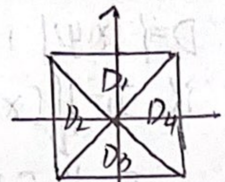
14讲 1000题

• 1000.14.1.

正方形 $\{(x,y) \mid |x|\le 1,\ |y|\le 1\}$ →



$I_k=\iint\limits_{D_k} ye^x\,dxdy$. $\max I_k = ?$ I?

[solution].

~~D_1左右~~: D_4 上下: 相消 $\iint=0$. $D_2=0$.

D_3. $e^x>0$ $y<0$. D_1 $y>0$ → D_1 max

• 1000.14.2

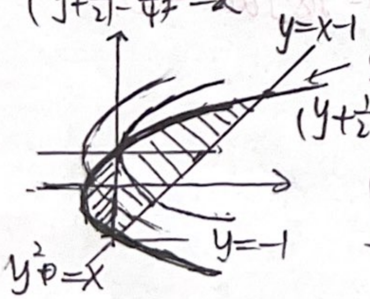
$$\int_{-\frac14}^{0}dx\int_{-\frac12-\sqrt{x+\frac14}}^{-\frac12+\sqrt{x+\frac14}} f(x,y)\,dy+\int_0^2 dx\int_{x-1}^{-\frac12+\sqrt{x+\frac14}} f(x,y)\,dy.$$

交换积分次序.

$y=-\frac12\pm\sqrt{x+\frac14}$　　$y=x-1$.　　$y+1=(y+\frac12)^2-\frac14$

$(y+\frac12)^2=x+\frac14$　　$y+1=y^2+(\frac14-\frac14)+y$

$(y+\frac12)^2-\frac14=x$　　$y=1$



$(y+\frac12)^2-\frac14=x$.

$$\int_{-1}^{1}dy\int_{(y+\frac12)^2-\frac14}^{y+1} f(x,y)\,dx$$

• 1000.14.3.

$I_k=\iint\limits_{x^2+y^2\le k^2} e^{-(x^2+y^2)}\sin\sqrt{x^2+y^2}\,dxdy$. $(k=1,2,3)$.

$\nearrow$

$\int_0^{2\pi}d\theta\int_0^k e^{-r^2}\cdot\sin r\cdot r\,dr$

$x^2+y^2=r^2$　$=2\pi\int_0^k -2r\,e^{-r^2}\sin r\,dr$

$=\pi-\pi\int_0^k \sin r\,d\,e^{-r^2}$

$=-\pi\sin r\cdot e^{-r^2}\Big|_0^k+\pi\int_0^k e^{-r^2}\cos r\,dr$

$=-\pi\sin k\cdot e^{-k^2}$

$=\cancel{\int}\ 2\pi\int_0^k r e^{-r^2}\sin r\,dr$

令 $\varphi(r)=re^{-r^2}\sin r=\frac{1}{2\pi}f'(r)$.

$r=1,2,3$, $\varphi(r)>0$ ∴ $f(r)\nearrow$.

∴ $I_1<I_2<I_3$

• 1000.14.4

$f(x)$ 连续函数, $f(0)=1$. 令 $F(t)=\iint\limits_{x^2+y^2\le t^2} f(x^2+y^2)\,d\sigma$ $(t\ge 0)$

则 $F''(0)=?$　　$r=t$, $r^2=t^2$

$F(t)=\int_0^{2\pi}d\theta\int_0^t f(r^2)\,r\,dr=\pi\int_0^t f(r^2)\,dr^2$

$=\cancel{\int_0^{t}}=\pi\int_0^{t^2}f(x)\,dx$.

$F'(x) = \pi f(x^2)\cdot 2x$

$F''(x) = \lim\limits_{x\to 0} \dfrac{\pi f(x^2)\cdot 2x - 0}{x-0} = \lim\limits_{x\to 0} 2\pi f(x^2) = 2\pi$

1000.14.5.

设双扭线 $(x^2+y^2)^2 = 2^2(x^2-y^2)$ 围区域 D

则 $\iint_D (x^2+y^2)\,d\sigma = ? \longrightarrow \iint_D r^2\,d\sigma$

虽然转了两圈，但区域只算个

$= 4\cancel{8}\int_0^{\frac{\pi}{4}} d\theta \int_0^{2\sqrt{\cos 2\theta}} r^2\, r\,dr$

$= 48\int_0^{\frac{\pi}{4}} d\theta \cdot \dfrac{r^4}{4}\Big|_0^{2\sqrt{\cos 2\theta}}$

$= \int_0^{\frac{\pi}{4}} \cdot 16\left(\sqrt{\cos 2\theta}\right)^2 d\theta$

$= 16 \int_0^{\frac{\pi}{4}} \cos^2(2\theta)\, d(2\theta)$

$= \cancel{8\pi}\, 2\pi$

$x = \rho\cos\theta$

$y = \rho\sin\theta$

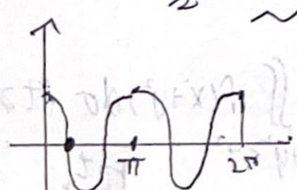
$x^2+y^2 = \rho^2$

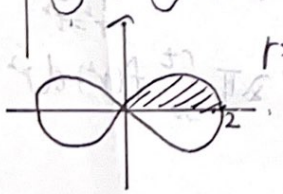
$x^2-y^2 = \rho^2(\cos^2\theta - \sin^2\theta)$.

$\Rightarrow 1 = \cos^2\theta - \sin^2\theta$

$= \dfrac{1+\cos 2\theta}{2} - \dfrac{1-\cos 2\theta}{2}$

$= \cancel{\dfrac{\cos 2\theta}{2}}\ \cos 2\theta$





$r = 2\sqrt{\cos 2\theta}$

1000.14.6.

$D = \{(x,y) \mid 1 < x \le e,\ 1 < y \le e\}$.

记 $I_1 = \iint_D [x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2}]\, \sin(\ln y)\, d\sigma$

$I_2 = \iint_D [y\ln(y+\sqrt{1+y^2}) - \sqrt{1+y^2}]\, \sin(\ln y)\, d\sigma$

I_1 I_2 大小：

~~$I_2 = \iint_D (y\ln(y+\sqrt{1+y^2}) - \sqrt{1+y^2})\, \sin(\ln x)\, d\sigma$~~

~~$x, y \in [1, e]$. $\to \ln x, \ln y \in [0, 1]$ $\sin\ln x\ \sin\ln y \in [0, \sin 1]$~~ ↗

$\therefore I_1 = \iint_D f(x)\,g(y)\,d\sigma$

$f(x) = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2}$ ↗

$f'(x) = \ln(x+\sqrt{1+x^2}) + \dfrac{1}{\sqrt{1+x^2}} > 0$.

$g(x) = \sin\ln x$ ↗

$I_2 = \iint_D f(y)\,g(y)\,d\sigma$

$I_1 - I_2 = \iint_D g(y)(f(x)-f(y))\,d\sigma$

$= \iint_D g(x)(f(y)-f(x))\,d\sigma$

$I_1 - I_2 = \frac{1}{2}\iint [f(x)-f(y)][g(y)-g(x)]\,d\sigma < 0$.

$\therefore I_1 < I_2$.

1000. 14.7.

$D=\{(x,y) \mid x^2+y^2 \le (\frac{\pi}{4})^2\}$

$M=\iint_D (x^3+y^3)\,dxdy \quad N=\iint_D \cos(x+y)\,dxdy \quad P=\iint_D (e^{-\sqrt{x^2+y^2}}-1)\,dxdy$

大小关系?

$m(x,y)=x^3+y^3. \quad M=\iint_D x^3 d\sigma + \iint_D y^3 d\sigma = 0.$

$N=\iint_D \cos(x+y)\, d\sigma > 0.$

$e^0=1 \quad e^{-\sqrt{x^2+y^2}}<1 \longrightarrow P<0.$

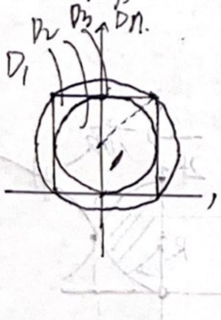
$\therefore N > M > P.$

1000. 14.8

设 D_1 在中心在 $(0,1)$ 点，边长为2且平行于坐标轴的正方形区域，D_2, D_3 分别为 D_1 的内切圆区域与外接圆区域

$f(x,y)=(2y-x^2-y^2)e^{-x^2-y^2}$

$I_n=\iint f(x,y)\,d\sigma \longrightarrow I_1\ I_2\ I_3$ 大小关系.



$f(x,y)=(2y-(x^2+y^2))\underbrace{e^{-(x^2+y^2)}}_{>0}.$

$g(x,y)=2y-(x^2+y^2)=0.$

$\longrightarrow x^2+(y-1)^2=1 \longrightarrow$ 即为内切圆.

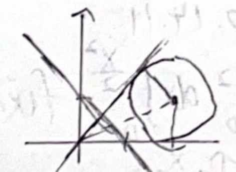
圆外. $2y-(x^2+y^2)<0.$

$D_2>D_1>D_3.$

1000. 14.9

$D=\{(x,y) \mid (x-2)^2+(y-1)^2 \le 1\}$

$I_1=\iint_D (x+y)^2 d\sigma \quad \iint_{D_2} (x+y)^3 d\sigma$



[solution]

$I_1=\int u^2 d\sigma \quad I_2=\int u^3 d\sigma$

$I_2-I_1=\iint_D (x+y)^2\underbrace{(x+y-1)}_{>0} d\sigma > 0.$

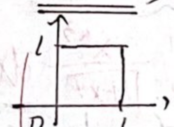
$y=-x+1 \quad y>0.$

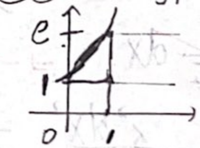
$(\rho\cos\theta-2)^2+(\rho\sin\theta-1)^2 \le 1$

$I_2>I_1$

1000. 14.10.

$\int_0^1 dy \int_0^1 \sqrt{e^{2x}-y^2}\,dx + \int_1^e dy \int_{\ln y}^1 \sqrt{e^{2x}-y^2}\,dx$





$y=e^x$

$\ln y < x < 1$

$1<y<e$

$=\int_0^1 dx \int_0^{e^x} \sqrt{e^{2x}-y^2}\,dy = \int_0^1 dx \int_0^{e^x} \sqrt{(e^x)^2-y^2}\,dy$

$=\int_0^1 \frac{e^{2x}}{2}\arcsin\frac{x}{e^x} + \frac{x}{2}\sqrt{e^x-x^2}\,dx$

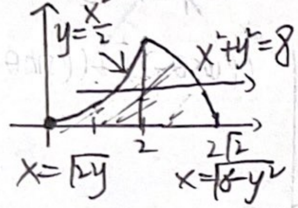
$\int_0^{e^x} \sqrt{e^{2x}-y^2}\,dy$ 半径为 e^x 的圆面积的 $\frac{1}{4}$.

$\frac{\pi}{4}e^{2x}$

$\therefore$ 原 $=\int_0^1 dx\ \frac{\pi}{4}\int_0^1 e^{2x}dx = \frac{\pi}{8}(e^2-1)$

- 1000.14.11.

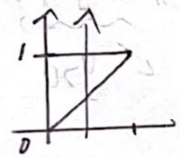
$I=\int_0^2 dx\int_0^{\frac{x^2}{2}} f(x,y)\,dy+\int_2^{2\sqrt2}dx\int_0^{\sqrt{8-x^2}} f(x,y)\,dy$



$\int_0^2 dy\int_{\sqrt{2y}}^{\sqrt{8-y^2}} f\,dx.$

- 1000.14.12

$\int_0^1 y^2dy\int_1^y \frac{1}{\sqrt{1+x^4}}dx$



$\frac{y^3}{3}\Big|_x^1$

$\frac13-\frac{x^3}{3}$

$=\int_0^1 dx\int_x^1 \frac{y^2}{\sqrt{1+x^4}}dy=\int_0^1\frac{1}{\sqrt{1+x^4}}dx\int_x^1 y^2dy$

$=\frac13\cancel{\int_0^1\frac{dx}{\sqrt{1+x^4}}}-\frac13\int_0^1\frac{x^3}{\sqrt{1+x^2}}dx$ $=-\frac13\int_0^1\frac{x^3}{\sqrt{1+x^4}}dx$

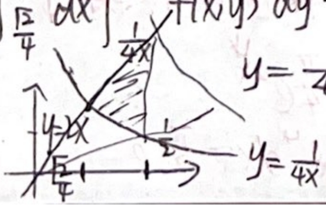
$=\cancel{\frac13\arcsin x\Big|_0^1-\frac16\int_0^1\frac{x^2dx^2}{\sqrt{1+x^2}}}$ $=-\frac{1}{12}\int_0^1\frac{dx+1}{\sqrt{1+x}}$

$=\cancel{\frac{\pi}{6}-\frac16\int_0^1\frac{xdx}{\sqrt{1+x}}}$ $=\frac{-1}{12}\,2x^{+\frac12}\Big|_1^2$

$=\cancel{\frac{1}{\sqrt2}\left(\frac{1}{\sqrt2}-1\right)}=\frac16(1-\sqrt2)$

- 1000.14.13

$\int_{\frac{\sqrt2}{4}}^{\frac12}dx\int_{\frac{1}{4x}}^{2x} f(x,y)\,dy+\int_{\frac12}^{\frac{\sqrt2}{2}}dx\int_x^{2x} f(x,y)\,dy$



$y=\frac{1}{4x}=\frac14\frac1x$ $\quad \frac{1}{\frac{\sqrt2}{4}\times4}=\frac{\sqrt2}{2}$

$2x=\frac{\sqrt2}{2}$

原 $=\int_{\frac12}^1 dy\int_{\frac{1}{4y}}^{\frac y2} f(x,y)\,dx$

$x=\frac12\longrightarrow\frac{1}{4x}=\frac12$ $\quad y=2x=2\times\frac12=1$

原 $=\int_{\frac{\pi}{4}}^{\arctan 2}d\theta\int_{\frac{1}{\sqrt{2\sin2\theta}}}^{\frac{1}{2\cos\theta}} f\,r\,dr$

$(\frac12,\frac12)$

$r\cos\theta+r\sin\theta=1$ $\quad 2r^2\sin2\theta=1$ $\quad r=\sqrt{\frac{1}{2\sin2\theta}}$

$y=2x$ $\quad 2\cos\theta=\sin\theta$ $\quad 2=\tan\theta$ $\quad \theta=\arctan 2$

$x=\frac12$ $\quad r\cos\theta=\frac12$

- 1000.14.14

$D=\{(x,y)\mid x^2+y^2\le1\}$ $\quad \iint_D e^{\lambda x}-e^{\lambda y}\,d\sigma$ ($\lambda\ne0$)的值

$\rho^2\le1$ $\quad \rho<1$ $\quad \iint_D \cancel{e^{\lambda\rho\cos x}-e^{\lambda\rho\sin y}}\,d\sigma$

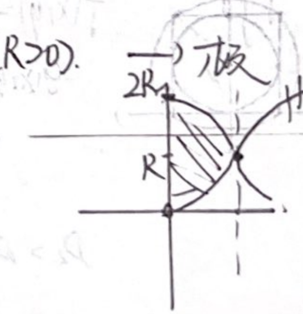
$\iint e^{\lambda x}-e^{\lambda y}\,d\sigma$ $=\iint_D \cancel{e^{\lambda\rho\cos y}-e^{\lambda\rho\sin x}}\,d\sigma$

$=\iint e^{\lambda y}-e^{\lambda x}\,d\sigma$

$=0.$

- 1000.14.15

$\int_0^{2R}dy\int_0^{\sqrt{2Ry-y^2}} f(x^2+y^2)\,dx$ $\quad (R>0).$ $\longrightarrow$ 了极



[solution]

$\int_0^{\frac{\pi}{2}}d\theta\int_0^{2R\sin\theta} f\,r\,dr$