§15. 1000

1000. 15.1

$x\,dy = (y - \sqrt{x^2+y^2})dx \quad (x>0).$ $y(1)=0$ 特解?

[solution]. $\frac{dy}{dx} = \frac{y}{x} - \sqrt{1+(\frac{y}{x})^2}$ $\boxed{\frac{y}{x}=u \text{ 齐次型}}$

$\frac{dy}{dx} = u + x\frac{du}{dx} = u - \sqrt{1+u^2}$

$\int \sqrt{1+u^2}\,du = \ln x \rightarrow \cancel{\frac{y}{x} = (\ln(x))^2 \rightarrow y = x(\ln cx)^2}$

$\cancel{0 = 1(\ln c)^2 \rightarrow c=1 \rightarrow y = x(\ln x)^2}$

$\ln(u+\sqrt{1+u^2}) = \ln x + C \quad u+\sqrt{1+u^2} = \frac{1}{x} + e^c$

$y+\sqrt{x^2+y^2} = e^c + e^c x$

$0+1 = 0+1 \rightarrow C=0.$

$\Rightarrow y+\sqrt{x^2+y^2} = x+1 \Rightarrow \quad y+\sqrt{x^2+y^2}=1$

1000.15.2

$y''+y'+y = e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x.$ 一个特解.

① $e^{-\frac{1}{2}x} \rightarrow \alpha = -\frac{1}{2} \quad \beta = \frac{\sqrt{3}}{2}x$ ③ $k=1$

$\lambda^2+\lambda+1=0 \quad \lambda_1\lambda_2 = \frac{-1\pm\sqrt{-3}}{2} = -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$\Delta = 1-4<0.$

② $a\sin\frac{\sqrt{3}}{2}x + b\cos\frac{\sqrt{3}}{2}x$

$\Rightarrow e^{-\frac{1}{2}}(a\sin\frac{\sqrt{3}}{2}x + b\cos\frac{\sqrt{3}}{2}x)\cdot x$

1000-15.3.

$y''(x) + p(x)y'(x) - y(x) = 0, \quad y(a)=y(b)=0.$

$y(a)=y(b)=0 \longrightarrow y'(\xi)=0,$

$\cancel{\lambda^2+p(x)\lambda-1}$

$y''(\xi) = y(\xi)$

① a.b.连续 → max, min

② $y(a)=y(b)=0 \rightarrow y'(\xi)=0$; 至少一个最值点

$\rightarrow y''(\xi)=y(\xi)$

$y(\xi)>0$. 极小→矛盾, $y(\xi)<0$ 极大→矛盾

$\Rightarrow y(x)\equiv 0.$

1000.15.4

$y''-4y'+5y = e^{2x}\sin x - 3x+2$ 特解

① $\lambda^2-4\lambda+5 = \cancel{(\lambda-5)(\lambda+1) \Rightarrow \lambda_1=5\ \lambda_2=-1}$ $\Delta=16-20<0!$ $\lambda_1\lambda_2 = \frac{4\pm\sqrt{-4}}{2} = 2\pm i$

② 则为 $e^{2x} \rightarrow \alpha=2 \rightarrow k=0,1$ $\beta=1$

③ $x^0=1$.

④ $\sin x \Rightarrow (a\sin x + b\cos x)$

$e^{2x}\sin x \downarrow y_1^* = e^{2x}\cdot(a\sin x+b\cos x)$

⑤ e^0 则为 $=1$.

⑥ $-3x+2 \rightarrow ax+b$

$y_2^* = cx+d$

⑦ $k=0.$

$\rightarrow y^* = e^{2x}(a\sin x+b\cos x) + cx+d$

1000.15.5

$y''-3y'+2y=xe^{2x}-2x+1$ 特解，

① $\lambda^2-3\lambda+2=(\lambda-1)(\lambda-2)=0$

$\Rightarrow \lambda_1=1 \quad \lambda_2=2$

② (1) e^{2x}，$\alpha=2$

(2). $x\to 1\to \quad ax+b$

(3). $\lambda_2=2\neq\lambda_1 \to k=1\to \quad x$

$y_1^*=e^{2x}(ax+b)x$

③ (1) e^{0x}，$\alpha=0$

(2). $-2x+1\to 1\to \quad cx+d$

(3). $\lambda_1\neq\lambda_2\neq 0 \to x^0$

$y_2^*=cx+d$

$\to y^*=e^{2x}(ax+b)x+cx+d$

1000.15.6

$p(x)$, $q(x)$, $f(x)$ 是 x 已知的连续函数，$y_1(x)$, $y_2(x)$, $y_3(x)$ 是

$y''+p(x)y'+q(x)y=f(x)$ 3个线性无关解．通解？

设 $\Delta>0$．$\cancel{C_1 x}$ $C_1e^{\alpha x}+C_2e^{\beta x}+y^*$

$C_1 \quad +y^*$

$C_2 \quad +y^*$

$\to$ 通解．$y_1(x)+y_2(x)-2y_3(x)$

1000.15.7

$y=(x+1)e^{-x}$ 是 $y''+ay'+by=c(x+1)e^x$ 解．

a,b,c？

[solution]

$y'=e^{-x}-(x+1)e^{-x}$ $\quad +b\cdot(x+1)e^{-x}$

$y''=-e^{-x}-e^{-x}+(x+1)e^{-x}$ ✓

$\underline{-2e^{-x}}\ \underline{+(x+1)e^{-x}}\ \underline{+ae^{-x}}\ \underline{-a(x+1)e^{-x}}=c(x+1)e^{x}$

$(a-2)+(x+1)(1-a+b)=c(x+1)e^{2x}$

$\to \underline{c=0}$ $\quad a=2 \quad 1-2+b=0\to \underline{b=1}$

1000.15.8

$f(x)-1=\int_0^x f(1-t)dt$，$f(x)=?$

$f(x)-1=\int_0^x f(1-t)dt \quad \underline{f'(x)=f(1-x)}$ $\quad 1-1+x$

$f''(x)=-f'(1-x)=-f(1-(1-x))$

$\to f''(x)=-f(x)$

$f''(x)+f(x)=0.$

$\lambda^2+0+1=0\to\lambda_1=\lambda_2=\pm i$

$f(x)=e^{\alpha x}[C_1\cos\beta_1x+C_2\,\cancel{\cos}\sin\beta_2x]$

$=C_1\cos x+C_2\sin x$

$C_2=\dfrac{1+\sin 1}{\cos 1}$

$f(0)-1=0\to f(0)=1\to f(0)=\underline{C_1=1}\to C_2=\dfrac{f(x)+\cancel{\cos x}\sin x}{\cos x\,\cancel{\sin x}}$

~~$f(1)-1=\int_0^1 f(0)$~~ $f'(1)=f(0)=1 \quad f'(x)=-\sin x+C_2\cos x$