Faculty of Electrical Engineering, Mathematics and Computer Science

Mathematical Methods For Physics: Finite–Element Analysis (AP3001-FE)

Lab Assignment

1 Problem Statement

We consider the heating of a chicken in a microwave oven. We begin by deriving the PDE that we will use to model this situation.

The electric and magnetic field in a microwave oven satisfy the harmonic Maxwell's equations:

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}, \tag{1}$$

$$\nabla \times \mathbf{H} = i\omega\varepsilon\mathbf{E} + \sigma\mathbf{E}, \tag{2}$$

where $\mathbf{E}$ denotes the electric field, $\mathbf{H}$ the magnetic field, and i is the imaginary unit. Moreover, ε is the permittivity, σ is the conductivity, μ is the permeability. Finally, ω denotes the wave number, which is given by $\omega = 2\pi f_m$ with f_m is the frequency of the microwave source in the microwave oven.

For simplicity, we will assume that the variation in the z direction is negligible. In other words, that $\dfrac{\partial \mathbf{E}}{\partial z} = 0$ and $\dfrac{\partial \mathbf{H}}{\partial z} = 0$. Under this assumption, the PDE simplifies to finding u such that

$$-\Delta u - \omega^2(\varepsilon - \frac{i\sigma}{\omega})u = 0, \quad \text{in } \Omega, \tag{3}$$

where Ω is now the two-dimensional region corresponding to the interior of the microwave oven (in the $xy - plane$), and u corresponds to the third component of the electric field E_3.

This equation is called the Helmholtz equation.

We note that the rate at which the chicken's temperature changes is proportional to the solution u from (3).

2 Assignment

1) Derive the weak formulation for the problem

$$\begin{cases} -\Delta u(x,y) - c(x,y)u(x,y) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\partial\Omega$ denotes the boundary enclosing Ω, and c is a bounded function.

Do not forget to specify the related functional space(s). (3 pt.)

2) Following the steps discussed in the lecture, write down the Galerkin discretization corresponding to the weak formulation found in Question 1).

Specify the size of the linear system and the entries of the resulting matrices and vectors. (3 pt.)

Hint: Do not forget the non-homogeneous Dirichlet boundary condition. For this, first enumerate the basis functions related to internal vertices and then the ones for vertices located in the boundary. You may use the notation n_i = number of internal vertices, n_b = number of vertices in the boundary, $n = n_i + n_b$.

3) Using your results from Question 2), compute the element matrix S^{e_k} for an (internal) triangle e_k. (3 pt.)

Now we continue to the implementation part of the lab assignment.

Instead of solving the $n_i \times n_i$ system $\mathbf{Sc} = \mathbf{f}$ as proposed in the lecture, we impose the non-homogeneous Dirichlet boundary conditions by considering the extended $n \times n$ linear system given by $\mathbf{Au} = \mathbf{b}$, with

$$\mathbf{b}[i] = \begin{cases} 0, & \text{for } i = 1, ..., n_i \\ g(x_i, y_i), & \text{for } i = n_i + 1, ..., n \end{cases}, \tag{5}$$

and

$$\mathbf{A} = \begin{pmatrix} \mathbf{S} & \mathbf{S}_b \\ \mathbf{0} & \mathbf{Id}_b \end{pmatrix}. \tag{6}$$

Here, $\mathbf{S}_b$ is the matrix built by

$$\mathbf{S}_b[i,j] = \mathbf{S}[i, j + n_i], \quad \text{for } i = 1, ..., n_i,\ j = 1, ..., n_b; \tag{7}$$

and $\mathbf{Id}_b$ is the $n_b \times n_b$ identity matrix.

This is already implemented for you in the `MATLAB` file `BuildMatricesandVectors_Dirichlet.m`

4) Complete the provided `MATLAB` implementation by modifying the file `GenerateElementMatrix.m` so it computes the element matrix S^{e_k} derived in Question 3). (3 pt.)

5) Take $c(x, y) = 0$ and $\Omega = [-1, 1]^2$. Run your code to solve the PDE with boundary conditions given by $g(x, y) = x + y$. What result should you get? Does your numerical solution agree with it?

   Plot your solution. (3 pt.)

6) Simulate now what happens when we put a chicken in the microwave.

   For this, we take $c(x, y) = \omega^2 \mu(\varepsilon - \dfrac{i\sigma}{\omega})$, where the values μ, ε and σ will depend whether (x, y) is in a region with chicken or with air.

   For simplicity, we set $\Omega = [-1, 1]^2$ and assume that the chicken occupies the region $D_c := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 0.5\}$, and that the rest of Ω is filled with air.

   The microwave source can be modeled via the Dirichlet boundary condition:

$$g(x, y) = \begin{cases} 100, & \text{if } x = 1, \text{ and } -0.5 \leq y \leq 0.5, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

   The microwave frequency for this domain is $f_m = 2.45e8$.

   In air, we have $\varepsilon_a = 8.85e-12$, $\sigma_a = 0$, and $\mu_a = \pi 4e-7$, while in chicken the values are $\varepsilon_c = 4.43e-11$, $\sigma_c = 3e-11$, and $\mu_c = \pi 4e-7$.

   Plot your solution using the initial mesh. What can you say about the solution? (4 pt.)

7) Modify the `MATLAB` file `AP3001Mesh.m` and plot your solution after one mesh refinement. Repeat this now with two mesh refinements. Compare their corresponding numerical solutions with the one you obtained in Question 6). Are the numerical solutions similar or different? Explain why this is the case.

What can you say about the approximation error? (3 pt.)

Hint: Remember that the solution u represents a time-harmonic wave.

8) Choose a different the geometry for Ω, adapt the boundary conditions accordingly and run your code. Plot your solution. What can you say about the solution?

   Do not forget to specify the geometry and the boundary condition that you chose. (2 pt.)

   Hint: The geometry is determined in the `MATLAB` file `AP3001Mesh.m`

To be submitted before November 15, 2021, 17:00 via Brightspace.

Remark: It is allowed to discuss and to collaborate with other participants, however, it is **not** *allowed to share your answers with other people. Online found, published answers will always be reported to the Exam Board.*