第三讲 1000题总结

1000.3.3.

设 $f(x)=\int_0^1 \ln\sqrt{x^2+y^2}\,dy$, $0\le x\le 1$，则 $f'_+(0)=?$

$$f(x)=\int_0^1 \ln\sqrt{x^2+y^2}\,dy = y\ln\sqrt{x^2+y^2}\Big|_{y=0}^{y=1} - \int_0^1 y\, d\ln\sqrt{x^2+y^2}$$

$$= \ln\sqrt{x^2+1} - \int_0^1 \frac{y}{\sqrt{x^2+y^2}}\cdot\frac{2y}{2\sqrt{x^2+y^2}}\,dy$$

$$=\ln\sqrt{x^2+1} - \int_0^1 \frac{2y^2}{2(x^2+y^2)}\,dy = \ln\sqrt{x^2+1} - \int_0^1 \frac{y^2}{x^2+y^2}\,dy$$

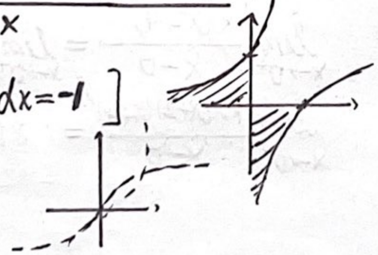
$$=\ln\sqrt{x^2+1} - \int_0^1 1-\frac{x^2}{x^2+y^2}\,dy = \ln\sqrt{x^2+1} -1 + \int_0^1\frac{x^2}{x^2+y^2}\,dy$$

$$=\ln\sqrt{x^2+1} -1 + x\arctan\frac{y}{x}\Big|_{y=0}^{y=1}$$

$$=\ln\sqrt{x^2+1}-1+x\arctan\frac1x$$

$$f'_+(0)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to0}\frac{\ln\sqrt{x^2+1}-1+x\arctan\frac1x+1}{x}$$

$$\left[\ f(0)=\int_0^1 \ln y\,dy = \int_{-\infty}^0 e^x\,dx = -1\ \right]$$



$$=\lim_{x\to0^+}\frac{\ln\sqrt{x^2+1}}{x}+\lim_{x\to0^+}\arctan\frac1x$$

$$=\lim_{x\to0^+}\frac{\frac12x^2}{x}+\frac\pi2=\frac\pi2.$$

1000.3.5.

$f(x)$可导 $y=f(x^3)$. x在$x=-1$处得增量$\Delta x=-0.1$时相应函数增量Δy的线性主部为0.3，$f'(-1)=?$

$$dy = y'\cdot\underbrace{(3x^2)}_{3}\cdot\underbrace{dx}_{-0.1}\Big|_{x=-1}=0.3 \Rightarrow y'=-1$$

1000.3.11

设$f(x)$在$(-\frac{\pi}{2a},\frac{\pi}{2a})$ $(a>0)$内有定义，且$f'(0)=a$，对$\forall x, y$, $x+y\in(-\frac{\pi}{2a},\frac{\pi}{2a})$，有$f(x+y)=\frac{f(x)+f(y)}{1-f(x)f(y)}$

求$f(x)$.

① $f(0)=\frac{2f(0)}{1-f^2(0)} \Rightarrow f(0)=0$.

② $$f'(x)=\lim_{\Delta x\to0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to0}\frac{\frac{f(x)+f(\Delta x)}{1-f(x)f(\Delta x)}-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to0}\frac{f(\Delta x)[1+f^2(x)]}{\Delta x[1-f(x)f(\Delta x)]}$$

$$=\lim_{\Delta x\to0}\frac{f(0+\Delta x)-f(0)}{\Delta x-0}\cdot\lim_{\Delta x\to0}\frac{1+f^2(x)}{1-f(x)f(\Delta x)}$$

$$=\overset{a}{f'(0)}\cdot\lim_{\Delta x\to0}\frac{1+f^2(x)}{1-f(x)\underset{0}{f(\Delta x)}} = a\cdot(1+f^2(x))$$

$$\Rightarrow a=\frac{f'(x)}{1+f^2(x)} \qquad \cancel{a=f'(x)\cdot\arctan f(x)}$$

$g(x)=\arctan f(x)$ $\Big|$ $g'(x)=\frac{f'(x)}{1+f^2(x)}=a$

$=ax+C$

$\because f(0)=0$ $\therefore \arctan 0=0+C \Rightarrow C=0$

$\Rightarrow f(x)=\tan ax$

1000.3.13

设 $y=f(x)$ 由方程 $\sin(xy)+\ln y-x=1$ 确定.

求 $\lim\limits_{n\to\infty} n[f(\frac{2}{n})-e]$

1) $x=0 \Rightarrow 0+\ln y-0=1 \Rightarrow y=e \Rightarrow f(0)=e.$

2) $\lim\limits_{n\to\infty} n[f(\frac{2}{n})-e] \overset{\frac{1}{n}=x}{=} \lim\limits_{x\to0}\frac{f(2x)-e}{x}=\lim\limits_{x\to0}2f'(2x)$

3). x' 的 $\sin(xy)+\ln y-x=1$

$\cos(xy)\cdot(y+xy')+\frac{y'}{y}-1=0.$

$(f(0)=e)$ 的 $\cos(0)\cdot(e+0)+\frac{y'}{e}-1=0.$

$y'=(1-e)e$

$\Rightarrow$ 原 $=2e(1-e)$

1000.3.15

设 $f(x)=\begin{cases} x^{\lambda}\sin\frac{1}{x}, & x>0 \\ 0, & x\le 0 \end{cases}$ 求 $f'(x)$ 并判定 $f'(x)$ 连续时 λ 范围

1). 导数存在.

左导数 $=0$. 右导数 $=\lim\limits_{x\to0^+}\frac{x^{\lambda}\sin\frac{1}{x}-0}{x-0}=\lim\limits_{x\to0^+}x^{\lambda-1}\sin\frac{1}{x}.$

$\Rightarrow \lambda-1>0 \Rightarrow \lambda>1$

2) $f'(x)=\lambda x^{\lambda-1}\sin\frac{1}{x}+x^{\lambda}\cos\frac{1}{x}\cdot\frac{-1}{x^2}$

$=\lambda x^{\lambda-1}\sin\frac{1}{x}-x^{\lambda-2}\cos\frac{1}{x}$

当 $x<0$ 时, $f'(x)=0$

当 ~~λ~~ $\lambda>2$ 时, $x\to0$ 时 $f'(x)=0 \Rightarrow \lambda>2$

1000.3_5.1

已知 $f(0)=0, f'(0)=2$, 则 $\lim\limits_{n\to\infty}[f(\frac{1}{n^2})-\frac{1}{n^2}+1]^{3n^2}=?$

[分析]

$\lim\limits_{x\to0}[f(x^2)-x^2+1]^{\frac{3}{x^2}}=\lim\limits_{x\to0}e^{\frac{3}{x^2}\ln[f(x^2)-x^2+1]}$

$=\lim\limits_{x\to0}e^{\frac{3\cdot[2xf'(x^2)-2x]}{2x[f(x^2)-x^2+1]}}=\lim\limits_{x\to0}e^{\frac{3[f'(x^2)-1]}{f(x^2)-x^2+1}}=e^{\frac{3(2-1)}{0-0+1}}=e^3$

1000.3_5.4

$f(x)$ 在 $(-\infty,+\infty)$ 有定义, 在 $[0,2]$ 上, $f(x)=x(x^2-4)$ $\forall x: f(x)=kf(x+2)$

(1) 写出 $f(x)$ 在 $[-2,0)$ 上的表达式.

$x\in[-2,0)$ $f(x)=kf(x+2)=k(x+2)[x^2+4x]=kx(x+2)(x+4)$

$x+2\in[0,2)$

(2) k 为何值, $f(x)$ 在 $x=0$ 处可导?

$f(x)=\begin{cases} x(x^2-4), & x\in[0,2] \\ kx(x+2)(x+4), & x\in[-2,0) \end{cases}$

$\lim\limits_{x\to0^+}\frac{x(x^2-4)}{x-0}=\lim\limits_{x\to0^+}(x^2-4)=-4$

$\lim\limits_{x\to0^-}\frac{kx(x+2)(x+4)}{x-0}=8k \Rightarrow k=-\frac{1}{2}$

习.3.3.

设 $f''(1)$ 存在，且 $\lim\limits_{x\to 1} \frac{f(x)}{x-1}=0$，记 $\varphi(x)=\int_0^1 f'[1+(x-1)t]dt$

求 $\varphi(x)$ 在 $x=1$ 的某个邻域内的导数，并讨论

$\varphi'(x)$ 在 $x=1$ 处连续性

[solution].

$\lim\limits_{x\to 1} \frac{f(x)-0}{x-1}=0 \Rightarrow f(1)=0 \quad f'(1)=0$

$\varphi(x)=\frac{1}{x-1}\left[f(x)-f(1)\right]=\frac{f(x)}{x-1}$

$\varphi'(x)$ $\cancel{g'(x)}=\frac{f'(x)(x-1)-f(x)}{(x-1)^2}$

$\lim\limits_{x\to 1} \overset{t=x-1}{=} \frac{f'(t+1)t-f(t+1)}{t^2}=\lim\limits_{t\to 0}\frac{\cancel{f'(t+1)}+f''(t+1)t-\cancel{f'(t+1)}}{2t}$

$=\frac{f''(1)}{2}$ $\quad\therefore\ \varphi'(x)$ 在 $x=1$ 连续