二、复合函数求导法.

1. 链式求导规则

eg $z <^{u}_{v} \bowtie ^{x}_{y}$ $\quad \frac{\partial z}{\partial x} = \left(\frac{\partial z}{\partial u}\ \ \frac{\partial z}{\partial v}\right)\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}\begin{bmatrix}dx\\ dy\end{bmatrix}$

2. 全导数.

eg $z <^{u}_{v} > x$ 一个.

$$\frac{dz}{dx} = \left(\frac{\partial z}{\partial u}\ \ \frac{\partial z}{\partial v}\right)\begin{bmatrix}\frac{du}{dx}\\ \frac{dv}{dx}\end{bmatrix}$$

3. 全微分形式不变性

$$dz = \frac{\partial z}{\partial x}\boxed{dx} + \frac{\partial z}{\partial y}\boxed{dy} = \frac{\partial z}{\partial u}\boxed{du} + \frac{\partial z}{\partial v}\boxed{dv}$$

例1 13.11

设 $z = f(\underbrace{x^2-y^2}_{1}, \underbrace{e^{xy}}_{2})$ 其中 f 具有连续二阶偏导,

$\frac{\partial z}{\partial x} = f_1'\cdot 2x + f_2'\cdot ye^{xy}$ $\quad z <^{x^2-y^2}_{e^{xy}} \bowtie ^{x}_{y}$

$\frac{\partial z}{\partial y} = f_1'\cdot(-2y) + f_2'\cdot xe^{xy}$

无论z对谁求导，求几阶导，求导后的新函数有与原来的相同结构

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial(\frac{\partial z}{\partial x})}{\partial y} = 2x\cdot\left(f_{11}''(-2y) + f_{12}''\cdot e^{xy}\cdot x\right)$$

$$(xye^{xy} + e^{xy})f_2' + xe^{xy}\left(f_{21}''(-2y) + f_{22}''e^{xy}\cdot x\right)$$

习 13.6 ★★★.

已知 $f(u,v)$ 具有二阶连续偏导数, $f(1,1)=2$ 是 $f(u,v)$ 极值, $z=f[x+y, f(x,y)]$ 求 $\frac{\partial^2 z}{\partial x\partial y}\Big|_{(1,1)}$.

不是通复合函数结构:

$z = f(x+y, f(x,y)) <^{1\ x+y}_{2\ f(x,y)} \bowtie ^{x}_{y}$ ✗

$z = f(x+y, f(x,y)) <^{1\ x+y}_{2\ f(x,y) <^{1\ x}_{2\ y}}$ $\to$ x, y ✓

$$\frac{\partial z}{\partial x} = f_1'(x+y, f(x,y))\cdot 1 + f_2'(x+y, f(x,y))\cdot f_1'(x,y)\cdot 1$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial y} = {} & f_{11}''(x+y, f(x,y))\cdot 1 + f_{12}''(x+y, f(x,y))\cdot f_2'(x,y)\cdot 1\\ & + f_1'(x,y)\left[f_{21}''(x+y, f(x,y))\cdot 1 + f_{22}''(x+y, f(x,y))\cdot f_2'(x,y)\cdot 1\right]\\ & + f_2'(x+y, f(x,y))\cdot\left[\cancel{f_{11}''(x,y)\cdot 1} + f_{12}''(x,y)\cdot 1\right]\end{aligned}$$

三、隐函数求导法.

1. 一个方程的情形

$F_z'(P_0) \neq 0$ $\to$ $z = z(x,y)$ 隐函数存在定理

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x'}{F_z'} \qquad \frac{\partial z}{\partial y} = -\frac{F_y'}{F_z'}}$$

注意 x与z是中间变量!(多元函数): 互不牵扯!

例13.7.

已知 $f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0.\end{cases}$

证明 $f(x,y)$ 在 $(0,0)$ 处偏导数不连续，但 $f(x,y)$ 在 $(0,0)$ 处可微.

①. $\cancel{\Delta z=}\ \Delta f=f(0+\Delta x,0+\Delta y)-f(0,0)$

$$=(\Delta x^2+\Delta y^2)\sin\frac{1}{\Delta x^2+\Delta y^2}-0$$

② $\Delta z=\cancel{\Delta f}\ A\Delta x+B\Delta y$

$$A=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-0}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x^2\sin\frac{1}{\Delta x^2}-0}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\Delta x\cdot\sin\frac{1}{\Delta x^2}=0\quad(\text{无穷小·有界}).$$

$$B=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-0}{\Delta y}=0.$$

$\Rightarrow \Delta z=0.$

③. $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{\overbrace{(\Delta x^2+\Delta y^2)\sin\frac{1}{\Delta x^2+\Delta y^2}}^{\Delta z-\Delta f}}{\underbrace{\sqrt{(\Delta x)^2+(\Delta y)^2}}_{\rho}}=0.$ 可微.

$A=\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{|x|}\ \dfrac{|x|}{x}=0$ 同理 $B=0$

(C). $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)}{\boxed{x^2+y^2}}=0$

$\underbrace{f(x,y)-f(0,0)}_{\Delta f}-\underbrace{0}_{A\Delta x+B\Delta y\ ?}=0$

取 $y=0$，$x\to 0$ 路径

例13.4.

$$\lim_{\substack{x\to 0\\ y\to 1}}\frac{f(x,y)-2x+y-2}{\sqrt{x^2+(y-1)^2}}=0.$$

则 $dz\big|_{(0,1)}=?$

② $f(0,1)=1$

①. $f(x,y)-\underbrace{2x+y-2}=o\left(\sqrt{(x-0)^2+(y-1)^2}\right)=o(\rho)$

③ $f(x,y)-f(0,1)-(A\Delta x-B\Delta y)=o(\rho).$

$=\underbrace{f(x,y)-f(0,1)}_{\Delta z}-\underbrace{(2(x-0)+(-1)(y-1))}=o(\rho).$

$\therefore dz\big|_{(0,1)}=2dx-dy$

⑤ 偏导数连续.

step1. 定义法求 $f_x'(x_0,y_0)$，$f_y'(x_0,y_0)$.

step2. 公式法求 $f_x'(x,y)$，$f_y'(x,y)$.

step3. $\lim\limits_{\substack{x\to x_0\\ y\to y_0}}f_x'(x,y)$，$\lim\limits_{\substack{x\to x_0\\ y\to y_0}}f_y'(x,y)$ step4

例13.8

二元函数 $f(x,y)$ 在 $(0,0)$ 可微的充分条件.

偏导数连续 $\longrightarrow$ 可微

(A). $\lim\limits_{(x,y)\to(0,0)}[f(x,y)-f(0,0)]=0.$

连续定义 连续 $\not\Longrightarrow$ 可微

用定义 ☆

(B). $\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x}=0.$

且 $\lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y}=0$ 偏导定义

偏导存在 $\not\Longrightarrow$ 可微

(D). $\lim\limits_{x\to 0}[f_x'(x,0)-f_x'(0,0)]=0$ 且 $\lim\limits_{y\to 0}[f_y'(0,y)-f_y'(0,0)]=0$

表达式错 — $\lim\limits_{\substack{x\to 0\\ y\to 0}}f_x'(x,y)=f_x'(0,0)$ 且 $\lim\limits_{\substack{x\to 0\\ y\to 0}}f_y'(x,y)=f_y'(0,0).$

例13.12

方程 $x+y^2+\sin(xy)=0$. 则在(0,0)某邻域内，方程

令 $F(x,y)=x+y^2+\sin(xy)$

则 $F_x'=1+y\cos(xy)$

$F_x'(0,0)=1\neq 0 \Rightarrow \frac{\partial F}{\partial y}=-\frac{F_y'}{F_x'}$ ✓ y ✓

$F_y'=2y+x\sin(xy)$

$F_y'(0,0)=0 \Rightarrow \frac{\partial F}{\partial x}=-\frac{F_x'}{F_y'}$ × × x

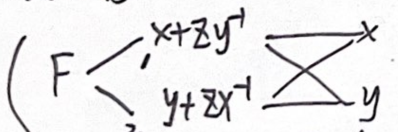
只能确定具有连续导数的隐函数 $y=y(x)$

例13.17 ☆☆☆

设 $z=z(x,y)$ 由 $F(x+zy^{-1}, y+zx^{-1})=0$ 所确定，其中F具有连续偏导，且 $xF_1'+yF_2'\neq 0$.

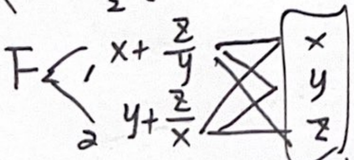
证明 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy$

[Solution]



z 怎么办

用公式法



中间变量

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}=-\frac{F_1'\cdot 1+F_2'\cdot\left(-\frac{z}{x^2}\right)}{F_1'\cdot\left(\frac{1}{y}\right)+F_2'\cdot\left(\frac{1}{x}\right)}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}=-\frac{F_1'\left(-\frac{z}{y^2}\right)+F_2'\cdot 1}{F_1'\left(\frac{1}{y}\right)+F_2'\left(\frac{1}{x}\right)}$$

$$\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy$$

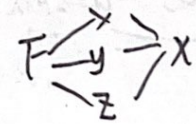
2. 方程组情形 (只听分析，不推)(只背[结论])

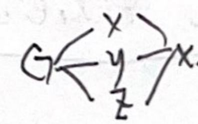
$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$$

↓ 简化

有 $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ 当 $\frac{\partial(F,G)}{\partial(y,z)}\neq 0$ 可确定 $\begin{cases} y=y(x) \\ z=z(x) \end{cases}$

$\Rightarrow$





且有 $\frac{dy}{dx}=-\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(y,z)}}$，$\frac{dz}{dx}=-\frac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}}$

- 例 13.13.

设 $y=y(x)$, $z=z(x)$ 是方程 $z=xf(x+y)$, $F(x,y,z)=0$.
所确定 且 $F_y'+xf'F_z'\neq 0$. 求 $\frac{dz}{dx}$

[solution].

$$\begin{cases} F(x,y,z)=0. \\ G(x,y,z)=xf(x+y)-z=0. \end{cases}$$

$$\frac{dz}{dx}=-\frac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}}=-\frac{\begin{vmatrix}\frac{\partial F}{\partial y} & \frac{\partial F}{\partial x}\\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial x}\end{vmatrix}}{\begin{vmatrix}\frac{\partial F}{\partial y} & \frac{\partial F}{\partial z}\\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z}\end{vmatrix}}=-\frac{\begin{vmatrix}F_y' & F_x'\\ xf' & f+xf'\end{vmatrix}}{\begin{vmatrix}F_y' & F_z'\\ xf' & -1\end{vmatrix}}$$

$$=-\frac{F_y'(f+xf')-xf'F_x'}{-F_y'-xf'F_z'}$$

- 习 13.8.

设 $u=f(x,y,z)$, $\varphi(x^2,e^y,z)=0$, $y=\sin x$. $\frac{\partial\varphi}{\partial z}\neq 0$.

$\frac{du}{dx}=?$

构造: $F(x,z,u)=f(x,\sin x,z)-u=0$.

$G(x,z,u)=\varphi(x^2,e^{\sin x},z)=0$.

$$\frac{du}{dx}=-\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(u,z)}}=-\frac{\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial z}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z}\end{vmatrix}}{\begin{vmatrix}\frac{\partial F}{\partial u} & \frac{\partial F}{\partial z}\\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial z}\end{vmatrix}}$$

- 例 13.16.

$F_x'\cdot F_y'\cdot F_z'\neq 0$. $F(x,y,z)=0 \rightarrow z=z(x,y)$, $y=y(z,x)$, $x=x(y,z)$

则 $\cancel{\frac{\partial z}{\partial x}}\ \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial x}\cdot\frac{\partial x}{\partial z}=?$ ← 这个不能约分

$$\frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'},\quad \frac{\partial y}{\partial x}=-\frac{F_x'}{F_y'},\quad \frac{\partial x}{\partial z}=-\frac{F_z'}{F_x'}$$

$\Rightarrow=-1.$