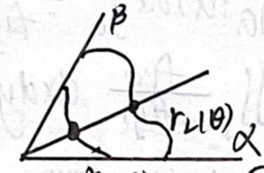
二. 计算

① 直角坐标系与换序

回路

② 极坐标系与换序



$$\iint_D f(x,y)\,d\sigma = \int_\alpha^\beta d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, \boxed{r\,dr}\ \cancel{d\theta}$$

③ 直极互化

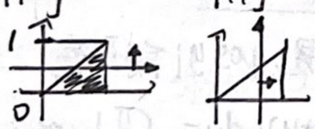
例 14.7 直角坐标系换序

$$I=\int_0^1 dy\int_y^1 \frac{y}{1+x^2+y^2}\,dx$$ ←"帮倒忙".

$$=\int_0^1 y\,dy\int_y^1 \frac{1}{\sqrt{1+y^2}^2+x^2}\,dx$$

$$=\int_0^1 y\cdot\frac{1}{\sqrt{1+y^2}}\arctan\frac{y}{\sqrt{1+y^2}}$$ ← 繁

换序!



$$原=\int_0^1 dx\int_0^x \frac{y}{1+x^2+y^2}\,dy$$

$$=\frac{1}{2}\int_0^1 dx\int_0^x \frac{d(1+x^2+y^2)}{1+x^2+y^2}=\frac{1}{2}\int_0^1 \ln(1+x^2+y^2)\Big|_{y=0}^{y=x} dx$$

$$=\frac{1}{2}\int_0^1 \ln\frac{1+2x^2}{1+x^2}\,dx \cancel{=\frac{1}{2}\left[\int_0^1 0+\ln\frac{x^2}{1+x^2}dx\right]}$$

$$\ln\left(\frac{1+2x^2}{1+x^2}\right)'=\frac{1+x^2}{1+2x^2}\cdot\frac{\overset{4x+\cancel{4x^3}-2x-\cancel{4x^3}}{4x(1+x^2)-(1+2x^2)2x}}{(1+x^2)^2}$$

$$=\frac{2x}{(1+2x^2)(1+x^2)}\ \cancel{=\ \frac{}{1+2x^2}\ \ \frac{}{1+x^2}}$$

$$\therefore 原=\cancel{\int_0^1}\ \frac{x}{2}\ln\frac{1+2x^2}{1+x^2}\Big|_0^1-\frac{1}{2}\int_0^1\frac{2x^2}{(1+2x^2)(1+x^2)}\,dx$$

$$=\frac{1}{2}\ln\frac{3}{2}+\int_0^1\frac{1}{1+2x^2}-\frac{1}{1+x^2}\,dx \qquad \frac{1}{2}\cdot\frac{1}{\frac{1}{2}+x^2}$$

$$=\frac{1}{2}\ln\frac{3}{2}+\left[\frac{1}{2}\cdot\sqrt{2}\arctan\sqrt{2}x-\arctan x\right]_0^1$$

$$=\frac{1}{2}\ln\frac{3}{2}+\left[\frac{\sqrt{2}}{2}\arctan\sqrt{2}-0-(\arctan 1-0)\right]$$

$$=\frac{1}{2}\ln\frac{3}{2}+\frac{\sqrt{2}}{2}\arctan\sqrt{2}-\frac{\pi}{4}$$

例 14.10. 极坐标系换序.

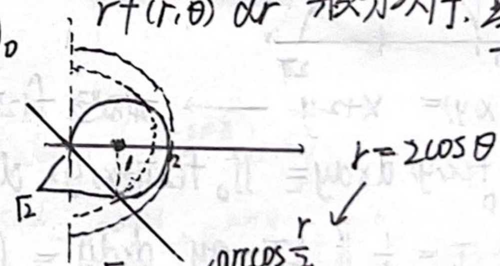
交换 $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta\int_0^{2\cos\theta} r f(r,\theta)\,dr$ 积分次序. 其中 $f(r,\theta)$.

$r=2\cos\theta$

$r^2=4\cos^2\theta$

$x^2+y^2=2x$

$(x-1)^2+y^2=1$



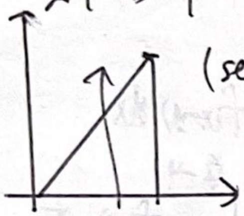
$$I=\int_0^{\sqrt{2}} \underline{r\,dr}\int_{-\frac{\pi}{4}}^{\arccos\frac{r}{2}} f(r\cos\theta, r\sin\theta)\,d\theta$$

$$+\int_{\sqrt{2}}^{2} r\,dr\int_{-\arccos\frac{r}{2}}^{\arccos\frac{r}{2}} f(r\cos\theta, r\sin\theta)\,d\theta$$

例14.12.

计算 $I=\iint_D \sqrt{1-r^2\cos2\theta}\quad r^2\sin\theta\, dr d\theta$.

其中 $D=\{(r,\theta)|0\le r\le\sec\theta,\ 0\le\theta\le\frac{\pi}{4}\}$



$(\sec\theta=\frac{1}{\cos\theta}=\frac{r_y}{r})$

$$I=\iint_D y\sqrt{1-x^2+y^2}\ d\sigma$$

$$=\int_0^1 dx\int_0^x y\sqrt{1-x^2+y^2}\,dy$$

$$=\frac12\int_0^1 dx\int_0^x (1-x^2+y^2)^{\frac12}\,d(1-x^2+y^2)$$

$$=\frac13\int_0^1 \cancel{dx}\ (1-x^2+y^2)^{\frac32}\Big|_{y=0}^{y=x} dx$$

$$=\frac13\int_0^1 [1-(1-x^2)^{\frac32}]\,dx$$

$x=\sin t$

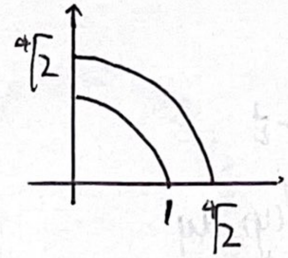
$$=\frac13-\frac13\int_0^{\frac{\pi}{2}} \sin^4 t\,dt=\frac13-\frac13\cdot\frac34\cdot\frac12\cdot\frac{\pi}{2}=\frac13-\frac{\pi}{16}$$

- 其他的前也有. §10 §14

例14.15. [取整函数]

$D=\{(x,y)|x^2+y^2\le\sqrt2,\ x\ge0\ y\ge0\}$, $[1+x^2+y^2]$ 表示不超过 $1+x^2+y^2$ 的最大整数, 计算二重积分

$\iint_D xy[1+x^2+y^2]\,dxdy$



① $0\le x^2+y^2<1$ $[1+x^2+y^2]=1$.

② $1\le x^2+y^2\le\sqrt2$ $[1+x^2+y^2]=2$

$$I=\iint_{D_1}xy\,d\sigma+2\iint_{D_2}xy\,d\sigma$$

$$=2\iint_D xy\,d\sigma-\iint_{D_1}xy\,d\sigma$$

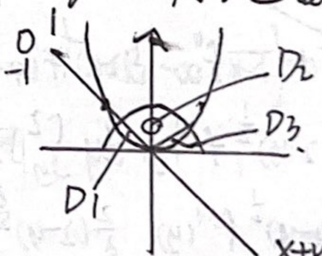
$$=2\int_0^{\frac\pi2}d\theta\int_0^{\sqrt[4]{2}} r^2\sin\theta\cos\theta\ r\,dr-\int_0^{\frac\pi2}d\theta\int_0^1 r^2\sin\theta\cos\theta\cdot r\,dr$$

$\cdots=\frac38$.

例14.16 [符号函数]

$I=\iint_D sgn(x+y)e^{x^2+y^2}\,dxdy$. 区域 $D=\{(x,y)|x^2\le y\le\sqrt{1-x^2}\}$ 其中 sgn 为符号函数.

$$sgn\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$$



x+y=0.

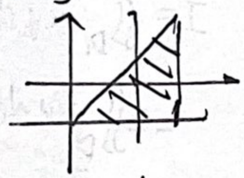
$$I=\iint_D=\cancel{\iint_{D_1}e^{x^2+y^2}d\sigma}+\iint_{D_2}e^{x^2+y^2}d\sigma+\cancel{\iint_{D_3}e^{x^2+y^2}d\sigma}=\iint_{D_2}e^{x^2+y^2}d\sigma$$

$$=2\int_{\frac\pi4}^{\frac\pi2}d\theta\int_0^1 e^{r^2}r\,dr=\frac{\pi}{4}(e-1)$$

例 14.17 【抽象函数】

$f(0)=f'(0)=f''(0)=-1$. $f(2)=-\frac{1}{2}$

$I=\int_0^2 dx\int_0^x \sqrt{(2-x)(2-y)}\, f'''(y)\,dy$



$=\int_0^2 dx\,[\sqrt{(2-x)(2-y)}$

$=\int_0^2\sqrt{2-x}\,dx\int_0^x\sqrt{2-y}\,d f''(y)$

$=\int_0^2\sqrt{2-x}\,dx\left[\sqrt{2-y}\,f''(y)\Big|_0^x+\frac{1}{2}\int_0^x(2-y)^{-\frac{1}{2}}f''(y)\,dy\right]$

$=\int_0^2\sqrt{2-x}\,dx\left[\sqrt{2-x}\,f''(x)+\frac{1}{2}\int_0^x(2-y)^{-\frac{1}{2}}\,df'(y)\right]$

$=\int_0^2\sqrt{2-x}\,dx\left[\sqrt{2-x}\,f''(x)+\frac{1}{2}(2-y)^{-\frac{1}{2}}f'(y)\Big|_0^x+\int_0^x f'(y)(2-y)^{-\frac{3}{2}}dy\right]$

$(-\frac{1}{4})$

$=\int_0^2\sqrt{2-x}\,dx\left[\sqrt{2-x}\,f''(x)-\frac{1}{2}(2-x)^{-\frac{1}{2}}f'(x)-\frac{1}{4}\int_0^x(2-y)^{-\frac{3}{2}}\,d f(y)\right]$

$=\int_0^2\sqrt{2-x}\,dx\left(\sqrt{2-x}\,f''(x)-\frac{1}{2}(2-x)^{-\frac{1}{2}}f'(x)-\frac{1}{4}(2-y)^{-\frac{3}{2}}f(y)\Big|_0^x+\frac{1}{4}\int_0^x f(y)\right.$.

$I=\int_0^2(2-y)^{\frac{1}{2}}f'''(y)\,dy\cdot\int_y^2\sqrt{2-x}\,dx$

$=\int_0^2(2-y)^{\frac{1}{2}}f'''(y)\ \frac{2}{3}(2-y)^{\frac{3}{2}}dy=\frac{2}{3}\int_0^2(2-y)^2f'''(y)\,dy$

$\frac{8}{3}+$ $=\frac{2}{3}(2-y)^2f''(y)\Big|_0^2+\frac{2}{3}\int_0^2 f''(y)\cdot 2(2-y)\,dy$

$\frac{8}{3}+=\frac{2}{3}\ 2(2-y)\cdot f'(y)\Big|_0^2-\frac{2}{3}\int_0^2 f'(y)(-2)\,dy$

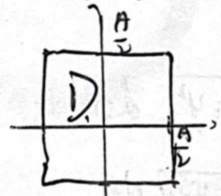
$=\frac{16}{3}+\frac{4}{3}\int_0^2 f'(y)\,dy=\frac{4}{3}f(y)\Big|_0^2=\cancel{-\frac{2}{3}}$ 6.

例 14.18 (2022 1)

设 $f(t)$ 连续 证明

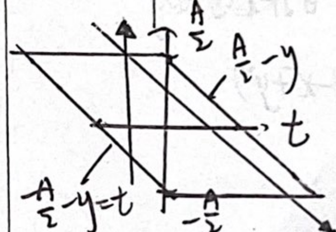
$\iint_D f(x-y)\,dxdy=\int_{-A}^{A} f(t)(A-|t|)\,dt$

D: $|x|\le\frac{A}{2}$, $|y|\le\frac{A}{2}$.



左 $=\int_{-\frac{A}{2}}^{\frac{A}{2}}dy\int_{-\frac{A}{2}}^{\frac{A}{2}}f(x-y)\,dx$

$x-y=t$ $\int_{-\frac{A}{2}}^{\frac{A}{2}}dy\int_{-\frac{A}{2}-y}^{\frac{A}{2}-y}f(t)\,dt$



$=\int_{-A}^{0}dt\int_{-\frac{A}{2}-t}^{\frac{A}{2}}f(t)\,dy$

$+\int_0^A dt\int_{-\frac{A}{2}}^{\frac{A}{2}-t}f(t)\,dt$

$=\int_{-A}^0 f(t)\cdot(A+t)\,dt+\int_0^A f(t)(A-t)\,dt$

$=\int_{-A}^0 f(t)(A-|t|)\,dt+\int_0^A f(t)(A-|t|)\,dt$

$=\int_{-A}^{A} f(t)(A-|t|)\,dt$

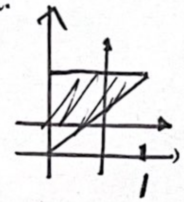
例14.19 (作业)

设连续 $f(x) = 1+\frac{1}{2}\int_x^1 f(y)f(y-x)\,dy$.

$I=\int_0^1 f(x)\,dx$.

(1). 证明 $I = 1+\frac{1}{2}\int_0^1 f(y)\,dy\int_0^y f(y-x)\,dx$.

(2). 求 I 的值.



$$I=\int_0^1 f(x)\,dx = 1+\frac{1}{2}\int_0^1 dx\int_x^1 f(y)f(y-x)\,dy$$

$$=1+\frac{1}{2}\int_0^1 dy\int_0^y f(y)f(y-x)\,dx$$

$$=1+\frac{1}{2}\int_0^1 f(y)\,dy\int_0^y f(y-x)\,dx$$

令 $t=y-x$,

$$I=1+\frac{1}{2}\int_0^1 f(y)\,dy\int_0^y f(t)\,dt$$

$$=1+\frac{1}{2}\int_0^1\left(f(y)\int_0^y f(t)\,dt\right)dy$$

$$=1+\frac{1}{2}\int_0^1\left[\int_0^y f(t)\,dt\right]d\left[\int_0^y f(t)\,dt\right]$$

$$=1+\frac{1}{4}\left[\int_0^y f(t)\,dt\right]^2\Big|_0^1 = 1+\frac{1}{4}I^2$$

$\Rightarrow I=2$

例14.20 (易).

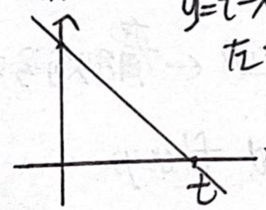
$f(x)$ 在 $[0,1]$ 连续导数, $f(0)=1$

$$\iint_{D_t} f'(x+y)\,dxdy = \iint_{D_t} f(t)\,dxdy$$

其中 $D_t=\{(x,t)\mid 0\le y\le t-x,\ 0\le x\le t\}$. $(0<t\le 1)$

求 $f(x)$?

$y=t-x$



左 $=\iint_{D_t} f'(x+y)\,dxdy = \cancel{\iint_{D_t} f'(t)\,d\sigma}$

$$=\int_0^t dx\int_0^{t-x} f'(x+y)\,dy$$

$$=\int_0^t\left[f(x+y)\Big|_{y=0}^{y=t-x}\right]dx$$

$$=\int_0^t [f(t)-f(x)]\,dx$$

$$=f(t)\cdot t-\int_0^t f(x)\,dx.$$

右 $=\iint_{D_t} f(t)\,dxdy$

$=S_t=\frac{1}{2}t^2 f(t)$

$f(t)\cdot t-\int_0^t f(x)\,dx = \frac{t^2}{2}f(t)$

$\rightarrow f'(t)\,t+\cancel{f(t)}-\cancel{f(t)} = t f(t)+\frac{t^2}{2}f'(t)$

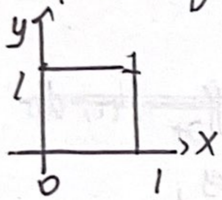
$(t-\frac{t^2}{2})f'(t) = t f(t)$. → 微分方程

例 14.21 (难). ★★★

已知 $f(x,y)$ 有二阶连续偏导数, $f(1,y)=0$, $f(x,1)=a$

$\iint_D f(x,y)\,dxdy = a$ $D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 1\}$

计算 $I=\iint_D xy\, f''_{xy}(x,y)\,dxdy$



$f''_{xy}(x,y)=\dfrac{\partial f'_x(x,y)}{\partial y}$

$f''_{xy}(x,y)=\dfrac{d f'_x(x,y)}{dy}$ ← 在积分号中.

$f''_{xy}(x,y)\,dy = d\, f'_x(x,y)$

∴

$I=\int_0^1 x\,dx\cdot\int_0^1 y\, d f'_x(x,y)$

$=\int_0^1 x\cdot\left[\, y f'_x(x,y)\Big|_{y=0}^{y=1} - \int_0^1 f'_x(x,y)\,dy\,\right]dx$

$=\int_0^1 x\left[\underbrace{f'_x(x,1)}_{0} - \int_0^1 f'_x(x,y)\,dy\right]dx$

$= -\int_0^1 x\left[\int_0^1 f'_x(x,y)\,dy\right]dx$

$= -\int_0^1 dy\int_0^1 x f'_x(x,y)\,dx$

$= -\int_0^1 dy\int_0^1 x\, d f(x,y) = -\int_0^1\left[\underbrace{x f(x,y)\Big|_0^1}_{0} - \int_0^1 f(x,y)\,dx\right]dy$

$= \int_0^1\int_0^1 f(x,y)\,dxdy = a$

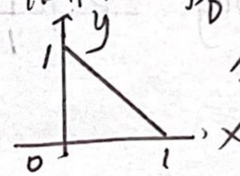
换元法.

eg. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\iint_{D_{xy}} f(x,y)\,dxdy = \iint_{D_{r\theta}} f(r\cos\theta, r\sin\theta)\left\|\begin{matrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{matrix}\right\| dr\,d\theta$

例 14.22

计算积分 $\iint_D \cos\dfrac{x-y}{x+y}\,d\sigma$ $D=\{(x,y)\mid x+y\le 1,\ x\ge 0,\ y\ge 0\}$



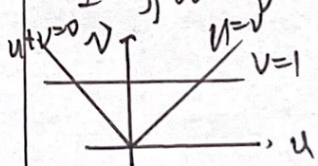
令 $\begin{cases} x-y=u \\ x+y=v \end{cases} \Rightarrow \begin{cases} x=\dfrac{u+v}{2} \\ y=\dfrac{v-u}{2} \end{cases}$

$I=\iint \cos\dfrac{u}{v}$

$d\sigma = |J|\,du\,dv$

$=\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$

$=\left\|\begin{matrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right\| = \left\|\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{matrix}\right\| = \frac{1}{2}$



$x=0 \rightarrow u=-v$

$y=0 \rightarrow u=v$

$x+y=1 \rightarrow v=1$

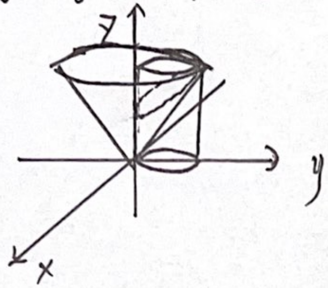
$I=\iint_{D_{uv}} \cos\dfrac{u}{v}\left(\dfrac{1}{2}\right)du\,dv = \dfrac{1}{2}\int_0^1 dv\int_{-v}^{v} v\cdot\cos\dfrac{u}{v}\,d\dfrac{u}{v}$

$=\dfrac{1}{2}\int_0^1 v\left(\sin\dfrac{u}{v}\right)\Big|_{u=-v}^{u=v} dv = \dfrac{1}{2}\sin 1$

例14.24.

求 $x^2+y^2=ay$ $(a>0)$，锥面 $z=\sqrt{x^2+y^2}$ 与平面 $z=0$

所围立体体积 V



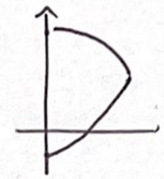
[solution].

$$V=\iint_{D:x^2+y^2\le ay}\sqrt{x^2+y^2}\,d\sigma$$

例14.27

曲线 $x^2+3y-5=0$，$x=\sqrt{y+1}$ 以及 $x=0$

所围均匀薄片，对 y 轴转动惯量.



$$I_y=\iint_D x^2\cdot\mu\,d\sigma=\frac{32\sqrt{2}}{45}\mu.$$

习题. all

例14.28

求由 $x^2+y^2=az$，$z=2a-\sqrt{x^2+y^2}$ 所围立体的体积

$$V=\iint\left[2a-\sqrt{x^2+y^2}-\frac{x^2+y^2}{a}\right]d\sigma$$