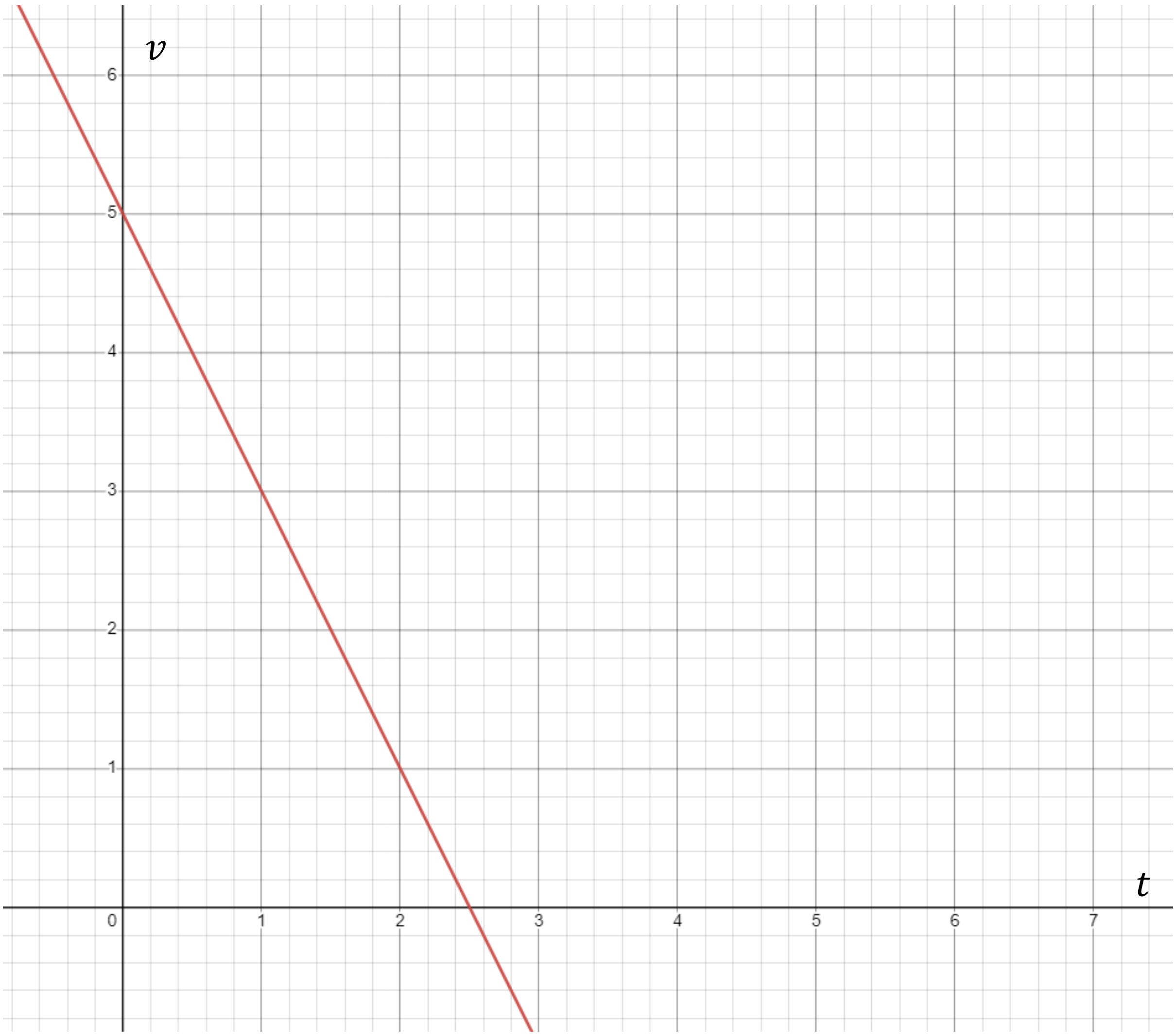


This is a displacement-time graph as it plots displacement against time. This graph could be differentiated using the power rule:

$$f'(t) = -2t + 5$$

But what does this derivative represent? Remember, the derivative is the gradient function of the original graph, meaning its change in d over change in t. In this case that would be $\frac{d}{t} = v$ meaning if we differentiate the displacement-time graph, we get a velocity-time graph.

I will now plot the graph $v = g(t) = -2t + 5$



This is a velocity-time graph as it plots velocity against time. If I know a velocity time, how could I work out its respective displacement time graph. In order to get from the displacement-time graph to the distance-time graph we differentiated, we found its derivative. This means that to get back to the displacement time graph, it reasons that one must differentiate, but in reverse. This is known as finding the functions antiderivative, also known as its indefinite integral.

To differentiate a polynomial with the power rule (with respect to x), you:

- Multiply by the term by the power of x
- Subtract one from the power of x

for each term. To find the antiderivative of a polynomial you to the reverse:

- Add one two the power of x
- Divide the term by the new power of x

for each term. For our function $g(x)$ the antiderivative would be:

$$\frac{-2t^{1+1}}{1+1}+\frac{5t^{0+1}}{0+1}$$

$$= -t^2 + 5t$$

which is what we expected. There is however one problem, if the entire displacement-time graph were moved up or down, (this could be described as the graph transformation $f(x) + c$ where c is the amount by which the graph was moved up, if c is negative then the graph has moved down) the gradient at a given point wouldn't change, meaning the derivative would be the same.

This means that if the antiderivative of a function is found to be equal to $f(x)$, $f(x) + c$ would also be an antiderivative of the same function as $f(x)$ and $f(x) + c$ would both differentiate to the same thing.

This means that the antiderivative of our velocity-time graph:

$$f(t) = -t^2 + 5t$$

should actually be:

$$-t^2 + 5t + c$$

where c is a constant.

It may seem that this makes the antiderivative useless. If for example I was given the above velocity-time graph and had to work out the total distance travelled when $t = 2.5$ I would not be able to. However this is not much of a problem because the question can be rephrased as: what is the total distance travelled between where $t = 0$ and where $t = 2.5$.

Given this question we can work out the distance travelled when $t = 2.5$ as $-(2.5)^2 + 5(2.5) + c$

$$= 6.25 + c$$

and the distance travelled when $t = 0$ as $-(0)^2 + 5(0) + c$

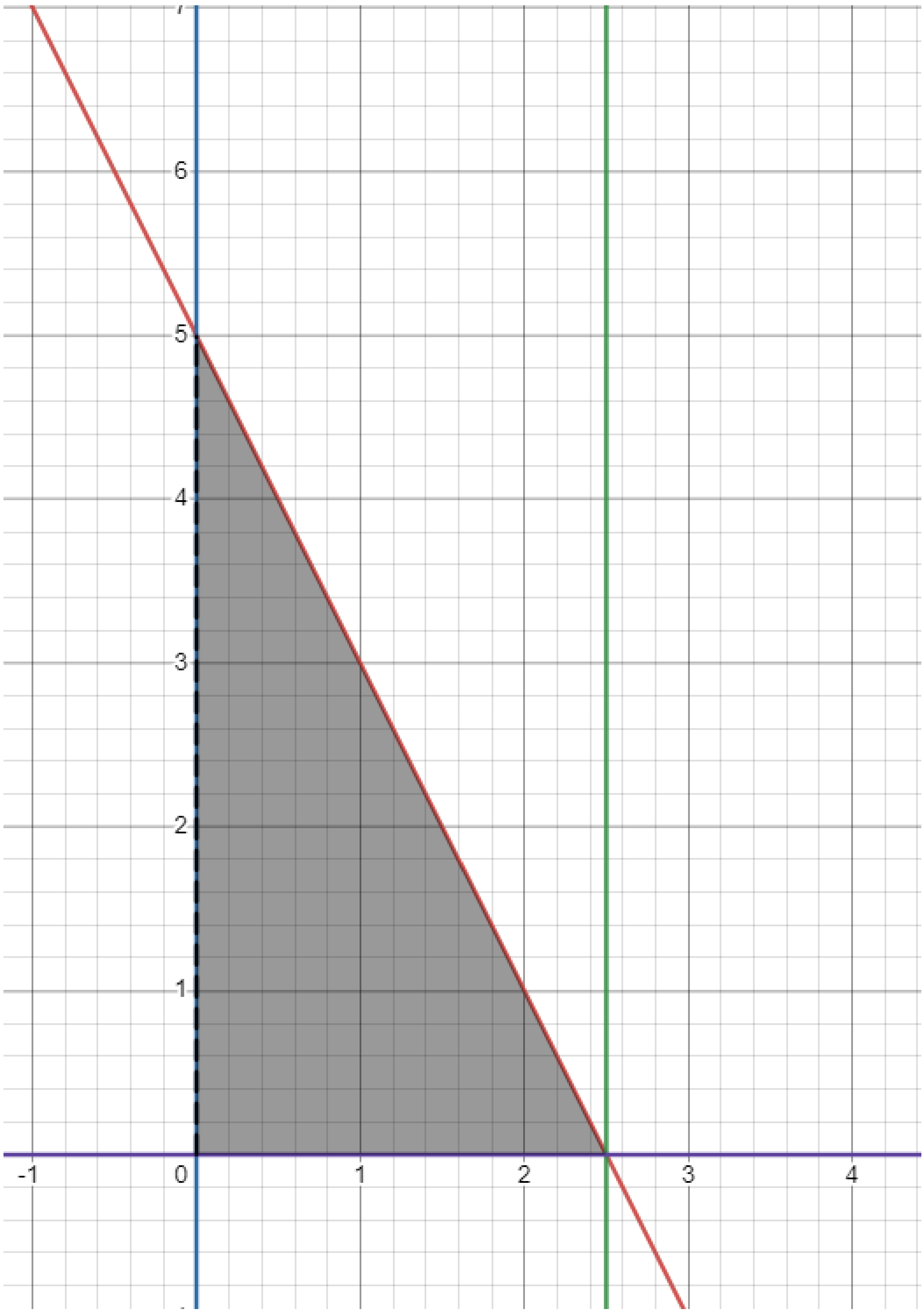
$$= c$$

and then calculate the difference between the two

$$6.25 + c - c = 6.25$$

That is how the distance travelled between two points in time can be calculated given a velocity-time graph by using integration.

There is however another way. Remember, $d = vt$. The velocity-time graph has axis v and axis t, meaning $vt = Area\ under\ the\ curve$ though in this particular case it was a straight line making it considerable easier. Using this method, the distance travelled between when $t = 0$ and $t = 2.5$ is equal to the area bound by the lines, $t = 0$, $t = 2.5$, $v = 0$ and $v = -2t + 5$. These line and shaded area are shown here:



This triangle's area represents the distance travelled between 0 and 2.5 seconds. Its $area = \frac{base \times height}{2}$ because its area is half that of a square with the same width and height. The height is 5 because that is where the y intercept is. We can set $y = 0$ to find the x-intercept.

$$0 = -2t + 5$$

$$2t = 5$$

$$t = 2.5$$

so, the base is 2.5

The area therefore

$$= \frac{5 \times 2.5}{2} = 6.25$$

Which is the same result as we got through integration. This means that integration can find the area under the curve $y = f(x)$ which means that if we let $F(x) = \int f(x)\, dx$

where $\int f(x) dx$ means, "the indefinite integral of $f(x)$ with respect to x":

$$The\ area\ bound\ by\ the\ lines:$$

$$y = f(x)$$

$$x = a$$

$$x = b$$

$$y = 0$$

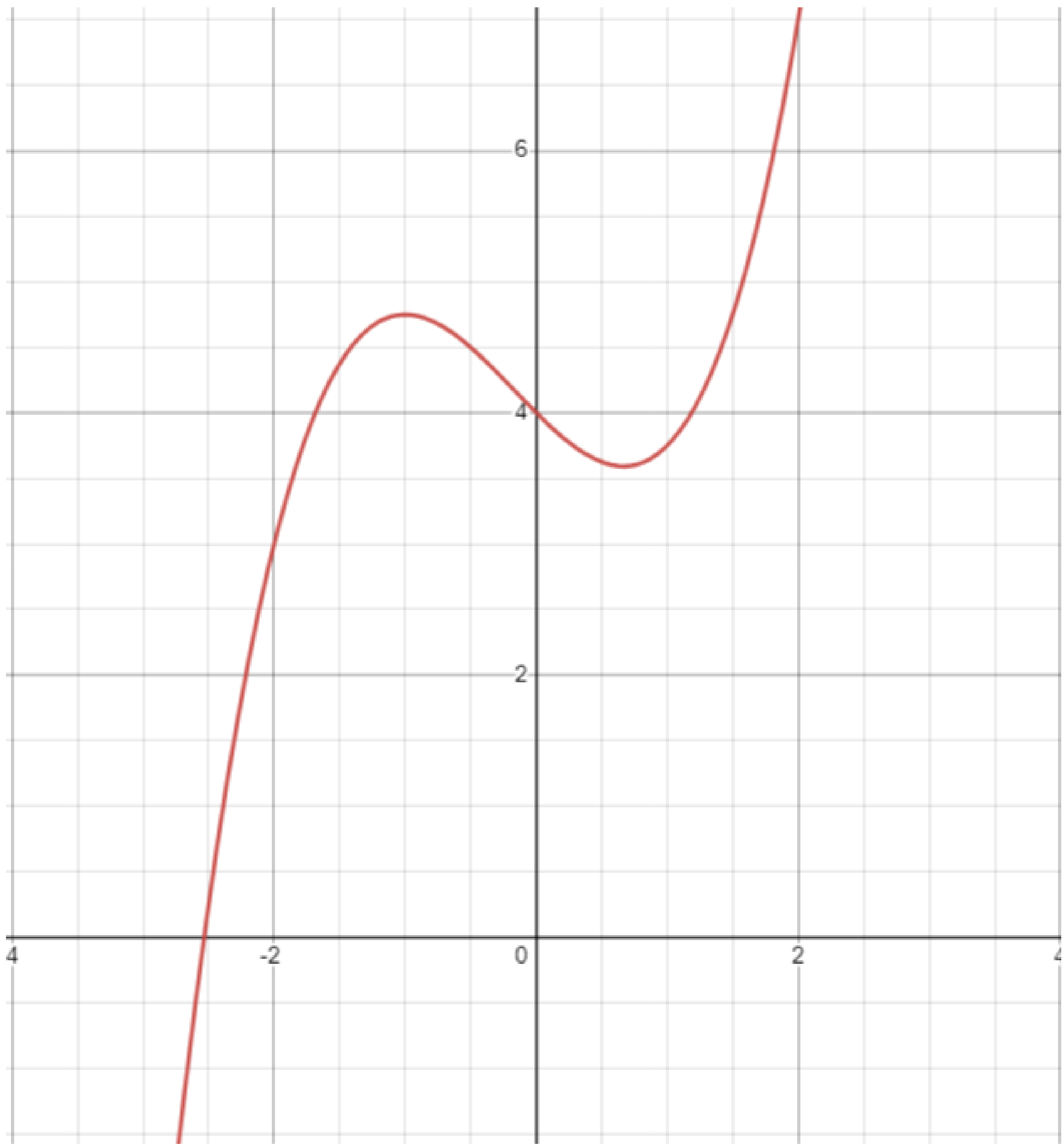
$$= F(b) - F(a)$$

$$where\ b > a$$

The meaning of the $\int$ symbol and the reason for the dx will be explained later.

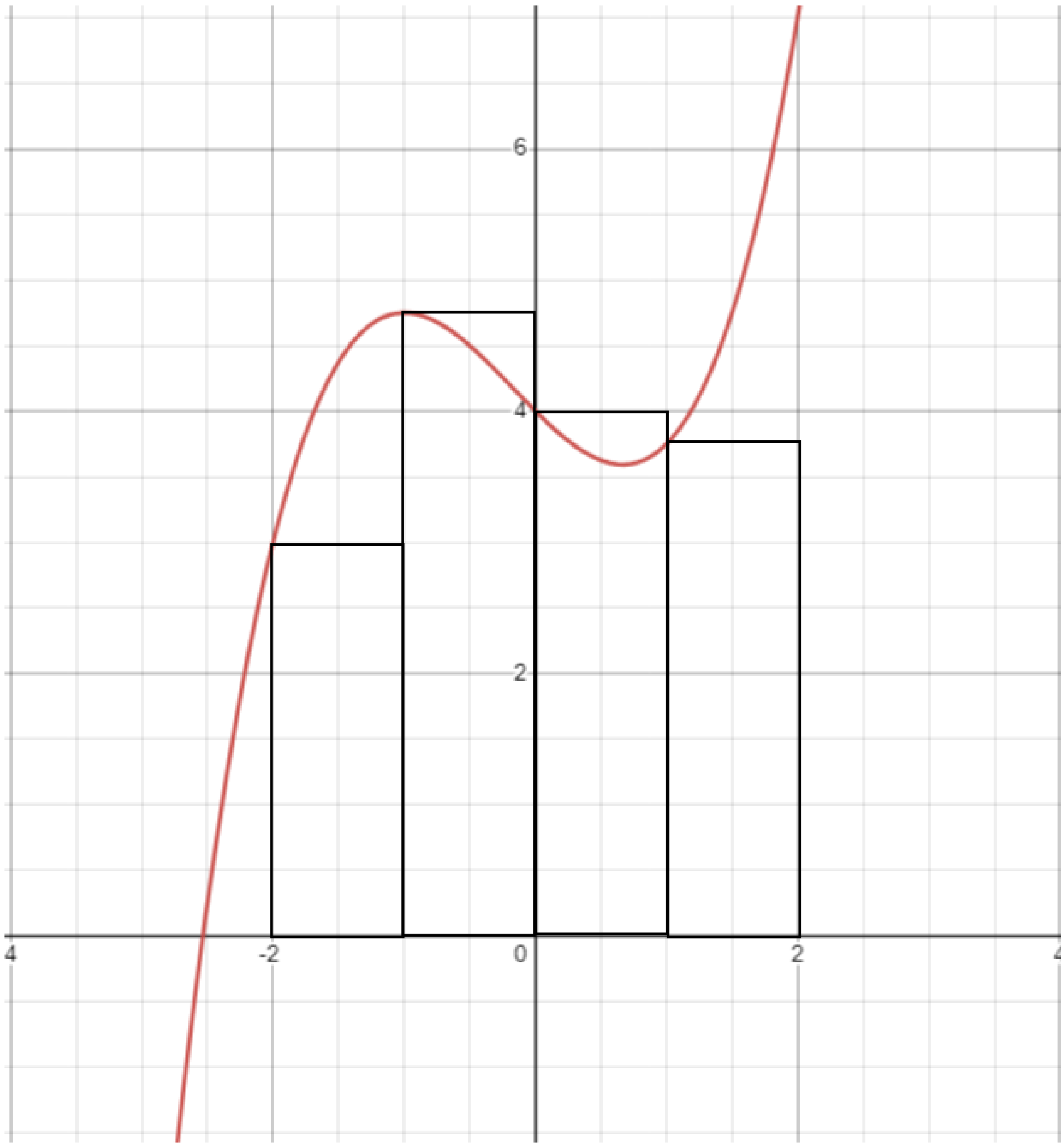
This method can be used to find the area under a curve, as long as we can find the indefinite integral of its function. There is one more thing which I must discuss before getting to more integration notation.

Here is a graph $y = f(x)$

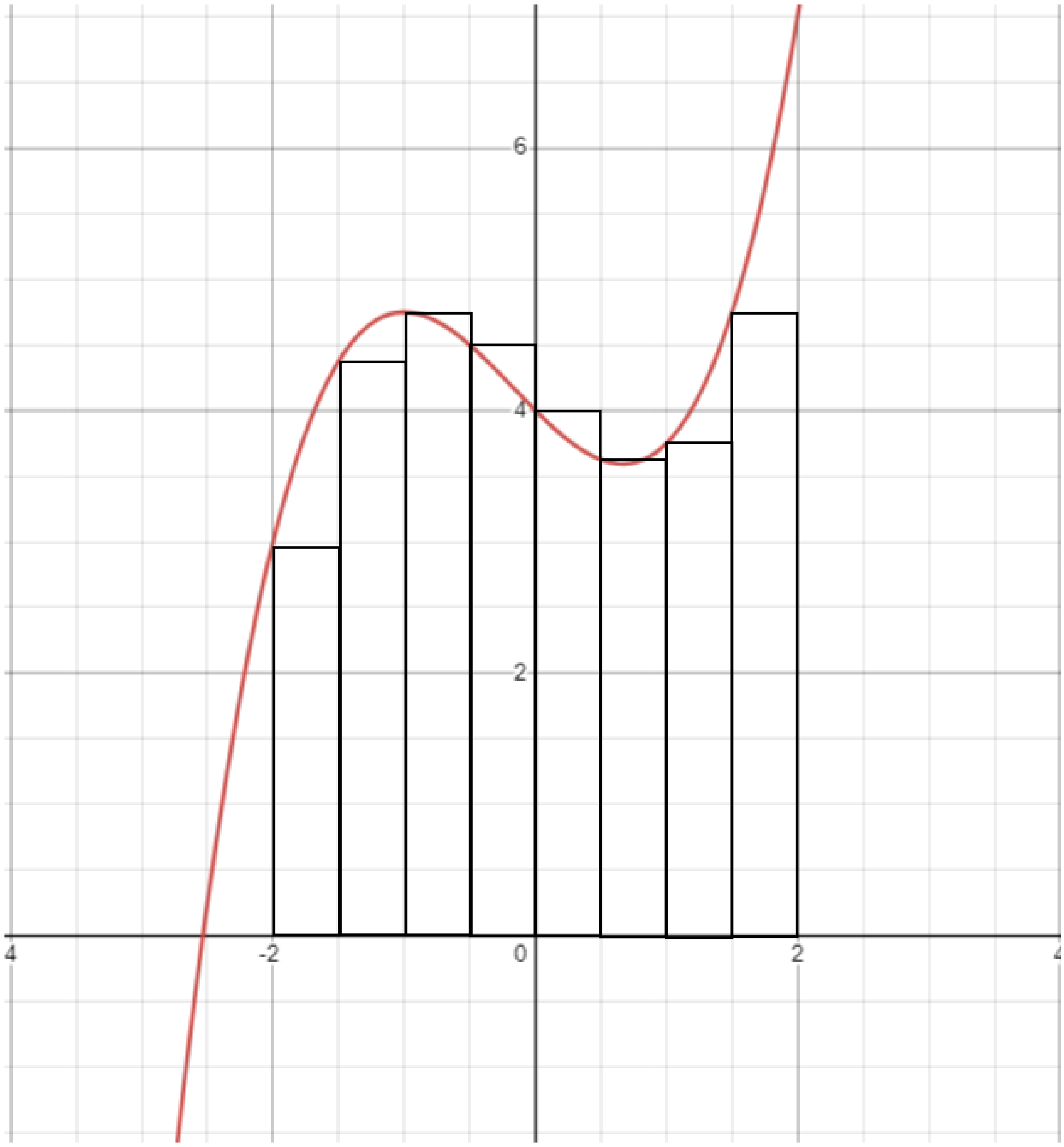


6
4
2
-4
-2
0
2
4

To find the area under the curve, without integration, we must approximate. We can do so by calculating the area of a series of rectangles, each with the same width underneath the curve and summing their areas together. To integrate from -2 to 2 you could calculate the areas of the rectangles as shown here.



This is only an approximation as there are empty spaces above the rectangles in some parts and other parts where the rectangles go above the line. The approximation can be improved by adding more rectangles.



This approximation is better because there is less area where the rectangles go above or below the curve. If there were an infinite number of rectangles, then their areas would sum to the true area under the curve. The area of each rectangle would equal to $base \times height$ where height would of course be $f(x)$, and the base would be dx. As mentioned in the chapter on differentiation, dx means infintesimally small change in x. The base of each rectangle would be dx because if there are an infinite number of rectangles, then the width of each base would be infinitesimally small. The notation for summing the area of the rectangles uses a tall "s" for sum which looks like this: $\int$ and was seen before in the notation of an indefinite integral. When integrating from one point to another, in the case of the above graph, from -2 to 2, the notation used looks like this:

$$\int_{-2}^{2} f(x)dx$$

Where $f(x)dx$ as mentioned is the area of each individual rectangle.

So $\int_a^b f(x)dx$ means: the sum of the areas of the infinte, infinitesimally small rectangles under the curve $y = f(x)$ ranging from x-coordinates, a to b, each with area $f(x)dx$. Or more simply:

$$\int_a^b f(x)dx$$

is the area under the curve from where $x = a$ to $x = b$.

a is always the smallest of the two.

An integral written like this is known as a definite integral and has an actual value, as opposed to an indefinite integral which is instead a function. The indefinite integral is needed to calculate the definite integral of a function between two points, if it is not possible to calculate this area using basic geometry.

If we $Let\ \int f(x)\,dx = F(x)$:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

I will now provide an example of integration in use:

Find the area under a sin wave from $x = 0$ to $x = \pi$.

$$Area = \int_0^\pi \sin(x)\,dx$$

Since $-\cos'(x) = \sin(x)$

$$\int \sin(x)dx = -\cos(x) + c$$

$$\int_0^\pi \sin(x)\,dx = (-\cos(\pi) + c) - (-\cos(0) + c) = -\cos(\pi) + c + \cos(0) - c$$

$$= -\cos(\pi) + \cos(0) = -(-1) + (1) = 1 + 1 = 2$$

$$\int_0^\pi \sin(x)\,dx = 2$$

Another interesting fact of integration is that if a curve goes beneath the x-axis, the area will be calculated as a negative which makes sense because if $f(x)$ is negative, then $f(x)dx$ will be negative.

This means that if I were to calculate the area under a sin wave from 0 to 2π, the result should be 0, as the area under the curve would cancel the area above the curve.

$$\int_0^{2\pi} \sin(x)\,dx = (-\cos(2\pi) + c) - (-\cos(0) + c) = -\cos(2\pi) + c + \cos(0) - c$$

$$= -\cos(2\pi) + \cos(0) = -1 + 1 = 0$$

$\int_0^{2\pi} \sin(x)\, dx = 0$ as expected.

Another interesting case is the graph $y = e^x$

As covered on the "self-differentiating function" chapter, $f(x) = e^x$ is its own derivative. This means that it is also its own indefinite integral.

$$\int e^x dx = e^x$$

With integration, we can not only integrate between two finite x values, but infinite ones too. For example, we can integrate $f(x) = e^x$ from $-\infty$ to x.

$$\int_{-\infty}^{x} e^x dx = \lim_{n\to\infty} (e^x - e^{-n}) = \lim_{n\to-\infty} \left(e^x - \frac{1}{e^n}\right) = \lim_{n\to\infty} \left(e^x - \frac{1}{n}\right) = e^x - 0$$

$$= e^x$$

Meaning that for any given x-coordinate, on the graph $y = e^x$, the y-coordinate equals the gradient which equals the area under the curve from $-\infty$ to x.

An integral to ∞ or from $-\infty$ is known as an improper integral and it is important to note that rather than letting $x = \infty$ or $-\infty$ we let x approach these values as a limit as shown in the above example.

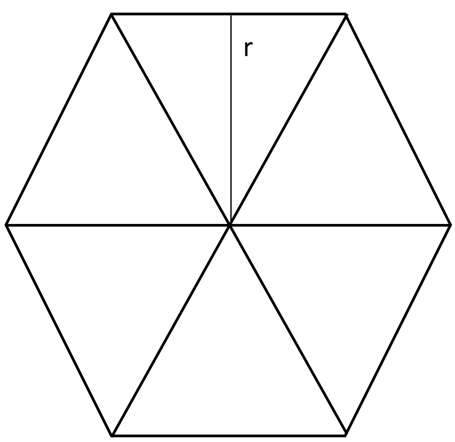
Circles

In this chapter I will show how to find the area of a circle as well as the equation of a circle. This equation can later be integrated which will be done in the next chapter.

If we wish to calculate π we will need to know some more about circles. We already know that $c = \pi d$

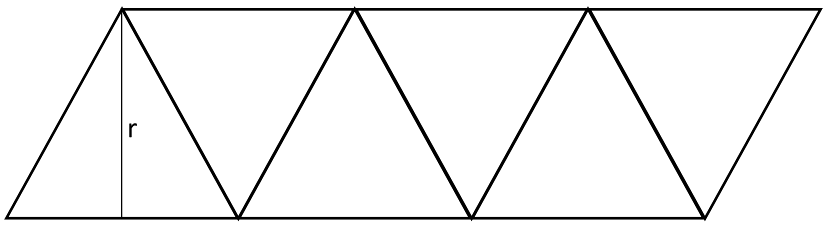
Finding a formula for the area of a circle may be slightly more difficult.

Here is a regular hexagon:



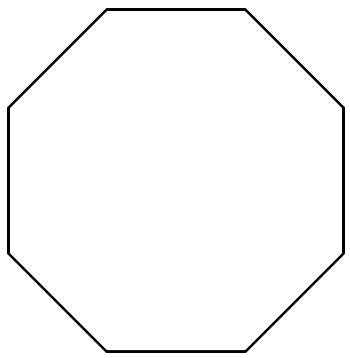
(The diagram may not be to scale)

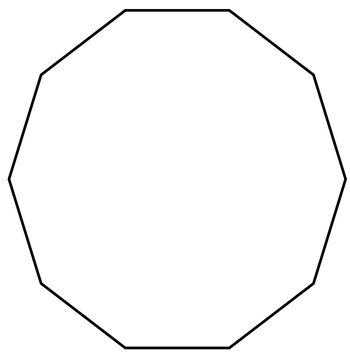
The hexagon has been split into six congruent (same shape and size) triangles, each with a height, r and a base equal to the length of a single side of the hexagon.



These triangles can be rearranged into a parallelogram with base $\frac{perimeter}{2}$ and height r.

Here is an octagon and a decagon:





You can tell that the more sides a regular polygon has, the closer its area is to that of a circle with the same radius. A similar parallelogram could be made for this octagon with base $\frac{perimiter}{2}$ and height r. As the shape chosen approaches a circle, the parallelogram approaches a rectangle, with base $\frac{circumference}{2}$ and height r.

The area of this rectangle would equal the area of the circle.

So, the area of a circle

$$= \frac{circumference}{2} \times r$$

$$\left\{\pi = \frac{c}{d} => c = \pi d = 2\pi r\right\}$$

$$= \frac{2\pi r}{2} \times r = \pi r^2$$

$$area\ of\ a\ circle = \pi r^2$$

This means that a unit circle (a circle with radius 1) would have an area equal to $\pi\ units^2$

There are many different methods to prove the area of a circle, for example Archimedes showed that a circle’s area was equal to that of a right-triangle with a base equal to the circumference and height equal to the radius of the circle. Its area therefore is $\frac{b\times h}{2}=\frac{r\times c}{2}=\frac{r\times 2\pi r}{2}=\pi r^2$. He did this in his book Measurement of a Circle of which I was unable to find a copy.

Now we need to know the equation of a circle. In order to do this, we must again review the definition of a circle. A shape consisting of a set of points equidistance from a fixed point (the centre), where the distance between the centre and a given point on the circle is the radius.

On a graph, if we had a circle, centred at the origin $(0,0)$ we could find the distance from the centre to a given point (x, y) using the Pythagorean theorem.

$$(x-0)^2+(y-0)^2=r^2$$

$$y^2+x^2=r^2$$

where r is the circles radius.

So, the equation of a unit circle would be:

$$y^2+x^2=1$$

Unfortunately, we cannot integrate this because in order to integrate, it must be in the form $y=f(x)$.

We can easily rearrange this equation into that form:

$$y^2+x^2=1$$

$$y^2=1-x^2$$

$$y=\sqrt{1-x^2}$$

$$y=(1-x^2)^{\frac{1}{2}}$$

Series Expansion of π

In this final chapter I will use integration, the binomial theorem and geometry to prove Newton’s Approximation of π series.

We know that the area of the unit circle is π. The function

$$y=(1-x^2)^{\frac{1}{2}}$$

can be turned into a polynomial using the Binomial Theorem. This will only give the half of the circle, above the x-axis because as mentioned, the Binomial Theorem will only return positive values when $n=\frac{1}{2}$.

The area of this semicircle $= \frac{\pi}{2}$ and could be found by integrating from -1 to 1, meaning:

$$\frac{\pi}{2} = \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}}\, dx$$

$$\pi = 2\int_{-1}^{1} (1 - x^2)^{\frac{1}{2}}\, dx$$

Alternatively, the area of the quarter circle from $x = 0$ to $x = 1$ would have the area $\frac{\pi}{4}$ and could be found by integrating from 0 to 1.

$$\frac{\pi}{4} = \int_{0}^{1} (1 - x^2)^{\frac{1}{2}}\, dx$$

$$\pi = 4\int_{0}^{1} (1 - x^2)^{\frac{1}{2}}\, dx$$

The latter of these two integrals would be slightly easier to calculate because integrating a polynomial from 0 is easier than from -1 because setting $x = 0$ for an indefinite integral of a polynomial will always return c, which is then subtracted from the integral at $x = 1$. It will later become clear precisely what this means.

$(1 - x^2)^{\frac{1}{2}}$ is already in the form $(1 + x)^n$ where

$$x \to -x^2$$

$$n \to \frac{1}{2}$$

substitute this into the expansion for $(1 + x)^n$

$$(1 - x^2)^{\frac{1}{2}}$$

$$= 1$$

$$+\frac{1}{2}(-x^2)$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)(-x^2)^2}{2!}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)(-x^2)^3}{3!}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} - 3\right)(-x^2)^4}{4!}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)(-x^2)^5}{5!}$$

$$+$$

$$.$$

$$.$$

$$.$$

$$=1$$

$$-\frac{1}{2}x^2$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)x^4}{2!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)x^6}{3!}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)x^8}{4!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)x^{10}}{5!}$$

$$+$$

$$.$$

$$.$$

$$.$$

We can now find the indefinite integral:

$$\int (1-x^2)^{\frac{1}{2}}\,dx$$

by using the reverse power rule.

$$\int (1-x^2)^{\frac{1}{2}}\,dx$$

$$=x$$

$$-\frac{\frac{1}{2}x^3}{3}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)x^5}{5\times 2!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)x^7}{7\times 3!}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)x^9}{9\times 4!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)(\frac{1}{2}-4)x^{11}}{11\times 5!}$$

$$+$$

$$.$$

$$.$$

$$.$$

$$+c$$

Now we can find the definite integral:

$$\int_0^1 (1-x^2)^{\frac{1}{2}}dx$$

$$=((1)$$

$$-\frac{\frac{1}{2}(1)^3}{3}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(1)^5}{5\times 2!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(1)^7}{7\times 3!}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)(1)^9}{9\times 4!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)(\frac{1}{2}-4)(1)^{11}}{11\times 5!}$$

$$+$$

$$.$$

$$.$$

$$.$$

$$+c)$$

$$-((0)$$

$$-\frac{\frac{1}{2}(0)^3}{3}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(0)^5}{5\times 2!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(0)^7}{7\times 3!}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)(0)^9}{9\times 4!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)(\frac{1}{2}-4)(0)^{11}}{11\times 5!}$$

$$+$$

$$.$$

$$.$$

$$.$$

$$+c)$$

$$=1$$

$$-\frac{\left(\frac{1}{2}\right)}{3}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{5\times 2!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{7\times 3!}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{9\times 4!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)(\frac{1}{2}-4)}{11\times 5!}$$

$$+$$

$$.$$

$$.$$

$$.$$

$$+c$$

$$-c$$

$$= 1-\frac{\left(\frac{1}{2}\right)}{3}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{5\times 2!}-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{7\times 3!}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{9\times 4!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)}{11\times 5!}+\cdots$$

$$= 1-\frac{\left(\frac{1}{2}\right)}{3\times 1!}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{5\times 2!}-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{7\times 3!}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{9\times 4!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)}{11\times 5!}+\cdots$$

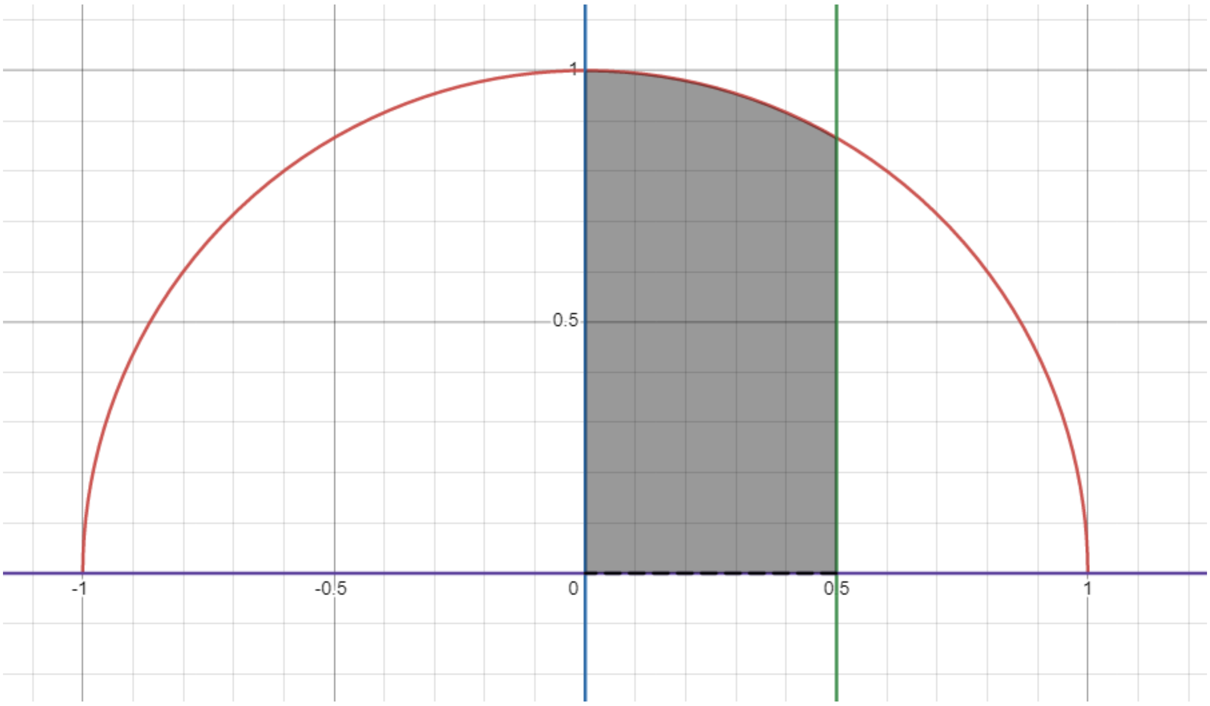
$$=\frac{\pi}{4}$$

Finally multiply both sides by 4.

$$\pi = 4-\frac{2}{3\times 1!}+\frac{2\left(\frac{1}{2}-1\right)}{5\times 2!}-\frac{2\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{7\times 3!}+\frac{2\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{9\times 4!}$$

$$-\frac{2\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)}{11\times 5!}+\cdots$$

This series could be used to calculate pi, although it is quite inefficient. Recall that when the binomial theorem has a fractional value for x is converges much faster. Newton had the idea to integrate from 0 to $\frac{1}{2}$ instead of 0 to 1 as the resulting series would converge much faster. The area calculated by this integral is shown here:

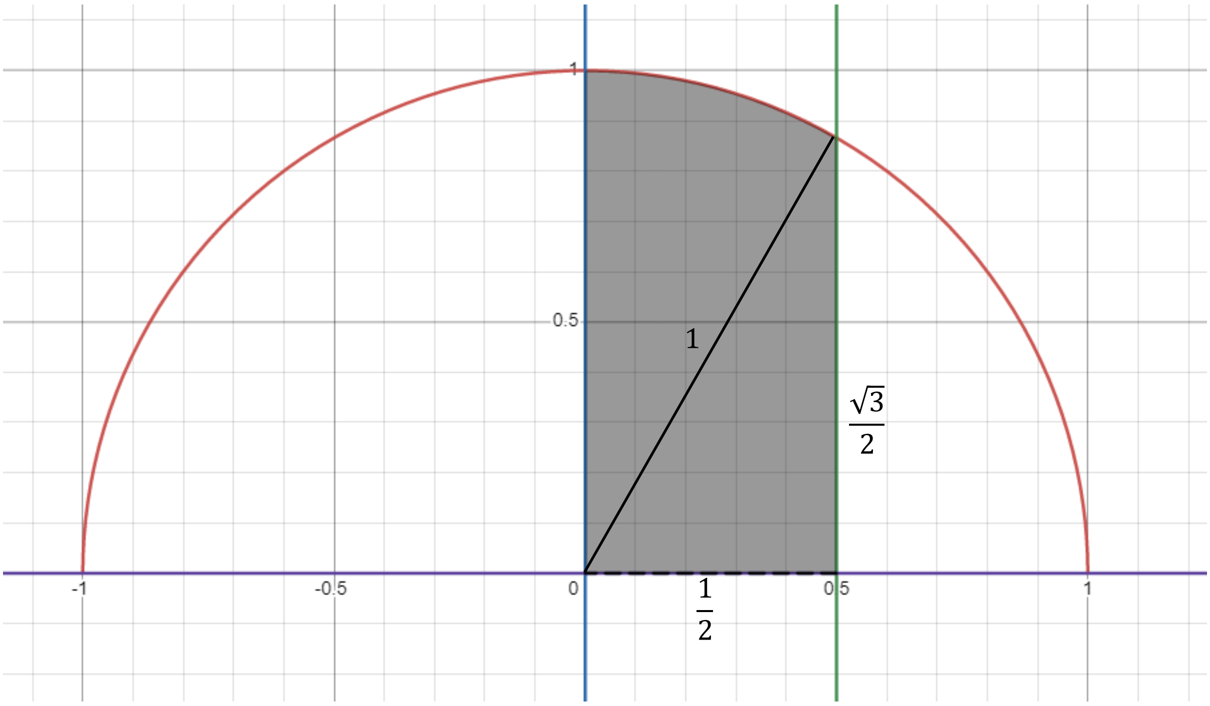


"(Exactly why he chose this particular semicircle may seem a complete mystery, but its special utility will in the end become clear.) As shown earlier in equation (*), the expression (1 - X)1/ 2 can be replaced by its binomial expansion, thus giving the equation of the semi· circle as $y = x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$ [Diagram]" – Dunham, W. (1991) "Journey Through Genius". p.147. New York: John Wiley & Sons, Inc. Available at: https://www.google.co.uk/books/edition/Journey_Through_Genius/_IbWAAAAMAAJ?hl=en&gbpv=0&bsq=journey%20through%20genius

The diagram which I have used is not exactly the same as the one shown in this source though they convey the same idea. Also, the equation of a circle is said in this source to be $y = x^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}$ as opposed to simply $y = (1-x^2)^{\frac{1}{2}}$ but this is simply because Newton's circle equation is shifted and scaled differently to the one I have drawn here as I believe the above diagram is easier to understand.

This shaded area could be calculated using geometry.

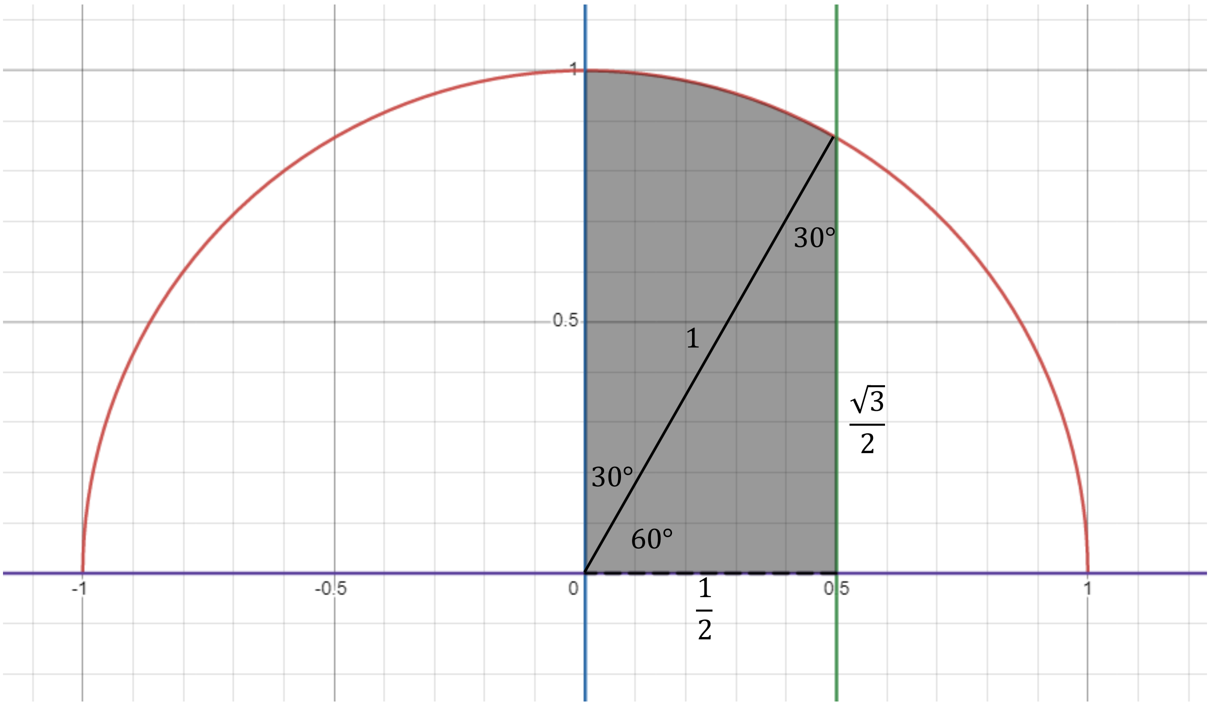
It can be split into a right-triangle and a sector as shown. The right triangle has base $\frac{1}{2}$ and hypotenuse 1, because the radius is 1. Using the Pythagorean Theorem, its height is $\sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$



The area of this triangle therefore $= \frac{\frac{1}{2}\times\frac{\sqrt{3}}{2}}{2} = \frac{\sqrt{3}}{8}$

Notice that this triangle is similar to the triangle in the "Trigonometric Functions: $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$" chapter, made by bisecting an equilateral triangle, but each side of the triangle above is half the size of that one. This means all angles in this triangle are the same as in that one.

This means that the angle of the sector $= 30° = \frac{\pi}{6}\ radians.$



$$\frac{\pi}{6} \div 2\pi = \frac{1}{12}$$

meaning this angle is $\frac{1}{12}$ of a full rotation. This means that the sector's area is $\frac{1}{12}$ that of the area of the circle, meaning the sector has area $\frac{\pi}{12}$

Therefore:

$$\int_0^{\frac{1}{2}} (1-x^2)^{\frac{1}{2}} dx = \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

$$-\frac{\sqrt{3}}{8} + \int_0^{\frac{1}{2}} (1-x^2)^{\frac{1}{2}} dx = \frac{\pi}{12}$$

$$\pi = -\frac{3\sqrt{3}}{2} + 12\int_0^{\frac{1}{2}} (1-x^2)^{\frac{1}{2}} dx$$

$$\int_0^{\frac{1}{2}} (1-x^2)^{\frac{1}{2}} dx$$

$$= \frac{1}{2}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}\right)^{3}}{3}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}\right)^{5}}{5\times 2!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}\right)^{7}}{7\times 3!}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}\right)^{9}}{9\times 4!}$$

$$-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)(\frac{1}{2}-4)\left(\frac{1}{2}\right)^{11}}{11\times 5!}$$

$$+$$

.

.

.

$$+c$$

$$-c$$

$$=\frac{1}{2}-\frac{\frac{1}{2}\left(\frac{1}{2}\right)^{3}}{3}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}\right)^{5}}{5\times 2!}-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}\right)^{7}}{7\times 3!}$$
$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}\right)^{9}}{9\times 4!}-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)\left(\frac{1}{2}\right)^{11}}{11\times 5!}+\cdots$$

$$\pi=-\frac{3\sqrt{3}}{2}+12\left(\frac{1}{2}-\frac{\frac{1}{2}\left(\frac{1}{2}\right)^{3}}{3}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}\right)^{5}}{5\times 2!}-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}\right)^{7}}{7\times 3!}\right.$$
$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}\right)^{9}}{9\times 4!}$$
$$\left.-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)\left(\frac{1}{2}\right)^{11}}{11\times 5!}+\cdots\right)$$

$$\pi = -\frac{3\sqrt{3}}{2} + 6 - \frac{6\left(\frac{1}{2}\right)^3}{3} + \frac{6\left(\frac{1}{2}-1\right)\left(\frac{1}{2}\right)^5}{5 \times 2!} - \frac{6\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}\right)^7}{7 \times 3!}$$
$$+ \frac{6\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}\right)^9}{9 \times 4!} - \frac{6\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)\left(\frac{1}{2}\right)^{11}}{11 \times 5!} + \cdots$$

This is a much more rapidly converging series. With just the first few terms shown here pi can be correctly calculated to 5 decimal places, at 3.141595445.

This then begs the question, what if we integrate from 0 to something smaller than $\frac{1}{2}$ such as $\frac{1}{4}$? Doing this would result in a right-triangle and a sector, though the area of the sector could not be written as a simple fraction of pi as it was here as integrating to $\frac{1}{2}$ was a special case where the angle of the sector was $30°$. Integrating to an arbitrarily small value would result in a sector with an irrational multiple of pi, which is not very useful.

That is how Sir Isaac Newton used integration to calculate π with more precision and ease than ever before.

Section 4: Data and Opinions

At first, I started searching online for data and surveys relevant to my question though I was unable to find anything of use. I instead created my own survey using Microsoft Forms and shared this with regular people by posting it on the social media site *Reddit*. I Created a second, identical version of the survey and distributed this among teachers at my school as well as various staff members and students at John Moore's University. For conciseness I will refer to the former as the L Survey, (Laymen Survey) and the latter as the P survey (Professional Survey). I will now provide the results of the non-word response questions (multiple choice) for each of these surveys. It is also worth mentioning that more people took part in the L Survey (16) than the P Survey (9).

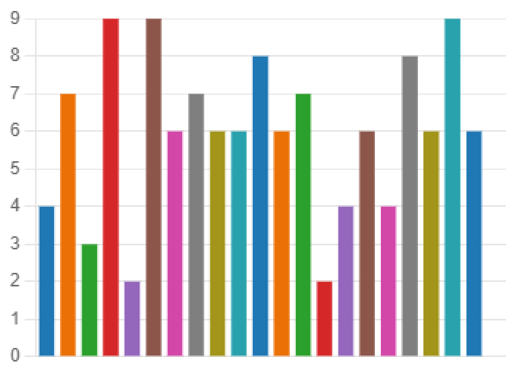
It is at this point also worth noting that the "laymen" are not an unbiased group as they were taken from various educational and certain mathematical online forums. The "professionals" are also not technically all professionals, as some are university students and some are non-maths teachers, (though they do teach closely related subjects).

P Survey:

2. which of the following terms/concepts/theorems are you familiar with? (in terms of Mathematics). Ask yourself whether or not you could explain or describe it to somebody else. If yes then check the box, if not then don't.

More Details

Axiom	4
Definition	7
Euclidean Geometry	3
Pythagorean Theorem	9
Gougu Rule	2
Trigonometric Functions (sin, co...	9
Addition Formulae	6
Calculus	7
Differentiation	6
Integration	6
Power Rule	8
Limit	6
Constant	7
Taylor/Maclaurin Series	2
Euler's Formula/Identity	4
Binomial Theorem	6
Combinatorics	4
Pascal's Triangle	8
e	6
pi	9
i (√-1)	6
None of the above	0

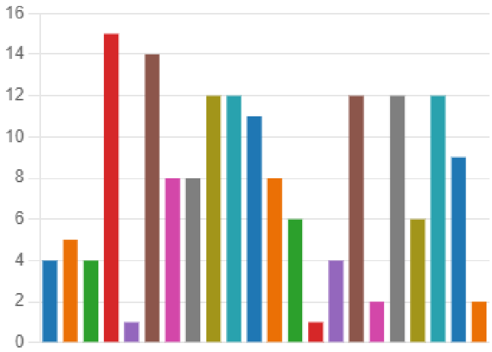


L Survey:

2. which of the following terms/concepts/theorems are you familiar with? (in terms of Mathematics). Ask yourself whether or not you could explain or describe it to somebody else. If yes then check the box, if not then don't.

More Details

Axiom	4
Definition	5
Euclidean Geometry	4
Pythagorean Theorem	15
Gougu Rule	1
Trigonometric Functions (sin, co...	14
Addition Formulae	8
Calculus	8
Differentiation	12
Integration	12
Power Rule	11
Limit	8
Constant	6
Taylor/Maclaurin Series	1
Euler's Formula/Identity	4
Binomial Theorem	12
Combinatorics	2
Pascal's Triangle	12
e	6
pi	12
i (√-1)	9
None of the above	2



These results show that, when it comes to more basic concepts and well-known ideas such as the Pythagorean theorem and pi, almost everybody knows what they are. The more complex ideas are not as well understood among laymen, for example only 1/16 laymen had heard of Taylor/Maclaurin Series, whereas 2/9 of the professionals had done.

It is also interesting almost everybody knew what the Pythagorean theorem was, yet most did not know what the Gougu Rule was. This is interesting, I think, because these are both different names for the same thing. The Pythagorean theorem is known in China as the "Gougu Rule" I think it is especially interesting that in the west, we have given this theorem its name after Pythagoras, even though he is known not to have been its original discoverer.

P Survey:

3. Which mathematical field, invention or discovery do you think has been the most influential, whether that be to the rest of mathematics or other fields such as physics or computer science. This can be as specific or as broad as you like.

More Details

Calculus	3
Complex Numbers	0
Geometry	3
Algebra	3
Other	0

L Survey:

3. Which mathematical field, invention or discovery do you think has been the most influential, whether that be to the rest of mathematics or other fields such as physics or computer science. This can be as specific or as broad as you like.

More Details

Calculus	4
Complex Numbers	1
Geometry	5
Algebra	6
Other	0

This result was particularly interesting to me both surveys show very similar results. It seems as though the most influential mathematical field of study has been either calculus, geometry or algebra. A single person from the laymen group did select complex numbers the reason being that they *"Allowed calculations that were thought to be impossible to be able to be done".*

For the most part, these results show that, perhaps there is not one single field of mathematics which has been the most influential or important, as they are all built on top of each other and are intricately connected. Without algebra, geometry would be much more difficult, and calculus would perhaps be impossible. This sentiment is further echoed by the reasons given as to the responses to question 3. Three such responses are:

Geometry *"has the most applications in other areas and people can connect it to real life more"*

Algebra *"underpins the other options and is an essential way of representing unknowns, variables etc."*

"Calculus is the application of maths across a multitude of different disciplines of science"

So maybe that part of the original question would be best unanswered. It may be a better idea to appreciate all of mathematics and how it works together. Comparing and trying to see which field or discovery is the "best" is a futile endeavour.

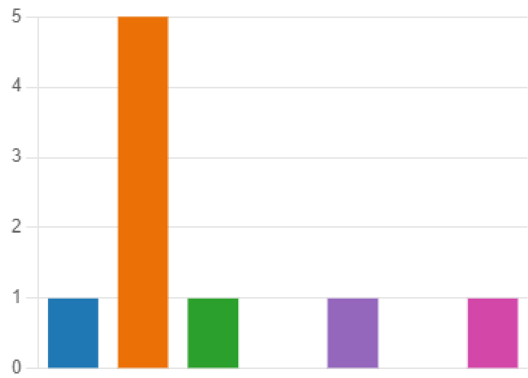
The most influential people and time periods is a different story however:

P Survey

5. Which mathematical figure (ancient or more recent) do you think has had the most influential on modern mathematics and on the world as a whole.

More Details

Archimedes	1
Isaac Newton	5
Gottfried Wilhelm Leibniz	1
Karl Friedrich Gauss	0
Leonhard Euler	1
Srinivasa Ramanujan	0
Other	1



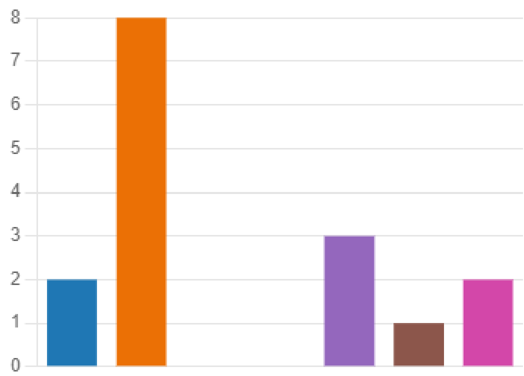
Other: Alan Turing

L Survey

5. Which mathematical figure (ancient or more recent) do you think has had the most influential on modern mathematics and on the world as a whole.

More Details

Archimedes	2
Isaac Newton	8
Gottfried Wilhelm Leibniz	0
Karl Friedrich Gauss	0
Leonhard Euler	3
Srinivasa Ramanujan	1
Other	2



Other: Descartes, Pythagoras

Whilst some members of both surveys did vote for various mathematicians, it is clear who most believe to have been the most influential, Isaac Newton. Each other name has been mentioned throughout this paper, with the exception of Gauss and Ramanujan. I will explain why this was the case in the concluding section.

Among both groups, Isaac Newton is seen to have been the most influential mathematician of all time. These are some reasons they gave as to why:

"He's the most well-known so it's some people's only introduction to a mathematician"

"The applications of calculus impact upon so much.... even Newton didn't realise how much… he was only interested in planets... although difficult to separate Newton from Leibniz as their work paralleled each other"

"His mathematical description of gravity to derive Kepler's laws of planetary motion, account for tides, the trajectories of comets, the precession of the equinoxes and other phenomena, eradicating doubt about the Solar System's heliocentricity."

There are certainly many valid reasons for people to believe that he was the most important.

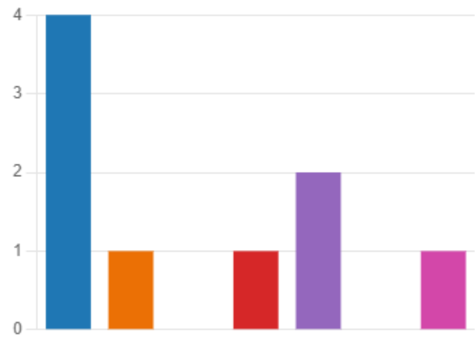
Note: [In the survey, Gauss' name was spelled "Karl", this should have been "Carl"]

P Survey

7. Which time period/setting was the most influential in mathematical history. (optional)

More Details

Ancient Greece	4
Ancient Babylon	1
Medieval Period	0
Renaissance	1
18th/19th centuries	2
Present	0
Other	1

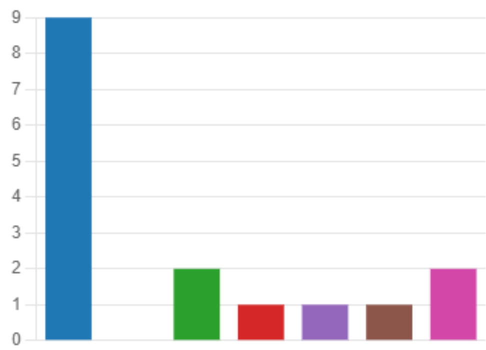


L Survey

7. Which time period/setting was the most influential in mathematical history. (optional)

More Details

Ancient Greece	9
Ancient Babylon	0
Medieval Period	2
Renaissance	1
18th/19th centuries	1
Present	1
Other	2



The results of the question are conclusive. It seems that in the minds of most, Ancient Greece was the most influential setting for mathematical development. It may be worth noting that those who selected “other” said that all periods were important and that it was hard to say, or something along those lines.

The reasons given as to why Ancient Greece was the most important time period for mathematics include:

“Lots of the fundamentals were founded during this time.”

“This was the time when mathematics was ‘seen’ in patterns and nature. For example, Pythagoras saw his theorem in a tiled floor.”

“This period has a lot of prevalent mathematicians”

Reasons given for other time periods include:

Ancient Babylon *“had an advanced number system”*

“Most of the mathematicians and models we use today to solve problems were formulated by renaissance mathematicians”

All time periods - *“The modern discoveries/inventions have the most effect, but they are only possible because of the discoveries/inventions from the past. 'If I have seen further, it is because I am stood on the shoulders of giants'”*

Overall, whilst all time periods have had their own important discoveries made and people lived, the *most* influential time period, I would have to say, way Ancient Greece due to the reasons listed above.

$\boldsymbol{Section} \sum_{\mathbf{n=0}}^{\infty} \frac{\left(\sqrt{-1}\left(-\frac{3\sqrt{3}}{2} + 12\int_0^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}dx\right)\right)^{\mathbf{n}}}{\mathbf{n}!} + 6$:

Conclusion

Throughout this project I have:

Demonstrated how complex ideas such as those discussed here can be derived from the most basic of observations, (axioms) and simple definitions.

Proven numerous trigonometric identities.

Delved into the world of both differential and integral calculus.

Proven Leonhard Euler's Identity:

$$e^{\pi i} = -1$$

Shown how Sir Isaac Newton was able to find his rapidly converging expansion:

$$\pi = -\frac{3\sqrt{3}}{2} + 6 - \frac{6\left(\frac{1}{2}\right)^3}{3} + \frac{6\left(\frac{1}{2}-1\right)\left(\frac{1}{2}\right)^5}{5 \times 2!} - \frac{6\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}\right)^7}{7 \times 3!}$$

$$+\frac{6\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}\right)^9}{9 \times 4!} - \frac{6\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)\left(\frac{1}{2}\right)^{11}}{11 \times 5!} + \cdots$$

which he used to calculate π more accurately than anybody before him.

I have proven each thing necessary to prove the above, and each thing necessary to prove those and so on until reaching mere definitions and axioms. Through this process I have developed a far deeper understanding of how mathematics is done, and how new mathematics can be invented.

I have researched which people have been responsible for which discoveries and when they were made.

I have created surveys to collect data and opinions from the public to help answer the posed question. Before I do though, I would like to address why certain mathematicians have not been mentioned here, such as Gauss and Ramanujan despite their contributions.

The simple reason is that they were alive after the two discoveries from which I was working down (the two above) were made. This meant that nothing which they, or anybody else from after the time of Euler (Alan Turing, Bernhard Riemann Carl Friedrich Gauss, Srinivasa

Ramanujan, etc.,) would be necessary, hence they were not mentioned. If I were to do this project again, that is the key thing I would change. That said, I doubt that the conclusion would have been any different as, looking at the surveys from the previous section, despite the options being present, Gauss got no votes, and Ramanujan only one.

To conclude and answer the initially posed question: Which mathematical figures and discoveries have been the most influential and important?

The most important figure in mathematical history was Sir Isaac Newton because of his work on calculus, π and physics to name a few. He I likely to be the most well-known mathematician of all time, though most don't know him as a mathematician, rather as the founder of modern physics, which is really an achievement when you think about it. His many contributions to mathematics which would have been the highlight of most mathematicians' careers, were to Newton simply "another thing he also did". His contributions to the subject have been immense. If I had to give a second-place medal, it would go to Euler as he contributed to many different areas of mathematics and has been referenced rather a lot throughout this particular project.

The most influential discovery? I would say that there cannot be one single influential discovery, because mathematics is not like the contents of a museum, where each individual artifact stands alone. NO! The discoveries of mathematics are not important or useful in a vacuum. They are only important as far as they are able to link together. Mathematics is the museum itself, the building. It has foundation, on top of which more mathematics is built, and then more on top of that and so on. Each part is integral, and one individual brick of the walls of this museum cannot be proclaimed to be the "most important".

The most important period of time? I would have to say that ancient Greece was the most important as the Greeks were the first to truly take an interest to mathematics, thanks to their many famous philosophers taking an interest in it. Ancient Greece gave us pi, much of geometry and truly began, what I believe to be the beginning of modern mathematics. The Babylonians before them gave some geometry and an advanced number system, but the Greeks, I believe, built upon this giving us what has now evolved into the rest of mathematics.

To Conclude:

Isaac Newton was the most influential mathematical figure.

Ancient Greece was the most influential mathematical period.

It is impossible to say that any individual discovery or field has been the most important.

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