费马定理

$\begin{cases} \text{主动给出条件} \\ \text{被动挖掘条件.} \end{cases}$ $f'(\xi)=0$ ☆

例6.13 12'

设$f(x)$在$[0,1]$二阶可导，且$f(0)=f(1)=0$.
在$[0,1]$上的最小值等于-1，证明存在一点$\xi\in(0,1)$.
使$f''(\xi)\geq 8$.

[分析]

f与$f^{(n)}(x)$ 用泰勒

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2$$

$[0,1]$内最小值，必是极小值 $\to$ $f'(a)=0$ $f(a)=-1$

$$f(x)=-1+0+\frac{f''(\xi)}{2}(x-a)^2$$

令$x=0$ $f(0)=-1+\frac{f''(\xi_1)}{2}a^2=0$. $\xi_1\in(0,a)$

令$x=1$ $f(1)=-1+\frac{f''(\xi_2)}{2}(1-a)^2=0$ $\xi_2\in(a,1)$.

$f''(\xi_1)=\frac{2}{a^2}$ $a\in(0,1)$ $\begin{cases} a\in(0,\frac{1}{2}) & f''(\xi_1)\geq 8 \\ a\in(\frac{1}{2},1) & f''(\xi_2)\geq 8. \end{cases}$

$f''(\xi_2)=\frac{2}{(1-a)^2}$

$\Rightarrow f''(\xi)\geq 8$

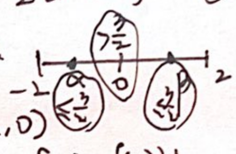
例6.15 (2022).

设$f(x)$在$[-2,2]$二阶可导
证明：若$|f(x)|\leq 1$，$x\in[-2,2]$，且$\frac{1}{2}[f'(0)]^2+[f(0)]^2>\frac{3}{2}$.
则必存在$x_0\in(-2,2)$使得$f''(x_0)+3[f(x_0)]^2=0$.

[证].

令$F(x)=\frac{1}{2}[f'(x_0)]^2+[f(x_0)]^3$，$F(0)>\frac{3}{2}$

~~f(x) =~~



$\exists$ $\alpha\in(-2,0)$

$$|f'(\alpha)|=\left|\frac{f(0)-f(-2)}{0-(-2)}\right|\leq\frac{|f(0)|+|f(-2)|}{2}\leq\frac{2}{2}=1$$

$$F(\alpha)=\frac{1}{2}[f'(\alpha)]^2+[f(\alpha)]^2\leq\frac{1}{2}\cdot 1+1=\frac{3}{2}$$

$\exists$ $\beta\in(0,2)$

$$|f'(\beta)|=\left|\frac{f(2)-f(0)}{2-0}\right|\leq\frac{|f(2)|+|f(0)|}{2}\leq 1$$

$$F(\beta)=\frac{1}{2}[f'(\beta)]^2+[f(\beta)]^2\leq\frac{1}{2}\cdot 1+1=\frac{3}{2}.$$

$\because$ $\left(\begin{array}{l} F'(\xi_1)=\frac{F(0)-F(\alpha)}{0-\alpha}>0. \\ F'(\xi_2)=\frac{F(\beta)-F(0)}{\beta-0}<0. \\ \text{由零点定理.}\quad F'(x_0)=(f''(x_0)+3[f(x_0)]^2)f'(x_0)=0. \end{array}\right)$

$F(x)$在$[\alpha,\beta]$上连续，且在(α,β)内取最值 $\xrightarrow{\text{记为}}$ x_0，
必为极大值，由费马定理 $\Rightarrow$ $F'(x_0)=0$.
即 $(f''(x_0)+3[f(x_0)]^2)f'(x_0)=0$.

如果$f'(x_0)=0$ $F(x_0)=[f(x_0)]^3\leq 1$ 矛盾 $f'(x_0)\neq 0$.

$\therefore$ $f''(x_0)+3[f(x_0)]^2=0$