空间曲线的切线与法平面

一、参数方程给出曲线

曲线 $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ $t \in I$

切向量 $\vec{\tau} = (x'(t_0), y'(t_0), z'(t_0))$

切线 $\dfrac{x-x_0}{x'(t_0)} = \dfrac{y-y_0}{y'(t_0)} = \dfrac{z-z_0}{z'(t_0)}$

法平面 $x'(t_0)(x-x_0) + y'(t_0)(y-y_0) + z'(t_0)(z-z_0) = 0$

二、用方程组给出曲线

曲线 $\begin{cases} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{cases}$ ← 用 x 当 t

求切向量 $\tau = (1, y'(x_0), z'(x_0))$

$$\frac{dy}{dx} = -\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(y,z)}} \qquad \frac{dz}{dx} = -\frac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}}$$

切线 $\dfrac{x-x_0}{1} = \dfrac{y-y_0}{y'(x_0)} = \dfrac{z-z_0}{z'(x_0)}$

法平面 $x'(x_0)(x-x_0) + y'(x_0)(y-y_0) + z'(x_0)(z-z_0) = 0$

☆ 背 $F(x,y,z)$，$G(x,y,z)$ 交出来的曲线

切线：$\begin{vmatrix} i & j & k \\ F'_x & F'_y & F'_z \\ G'_x & G'_y & G'_z \end{vmatrix}_{P_0}$

例 17.1

求切线与法平面 在 $t = \frac{\pi}{2}$

1) $\begin{cases} x = \cos t + \sin^2 t \\ y = \sin t(1-\cos t) \\ z = \cos t \end{cases}$

$x'_t = -\sin t + 2\sin t\cos t$

$y'_t = \cos t(1-\cos t) + \sin^2 t$

$z'_t = -\sin t$

切线 $\dfrac{x-x_0}{x'_{t_0}} = \dfrac{y-y_0}{y'_{t_0}} = \dfrac{z-z_0}{z'_{t_0}}$

令 $t = \frac{\pi}{2} \Rightarrow \vec{\tau} = (x'_t, y'_t, z'_t) = (-1, 1, -1)$

$(x_0, y_0, z_0) = (1, 1, 0)$

切线 $\dfrac{x-1}{-1} = \dfrac{y-1}{1} = \dfrac{z-0}{-1}$

法平面 $-(x-1) + (y-1) - z = 0$

2) $\begin{cases} x = y^2 \\ z = x^2 \end{cases}$ 在点 $(1,1,1)$

令$t=y$.

$$\Rightarrow\begin{cases}x=t^2\\ y=t\\ z=t^4\end{cases}\quad\begin{cases}x'_t=2t\\ y'_t=1\\ z'_t=4t^3\end{cases}\quad\xrightarrow{t=1}\quad\begin{cases}x'_t=2\\ y'_t=1\\ z'_t=4\end{cases}\quad\begin{cases}x_0=\cancel{2}1\\ y_0=1\\ z_0=1\end{cases}$$

切. $\dfrac{x-1}{2}=\dfrac{y-1}{1}=\dfrac{z-1}{4}$

法平. $2(x-1)+(y-1)+4(z-1)=0$.

例 17.2

$f'_x(0,0)=3$, $f'_y(0,0)=1$. $z=f(x,y)$在$(0,0)$附近有定义

$\longrightarrow dz|_{(0,0)}=3dx+dy$ 错.

$\longrightarrow$ 令 $\begin{cases}x=t\\ y=0\\ z=f(t,0)\end{cases}$ $\longrightarrow \vec{\tau}=\{1,0,f'_x(0,0)\}$

$\therefore$ 切向量是 $(1,0,3)$

曲线$\begin{cases}z=f(x,y)\\ y=0\end{cases}$ 切线向. 写参数方程

~~例 17.~~

二. 空间曲面的切平面与法线.

(1). 用隐式方程给出曲面. $F(x,y,z)=0$.

法向量 $(F'_x|_{P_0}, F'_y|_{P_0}, F'_z|_{P_0})$.

切平面. $F'_x|_{P_0}(x-x_0)+F'_y|_{P_0}(y-y_0)+F'_z|_{P_0}(z-z_0)=0$.

法线 $\dfrac{x-x_0}{F'_x|_{P_0}}=\dfrac{y-y_0}{F'_y|_{P_0}}=\dfrac{z-z_0}{F'_z|_{P_0}}$

(2). 显式方程给出曲面.

$z=f(x,y) \Longrightarrow f(x,y)-z=0$

法向量. $\vec{n}=(f'_x(x_0,y_0), f'_y(x_0,y_0), -1)$

$\vec{n}$向下

切平面. $f'_x(x_0,y_0)(x-x_0)+f'_y(x_0,y_0)(y-y_0)-(z-z_0)=0$. 下的情况

法线 $\dfrac{x-x_0}{f'_x(x_0,y_0)}=\dfrac{y-y_0}{f'_y(x_0,y_0)}=\dfrac{z-z_0}{-1}$

(3). 参数方程.

$$\begin{cases}x=x(\cancel{t})u,v)\\ y=y(\cancel{t})u,v)\\ z=z(\cancel{t})u,v).\end{cases}\longrightarrow \vec{n}=\begin{vmatrix}i & j & k\\ x'_u & y'_u & z'_u\\ x'_v & y'_v & z'_v\end{vmatrix}=(A,B,C)$$

例 17.3. 考得最多

曲面 $z=x^2(1-\sin y)+y^2(1-\sin x)$ 在 $(1,0,1)$ 切平面

[solution]

$F(x,y,z)=x^2(1-\sin y)+y^2(1-\sin x)-z$

$$\begin{cases} F'_x=2x(1-\sin y)-y^2\cos x \\ F'_y=-x^2\cos y+2y(1-\sin x) \\ F'_z=-1 \end{cases}$$ 在 $(1,0,1)$ $(2,-1,-1)$

切平面 $2(x-1)-1(y-0)-1(z-1)=0.$

$2x-y-z=1$

例 17.4.

设 $f(u,v)$ 可微，证明 $f(ax-bz,\ ay-cz)=0.$

上任一点、的切平面都与某一定直线平行，→ 柱面

a,b,c 是不同为零的常数 法向量→垂直

$f \begin{cases} 1 & ax-bz \\ 2 & ay-cz \end{cases}$ → x, y, z

$\vec{n}=(f'_x,\ f'_y,\ f'_z)$

$=(af'_1,\ af'_2,\ -bf'_1-cf'_2)$

取 $\vec{\tau}=\{b,c,a\}$ 使 $\vec{n}\cdot\vec{\tau}=0.$

例 17.5 ☆（未来性） 柱面

证明 曲面 $e^{2x-z}=f(\pi y-\sqrt{2}z)$ 是柱面，f 可微.

令 $F(x,y,z)=f(\pi y-\sqrt{2}z)-e^{2x-z}$

$\vec{n}=(F'_x,\ F'_y,\ F'_z).$

$=(-2e^{2x-z},\ \pi f',\ -\sqrt{2}f'+e^{2x-z})$

取 $\vec{\tau}=(a,b,c).$

s.t. $\vec{n}\cdot\vec{\tau}=0.$ $0=-2ae^{2x-z}+\pi bf'-\sqrt{2}cf'+ce^{2x-z}$

$c-2a=0.$ $\sqrt{2}c-\pi b=0.$

令 $c=1$

则 $a=\frac{1}{2}$

$b=\frac{\sqrt{2}}{\pi}$

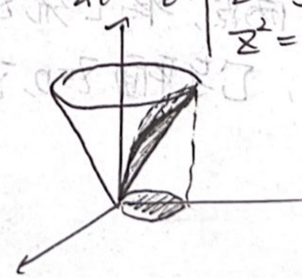
、如是 $\vec{\tau}=\begin{cases} a=\frac{1}{2} \\ b=1 \\ c=\frac{\sqrt{2}}{\pi} \end{cases}$ 则 $\vec{n}\cdot\vec{\tau}=0.$

三、空间曲线在坐标面上的投影

投影，没有哪个字母就消去哪个字母.

例 17.6

求 $\Gamma:\begin{cases} z=\sqrt{x^2+y^2} \\ z^2=2x \end{cases}$ 在 xOy 面上的投影曲线方程



Step 1. 消 z.

$x^2+y^2=2x$

Step2. 联立 $\begin{cases} x^2+y^2=2x \longrightarrow (x-1)^2+y^2=1 \\ z=0 \end{cases}$

IV. 旋转曲面

1. 任意一曲线 绕另一曲线

① 曲线 $\Gamma = \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ 绕 $\dfrac{x-x_0}{m} = \dfrac{y-y_0}{n} = \dfrac{z-z_0}{l}$

$\vec{MP} \perp l$

② $m(x-x_1)+n(y-y_1)+l(z-z_1)=0$

$MO=PO$

③ $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = (x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2$

eg. 绕 z 轴 $\longrightarrow$ $m, n, l = 0, 0, 1$.

$\Downarrow$

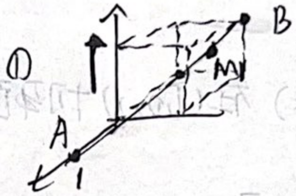
得 $\begin{cases} F(x_1,y_1,z)=0 \\ G(x_1,y_1,z)=0 \\ x^2+y^2=x_1^2+y_1^2 \end{cases}$ 消去 x_1, y_1 得到方程

例18.11

设 l 过 $A(1,0,0)$, $B(0,1,1)$ 两点，将 L 绕 z 轴旋转一周得到曲面 Σ，Σ 与平面 $z=0$, $z=2$ 所围成的立体为 Ω.

(1). 求曲面 Σ 的方程

①



L: ① $\dfrac{x-1}{1} = \dfrac{y-0}{-1} = \dfrac{z-0}{-1}$

过 $A(1,0,0)$ $\vec{\tau}=(1,-1,-1)$.

$M_1: (x_1, y_1, z_1)$. ②

1) 到原点距离相等. $x^2+y^2+z^2 = x_1^2+y_1^2+z_1^2$

2) 垂直 $0(x-x_1)+0(y-y_1)+1(z-z_1)=0$

$\Rightarrow$ ② $z=z_1$

$\Rightarrow$ $\cancel{x^2+y^2} = \begin{cases} x_1=1-z \\ y_1=z \\ z_1=z \end{cases}$

五. 向量运算及应用

1) 内积.

- $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$ ← 位置关系用到它
- $proj_b a = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

2) 外积 $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_x & \cancel{b_x} a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

3) 混合积 $[abc] = (a\times b)\cdot c = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$

☆ 方向余弦.

$\cos\alpha = \dfrac{a_x}{|a|}$ $\cos\beta = \dfrac{a_y}{|a|}$ $\cos\gamma = \dfrac{a_z}{|a|}$

六 平面,直线与位置关系

一般式: $Ax + By + Cz + D = 0.$

点法式: $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$

三点式 $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{vmatrix} = 0.$ (三点不共面).

平面束方程.

$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$

$\longrightarrow \mu(A_1x + B_1y + C_1z + D_1) + \lambda(A_2x + B_2y + C_2z + D_2) = 0$

位置关系.

(1) ☆点到平面距离:

$P_0(x_0, y_0, z_0)$ 到 $Ax + By + Cz + D = 0.$

$d = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

(2) 直线与直线

$\tau_1 = (l_1, m_1, n_1)$. $\tau_2 = (l_2, m_2, n_2)$ 分别是 L_1, L_2 的方向向量.

① $L_1 \perp L_2 \iff \tau_1 \perp \tau_2 \iff l_1l_2 + m_1m_2 + n_1n_2 = 0$

② $l_1 \parallel l_2 \iff \tau_1 \parallel \tau_2 \iff \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

③ $L_1 L_2$ 夹角. $\theta = \arccos \dfrac{|\tau_1\cdot\tau_2|}{|\tau_1||\tau_2|}$

(3) 平面与直线 (4). 平面与平面.

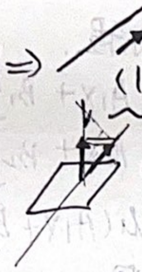
· 例17.9

设有直线 L_1: $\frac{x-1}{1}=\frac{y-5}{-2}=\frac{z+8}{1}$ ⇒ $(1,-2,1)$

与 L_2: $\begin{cases} x-y=6 \\ 2y+z=3 \end{cases}$ $(-1,-1,2)$

$\begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = -i-j+2k$

$\cos\varphi = \frac{|n_1\cdot n_2|}{|n_1||n_2|} = \frac{1}{2} \longrightarrow \varphi=\frac{\pi}{3}$



· 例17.8 ★

求过 L: $\begin{cases} x=1+t \\ y=1+2t \\ z=1+3t \end{cases}$ 且过点 $(2,2,2)$ 的距离为 $d=\frac{1}{\sqrt{3}}$ 的平面方程.

由 $t=x-1 \longrightarrow \begin{cases} 2x-y-1=0 & ① \\ 3x-z-2=0 & ② \end{cases}$

② 与 $(2,2,2)$ 距离

$\boxed{d=\frac{|Ax_0+By_0+Cz_0+D_0|}{\sqrt{A^2+B^2+C^2}}} = \frac{|3\times2+0-2-2|}{\sqrt{3^2+0^2+(-1)^2}} = \frac{2}{\sqrt{10}} \neq \frac{1}{\sqrt{3}}$

∴ 不是所求平面

⇒ 设平面方程: $\boxed{2x-y-1+\lambda(3x-z-2)=0}$

$\longrightarrow (2+3\lambda)x-y-\lambda z-(1+2\lambda)=0$

$\longrightarrow d=\frac{1}{\sqrt{3}}=\frac{|\quad|}{\sqrt{\quad}}$ ⇒ $\begin{cases} 5x-y-z-3=0 \\ 或\ x+y-z-1=0 \end{cases}$

七. 场论论步

1. 方向导数

1) $\begin{cases} x-x_0=\Delta x=t\cos\alpha \\ y-y_0=\Delta y=t\cos\beta \\ z-z_0=\Delta z=t\cos\gamma \end{cases}$

$t=\sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}$

$\lim_{t\to0^+}\frac{u(P)-u(P_0)}{t}$ ★ 单侧极限

$=\lim_{t\to0^+}\frac{u(x_0+t\cos\alpha,\ y_0+t\cos\beta,\ z_0+t\cos\gamma)-u(x_0,y_0,z_0)}{t}$

定义法

2). $\frac{\partial u}{\partial l}\Big|_{P_0} = u'_x(P_0)\cos\alpha + u'_y(P_0)\cos\beta + u'_z(P_0)\cos\gamma$

$= \overrightarrow{\text{grad}\,u}\cdot \overrightarrow{l_0}$

公式法

$\frac{1}{3},\frac{2}{3},\frac{2}{3}$

1, 2, 3

$\sqrt{1^2+2^2+2^2}$

· 例17.11 (公式法)

$f(x,y,z)=x^2y+z$ 在 $(1,2,0)$ 处沿向量 $\vec{n}=(1,2,2)$ 的方向导数为?

$\frac{\partial f}{\partial n}\Big|_{(1,2,0)} = \frac{\partial f}{\partial x}\Big|_{P_0}\cos\alpha + \frac{\partial f}{\partial y}\Big|_{P_0}\cos\beta + \frac{\partial f}{\partial z}\Big|_{P_0}\cos\gamma$

$=4\cdot\frac{1}{3}+1\cdot\frac{2}{3}+0\cdot\frac{2}{3} = 2$

例17.12 (2022)

$$f(x,y)=\begin{cases} x+y+\dfrac{x^3y}{x^4+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

求 $f(x,y)$ 在 $P(0,0)$ 处沿 $l=(\cos\alpha,\cos\beta)$ 的方向导数

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta f-(A\Delta x+B\Delta y)}{\rho}=0. \qquad f(0+\Delta x,0+\Delta y)-f(0,0)=f'_x(0,0)\Delta x+f'_y(0,0)\Delta y+o(\rho).$$

$$f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{x}{x}=1$$

$$f'_y(0,0)=1$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{(\Delta x)^3\Delta y}{(\Delta x)^4+(\Delta y)^2}\cdot\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=$$

取 $\Delta y=(\Delta x)^2 \to \lim\limits_{\substack{\Delta x\to 0\\ \Delta y=(\Delta x)^2}} \dfrac{(\Delta x)^5}{2(\Delta x)^4\,|\Delta x|\sqrt{1+(\Delta x)^2}}\neq 0.$

$\Rightarrow$ 不可微

令 $\Delta x=\rho\cos\alpha \quad \Delta y=\rho\cos\beta \quad \rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$

$$\left.\frac{\partial f}{\partial l}\right|_P=\lim_{\rho\to 0^+}\frac{f(\Delta x,\Delta y)-f(0,0)}{\rho}$$

$$=\lim_{\rho\to 0^+}\frac{1}{\rho}\left[\rho\cos\alpha+\rho\cos\beta+\overset{0}{\frac{\rho^4\cos^3\alpha\cos\beta}{\rho^2(\rho^2\cos^4\alpha+\cos^2\beta)}}\right]$$

$$=\cos\alpha+\cos\beta$$

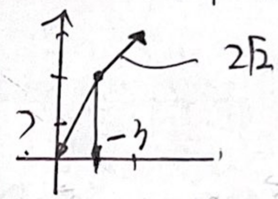
例17.14 ☆

已知 $z=f(x,y)$ 可微.

在 $P_0(1,2)$ 从 P_0 到 $P_1(2,3)$ 方向导数 $2\sqrt{2}$,

从 P_0 到 $P_2(1,0)$ 方向导数为 -3. 则 z 在

P_0 处沿从 P_0 到原点 O 方向的方向导数?



$\vec{l}_1=(\cos\alpha_1=\frac{1}{\sqrt{2}},\cos\beta_1=\frac{1}{\sqrt{2}})$

$\vec{l}_2=(\cos\alpha_2=0 \quad \cos\beta_2=-1)$

$\vec{l}_3=(\cos\alpha_3=-\frac{1}{\sqrt{5}},\cos\beta_2=-\frac{2}{\sqrt{5}})$

$$\begin{cases} \left.\dfrac{\partial f}{\partial x}\right|_{P_0}\cdot\dfrac{1}{\sqrt{2}}+\left.\dfrac{\partial f}{\partial y}\right|_{P_0}\cdot\dfrac{1}{\sqrt{2}}=2\sqrt{2} \\ \left.\dfrac{\partial f}{\partial x}\right|_{P_0}\cdot 0+\left.\dfrac{\partial f}{\partial y}\right|_{P_0}(-1)=-3 \end{cases}$$

$$\Rightarrow \begin{cases} z'_x(P_0)=1 \\ z'_y(P_0)=3 \end{cases} \longrightarrow -\frac{1}{\sqrt{5}}\cdot 1-\frac{2}{\sqrt{5}}\cdot 3$$

2. 梯度

$$\mathrm{grad}\,u\big|_{P_0} = \left(u'_x(P_0),\ u'_y(P_0),\ u'_z(P_0)\right)$$

$$\frac{\partial u}{\partial l}\Big|_{P_0} = \mathrm{grad}\,u\big|_{P_0} \cdot l^0 = \left|\mathrm{grad}\,u\big|_{P_0}\right| \cdot \cos\theta$$

$\cos\theta = 1$ 时 $\frac{\partial u}{\partial l}\Big|_{P_0}$ max

4. 散度

$$A(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

$$\mathrm{div}\,A = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

5. 旋度

$$\mathrm{rot}\,A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$