§11 习题总结.

1000.11.4.

$$\lim_{n\to\infty}\int_{1}^{2}(\arctan nx)^3\,dx$$

$$=\lim_{n\to\infty}\int_1^2(\arctan nx)^3\,dx$$

$$\overset{u=nx}{=\!=}\lim_{n\to\infty}\frac{1}{n}\int_n^{2n}(\arctan u)^3\,du$$

$$=\lim_{n\to\infty}\frac{1}{n}(2n-n)(\arctan\xi)^3 \quad \leftarrow \text{积分中值定理.}$$

$$=\lim_{n\to\infty}(\arctan\xi)^3,\ \xi\in(n,2n).$$

$$\lim_{n\to\infty}\arctan n=\frac{\pi}{2},\ \lim_{n\to\infty}\arctan 2n=\frac{\pi}{2}.$$

$\therefore$ 原 $=\left(\frac{\pi}{2}\right)^3=\frac{\pi^3}{8}$ (夹逼).

1000.11.6.

证明 $\underbrace{\int_1^{e^2}\frac{\ln x}{1+x}dx}_{I_1}+\underbrace{\int_1^{\frac{1}{e^2}}\frac{\ln x}{1+x}dx}_{I_2}=\int_1^{e^2}\frac{\ln x}{x}dx$

$$I_2\overset{令t=\frac{1}{x}}{=\!=}\int_1^{e^2}\frac{-\ln t}{1+\frac{1}{t}}\,d\frac{1}{t}=\int_1^{e^2}\frac{\ln t}{t^2+t}\,dt$$

$$=\int_1^{e^2}\frac{\ln x}{x(x+1)}\,dx.$$

$$I_1+I_2=\int_1^{e^2}\frac{(x+1)\ln x}{x(x+1)}dx=\int_1^{e^2}\frac{\ln x}{x}dx.$$

1000.11.7

已知 $f(x)$在$[a,b]$连续单增,

证明 $\int_a^b\left(\frac{b-x}{b-a}\right)^n f(x)dx\le\frac{1}{n+1}\int_a^b f(x)dx.\ (n\in N).$

solution.

$\longrightarrow$ 即证 $(n+1)\int_a^b(b-x)^n f(x)dx\le(b-a)^n\int_a^b f(x)dx$

令 $F(x)=(n+1)\int_a^b(b-x)^n f(x)dx-(b-a)^n\int_a^b f(x)dx$

$\cancel{F'(x)=(n+1)\int_a^x(b-x)^n f(x)dx-(n+1)\int_b^x(b-x)^n f(x)dx}$ — ①

$\cancel{F'(x)=(n+1)(b-x)^n f(x)}$

① 令 $\frac{b-x}{b-a}=t$ $\quad x=-(b-a)t+b$

即证 $\int_0^1 t^n f(b-(b-a)t)\,(b-a)\,dt-(b-a)^n\int_0^1 f(b-(b-a)t)(b-a)dt$

$I_1=(b-a)\int_0^1 t^n f(b-(b-a)t)\,dt$ $\quad f(x)\nearrow\quad t\in[0,1]$

② $t\nearrow\quad b-(b-a)t\searrow$

$$\le(b-a)\int_0^1 t^n f(b-(b-a)t^{n+1})\,dt$$

$$=(b-a)\int_0^1 f(b-(b-a)t^{n+1})\frac{1}{n+1}\,dt^{n+1}\ ②$$

$$=-\frac{1}{n+1}\int_0^1 f(b-(b-a)t^{n+1})\,d(b-(b-a)t^{n+1})\ ③$$

$b-(b-a)t^{n+1}=u$ ③

$$=-\frac{1}{n+1}\int_b^a f(u)du=\frac{1}{n+1}\int_a^b f(x)dx$$

1000.11.8

设 $f(x)$在$[a,b]$连续，且 $f(x)>0$.

证明: $\ln\left[\frac{1}{b-a}\int_a^b f(x)dx\right] \geq \frac{1}{b-a}\int_a^b \ln f(x)dx$.

记 $A=\frac{1}{b-a}\int_a^b f(x)dx$.

原 $\Rightarrow$ $\ln A \geq \frac{1}{b-a}\int_a^b \ln f(x)dx$.

即证 $\int_a^b [\ln f(x) - \ln A]dx \leq 0$.

$\ln f(x) - \ln A = \ln\left[1+\left[\frac{f(x)}{A}-1\right]\right] \leq \frac{f(x)}{A}-1$

$\therefore \int_a^b[\cdots]dx \leq \int_a^b\left[\frac{f(x)}{A}-1\right]dx$

$=\frac{1}{A}\int_a^b f(x)dx-(b-a)=0$.

1000.11.9

设 $f(x)$在$[0,1]$有二阶导且 $f(\frac{1}{2})=1$. $f''(x)>0$.

证明 $\int_0^1 f(x)dx \geq 1$.

$\cancel{\int_0^1 f(x)dx}$ $f(x)=f(\frac{1}{2})+f'(\frac{1}{2})(x-\frac{1}{2})+\frac{f''(\frac{1}{2})}{2!}(x-\frac{1}{2})^2$

$\int_0^1 f(x)dx = 1+\int_0^1 \underbrace{f'(\frac{1}{2})(x-\frac{1}{2})}_{0}+\underbrace{\frac{f''(\frac{1}{2})}{2!}(x-\frac{1}{2})^2}_{>0} dx > 1$

1000.11.10.

设$f(x)$在$[0,1]$可导，且 $f(1)=4\int_0^{\frac{1}{4}} x^3 f(x)dx$.

证明 $\exists\xi\in(0,1)$ s.t. $f'(\xi)=-\frac{3f(\xi)}{\xi}$

$\to f(1)=\frac{\int_0^{\frac{1}{4}}x^3f(x)dx}{\frac{1}{4}-0}=\eta^3 f(\eta)$ $\eta\in(0,\frac{1}{4})$.

令 $F(x)=x^3f(x)$

$F(1)=1^3f(1)=\eta^3f(\eta)=F(\eta)$.

罗尔 $\frac{F(\eta)-F(1)}{\eta-1}=\frac{0}{\eta-1}=F'(\xi)$.

$0=F'(\xi)=3\xi^2f(\xi)+\xi^3f'(\xi) \longrightarrow f'(\xi)=-\frac{3f(\xi)}{\xi}$

1000.11七.1

$f(x)$在$(-\infty,+\infty)$连续，且 $f(x)=\int_0^x f(x-t)\sin t\,dt + x$

1) 区间再现 $\longrightarrow f(x)=\int_0^x f(t)\sin(x-t)dt+x$

$\to f(x)=\sin x\int_0^x f(t)\cos t\,dt - \cos x\int_0^x f(t)\sin t\,dt + x$

$\to f'(x)=\cos x\int_0^x f(t)\cos t\,dt+\sin x\int_0^x f(t)\sin t\,dt + 1$

$\to f''(x)= f(x) - f(x) + x = x$

$\longrightarrow f(x)=\frac{x^3}{6}+C_1x+C_2$ $f(0)=0$ $f'(0)=1$

$\to f(x)=x(\frac{x^2}{6}+1)$.

$f(x)$与x同号.

1000.11t.2

$\lim\limits_{n\to\infty}\int_0^1 (n+1)x^n\ln(1+x)\,dx$

$= \lim\limits_{n\to\infty}\int_0^1 \ln(1+x)\,dx^{n+1} = \cdots = \lim\limits_{n\to\infty} +\ln 2 - \int_0^1 \frac{x^{n+1}}{1+x}\,dx.$

对 $\int_0^1 \frac{x^{n+1}}{1+x}\,dx$ 放缩.

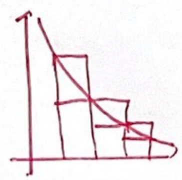
$0 < \lim\limits_{n\to\infty}\int_0^1 \frac{x^{n+1}}{1+x}\,dx < \lim\limits_{n\to\infty}\int_0^1 \frac{x^{n+1}}{x}\,dx = \lim\limits_{n\to\infty}\int_0^1 x^n\,dx = \lim\limits_{n\to\infty}\frac{x^n}{n+1}\Big|_0^1 = 0.$

∴ 原 $= +\ln 2$

1000.11t.3

证明不等式

$\ln(n+1) < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < 1+\ln n$



$\int_k^{k+1}\frac{1}{x}\,dx < \frac{1}{k} < \int_{k-1}^{k}\frac{1}{x}\,dx$

$\therefore \ln(n+1) = \int_1^{n+1}\frac{1}{x}\,dx < \sum_{k=1}^{n}\frac{1}{k}$

又∵ $\sum_{k=1}^{n-1}\frac{1}{k+1} < \int_1^n \frac{1}{x}\,dx.$

∴ $\frac{1}{2}+\cdots+\frac{1}{n} < \ln n. \to \sum_{k=1}^{n}\frac{1}{k} < 1+\ln n$

2、证明数列 $a_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln(n+1)$ ↗, 且 $0<a_n<1$

① $a_{n+1}-a_n = \frac{1}{n+1}-\ln(n+2)+\ln(n+1)$

$= \frac{1}{n+1}-\ln\left(1+\frac{1}{n+1}\right) \to f(x)=x-\ln(1+x).$

$\to a_{n+1} > a_n$ $\quad$ $x>0$时 $x>\ln(1+x)$

② 由1) $0<a_n < 1+\ln n-\ln(n+1)$

$= 1+\ln\frac{n}{n+1} < 1$

1000.11t.4

求极限 $\lim\limits_{n\to\infty}\dfrac{\sum_{k=1}^{n}\frac{1}{k}}{\ln n}$

由1000.11t.3,

$\ln(n+1) < \sum_{k=1}^{n}\frac{1}{k} < 1+\ln n$

$\lim\limits_{n\to\infty}\frac{\ln(n+1)}{\ln n} = 1 = \lim\limits_{n\to\infty}\frac{1+\ln n}{\ln n} \to$ 原式 $=1.$