§12.

1000.12.1.

曲线 $\begin{cases} x=\cos^3 t \\ y=\sin^3 t. \end{cases}$ 每一点线密度 = 该点到原点距离立方. G,引力常数.

求曲线第一象限部分对原点处引力在X轴分量大小

[solution].

$$F=\int_0^{\frac{\pi}{2}} G\frac{(\cos^6 t+\sin^6 t)^{\frac{3}{2}}}{r^2}\cdot\sqrt{(x')^2+(y')^2}\,dt$$

$$=\int_0^{\frac{\pi}{2}} G3\frac{(\cos^6 t+\sin^6 t)^{\frac{3}{2}}}{r^2}\cdot\sin t\cos t\,dt.$$

$$F_x=\int_0^{\frac{\pi}{2}} 3G\frac{(\cos^6 t+\sin^6 t)^{\frac{3}{2}}}{r^2}\cdot\sin t\cos t\,\frac{x}{\sqrt{x^2+y^2}}\,dt$$

$$=\int_0^{\frac{\pi}{2}} 3G\frac{(\cos^6 t+\sin^6 t)^{\frac{3}{2}}}{r^2}\cdot\sin t\cos t\,\frac{\cos^3 t}{(\cos^6 t+\sin^6 t)^{\frac{1}{2}}}\,dt$$

$$=\int_0^{\frac{\pi}{2}} 3G\frac{(\cos^6 t+\sin^6 t)\sin t\cos^4 t}{r^2}\,dt$$

$$\cancel{=-3G\int_0^{\frac{\pi}{2}}[\cos^6 t+(1-\cos^2 t)^3]\cdot\cos^4 t\,d\cos t}$$

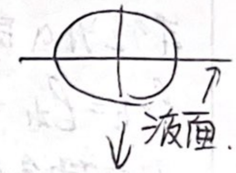
$$\cancel{=+3G\int_0^1[x^6+(1-x^2)^3]\cdot x^4\,dx}$$

$$\cancel{=9G\int_0^1(x^8-x^6)\,dx=\frac{6}{7}G}$$

$$=\int\frac{3}{5}G$$

1000.12.2

椭圆形薄板,长、短半轴,a与b.



设液体比重为γ,则液体对薄板的侧压力为?

$$dP=2\gamma yx\,dy.\qquad F=\overset{\gamma}{\rho g}\overset{y}{h}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad x=a\sqrt{1-\frac{y^2}{b^2}}=\frac{a\sqrt{b^2-y^2}}{b}$$

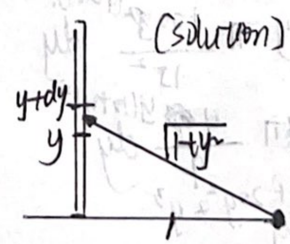
$$P=\int dP=2\gamma\int_0^b yx\,dy=\frac{2\gamma a}{b}\int_0^b y\sqrt{b^2-y^2}\,dy$$

$$=\frac{\gamma a}{b}\int_0^b\sqrt{b^2-y^2}\,dy^2=\frac{-\gamma a}{b}\int_0^{b^2}\sqrt{b^2-x}\,d(b^2-x)$$

$$=\frac{-\gamma a}{b}(b^2-x)^{\frac{3}{2}}\cdot\frac{2}{3}\Big|_0^{b^2}=\frac{2}{3}\gamma ab^2$$

1000.12.3.

设沿y轴上区间[0,1]放置一长度为1且线密度为ρ的均匀细杆,在X轴上x=1处有一单位质点,则该细杆对此质点的引力X轴分量



(solution) $dF=\frac{G\cdot 1\cdot\rho}{1+y^2}\,dy$

$$dF_x=\frac{G\rho}{1+y^2}\cdot\frac{1}{\sqrt{1+y^2}}\,dy\qquad 令y=\tan t$$

$$\int dF_x=\int_0^1\frac{G\rho\,dy}{(1+y^2)^{\frac{3}{2}}}=-\frac{\sqrt{2}}{2}G\rho$$

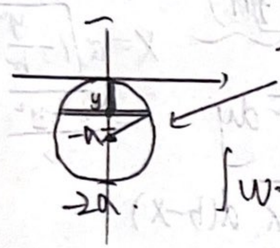
1000.12.4

半径为a的球沉入水中，球顶与水平面相齐平，

$\rho_{球}=\rho_{水}=\rho$，g，球从水中提出

至少做多少功？

[solution]



球：$x^2+(y+a)^2=a^2$.

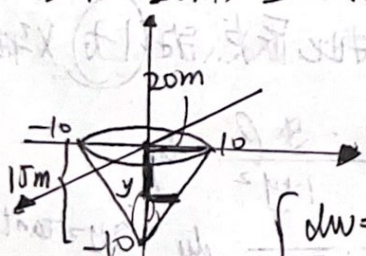
$dW=\rho\cdot g\cdot \cancel{2a}[a^2-(y+a)^2]\pi\cdot dy$

$\int W=\int_{-2a}^{0}\rho g\cdot\cancel{2a}[a^2-y^2-a^2-2ay]\pi\,dy$

$=\cancel{2a}\rho g\pi\int_{-2}^{0}-2ay-y^2\,dy=a^4\rho g\pi(\cancel{4a}\ \tfrac{8}{3})\ \tfrac{4}{3}$

1000.12.5.

设有一锥形贮水池（锥顶朝下），深15m，口径20m，盛满水，用水泵将水抽尽，功？



$\frac{15}{-10-y}=\frac{10}{R}$

$dW=\cancel{\rho g}\ \rho g(0-y)\,dV$

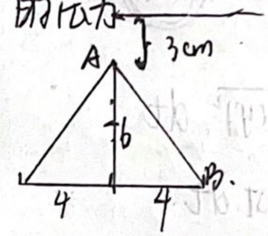
$=-\rho g y\,\pi\,\frac{100(10+y)^2}{15^2}\,dy$

$\int dW=\int_{-15}^{0}-\rho g\pi\,\frac{4y(10+y)^2}{9}\,dy$

$=-\rho g\pi\frac{4}{9}\int_{-15}^{0}100y+20y^2+y^3\,dy$

1000.12.6

一底为8cm，高为6cm的等腰三角形片，铅直地沉没在水中，顶在上，底在下且与水平平行，而顶离水面3cm，求一侧所受的压力



$y=\frac{2}{3}x-2$

$dF=0.0098\ x\cdot2(\frac{2}{3}x-2)\,dx$

$F=\int_{3}^{9}2\times0.0098\,x(\frac{2}{3}x-2)\,dx$

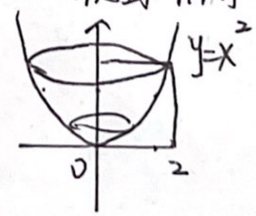
$=1.6464\,N$

1000.12.7

一容器的内表面是由 $y=x^2$ $(0\le x\le2)$ 绕y轴旋转一周所得到的曲面，现以 $2\ m^3/min$ 注入液体。

(1). 容器体积？

(2). 液面升高到1m时液面上升速度。



$\int_{0}^{2}\pi(x)^2\,dx^2$

$=\int_{0}^{2}2\pi x^3\,dx=2\pi\frac{x^4}{4}\Big|_0^2=8\pi\ m^3$

(2) $\cancel{\frac{d\,y}{dt} = d}$

$$V=\int_0^x \pi x^2\, d x^2 = \int_0^x 2\pi x^3\, dx = 2\pi \frac{x^4}{4}\Big|_0^x = \frac{\pi}{2}x^4$$

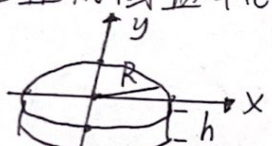
$$V=\frac{\pi}{2}y^2 \quad \Downarrow \qquad \frac{dy}{dt}=\frac{dy}{dV}\frac{dV}{dt} = \sqrt{\frac{2}{\pi}}\,\frac{1}{2}V^{-\frac{1}{2}}\cdot \alpha \Big|_{V=\frac{\pi}{2}}$$

$$=\frac{2}{\pi} \quad (m/min)$$

• 1000.Ut.1

1.(1). 设圆盘半径为R，厚为h，点密度为该点到与圆盘垂直的圆盘中心轴的距离平方，求该圆盘m.

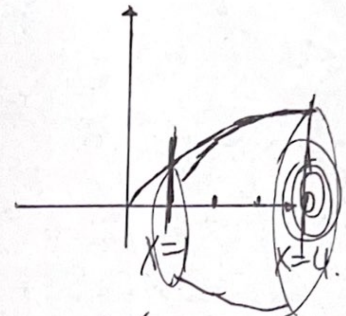
[solution]



$$m=\int_0^h dh \cancel{\int_0^R}\int_0^{2\pi}d\theta\int_0^R \rho\, r\, dr$$

$$=2\pi h\int_0^R r^3\, dr = 2\pi h \frac{r^4}{4}\Big|_0^R = \frac{\pi h R^4}{2}.$$

(2) 将以曲线 $y=\sqrt{x}$，$x=1$，$x=4$ 及x轴围成的曲边梯形绕x轴转一圈围成的旋转体记为V，设V的点密度为该点到旋转轴的距离的平方，求M.



$y=\sqrt{x}$ $\quad \pi r^2=\pi x$

$$M=\int_1^4 \cancel{\pi x}\left(\int_0^{2\pi}\int_0^{\sqrt{x}} r^2\, r\,dr\right) d\theta\, dx$$

$$=2\pi\int_1^4 dx\int_0^{\sqrt{x}} r^3\,dr = 2\pi\int_1^4 \frac{x^2}{4}\,dx = 2\pi \frac{x^3}{12}\Big|_1^4$$

$$=\frac{63}{6}\pi = \frac{21}{2}\pi.$$