

第三讲 1000题总结

1000.3.3.

设 $f(x) = \int_0^1 \ln \sqrt{x^2+y^2} dy$, $0 \leq x \leq 1$, 则 $f'(0) = ?$

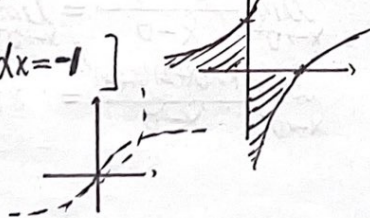
$$\begin{aligned} f(x) &= \int_0^1 \ln \sqrt{x^2+y^2} dy = y \ln \sqrt{x^2+y^2} \Big|_{y=0}^{y=1} - \int_0^1 y d \ln \sqrt{x^2+y^2} \\ &= \ln \sqrt{x^2+1} - \int_0^1 \frac{y}{\sqrt{x^2+y^2}} \cdot \frac{2y}{2\sqrt{x^2+y^2}} dy \\ &= \ln \sqrt{x^2+1} - \int_0^1 \frac{y^2}{x^2+y^2} dy = \ln \sqrt{x^2+1} - \int_0^1 \frac{y^2}{x^2+y^2} dy \\ &= \ln \sqrt{x^2+1} - \int_0^1 \left(1 - \frac{x^2}{x^2+y^2} \right) dy = \ln \sqrt{x^2+1} - 1 + \int_0^1 \frac{x^2}{x^2+y^2} dy \\ &= \ln \sqrt{x^2+1} - 1 + x \arctan \frac{y}{x} \Big|_{y=0}^{y=1} \\ &= \ln \sqrt{x^2+1} - 1 + x \arctan \frac{1}{x} \end{aligned}$$

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\ln \sqrt{x^2+1} - 1 + x \arctan \frac{1}{x} + 1}{x}$$

$$[f(0) = \int_0^1 \ln y dy = \int_{-1}^0 e^x dx = -1]$$

$$= \lim_{x \rightarrow 0} \frac{\ln \sqrt{x^2+1}}{x} + \lim_{x \rightarrow 0} \arctan \frac{1}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{2} x^2}{x} + \frac{\pi}{2} = \frac{\pi}{2}$$



1000.3.5.

$f(x)$ 可导 $y=f(x)$. x 在 $x=-1$ 处得增量 $\Delta x = -0.1$ 时
相应函数增量 Δy 的线性主部为 0.3, $f'(-1) = ?$

$$\Delta y = y' \cdot \underbrace{(\Delta x)}_{-0.1} \cdot \underbrace{dx}_{x=-1} = 0.3 \Rightarrow y' = -1$$

1000.3.11

设 $f(x)$ 在 $(-\frac{\pi}{2a}, \frac{\pi}{2a})$ ($a > 0$) 内有定义, 且 $f(0) = a$,
对 $\forall x, y, x+y \in (-\frac{\pi}{2a}, \frac{\pi}{2a})$ 有 $f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$
求 $f(x)$.

$$\textcircled{1} f(0) = \frac{2f(0)}{1-f(0)^2} \Rightarrow f(0) = 0.$$

$$\textcircled{2} f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{f(x)+f(\Delta x)}{1-f(x)f(\Delta x)} - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x)[1+f(x)]}{\Delta x[1-f(x)f(\Delta x)]}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{f(0+\Delta x) - f(0)}{\Delta x - 0} \cdot \lim_{\Delta x \rightarrow 0} \frac{1+f(x)}{1-f(x)f(\Delta x)}$$

$$= f'(0) \cdot \lim_{\Delta x \rightarrow 0} \frac{1+f(x)}{1-f(x)f(\Delta x)} = a \cdot (1+f(x))$$

$$\Rightarrow a = \frac{f'(x)}{1+f(x)} \quad a = f'(x) \cdot \arctan f(x)$$

$$g(x) = \arctan f(x) \quad \begin{cases} g'(x) = \frac{f'(x)}{1+f(x)} = a \\ = ax + C \end{cases}$$

$$\because f(0) = 0 \therefore \arctan 0 = 0 + C \Rightarrow C = 0$$

$$\Rightarrow f(x) = \tan ax$$

1000.3.13

设 $y=f(x)$ 由方程 $\sin(xy) + \ln y - x = 1$ 确定.

求 $\lim_{n \rightarrow \infty} n[f(\frac{1}{n}) - e]$

1) $x=0 \Rightarrow 0 + \ln y - 0 = 1 \Rightarrow y = e \Rightarrow f(0) = e.$

2) $\lim_{n \rightarrow \infty} n[f(\frac{1}{n}) - e] \stackrel{f(x)}{=} \lim_{x \rightarrow 0} \frac{f(x) - e}{\frac{1}{n}} = \lim_{x \rightarrow 0} 2f'(2x)$

3) $x' \begin{cases} \sin(xy) + \ln y - x = 1 \\ \cos(xy) \cdot (y + xy') + \frac{y'}{y} - 1 = 0. \end{cases}$

$f(0) \Rightarrow \cos(0) \cdot (e + 0) + \frac{y'}{e} - 1 = 0.$

$y' = (1-e)e$

$\Rightarrow \text{原} = 2e(1-e)$

1000.3.15

设 $f(x) = \begin{cases} x^\lambda \sin \frac{1}{x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$ 求 $f'(x)$ 并判定 $f'(x)$ 连续时 λ 范围

1) 导数存在.

左导数 $= 0$. 右导数 $= \lim_{x \rightarrow 0^+} \frac{x^\lambda \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \rightarrow 0^+} x^{\lambda-1} \sin \frac{1}{x}.$

$\Rightarrow \lambda - 1 > 0 \Rightarrow \lambda > 1$

2) $f'(x) = \lambda x^{\lambda-1} \sin \frac{1}{x} + x^\lambda \cos \frac{1}{x} \cdot \frac{-1}{x^2}$
 $= \lambda x^{\lambda-1} \sin \frac{1}{x} - x^{\lambda-2} \cos \frac{1}{x}$

当 $x < 0$ 时, $f'(x) = 0$

当 $\lambda > 2$ 时, $x > 0$ 时 $f'(x) = 0 \Rightarrow \lambda > 2$

1000.3.5.1

已知 $f(0)=0, f'(0)=2$, 求 $\lim_{n \rightarrow \infty} [f(\frac{1}{n^2}) - \frac{1}{n^2} + 1]^{3n^2} = ?$

[分析]

$\lim_{x \rightarrow 0} [f(x^2) - x^2 + 1]^{\frac{3}{x^2}} = \lim_{x \rightarrow 0} e^{\frac{3}{x^2} \ln [f(x^2) - x^2 + 1]}$
 $= \lim_{x \rightarrow 0} e^{\frac{3 \cdot [2x f'(x) - 2x]}{2x [f(x^2) - x^2 + 1]}} = \lim_{x \rightarrow 0} e^{\frac{3 [f'(x^2) - 1]}{f(x^2) - x^2 + 1}} = e^{\frac{3(2-1)}{0-0+1}} = e^3$

1000.3.5.4

$f(x)$ 在 $(-\infty, +\infty)$ 有定义, 在 $[0, 2]$ 上, $f(x) = x(x^2 - 4)$ $\forall x: f(x) = k f(x+2)$

1) 写出 $f(x)$ 在 $[-2, 0]$ 上的表达式.

$x \in [-2, 0] \quad f(x) = k f(x+2) = k(x+2)[x^2 + 4x] = kx(x+2)(x+4)$

$x+2 \in [0, 2]$

2) k 为多少时, $f(x)$ 在 $x=0$ 处可导?

$f(x) = \begin{cases} x(x^2 - 4), & x \in [0, 2] \\ kx(x+2)(x+4), & x \in [-2, 0] \end{cases}$

$\lim_{x \rightarrow 0^+} \frac{x(x^2 - 4)}{x - 0} = \lim_{x \rightarrow 0^+} (x^2 - 4) = -4$

$\lim_{x \rightarrow 0^-} \frac{kx(x+2)(x+4)}{x - 0} = 8k \Rightarrow k = -\frac{1}{2}$