第8讲 习题(1000题).

1000.8.5

$f(x)$ 以2为周期连续函数, $G(x)=2\int_0^x f(t)dt - x\int_0^2 f(t)dt$

1) $G(x+2) = 2\int_0^{x+2} f(t)dt - (x+2)\int_0^2 f(t)dt$

$\overset{t=u+2}{=} 2\int_{-2}^{x} f(u+2)du - (x+2)\int_{-2}^{0} f(u+2)du$

$= 2\int_{-2}^0 f(u)du + 2\int_0^x f(u)du - x\int_{-2}^0 f(t)dt - 2\int_0^2 f(t)dt$

$= G(x) + 2\int_{-2}^0 f(u)du - 2\int_0^2 f(t)dt = G(x)$

$G(x)$ 以2为周期

2) $G'(x) = 2f(x) - \int_0^2 f(t)dt$ $\cancel{=2f(x)}$ 是周期函数

$= \cancel{(2-x)f(x) - \int_0^2 f(t)dt}$

1000.8.7

$f(x)$ 在 $[-1,1]$ 二阶可导, $f''(x)>0$, 且 $|f(x)|\le x^2$.

$I=\int_{-1}^1 f(x)dx$, $I >< = 0$?

[solution].

$x^2=g(x)$

$f(x) = f(0) + f'(0)(x-0) + \frac{f''(\xi)}{2}(x-0)^2$

$|f(x)| = |f'(0)x| + \frac{x^2}{2}f''(\xi)$ $\cancel{> x^2 = 0+}$ $> |f'(0)x|$

$I=\int f(x)dx = \int 0 + 0 > \int f'(0)x\,dx = f'(0)\int_{-1}^1 x\,dx = 0.$

$\Rightarrow I>0.$

1000.8.11

设 $f(x)$ 在 $[0,2]$ 连续, 且 $f(x)=\int_0^1 e^{x+t} f(t)dt + x$

则 $\frac{f(0)}{f(2)}=$?

[solution]

$f(x) = e^x \underbrace{\int_0^1 e^t f(t)dt}_{a} + x$

① $a=\int_0^1 e^t f(t)dt$

$f(x) = ae^x + x$ $(0\le x\le 2)$.

② $a=\int_0^1 e^x f(t)dt = a\int_0^1 e^{2x}dx + \int_0^1 xe^x dx$

$= \frac{a}{2}(e^2-1) + (x-1)e^x\Big|_0^1 = \frac{a}{2}(e^2-1)+1 = a$

$\Rightarrow a = \frac{2}{3-e^2}$ $\quad \frac{f(0)}{f(2)} = \frac{a}{ae^2+2} = \frac{1}{3}.$

1000.8.13

m 与 n 常数, 若 $\int_0^{+\infty} \frac{x^n(1-e^{-x})}{(1+x)^m}dx$ 收敛.

则 m 与 n 范围( )

[solution].

$I=\int_0^{+\infty}\frac{x^n(1-e^{-x})}{(1+x)^m}dx = \int_0^1 \frac{x^n(1-e^{-x})}{(1+x)^m}dx + \int_1^{+\infty}\frac{x^n(1-e^{-x})}{(1+x)^m}dx$

x^n 收敛. $\frac{1}{x^{-n}}$ $\qquad \frac{0}{x^{m-n}}$ $m-n>1$

$-n > 2$ $\qquad m>n+1$

$n<-2$

• 1000.8.14

$$\lim_{n\to\infty} \ln \sqrt[n]{(1+\frac{1}{n})^2(1+\frac{2}{n})^2\cdots(1+\frac{n}{n})^2} = ?$$

[solution].

$$\lim_{n\to\infty}\frac{2\ln(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{n}{n})}{n}$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[\ln(1+\frac{1}{n})+\ln(1+\frac{2}{n})+\cdots+\ln(1+\frac{n}{n})\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln(1+\frac{i}{n}) = 2\int_0^1 \underset{u=1+x}{\underbrace{\ln(1+x)}}\,dx$$

$$=2\int_1^2 \ln(x)\,dx = 2(2\ln 2-1).$$

• 1000.8.18.

设f(x)可导且 $f(x)=x+x\int_0^1 f(x)dx + x^2\lim_{x\to0}\frac{f(x)}{x}$ 求f(x)

① $\lim_{x\to0}\frac{f(x)}{x} = \lim_{x\to0}\frac{f(x)-A}{x-0} \Rightarrow f'(0)=A$

$\int_0^1 f(x)\,dx = B$

$\Rightarrow f(x)=x+Bx+Ax^2$

② $f'(x)=1+B+2xA$

$f'(0)=A=1+B.$

$f'(x)=1+B+2x(1+B) \Rightarrow f(x)=(1+B)x+(1+B)x^2$

③ $B=\int_0^1 f(x)dx = \cancel{\int_0^1 (1+B)+2x(1+B)dx}$

$\cancel{=(1+B)+x^2(1+B)x^2\Big|_0^1 = 2+2B=B \Rightarrow B=-2}$

$$=\int_0^1 (1+B)x+(1+B)x^2\,dx=(1+B)\left(\frac{x^2}{2}+\frac{x^3}{3}\right)\Big|_0^1=\frac{5(1+B)}{6}=B.$$

$\Rightarrow B=5\quad A=6.$

$f(x)=6x+6x^2$

• 1000.8.20.

证明 $\int_0^1\frac{x\sin\frac{\pi}{2}x}{1+x}dx > \int_0^1\frac{x\cos\frac{\pi}{2}x}{1+x}dx$

构造 $\int_0^1\frac{x(\sin\frac{\pi}{2}x-\cos\frac{\pi}{2}x)}{1+x}dx.$

$\cancel{g(x)=\sin\frac{\pi}{2}x-\cos\frac{\pi}{2}x,\quad g'(x)=\frac{-\pi}{2}(+\cos\frac{\pi}{2}x + \sin\frac{\pi}{2}x)} < 0$

$=\int_0^1\frac{x\cdot\sqrt{2}\cdot\sin(\frac{\pi}{2}x-\frac{\pi}{4})}{1+x}$

令 $u=\frac{\pi}{2}x-\frac{\pi}{4}$ $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sqrt{2}\sin u\ \frac{2}{\pi}(u+\frac{\pi}{4})}{\frac{2}{\pi}(u+\frac{\pi}{4})+1}\frac{2}{\pi}dx$

$$=\frac{\sqrt{2}}{2\pi^3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{(\sin u)(3\pi+4u-2\pi)}{3\pi+4u}dx=\frac{\sqrt{2}}{2\pi^3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sin u-\sin u\frac{2\pi}{3\pi+4u}du$$

$$=\frac{-\sqrt{2}}{\pi^2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sin u}{3\pi+4u}du=\frac{-\sqrt{2}}{\pi}\int_0^{\frac{\pi}{4}}\sin u\frac{6\pi}{9\pi^2-16u^2}du$$

$-\frac{3\sqrt{2}}{8\pi}=\int_0^{\frac{\pi}{4}}\sin u\cdot\frac{1}{\frac{9\pi^2}{16}-u^2}du$

$$=\frac{-\sqrt{2}}{4\pi^2}\int_0^{\frac{\pi}{4}}\sin u\ d\ln\left|\frac{x+\frac{3}{4}\pi}{x-\frac{3}{4}\pi}\right|$$

对于 I_1, $x\to0^+$ 时 $x\in(0,\frac{1}{2})$, $(1-x)^b$ 不参与讨论.

$a>0$ 时 $\lim\limits_{x\to0^+}x^a(1-x)^b\ln x=0$. $\Rightarrow$ 不是瑕点.

$a=0$ 时 $\int_0^{\frac{1}{2}}(1-x)^b\ln x\,dx$ 与 $\int_0^{\frac{1}{2}}\ln x\,dx$ 同敛散. 收敛

$a\in(-1,0)$ 时 $\lim\limits_{x\to0^+}\dfrac{x^a\ln x}{\frac{1}{x^\varepsilon}}=\lim\limits_{x\to0^+}x^{a+\varepsilon}\ln x$ $\Rightarrow$ 收敛.

取 $0<-a<\varepsilon<1$ $\Rightarrow a+\varepsilon>0$

$a\le-1$ 时 $\lim\limits_{x\to0^+}\dfrac{x^a\ln x}{\frac{1}{x}}=\lim\limits_{x\to0^+}\underset{\infty}{x^{a+1}}\underset{\infty}{\ln x}=\infty$.

$\Rightarrow \underline{a>-1}$

对于 I_2, $x\to1^-$ 时 $x\in(\frac{1}{2},1)$ x^a 不参与讨论.

等价于讨论. $\int_{\frac{1}{2}}^1(1-x)^b\ln x\,dx$.

$\lim\limits_{x\to1}\ln(x)=\lim\limits_{x\to1}\ln(x-1+1)=\lim\limits_{x\to1}x-1$

P_2 讨论 $\lim\limits_{x\to1}(1-x)^{b+1}$ $\to 0<-(b+1)<1$

$b+1>-1$

$\Rightarrow \underline{b>-2}$

1000.8t.12

判别 $\int_1^{+\infty}[\ln(1+\frac{1}{x})-\frac{1}{1+x}]dx$ 敛散性.

$x\to0$ 时 $0\le\ln(1+\frac{1}{x})-\frac{1}{1+x}\le\frac{1}{x}-\frac{1}{1+x}=\frac{1}{x(1+x)}\le\frac{1}{x^2}$

$\int_1^{+\infty}\frac{1}{x^2}dx$ 收敛. 故 $\int_1^{+\infty}[\ln(1+\frac{1}{x})-\frac{1}{1+x}]dx$ 收敛

1000.8t.7

极限 $\lim\limits_{n\to\infty}\dfrac{(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n})(1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}})}{(n+1)(n+2)}$

$=\lim\limits_{n\to\infty}\dfrac{\sum_{i=1}^n\sqrt{i}\cdot\sum_{i=1}^n\frac{1}{\sqrt{i}}}{(n+1)(n+2)}$

~~$=\lim\limits_{n\to\infty}\sum_{i=1}^n i^{\frac{1}{2}}\cdot\sum_{i=1}^n i^{-\frac{1}{2}}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$~~

$=\lim\limits_{n\to\infty}\dfrac{n^2}{(n+1)(n+2)}\cdot\lim\limits_{n\to\infty}\sum_{i=1}^n\frac{1}{n}\left(\frac{i}{n}\right)^{\frac{1}{2}}\cdot\lim\limits_{n\to\infty}\frac{1}{n}\left(\frac{i}{n}\right)^{-\frac{1}{2}}$

$=1\cdot\int_0^1\sqrt{x}\,dx\cdot\int_0^1\frac{1}{\sqrt{x}}dx=\frac{4}{3}$

1000.8t.

$\lim\limits_{n\to\infty}\sum_{i=1}^n\dfrac{1}{n+\frac{(i-1)^2+1}{n}}$

$=\lim\limits_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{n}\dfrac{1}{1+\frac{i^2+1}{n^2}}=A$

$\lim\limits_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{n}\dfrac{1}{1+\left(\frac{i+1}{n}\right)^2}<A<\lim\limits_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{n}\dfrac{1}{1+\left(\frac{i}{n}\right)^2}$

$\qquad\| \qquad\qquad\qquad\qquad \|$

$\int_0^1\frac{1}{1+x^2}dx \qquad \int_0^1\frac{1}{1+x^2}dx$

$A=\int_0^1\frac{1}{1+x^2}dx=\arctan x\big|_0^1=\frac{\pi}{4}$

[solution].

$$左-右=\int_0^1 \frac{x}{1+x}(\sin\frac{\pi}{2}x-\cos\frac{\pi}{2}x)dx$$

$$=\int_0^{\frac{1}{2}} \frac{x}{1+x}(\sin\frac{\pi}{2}x-\cos\frac{\pi}{2}x)dx+\int_{\frac{1}{2}}^1 \frac{x}{1+x}(\sin\frac{\pi}{2}x-\cos\frac{\pi}{2}x)dx$$

$$\xrightarrow[令x=1-t]{第二项} \int_0^{\frac{1}{2}} \frac{x}{1+x}(\sin\frac{\pi}{2}x-\cos\frac{\pi}{2}x)dx-\int_{\frac{1}{2}}^0 \frac{1-t}{1+(1-t)}(\cos\frac{\pi}{2}t-\sin\frac{\pi}{2}t)dt$$

$$=\int_0^{\frac{1}{2}}\left(\frac{x}{1+x}-\frac{1-x}{2-x}\right)(\sin\frac{\pi}{2}x-\cos\frac{\pi}{2}x)dx$$

$$=\int_0^{\frac{1}{2}}\frac{2x-1}{(1+x)(2-x)}(\sin\frac{\pi}{2}x-\cos\frac{\pi}{2}x)dx$$

$(0,\frac{1}{2})$ $2x-1<0$. $(1+x)(2-x)>0$. $\sin\frac{\pi}{2}x-\cos\frac{\pi}{2}x<0$.

$\therefore$ 不等式成立

1000.8t.2

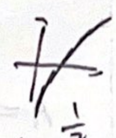
反常积分 $I=\int_1^2\left[\frac{1}{x\ln^2 x}-\frac{k}{(x-1)^2}\right]dx$.

何时收敛?

[solution].

$$I=\int_1^2\left[\frac{1}{x\ln^2 x}-\frac{k}{(x-1)^2}\right]dx=\left[\frac{-1}{\ln x}+\frac{1}{x-1}\right]\Bigg|_1^2$$

$$=\left(\frac{-1}{\ln 2}+1\right)-\lim_{x\to1^+}\left(\frac{-1}{\ln x}+\frac{1}{x-1}\right)$$



$$=1-\frac{1}{\ln 2}-\lim_{x\to1^+}\frac{\ln x-(x-1)}{(x-1)\ln x}=1-\frac{1}{\ln 2}-\lim_{x\to1^+}\frac{\frac{1}{x}-1}{\ln x+\frac{x-1}{x}}$$

$$=\frac{3}{2}-\frac{1}{\ln 2}.$$

$\therefore k=1$时收敛

1000.8t.3

$$\int_0^2\frac{dx}{\sqrt{x^2+x^3}}=\int_0^2\frac{dx}{x\sqrt{1+x}}.$$

$$\lim_{x\to0}\frac{x}{x\sqrt{1+x}}=1 \quad \therefore x\sim x\sqrt{1+x}$$

$\int_0^2\frac{1}{x}dx$ 发散 $\to$ $\int_0^2\frac{1}{x\sqrt{1+x}}dx$ 发散.

1000.8t.5

$$f(x)=\frac{\ln(1+\sin\frac{1}{x^a})}{x^b\ln\cos\frac{1}{x}} \quad 1<x<+\infty. \quad a>0$$

若 $\int_0^{+\infty}f(x)dx$ 收敛, 则. ______

[solution].

$\ln(1+\sin\frac{1}{x^a})\sim\sin\frac{1}{x^a}\sim x^{-a}$.

$\ln\cos\frac{1}{x}=\ln(\cos\frac{1}{x}+1-1)\sim\cos\frac{1}{x}+1\sim-\frac{1}{2}\frac{1}{x^2}=-\frac{1}{2}x^{-2}$

原 $\sim-\frac{1}{2}x^{-a-b}x^{+2}$ $\cancel{-(a+b-2)>1}$

$\cancel{a+b<-3}$

$\frac{1}{x^{a+b-2}}$ $a+b-2>1$ $a+b>3$.

1000.8t.6.

$\int_0^1 x^a(1-x)^b\ln x\,dx$ 收敛. , a,b 范围

[solution] $\int_0^1 x^a(1-x)^b dx=\int_0^{\frac{1}{2}}x^a(1-x)^b dx+\int_{\frac{1}{2}}^1 x^a(1-x)^b dx$

$I_1 + I_2$