第6讲 习题总结

- 1000.6.4

$f(x)$ 在 $[0,1]$ 可导，$f(0)=0$. $|f'(x)| \le |f(x)|$，$f(1)=?$

[solution]. $|f(x)| = |f(x)-f(0)|$ $\le \le \cdots x^n(\ )$

$x \in (0,1)$. $\le x \cdot |f(\xi_1)|$

$0 < |f(x)| = |f(x)-f(0)| = (x-0)|f'(\xi_1)|$ $\cancel{\ge |f'(x)|}$

$= \cancel{\le}\ x \cdot |f(\xi_1)-f(0)| = x^2|f'(\xi_2)| \le x^2|f(\xi_2)|$

$\cdots \le x^n|f(\xi_n)|$

可导 $\to$ 有界. $x^n \longrightarrow 0$. $\lim_{n\to\infty} x^n|f(\xi_n)| = 0$.

$x \in (0,1)$ $f(x)=0$. ，左连续 $\longrightarrow$ $f(1)=0$.

- 1000.6.0.

$f(x)$，$[0,1]$ 连续，$(0,1)$ 可导，$f(0)=0$ $f(1)=1$.

(1) 证明 $x_0 \in (0,1)$. $f(x_0) = 2-3x_0$.

$\to g(x) = \frac{f(x)+3x}{2}$ $g(0)=0$ $g(1)=1$ $\xrightarrow{拉} g'(\xi)=1$

(2) 证明 $a,b \in (0,1)$. $a \ne b$，$[1+f'(a)][1+f'(b)] = 4$.

由 $f(x_0) = 2-3x_0$.

$f'(a) \overset{拉}{=} \frac{-1+3x_0}{1-x_0}$ $f'(b) \overset{拉}{=} \frac{2-3x_0}{x_0-0}$

$[f'(a)+1][f'(b)+1] = 4$.

- 1000.6.8

$[a,b]$ 连续 (a,b) 可导.

证明 ξ. $\frac{1}{a-b}\begin{vmatrix} a & b \\ f(a) & f(b) \end{vmatrix} = f(\xi)-\xi f'(\xi)$ 成立.

[solution]. $\frac{1}{a-b}\begin{vmatrix} a & b \\ f(a) & f(b) \end{vmatrix} = \frac{af(b)-bf(a)}{a-b} = \boxed{\dfrac{\frac{f(b)}{b}-\frac{f(a)}{a}}{\frac{1}{b}-\frac{1}{a}}}$ 柯西

$g(x) = \frac{f(x)}{x}$ $\left(\frac{g(b)-g(a)}{b-a} = g'(\xi) = \frac{f'(\xi)\xi - f(\xi)}{\xi^2}\right)$

$\varphi(x) = \frac{1}{x}$

柯西: $\frac{g(b)-g(a)}{\varphi(b)-\varphi(a)} = \frac{g'(\xi)}{\varphi'(\xi)} = -(f'(\xi)\xi - f(\xi))$

- 1000.6.9

$f(x)$ 在 $[1,2]$ 可导. 证明 $\xi \in (1,2)$. $f(2)-2f(1) = \xi f'(\xi) - f(\xi)$.

[solution]. $xf'(x) - f(x) = f(2)-2f(1)$

$\to \left[\frac{f(x)}{x}\right]' = \left(-\frac{1}{x}\right)' \cdot [f(2)-2f(1)]$

证 $[F(x)]' = \left[\frac{f(x)+f(2)-2f(1)}{x}\right]' = 0$.

$F(1) = f(1)+f(2)-2f(1) = f(2)-f(1)$

$F(2) = 2(f(2)-f(1))/2 = f(2)-f(1)$

$F'(\xi) = \frac{F(2)-F(1)}{2-1} = 0$.

1000.6.10.(2).

$f(x)$，$g(x)$ 在 $[a,b]$ 二阶可导，$g''(x)\neq 0$．$g(x)\neq 0$．

$f(a)=f(b)=g(a)=g(b)=0$

证 ξ．$\dfrac{f(\xi)}{g(\xi)}=\dfrac{f''(\xi)}{g''(\xi)}$ $\longrightarrow$ $f(x)g''(x)-f''(x)g(x)=0$.

$\nearrow F'(x)$

令 $F(x)=f(x)g'(x)-f'(x)g(x)$

$F(a)=F(b)=0$ $\quad F'(\xi)=0$.

1000.6.11.

$[a,b]$ 连续，(a,b) 可导．$f'(x)\neq 0$.

证 $\dfrac{f'(\xi)}{f'(\eta)}=\dfrac{e^b-e^a}{b-a}e^{-\eta}$.

[solution]. 柯西 $\dfrac{f(b)-f(a)}{e^b-e^a}=f'(\eta)\cdot e^{-\eta}$ ①

拉中 $\dfrac{f(b)-f(a)}{b-a}=f'(\xi)$ ②

$\dfrac{②}{①}\Rightarrow\dfrac{f'(\xi)}{f'(\eta)}=\dfrac{e^b-e^a}{b-a}e^{-\eta}$

1000.6.19

证明 $e^x+e^{-x}\geqslant 2x^2+2\cos x$ $\quad(-\infty<x<+\infty)$

$F(x)=e^x+e^{-x}-2x^2+2\cos x$

$F'(x)=e^x-e^{-x}-4x-2\sin x$

$F''(x)=e^x+e^{-x}-4-2\cos x$

$F'''(x)=e^x-e^{-x}+2\sin x$ $\quad F''''(x)=e^x+e^{-x}+2\cos x$

$f(x)$ 为偶函数，只证 $x>0$ 部分即可．

$f^{(4)}(x)=e^x+e^{-x}+2\cos x\geqslant 2\sqrt{e^x\cdot e^{-x}}+2\cos x=2(1+\cos x)\geqslant 0.$

$f(0)=f'(0)=f''(0)=f'''(0)=0.$

$\Rightarrow$ 泰勒 $f(x)=f(0)+f'(0)x+\dfrac{f''(0)}{2}x^2+\dfrac{f'''(0)}{3!}x^3+\dfrac{f^{(4)}(\xi)}{4!}x^4$

$=\dfrac{f^{(4)}(\xi)}{4!}x^4\geqslant 0.$

△?

1000.6.20

$f(x)$ 在 $[0,+\infty)$ 二阶导数，且 $|f(x)|\leq 1$,

$0<|f''(x)|\leq 2$ $\quad(0\leq x\leq+\infty)$

$f(x+h)=f(x)+f'(x)h+\frac{1}{2!}f''(\xi)h^2.$

$\xi\in(x,x+h).$

$\Rightarrow f'(x)=\frac{1}{h}[f(x+h)-f(x)]-\frac{h}{2}f''(\xi).$

$|f'(x)|\leq\frac{1}{h}[|f(x+h)|+|f(x)|]+\frac{h}{2}|f''(\xi)|\leq\frac{2}{h}+h$

令 $g(h)=\frac{2}{h}+h$ $(h>0)$，求其最小值

令 $g'(h)=-\frac{2}{h^2}+1=0$，得到驻点 $h_0=\sqrt{2}$，$g''(h)=\frac{4}{h^3}>0.$

$g(h)$，$h_0=\sqrt{2}$．$min\rightarrow$ $g(h_0)=2\sqrt{2}.$

$\therefore|f'(x)|\leq 2\sqrt{2}$ $\quad(x\in[0,+\infty))$

1000.6.22

$f(x)$ 在 $[0,1]$ 二阶可导. $f(0)=0$ $f(1)=1$.

存在 $\xi\in(0,1)$ s.t. $\xi f''(\xi)+(1+\xi)f'(\xi)=1+\xi$.

$xf''(x)+(1+x)f'(x)=1+x$

$xf''(x)+(1+x)f'(x)-1-x=0 \Rightarrow e^x[xf''(x)+(1+x)f'(x)-1-x]=0$

辅助函数: $xe^x[f'(x)-1]=F(x)$

$F(0)=0$ $F(1)=1$. $F'(\xi)=1$.

1000.6.27

$p\geq 0$. 证明不等式 $\frac{p-1}{p}a+\frac{1}{p}a^{1-p}b^p\geq b$

对一切 $a,b>0$ 都成立. 把a当成x].

[solution].

$p=1$时. $a+b\geq b$

$p>1$时. $f(x)=\frac{p-1}{p}x+\frac{1}{p}x^{1-p}b^p$ $(x>0)$.

$f'(x)=\frac{p-1}{p}+\frac{1-p}{p}x^{-p}b^p \stackrel{\text{令}}{=} 0 \Rightarrow x=b$. 极小值

$f''(x)=\cdots>0$. $(x>0)$

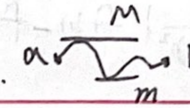
$\therefore f(a)\geq f(b)=b$.

1000.06.1

$f(x)$在$[a,b]$二阶可导, $f(a)=f(b)=0$.

$f''(x)+\cos f'(x)=e^{f(x)}$, $f(x)$ 不小于/不大于/恒为0?

[solution].

$f(x)$有 m, M.  (设) 极 m,M.)

极大值点: $f'(c)=0$ $f''(c)=e^{f(c)}-1<0$.

又$\because f(c)>0$ 矛盾.

极小值点: $f'(d)=0$. $f''(d)=e^{f(d)}-1>0$.

又$\because f(d)<0$. 矛盾.

$\Rightarrow$ 恒为0.

1000.06.2

设$f(x)$在$[0,+\infty)$上可导, $f(0)=0$, 且存在常数$k>0$,

使得 $|f'(x)|\leq k|f(x)|$ 在$[0,+\infty)$ 成立.

则在$(0,+\infty)$上

[solution]

设 $x_0\in[0,\frac{1}{2k}]$ $|f(x_0)|$是 $|f(x)|$在$[0,\frac{1}{2k}]$ max

拉中: $f(x_0)-f(0)=f'(\xi)(x_0-0)$, $\xi\in(0,x_0)\subset[0,\frac{1}{2k}]$

$|f(x_0)|=|f(0)+f'(\xi)x_0|=|f'(\xi)|x_0\leq k|f(\xi)|\frac{1}{2k}$

$\leq k|f(x_0)|\cdot\frac{1}{2k}=\frac{1}{2}|f(x_0)|$ $\Rightarrow |f(x_0)|=0$.

$x\in[0,\frac{1}{2k}]$时 $f(x)$恒为0

- 1000. 6t. 4

$f(x)$ 在 $[a,b]$ 二阶可导，且 $f'(a)=f'(b)=0$.

证明存在 $\xi\in(a,b)$ 使.

$$|f''(\xi)| \geq \frac{4}{(b-a)^2}|f(b)-f(a)|$$

[solution]

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2}(x-x_0)^2$$

在 $x_0=a$ 与 b 处:

$$\begin{cases} f(x)=f(a)+\frac{f''(\xi_1)}{2}(x-a)^2 & \xi_1\in(a,x). \ ① \\ f(x)=f(b)+\frac{f''(\xi_2)}{2}(x-b)^2 & \xi_2\in(x,b). \ ② \end{cases}$$

$x=\frac{a+b}{2}$ ② − ①

$$0=f(b)-f(a)+\frac{(b-a)^2}{4}\cdot\frac{1}{2}[f''(\xi_2)-f''(\xi_1)]$$

$$\Rightarrow \frac{4|f(b)-f(a)|}{(b-a)^2}=\frac{1}{2}|f''(\xi_2)-f''(\xi_1)|\leq\frac{1}{2}\left[|f''(\xi_1)|+|f''(\xi_2)|\right]$$

令 $|f''(\xi)|=\max\{|f''(\xi_1)|,|f''(\xi_2)|\}$. $\xi\in(a,b)$.

$$\frac{1}{2}[|f''(\xi_1)|+|f''(\xi_2)|]\leq\frac{1}{2}\times2\times|f''(\xi)|=|f''(\xi)|$$

- 1000. 6t. 5

设 $f(x)$ 在 $[a,b]$ 上具有二阶导数, 且 $f''(x)>0$.

证明: $f(\frac{a+b}{2})<\frac{1}{b-a}\int_a^b f(t)dt<\frac{1}{2}[f(a)+f(b)]$

[solution]. 将上限令成 x 改成了变上限积分.

① 令 $\varphi(x)=$ ~~$\frac{b-a}{}$~~ $(x-a)f(\frac{a+x}{2})-\int_a^x f(t)dt.$

$(\varphi(a)=0)$.

$$\varphi'(x)=f(\frac{a+x}{2})+\frac{(x-a)}{2}f'(\frac{a+x}{2})-\cancel{f(a)}\,f(x)$$

$$=\frac{x-a}{2}\cdot f'(\frac{a+x}{2})+\left[f(\frac{a+x}{2})-f(x)\right]$$

$$=\frac{x-a}{2}\cdot f'(\frac{a+x}{2})+f'(\xi)(\frac{a+x}{2}-x)$$

$$=\frac{x-a}{2}\left[f'(\frac{a+x}{2})-f'(\xi)\right]\quad \xi\in(\frac{a+x}{2},x)$$

$$=\underbrace{\frac{x-a}{2}}_{>0}\cdot\underbrace{f''(\xi_2)}_{>0}\underbrace{(\frac{a+x}{2}-\xi)}_{>0}>0.\quad \xi_2\in(\frac{a+x}{2},\xi)$$

$\therefore \varphi(x)>0\quad (x\in(a,b))$

$$\Rightarrow f(\frac{a+b}{2})<\frac{1}{b-a}\int_a^b f(t)dt.$$

② 令 $\psi(x)=\frac{x-a}{2}[f(a)+f(x)]-\int_a^x f(t)dt$

$$\psi'(x)=\frac{1}{2}[f(a)+f(x)]+\frac{x-a}{2}f'(x)-f(x)$$

$$=\frac{1}{2}[f(a)-f(x)]+\frac{x-a}{2}f'(x)\qquad \xi_2\in(a,\xi)$$

$$=\frac{1}{2}f'(\xi)\cdot(a-x)+\frac{x-a}{2}f'(x)\qquad \xi\in(a,x).$$

$$=\frac{x-a}{2}(f'(x)-f'(\xi))=\frac{x-a}{2}f''(\xi_2)(x-\xi)>0.$$

$\psi(a)=0\Rightarrow$ 得证

(1000. 6t. 6)

证明 $\cos\sqrt{2}x < -x^2 + \sqrt{1+x^4}$ $\quad x\in(0, \frac{\sqrt{2}}{4}\pi)$.

$f(x) = \cos\sqrt{2}x + x^2 - \sqrt{1+x^4}$

$f(0) = 1+0-1 = 0.$

$f'(x) = -\sqrt{2}\sin\sqrt{2}x + 2x - \frac{1}{2}(1+x^4)^{-\frac{1}{2}}\cdot 4x^3$

$f'(x) = -\sqrt{2}\sin\sqrt{2}x + 2x - 2x^3(1+x^4)^{-\frac{1}{2}}$

$= \sqrt{2}(\sqrt{2}x - \sin\sqrt{2}x) - 2x^3(1+x^4)^{-\frac{1}{2}}$

$g(t) = t - \sin t < 0$ ~~$g'(t) = 1-\cos t = 0$ $t = 2\pi k$ $k=0,1,2\ldots$~~

~~$x\in(0,\frac{\sqrt{2}}{4}\pi)$, $t\in(0,\frac{\sqrt{2}}{2}\pi)$, $\frac{\sqrt{2}}{2}\pi < 2\pi$, $g'(t)$~~

$\therefore f'(x) < 0.$

得证

1000. 6t. 7 　6t. 5 → 变上限积分 ⇒ 6t. 7 拆成变上限积分.

$F(x) = \int_{-1}^{1} |x-t| e^{t^2} dt - \frac{1}{2}(e^{-1}+1)$, 讨论 $F(x)$ 在 $[-1,1]$ 零点个数.

~~$F'(x)=$~~ $F(x) = \int_{-1}^{x} \underbrace{|x-t|}_{x>t} e^{t^2}dt + \int_{x}^{1} \underbrace{|x-t|}_{x<t} e^{t^2}dt - \frac{1}{2}(e^{-1}+1)$

$F(x) = x\int_{-1}^{x} e^{t^2}dt - \int_{-1}^{x} te^{t^2}dt - x\int_{x}^{1} e^{t^2}dt + \int_{x}^{1} te^{t^2}dt - \frac{1}{2}(e^{-1}+1)$

$= x\int_{-1}^{x} e^{t^2}dt - \int_{-1}^{x} te^{t^2}dt + x\int_{1}^{x} e^{t^2}dt - \int_{1}^{x} te^{t^2}dt - \frac{1}{2}(e^{-1}+1)$

$F'(x) = \int_{-1}^{x} e^{t^2}dt + \cancel{xe^{x^2}} - \cancel{xe^{x^2}} + \cancel{xe^{x^2}} + \int_{1}^{x} e^{t^2}dt - \cancel{xe^{x^2}}$

$= \int_{-1}^{x} e^{t^2}dt + \int_{1}^{x} e^{t^2}dt = \int_{0}^{x} e^{t^2}dt + \underbrace{\int_{-1}^{0} e^{t^2}dt + \int_{1}^{x} e^{t^2}dt}_{=\int_0^1}$

$= 2\int_{0}^{x} e^{t^2}dt$ ~~>0~~.

$F(x)$ 在 $[-1,0]$ ↘ 　$[0,1]$ ↗.

$F(-1) = 0 + 2\int_{0}^{1} e^{t^2}dt - \frac{1}{2}(e^{-1}+1)$

$> 2\int_{0}^{1} e^{-t}dt - \frac{1}{2}(e^{-1}+1) = \frac{3}{2} - \frac{5}{2}e^{-1} > 0$

$F(0) = \frac{1}{2} - \frac{3}{2}e^{-1} < 0.$

零点单调 + 對称性 　2个.

1000. 6t. 8.

$f(x_0) = x_0$, x_0 为 I 区间 I 上不动点.

$f(x) = 3x^2 + \frac{1}{x^2} - \frac{18}{25}$, 则 $f(x)$ 在 $(0,+\infty)$ 是否有不动点.?

[solution] 　　$-x^{-2}$

$g(x) = 3x^2 + \frac{1}{x^2} - \frac{18}{25} - x$ 　　$g(x)=0$. 不动点. $x>0$.

$g'(x) = 6x - 1 - 2x^{-3}$ ~~$= 0$~~ $= \frac{6x^4 - x^3 - 2}{x^3}$

$\Rightarrow 6x^4 - x^3 - 2 = 0.$

$u(x) = 6x^4 - x^3 - 2$ 　$u'(x) = 24x^3 - 3x^2 = 3x^2(8x-1)$

↘ $x_1 = 0$ ↘ $x_2 = \frac{1}{8}$ ↗

$u(\frac{1}{8}) = \frac{6}{4096} - \frac{1}{512} - 2 < 0.$ 　$u(0)=0$. 有唯一零点

$6x^4 - x^3 - 2 = 0$

$\therefore u(x) > 0$ $(x>0)$.

$\therefore$ ~~$g'(x) > 0.$ $(x>0)$ 　$g(0) = -\frac{18}{25}$~~ 　估算 $x_0\in(\frac{1}{2},1)$

~~$\lim_{x\to+\infty} g(x) = +\infty$~~ 　$g(x)$ 在 $(0,x_0)$ ↘ 　$(x_0,+\infty)$ ↗

~~有不动点, 一个.~~

~~$3x^2 - \frac{1}{x} - \frac{18}{25} - x = 0$~~

$\min\{g(x)\} = g(x_0) = 3x_0^2 + \frac{1}{x_0^2} - x_0 - \frac{18}{25} > \frac{3}{4} - \frac{18}{25} = \frac{3}{100} > 0.$ 无不动点

$\geq 2\sqrt{3} - x_0 \geq 2\sqrt{3} - 1 - \frac{15}{4} > 0.$