泰勒公式. $f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2}(x-x_0)^2$

例 6.32

设 $f(x)$ 在 $[0,1]$ 二阶可导，且 $\int_0^1 f(x)dx=0$ 则

(A) 当 $f(x)<0$ 时，$f(\frac{1}{2})<0$.

(B) $f'(x)<0$ $<$

(C) $f'(x)>0$ $<$

(D) $f''(x)>0$ $<$

[solution].

$f(x)=f(\frac{1}{2})+f'(\frac{1}{2})(x-\frac{1}{2})+\frac{f''(\xi)}{2}(x-\frac{1}{2})^2$

$0=\int_0^1 f(x)dx=f(\frac{1}{2})+\underbrace{\int_0^1 f'(\frac{1}{2})(x-\frac{1}{2})dx}_{0}+\int_0^1\frac{f''(\xi)}{2}(x-\frac{1}{2})^2dx$

$\Rightarrow f(\frac{1}{2})=-\int_0^1\frac{f''(\xi)}{2}(x-\frac{1}{2})^2dx$ $\Rightarrow$ D.

例 6.14 ☆

设 $f(x)$ 在 $[a,b]$ 二阶可导，$|f'(x)|\leq\frac{1}{2}$,

$f'(x_0)=0$. $f''(x_0)=c\neq0$, $x_0\in(a,b)$.

且满足 $x_0=f(x_0)$.

(1) $\forall x_1\in[a,b]$, $x_{n+1}=f(x_n)$ $(n=1,2\cdots)$

证明 $\lim\limits_{n\to\infty}x_n$ 存在，且 $\lim\limits_{n\to\infty}x_n=x_0$

(2) ~~求~~ 求 $\lim\limits_{n\to\infty}\frac{x_{n+1}-x_0}{(x_n-x_0)^2}$

$f(x)=\cancel{f(x_0)}\ f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2}(x-x_0)^2$

~~$x_0=f(x_0)=f(x_0)+$~~

(先斩后奏: $\cancel{\lim\limits_{n\to\infty}}$ $x_0=f(x_0)$. 再用定义法做.

$\lim\limits_{n\to\infty}|x_{n+1}-x_0|\overset{\geq0}{=}\lim\limits_{n\to\infty}|f(x_n)-f(x_0)|$

$=\lim\limits_{n\to\infty}\left|\frac{f'(\xi)|x_n-x_0|}{\cancel{x_n-x_0}}\right|$

$\leq\frac{1}{2}\lim\limits_{n\to\infty}|x_n-x_0|\cdots\leq(\frac{1}{2})^n(x_1-x_0)\to0$.

$\Rightarrow\lim\limits_{n\to\infty}|x_{n+1}-x_0|=0$. $\lim\limits_{n\to\infty}x_n=x_0$.

(2). $\lim\limits_{n\to\infty}\frac{x_{n+1}-x_0}{(x_n-x_0)^2}$ $f(x_n)=f(x_0)+\overset{0}{f'(x_0)(x-x_0)}$

$=\lim\limits_{n\to\infty}\frac{f(x_n)-f(x_0)}{(x_n-x_0)^2}$ $+\frac{f''(x_0)}{2}(x-x_0)^2+o[(x-x_0)^2]$

$=\cancel{\lim\limits_{n\to\infty}\frac{f'(x_0)}{x_n-x_0}}+\lim\limits_{n\to\infty}\frac{f''(x_0)}{2}=\frac{c}{2}$

例 6.12

设 $f(x)$ 在区间 $[-a,a]$ $(a>0)$ 上具有二阶连续导数，$f(0)=0$.

(1). 写出 $f(x)$ 带拉格朗日余项的一阶麦克劳林公式.

(2). 证明: 在 $[-a,a]$ 上至少存在一点 η, 使 $a^3f''(\eta)=3\int_{-a}^{a}f(x)dx$

(1). $f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2!}(x-x_0)^2$

$x_0=0$: $f(x)=f'(0)x+\frac{f''(\xi)}{2}x^2$

(2). $\int_{-a}^{a}f(x)dx=\underbrace{\int_{-a}^{a}f'(0)x\,dx}_{0}+\int_{-a}^{a}\frac{f''(\xi)}{2}x^2dx$

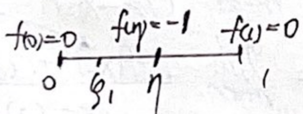
$\int_{-a}^{a}f(x)dx=\int_{-a}^{a}\frac{f''(\xi)}{2}x^2dx=\frac{f''(\xi)}{3}a^2$

$f''(\xi)a^3=3\int_{-a}^{a}f(x)dx$

例1 6.13

设$f(x)$在$[0,1]$上二阶可导，且 $f(0)=f(1)=0$，$f(x)$在$[0,1]$上最小值$=-1$. 证明：至少存在一点 $\xi\in(0,1)$，使$f''(\xi)\geq 8$.

[Solution].



~~由~~ $\cancel{\frac{f(\eta)-f(0)}{\eta-0}=\frac{-1-0}{\eta}=\frac{-1}{\eta}=f'(\xi_1)}$.

$\cancel{\frac{f(1)-f(\eta)}{1-\eta}=\frac{1}{1-\eta}=f'(\xi_2)}$. $\cancel{F(x)=f(1)-}$

$\cancel{\frac{f'(\xi_2)-f'(\xi_1)}{\xi_2-\xi_1}}$

设$x=a$处最小值

$f(x)=f(a)+\underbrace{f'(a)}(x-a)+\frac{f''(\xi)}{2}(x-a)^2$

$f(x)=-1+\frac{f''(\xi)}{2}(x-a)^2$ ~~ξ∈(0,a)~~ ξ介于a与x之间

$x=0$: $f(0)=-1+\frac{f''(\xi_1)}{2}a^2=0$ $\xi_1\in(0,a)$

$x=1$ $f(1)=-1+\frac{f''(\xi_2)}{2}(1-a)^2=0$ $\xi_2\in(a,1)$

$f''(\xi_1)=\frac{2}{a^2}$ $f''(\xi_2)=\frac{2}{(1-a)^2}$

当 $a\in(0,\frac{1}{2})$ $f''(\xi_1)\geq 8$ 当$a\in(\frac{1}{2},1)$ $f''(\xi_2)\geq 8$