§11

例 11.1 设 $f(x)$ 是连续的偶函数，且是以 T 为周期的周期函数.

(1). 证明：$\int_0^{nT} xf(x)dx = \frac{n^2T}{2}\int_0^T f(x)dx \quad (n=1,2,3,\cdots)$.

(2). 利用(1)计算 $I=\int_0^{n\pi} x|\sin x| \, dx$

[分析].

(1) $\begin{cases} f(x)=f(-x) \\ f(x+T)=f(x). \end{cases}$

$\int_0^{nT} xf(x)dx \xlongequal{\text{区间再现}} \int_0^{nT}(nT-x)f(nT-x)dx$

$=\int_0^{nT}(nT-x)f(x)dx$

$=nT\int_0^{nT}f(x)dx-\int_0^{nT}xf(x)dx$

$\int_0^{nT}xf(x)dx=\frac{nT}{2}\int_0^{nT}f(x)dx = \frac{n^2T}{2}\int_0^T f(x)dx$

(2). $I=\int_0^{n\pi}x|\sin x|\,dx$

$f(x)=|\sin x|$. 偶, $T=\pi$.

$I=\frac{\pi n^2}{2}\int_0^{\pi}|\sin x|\,dx = n^2\pi$

例 11.2

$f(x), g(x)$ 在 $[-a,a]$ 上连续，$g(x)$ 偶，

$f(x)+f(-x)=A$ (A 为常数).

(1). 证明 $\int_{-a}^{a} f(x)g(x)\,dx = A\int_0^a g(x)dx$.

(2). 利用(1)，证 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x| \arctan e^x \, dx$

[solution].

$\int_{-a}^{a}f(x)g(x)dx=\int_{-a}^{a}f(-x)g(-x)dx$

$=\int_{-a}^{a}[A-f(x)]g(x)dx$

$=A\int_{-a}^{a}g(x)dx-\int_{-a}^{a}f(x)g(x)dx$

$\int_{-a}^{a}f(x)g(x)dx=\frac{A}{2}\int_{-a}^{a}g(x)dx = A\int_0^a g(x)dx.$

(2). $\arctan e^x+\arctan e^{-x}=A$

$\arctan 1+\arctan 1 = A = \frac{\pi}{2}$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x| \arctan e^x$

$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}|\sin x|\,dx = \frac{\pi}{2}$

例 11.3 ☆

设 $f(x)$ 在区间 $[-1,1]$ 连续，且 $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} f(x)\tan x\,dx = 0$.

证明在区间 $(-1,1)$ 内至少存在互异两点 ξ_1, ξ_2.

使得 $f(\xi_1) = f(\xi_2) = 0$.

[solution]

令 $\boxed{F(x) = \int_{-1}^{x} f(t)\,dt}$ $F(1)=0$ $F(-1)=0$.

step1. 分部积分法.

$\int_{-1}^{1} f(x)\tan x\,dx = \int_{-1}^{1} \tan x\,dF(x)$.

$= \tan x\,F(x)\Big|_{-1}^{1} - \int_{-1}^{1} F(x)\cdot\sec^2 x\,dx = -\int_{-1}^{1} F(x)\sec^2 x\,dx$.

☆ step2. $0 = \int_{-1}^{1} F(x)\sec^2 x\,dx$.

如果 $F(x)$ 恒 >0. $\int_{-1}^{1} F(x)\sec^2 x\,dx > 0$ 矛盾

$F(x)$ 恒 <0. <0 矛盾.

$\therefore F(x)$ 在 $(-1,1)$ 有正有负 $\longrightarrow$ $F(\xi)=0$.

step3 $F(-1) = F(\xi) = F(1) = 0$ 罗尔

$f(\xi_1) = f(\xi_2) = 0$

例 11.4

设 $f(x)$, $g(x)$ 在 $[a,b]$ 上连续且 $g(x)$ 不变号.

证明 $\xi\in[a,b]$: $\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$

推广积分中 $\Rightarrow$ $\int_a^b f(x)\cdot 1\,dx = f(\xi)(b-a)$

[solution].

$g(x)=0$ 时 左=右=0.

$g(x)\neq 0$ 时. 设 $g(x)>0$

$m \le f(x) \le M$.

$\times g(x)$

$\Rightarrow$ $m g(x) \le f(x)g(x) \le M g(x)$

$\int_a^b$

$\Rightarrow m\int_a^b g(x)\,dx = \int_a^b m g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le \int_a^b M g(x)\,dx$

$= M\int_a^b g(x)\,dx$

由于 $\int_a^b g(x)\,dx > 0$ $= \mu$

得 $m \le \dfrac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \le M$

介值定理: $f(\xi) = \mu$

$\Rightarrow$ $\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$.

例11.5 ☆ 考.

$f(x), g(x)$ 在 $[a,b]$ 连续，且 $g(x)$ 在 $[a,b]$ 不变号.

证明 $\xi \in (a,b)$　$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$.

开区间成立，闭区间必然成立

[solution].

$g(x) \equiv 0$. 成立

$g(x) \not\equiv 0$.　$g(x)$ 不变号，不妨设 $g(x) > 0$.

$F(x) = \int_a^x f(t)g(t)dt$　$G(x) = \int_a^x g(t)dt$

在 $[a,b]$ 上用柯西中值定理

$$\frac{\int_a^b f(x)g(x)dx - 0}{\int_a^b g(x)dx - 0} = \frac{f(\xi)g(\xi)}{g(\xi)} = f(\xi)$$

$$\Rightarrow \int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$$

例11.6

设 $f(x)$ 在 $[0,1]$ 可导. $0 \le f'(x) \le 1$ 且 $f(0)=0$.

证明 $[\int_0^1 f(x)dx]^2 \ge \int_0^1 f^3(x)dx$

(solution)

$$F(x) = [\int_0^x f(t)dt]^2 - \int_0^x f^3(t)dt$$

$$F'(x) = (2\int_0^x f(t)dt)\,f(x) - f^3(x)$$

$$= f(x)[\underline{2\int_0^x f(t)dt - f^2(x)}]$$

$$G(x) = 2\int_0^x f(t)dt - f^2(x)$$

$$G'(x) = 2f(x) - 2f(x)f'(x) = 2f(x)(1-f'(x)) \ge 0.$$

$G(0) = 0-0=0$　$G(x) \ge 0$.

$\Rightarrow F'(x) \ge 0$.　$F(0)=0$.

$\therefore F(x) \ge 0 \Rightarrow [\int_0^1 f(x)dx]^2 \ge \int_0^1 f^3(t)dt$

例11.7

设 $f(x)$ 在 $[a,b]$ 单增且连续,

证明 $\int_a^b x f(x)dx \ge \frac{a+b}{2}\int_a^b f(x)dx$

(solution)

$$F(x) = \int_a^x t f(t)dt - \frac{a+x}{2}\int_a^x f(t)dt$$

$$F'(x) = x f(x) - (\frac{1}{2}\int_a^x f(t)dt + \frac{a+x}{2}f(x))$$

$$= -(\frac{a-x}{2}f(x) + \frac{1}{2}\int_a^x f(t)dt)$$

$$= -\frac{1}{2}\int_a^x \underline{f(t) - f(x)}\,dt \quad > 0.$$

$F(a) = 0.$　$\Rightarrow \int_a^b x f(x) \ge \frac{a+b}{2}\int_a^b f(x)dx.$

$F(x) \ge 0$

(2) 处理被积函数.

例11.8.

设f(x)在[a,b]连续，且∀t∈[0,1]，∀$x_1,x_2\in[a,b]$

满足：$f[tx_1+(1-t)x_2]\leq tf(x_1)+(1-t)f(x_2)$

证明：$\boxed{f(\frac{a+b}{2})\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}}$

☆[分析] 令 $x=ta+(1-t)b$

$$\Rightarrow\int_a^b f(x)dx=\int_1^0 f(ta+(1-t)b)\cdot(a-b)dt$$

$$=(b-a)\int_0^1 f(ta+(1-t)b)\,dt$$

$$=(b-a)\frac{f(a)+f(b)}{2}\quad\leq(b-a)\int_0^1[tf(a)+(1-t)f(b)]dt$$

$$\Rightarrow\int_a^b f(x)dx=\int_a^{\frac{a+b}{2}}(f(a+b-x)+f(x))dx$$

由 $\int\frac{f(x_1)+f(x_2)}{2}\geq\int f(\frac{x_1+x_2}{2})$

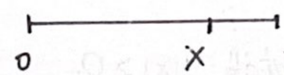
$$\therefore\ \cdots\geq\int_a^{\frac{a+b}{2}}f(\frac{a+b}{2})dx=f(\frac{a+b}{2})\cdot\frac{b-a}{2}$$

例11.9 [用拉格朗日中值定理]

设f(x)在[0,2]连续，在(0,2)可导，$f(0)=f(2)=1$.

$|f'(x)|\leq1$ 证明 $\int_0^2 f(x)dx<3$.

[solution].



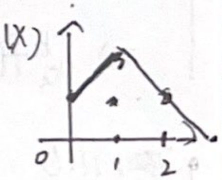
$$\begin{cases}f(x)-f(0)=f'(\xi_1)\cdot x\\ f(2)-f(x)=f'(\xi_2)(2-x)\end{cases}$$

$$\begin{cases}f(x)=1+f'(\xi_1)\cdot x\leq1+x\\ f(x)=1-f'(\xi_2)(2-x)\leq3-x\end{cases}$$

$-1\leq f'(\xi_2)\leq1$

$2-x\geq f'(\xi_2)(x-2)\geq x-2$，$x-2<0$

令 $g(x)=\begin{cases}1+x, & 0\leq x\leq1\\ 3-x, & 1\leq x\leq2\end{cases}$ 则 $f(x)\leq g(x)$



则 $f(x)\leq g(x)$

$$\int_0^2 f(x)dx\leq\int_0^2 g(x)dx=3$$

注(1). 证明 $f(x)<g(x)$. 其中 $g(x)=\begin{cases}x+1, & 0\leq x\leq1\\ 3-x, & 1<x\leq2\end{cases}$

(2). $\int_0^2 f(x)dx<3$

例 11.10. [用泰勒]

$f(x)$ 二阶可导，且 $f''(x) \geq 0$，$u(t)$ 为任一连续函数，$a>0$，证明 $\frac{1}{a}\int_0^a f[u(t)]\,dt \geq f[\frac{1}{a}\int_0^a u(t)\,dt]$

$\rightarrow \frac{1}{a}\int_0^a f(u)\,dt \geq f(\bar{u})$

泰勒：$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2}(x-x_0)^2$

$\geq f(x_0) + f'(x_0)(x-x_0)$

取 $x_0 = \frac{1}{a}\int_0^a u(t)\,dt$，$x = u(t)$

$f[u(t)] \geq f[\underbrace{\frac{1}{a}\int_0^a u(t)\,dt}_{常数}] + f'(x_0)(u(t)-x_0)$

两端从0至a积分

$\int_0^a f[u(t)]\,dt \geq a f[\frac{1}{a}\int_0^a u(t)\,dt] + f'(x_0)[\int_0^a u(t)\,dt - a x_0]$

$= a f[\frac{1}{a}\int_0^a u(t)\,dt]$

$\therefore \frac{1}{a}\int_0^a f[u(t)]\,dt \geq f[\frac{1}{a}\int_0^a u(t)\,dt]$

① $f'(x_0)=0$.

② $\int_0^2 (x-1)\,dx = 0$.

③ $\int_0^a (u-\bar{u})\,dt = 0$

例 11.11 [用放缩法].

例 11.11 设 $f(x) = \int_x^{x+1} \sin e^t\,dt$

(1). $f(x) = \frac{\cos e^x}{e^x} - \frac{\cos e^{x+1}}{e^{x+1}} - \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\,du$

(2). $e^x|f(x)| \leq 2$

[solution]

① 令 $e^t = u$. $f(x) = \int_{e^x}^{e^{x+1}} \frac{1}{u}\sin u\,du = \int_{e^x}^{e^{x+1}} \frac{-1}{u}\,d\cos u$

$t = \ln u$. $dt = \frac{1}{u}du$

② 分部积分法

$f(x) = -\frac{1}{u}\cos u\Big|_{e^x}^{e^{x+1}} - \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\,du$

$= \frac{\cos e^x}{e^x} - \frac{\cos e^{x+1}}{e^{x+1}} - \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\,du$

③ $|f(x)| \leq \left|\frac{\cos e^x}{e^x}\right| + \left|\frac{\cos e^{x+1}}{e^{x+1}}\right| + \left|\int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\,du\right|$

$\leq \left|\frac{1}{e^x}\right| + \left|\frac{1}{e^{x+1}}\right| + \int_{e^x}^{e^{x+1}} \frac{\cancel{\cos u}\ 1}{u^2}\,du$

$e^x|f(x)| \leq 1 + \frac{1}{e} + \cancel{e}\, e^x \left[-\frac{1}{u}\right]_{e^x}^{e^{x+1}} = 2$

例11.12 [用分部积分法]

设 $f(x)=\int_x^{x+1}\sin t^2\,dt$ 证明：当 $x>0$ 时，$|f(x)|\le\frac{1}{x}$

(1). $f(x)=\frac{1}{2}\left[\frac{\cos x^2}{x}-\frac{\cos(x+1)^2}{x+1}\right]-\frac{1}{2}\int_x^{x+1}\frac{\cos t^2}{t^2}dt$

(2). $|f(x)|\le\frac{1}{x}$

(1) $f(x)=\int_x^{x+1}\sin t^2\,dt$

$$=\int_x^{x+1}\frac{\sin t^2}{2t}\cdot 2t\,dt=\int_x^{x+1}\frac{\sin t^2}{2t}\,dt^2$$

$$=\int_x^{x+1}\frac{d\cos t^2}{2t}=\frac{\cos t^2}{2t}\Big|_x^{x+1}-\frac{1}{2}\int_x^{x+1}\frac{\cos t^2}{t^2}dt$$

(2)

$$f(x)=\frac{1}{2}\left[\frac{\cos x^2}{x}-\frac{\cos(x+1)^2}{x+1}\right]-\frac{1}{2}\int_x^{x+1}\frac{\cos t^2}{t^2}dt$$

$$|f(x)|\le\frac{1}{2}\left[\frac{1}{x}\right]+\frac{1}{2}\left[\frac{1}{x+1}\right]+\frac{1}{2}\int_x^{x+1}\frac{dt}{t^2}$$

$$=\frac{1}{2}\frac{1}{x}+\frac{1}{2}\frac{1}{x+1}+\left[-\frac{1}{t}\right]_x^{x+1}=\frac{1}{x}$$

例11.13 ☆(每次未出3) [换元法].

设 $|f(x)|\le\pi$，$f'(x)\ge m>0$. $(a\le x\le b)$

证明 $\left|\int_a^b\sin f(x)\,dx\right|\le\frac{2}{m}$

令 $f(x)=t$ 则 $x=g(t)$ $\quad\left(f'(x)\cdot g'(t)=1\right)$

$$\left|\int_a^b\sin f(x)\,dx\right|=\left|\int_{f(a)}^{f(b)}\sin t\cdot g'(t)\,dt\right|$$

$f'_x\ge m>0.\Rightarrow 0<g'(t)=\frac{1}{f'_x}\le\frac{1}{m}$ ☆

$$\le\left|\int_{f(a)}^{f(b)}\sin t\cdot\frac{1}{m}\,dt\right|\le\int_{f(a)}^{f(b)}\frac{|\sin t|}{m}dt$$

$$\le\frac{1}{m}\int_0^{\pi}\sin t\,dt.$$

$$=\frac{2}{m}$$

$f(a)$、$f(b)$ 最后为让 $\int_a^b\sin t\,dt$ 取一技 $(S=2)$

例 11.14 [用夹逼准则].

例1 求 $\lim\limits_{n\to\infty}\int_0^{\frac{\pi}{4}}\tan^n x\,dx$

→ (1). 设 $f(n)=\int_0^{\frac{\pi}{4}}\tan^n x\,dx\ (n>2)$.

先证明 $\frac{n}{2(1+n)}\le nf(n)\le\frac{n}{2(n-1)}$

再求 $\lim\limits_{n\to\infty} n\int_0^{\frac{\pi}{4}}\tan^n x\,dx$

☆联 $\tan^n x$ 和 $\tan^{n+2}x$

$\tan^n x+\tan^{n+2}x = \tan^n x(1+\tan^2 x)$

$\Rightarrow f(n)+f(n+2)=\int_0^{\frac{\pi}{4}}\tan^n x\,d\tan x=\frac{\tan^{n+1}x}{1+n}\Big|_0^{\frac{\pi}{4}}=\frac{1}{n+1}$

$0\le x\le\frac{\pi}{4}$ 时 $\tan^{n+2}x\le\tan^n x\le\tan^{n-2}x$

$\therefore f(n+2)\le f(n)\le f(n-2)$.

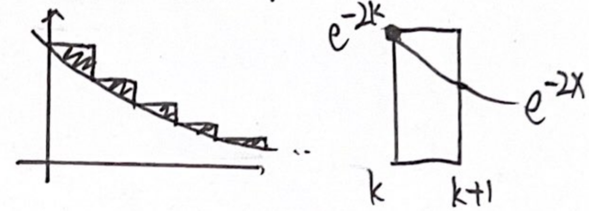
$\frac{1}{n+1}=f(n)+f(n+2)\le 2f(n)\le f(n-2)+f(n)=\frac{1}{n-1}$

故有 $\frac{n}{2(n+1)}\le nf(n)\le\frac{n}{2(n-1)}$.

夹逼 $\lim\limits_{n\to\infty}nf(n)=\frac{1}{2}$

例 11.15 ~~[illegible]~~.

当 $x>0$ 时 在 $y=e^{-2x}$ 上面作一个台阶曲线, 台阶宽度皆为1. 求S.



$S_k=e^{-2k}-\int_k^{k+1}e^{-2x}dx=(\frac{1}{2}+\frac{1}{2e^2})e^{-2k}$

$\sum\limits_{k=0}^{n}(\frac{1}{2}+\frac{1}{2e^2})e^{-2k}=\frac{e^2+1}{2(e^2-1)}$

习题都重要.