1. 多元函数极最值

① 泰勒公式

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+R_2$$

$$f(x,y)=f(x_0,y_0)+[f_x', f_y']_{x_0}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}+\frac{1}{2!}(\Delta x\ \Delta y)\begin{bmatrix}f_{xx}'' & f_{xy}''\\ f_{yx}'' & f_{yy}''\end{bmatrix}_{x_0}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}+R_2$$

$$=\Delta x^2 f_{xx}''+\Delta y^2 f_{yy}''+2\Delta x\Delta y f_{xy}''\Big|_{x_0}$$

例 13.18.

二元函数 $f(x,y)=x^y$ 在 $(e,0)$ 处附 二阶泰勒展开式.

$$f(x,y)=f(e,0)+[f_x'(e,0)\ \ f_y'(e,0)]\begin{bmatrix}x-e\\ y-0\end{bmatrix}$$

$$+\frac{1}{2}\left[f_{xx}''(e,0)\cdot(x-e)^2+f_{yy}''(y-0)^2+2f_{xy}''(e,0)(x-e)(y-0)\right]$$

② 无条件极值、— 偏导数为 0.

$$f(x)=f(x_0)+\underbrace{f'(x_0)(x-x_0)}_{0}+\frac{1}{2!}f''(x_0)(x-x_0)^2+R_2$$

$f(x)-f(x_0)=\frac{1}{2!}f''(x_0)(x-x_0)^2 \Rightarrow$ 证出 $f''>0$ 极大

$<$ 极小

(1) 正定:

$\Rightarrow$ 同理. $f_{xx}''|_{x_0}>0$. 且 $\begin{vmatrix}f_{xx}'' & f_{xy}''\\ f_{yx}'' & f_{yy}''\end{vmatrix}>0$, 极小值

记 $A>0$ — f_{xy}'' 记 $B>0$ — f_{yy}'' 记为 C. $\Rightarrow \Delta=AC-B^2>0$.

(2) 负定: $f_{xx}''|_{x_0}<0$, $\begin{vmatrix}f_{xx}'' & f_{xy}''\\ f_{yx}'' & f_{yy}''\end{vmatrix}>0$. 即

$$\left.\begin{cases}f_{xx}''=A<0\\ f_{xy}''=f_{yx}''=B\\ f_{yy}''=C\end{cases}\right\}\rightarrow AC-B^2>0 \text{ 时、极大值}$$

(3). $AC-B^2<0$ 时 非极值点

例 13.22

$f(x,y)=x^2(2+y^2)+y\ln y$ 极值

step1 $\begin{cases}f_x'=0\\ f_y'=0\end{cases}$ — 驻点

step2 $\begin{cases}A=f_{xx}''|_{P_0}\\ B=f_{xy}''|_{P_0}\\ C=f_{yy}''|_{P_0}\end{cases}$ $\Delta=AC-B^2$

$\begin{cases}\Delta>0 \begin{cases}A>0 & \text{极大}\\ A<0 & \text{极小}\end{cases}\\ \Delta<0 \ \text{非极值}\end{cases}$

[solution] ① $f_x'=2x(2+y^2)\overset{令}{=}0$

$f_y'=2yx^2+\ln y+1\overset{令}{=}0$, $\Rightarrow \begin{cases}x=0\\ y=e^{-1}\end{cases}$

② $\begin{cases}A=f_{xx}''|_{P_0}\cdot\ f_{xx}''=2(2+y^2)\\ B=f_{xy}''|_{P_0}\ \ f_{xy}''=2x+2y\\ C=f_{yy}''|_{P_0}\ \ f_{yy}''=2x^2+\frac{1}{y}\end{cases}$ $A=2(2+e^{-2})$ $B=0$ $C=e$ $\rightarrow \Delta=2(2+e^{-2})e^{*}>0$. $A>0$. 极小值

三.条件极值与拉氏函数法

目标函数 $f(x,y)$， 约束 $\varphi(x,y)=0$.

令 $F(x,y,\lambda) = f(x,y)+\lambda\varphi(x,y)$

则 $\begin{cases} F_x'=0 \\ F_y'=0 \\ F_\lambda'=0 \end{cases}$ 解 $\Rightarrow$ $P_0(x_0,y_0)$ $P_1(x_1,y_1)\cdots$

目标函数 $f(x,y,z)$ 约束 $\varphi_1(x,y,z)=0$ $\varphi_2(x,y,z)=0$

例 13.24

求 $u=x^2+y^2+z^2$，在 $\begin{cases} z=x^2+y^2 \\ x+y+z=4 \end{cases}$ 下最值

(solution)

令 $F(x)= x^2+y^2+z+\lambda(x^2+y^2-z^2)+\mu(x+y+z-4)$

$$\begin{cases} F_x'= 2x+2\lambda x+\mu=0 \\ F_y'= 2y+2\lambda y+\mu=0 \\ F_z'= 1-2\lambda z+\mu=0 \\ F_\lambda'= x^2+y^2-z=0 \\ F_\mu'= x+y+z-4=0. \end{cases}$$

三种求解方法.

1). 消元代入

2) 观察法

3) 得到轮换对称 $\longrightarrow$ $x^2=y^2$ 与 $|x|=|y|$

$\Rightarrow x=y$ 或 $x=-y$

解之，得 $P_1(1,1,2)$ $P_2(-2,-2,8)$

$u_1=6$ $u_2=72$

$u_{max}=72$ $u_{min}=6$.

注带着 绝对值、根号等：

$d=\frac{1}{\sqrt{13}}|2x+3y-6|$ $\longrightarrow$ 用 u^2

再如 $u=\sqrt{x^2+y^2}$ $u=\sqrt[3]{x^2+y^2}$ $\longrightarrow$ 用 x^2+y^2

四. 偏微分方程

例 13.28.

已知 $\frac{\partial z}{\partial x}=-\sin y+\frac{1}{1-xy}$，且 $z(0,y)=2\sin y+y^2$

求 $z(x,y)$

对 x 积分. $\int\frac{1}{1-xy}dx=-\frac{1}{y}\ln|1-xy|$

$\begin{cases} z(x,y)= -x\sin y-\frac{1}{y}\ln|1-xy| + \varphi(y) \\ z(0,y)= 0+0+\varphi(y)= 2\sin y+y^2 \end{cases}$

例13.29. ☆☆☆.

已知二元函数 $z=f(x,y)$ 可微，两个偏增量

$\Delta_x z=(2+3x^2y^2)\Delta x+3xy^2(\Delta x)^2+y^2(\Delta x)^3$,

$\Delta_y z=2x^3y\Delta y+x^3(\Delta y)^2$. 且 $f(0,0)=1$.

求 $f(x,y)$.

$\Delta_x z=f(x+\Delta x,y_0)-f(x_0,y_0)$

$\Delta_y z=f(x_0,y_0+\Delta y)-f(x_0,y_0)$ 偏增量

$\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x} \Rightarrow \frac{\Delta_x z}{\Delta x}=\frac{\partial z}{\partial x}+\alpha$

$\Rightarrow \Delta_x z=\frac{\partial z}{\partial x}\Delta x+o(\Delta x)$.

$\Rightarrow \frac{\partial z}{\partial x}=2+3x^2y^2$ 对x积分 $z(x,y)=x^2+y^2x^3+\varphi(y)$.

同理 $\frac{\partial z}{\partial y}=2x^3y \Rightarrow z(x,y)=x^3y^2+\mu(x)$

$\Rightarrow \varphi'(y)=0$.

$\Rightarrow z(x,y)=x^2+y^2x^3+C$ $\Rightarrow \varphi(y)=C$

例13.30 偏微分方程 ⇒ 常微分方程

设 $f(u)$ 有二阶连续导数. $z=f(e^x\cos y)$.

满足 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(4z+e^x\cos y)e^{2x}$.

若 $f(0)=0$, $f'(0)=0$. 求 $f(u)$

$\frac{\partial z}{\partial x}=f'(e^x\cos y)\cdot e^x\cos y$

$\frac{\partial^2 z}{\partial x^2}=f''(e^x\cos y)\cdot e^{2x}\cos^2 y+f'(e^x\cos y)e^{x}\cos y$

$\frac{\partial z}{\partial y}=-f'(e^x\cos y)\cdot e^x\sin y$

$\frac{\partial^2 z}{\partial y^2}=f''(e^x\cos y)\cdot e^{2x}\sin^2 y-f'(e^x\cos y)e^x\cos y$

$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=f''(e^x\cos y)\cdot e^{2x}=(4z+e^x\cos y)e^{2x}$

$\Rightarrow f''(e^x\cos y)=4z+e^x\cos y$

$F(x,y,z)=f''(e^x\cos y)-4z-e^x\cos y=0$.

令 $e^x\cos y=u$

$F(u,z)=f''(u)-4f(u)-u=0$

解 $\Longrightarrow f(u)=\frac{1}{16}e^{2u}-\frac{1}{16}e^{-2u}-\frac{u}{4}$.

例13.31 ☆☆☆. 模下来

设 $f(x,y)$ 是一阶偏导数连续的正值函数

满足 $f_x'(x,y)+f(x,y)=0$. 又 $f_y'(x,y)=\tan y$.

$f(0,0)=1$, 求 $f(x,y)$

$(\ln f(x))'=\frac{f'(x)}{f(x)}$ $(\ln(f(x,y)))_x'=\frac{f_x'(x,y)}{f(x,y)}$ ☆.

$f_x'(x,y) + f(x,y) = 0.$

$\Rightarrow \frac{f_x'(x,y)}{f(x,y)} = -1 \Rightarrow \ln f(x,y) = -x + \varphi(y).$

$\Rightarrow f(x,y) = e^{-x} \cdot e^{\varphi(y)}$

$f(0,0) = e^{-0} \cdot e^{\varphi(0)} = 1 \Rightarrow \varphi(0) = 0$

$f_y'(0,y) = [f(0,y)]_y' = (e^{\varphi(y)})_y' = e^{\varphi(y)} \cdot \varphi'(y) = -\tan y.$

对y积分 $\Rightarrow$ $e^{\varphi(y)} = \int \tan y \, dy = -\ln|\cos y| + C.$

$\Rightarrow$ $y=0$时 $e^{\varphi(0)} = 1 \Rightarrow C = 1$

$\Rightarrow f(x,y) = e^{-x}(-\ln|\cos y| + 1).$

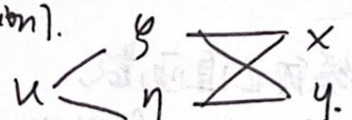
例 13.32

设. $4\frac{\partial^2 u}{\partial x^2} + 12\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2} = 0.$

确定 a.b. 值

使等式在变换 $\xi = x + ay$, $\eta = x + by$ 下简化 $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$

[solution].



$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} \left(+\frac{\partial^2 u}{\partial \eta \partial \xi}\right) + \frac{\partial^2 u}{\partial \eta^2}$

$\frac{\partial^2 u}{\partial x \partial y} = a\frac{\partial^2 u}{\partial \xi^2} + \overset{(a+b)}{b}\frac{\partial^2 u}{\partial \xi \partial \eta} + \left(a\frac{\partial^2 u}{\partial \eta \partial \xi}\right) + b\frac{\partial^2 u}{\partial \eta^2}$

$\frac{\partial^2 u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial \xi^2} + 2ab\frac{\partial^2 u}{\partial \xi \partial \eta} + b^2\frac{\partial^2 u}{\partial \eta^2}$

$\Rightarrow (4 + 12a + 5a^2)\frac{\partial^2 u}{\partial \xi^2} + (8 + 12(a+b) + 10ab)\frac{\partial^2 u}{\partial \xi \partial \eta} + (4 + 12b + 5b^2)\frac{\partial^2 u}{\partial \eta^2} = 0.$

$\because$ 使 $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \Rightarrow \begin{cases} 4 + 12a + 5a^2 = 0 & (5a+2)(a+2) = 0 \\ 4 + 12b + 5b^2 = 0. & (5b+2)(b+2) = 0. \end{cases}$

$8 + 12(a+b) + 10ab \neq 0.$

$\begin{cases} a = -2 \\ b = -2 \end{cases}$ 舍 $\quad \begin{cases} a = -\frac{2}{5} \\ b = -2 \end{cases} \quad \begin{cases} a = -2 \\ b = -\frac{2}{5} \end{cases} \quad \begin{cases} a = -\frac{2}{5} \\ b = -\frac{2}{5} \end{cases}$ (舍).

$8 - 48 + 40 = 0 \qquad 8 - \frac{48}{5} + 10 \cdot \frac{4}{25} = 0.$

习题都重要.

更重要 9.10.15.16.17