Multiscale SVD

Estimation of the intrinsic dimensionality

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Outline

- Introduction to the problem
- Multiscale SVD
- Analysis on a new dataset
- Conclusion and questions

Introduction to the problem

Estimation of intrinsic dimension

- Occurs in mamy scientific problems
  - Number of variables in statistical models
  - Number of degree of freedom in dynamical systems
  - Estimation of probability distribution highly concentrated around low dimensional manifold
- Input of algorithms in many scientific fields
  - Signal processing, economics, genomics ...

Defining our datasets and notations

- Let M be a smooth k-dimensional non-linear manifold, then:
  - Let $X = \{x_i\}_{i=1}^{n}$ be a set of be a set of uniformly distributed random sample points of M
  - Let $\tilde{X} = \{x_i + \sigma\eta_i\}_{i=1}^{n}$ be the noisy samples, where η_i is a centered white noise with σ as its standard deviation
- Given a set $\tilde{X}$ of n sample points embedded in $\mathbb{R}^D$ - they will be represented by a $n \times D$ matrix

Notions of dimensionality

- Ambient dimension : the dimensions where the manifold is embedded (in the matrix representation : the number of columns)
- Extrinsic dimension : minimum number of dimensions in which the shape of the manifold can be embedded (in the matrix representation : the rank of the matrix)
- Intrinsic dimension : the number of parameters needed to generate the manifold

EXAMPLE

Ambient dimension : 3

> The circle is embedded in R^3

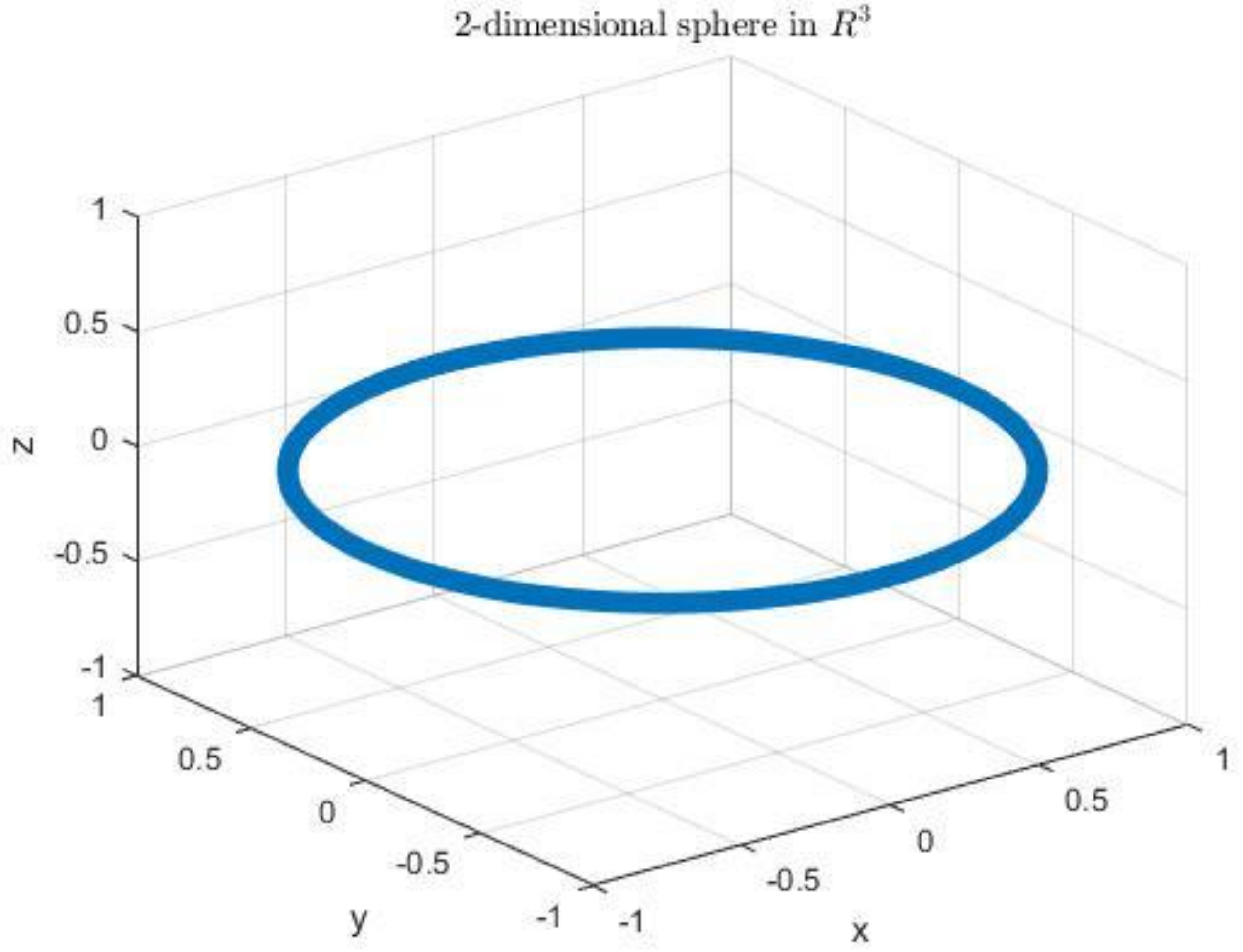
Extrinsic dimension : 2

> This is a 2 dimensional circle

Intrinsic dimension : 1

> The circle is generated with only one parameter

$(\cos(\alpha), \sin(\alpha), 0)$



Multiscale SVD

Estimation with SVD

The SVD is a good estimator of the extrinsic dimension, and is most often used to perform dimensionality estimation

Linear manifolds	Non linear manifolds
› e.g. Hyperplan › Intrinsic = Extrinsic › SVD is a good estimator of the intrinsic dimensionality › Robust to noise	› e.g Sphere › Intrinsic < Extrinsic › SVD over estimates the intrinsic dimension

LOCALLY VS GLOBALLY

Curvature causes

over-estimation of the intrinsic dimension

SVD is performed globally

The manifold is locally linear and can be approximated by a tangent plane.

Local SVD should approximate the local extrinsic dimension (that is also the local intrinsic dimension)

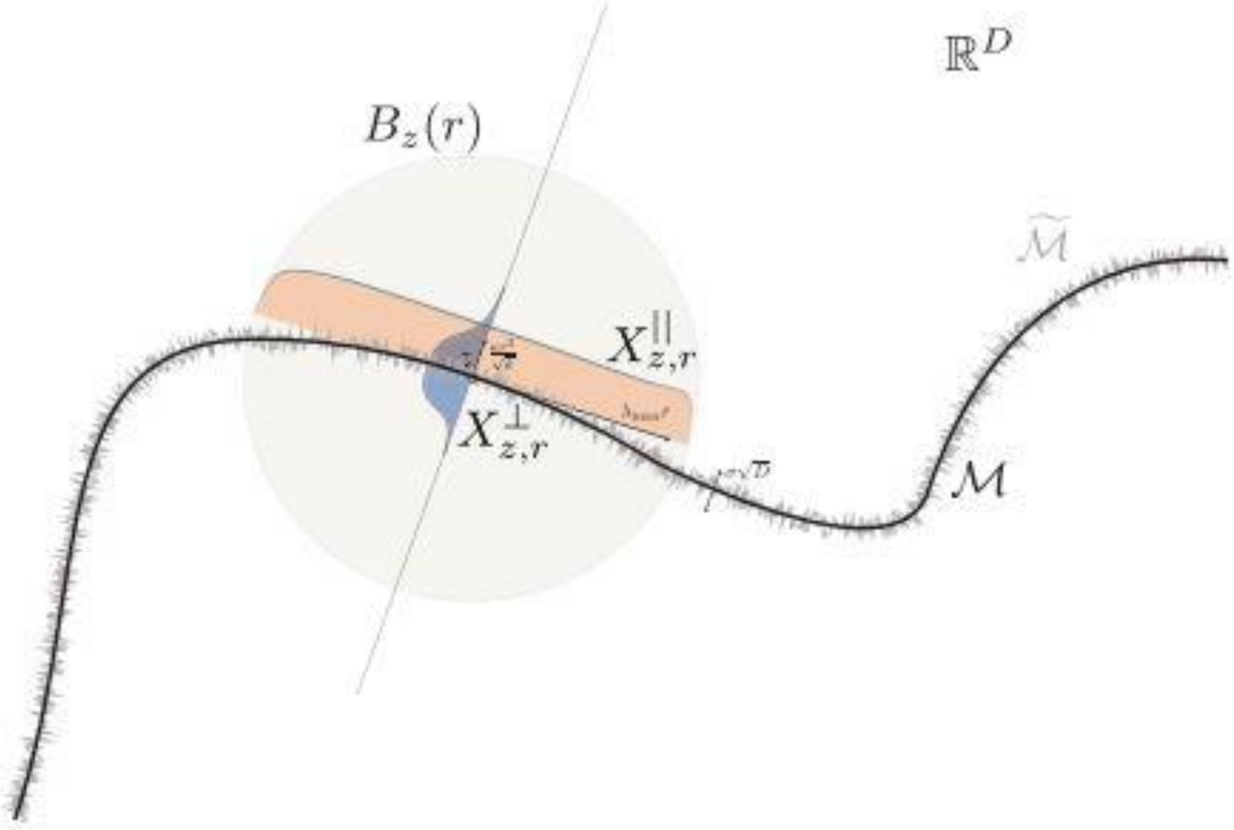


Figure taken from : *Little, A. V., Maggioni, M., & Rosasco, L. (2011). Multiscale geometric methods for data sets I: Intrinsic dimension.*

MSVD

Compute the singular values for X intersected with each ball centered around point z with radius r.

This process is performed over a wide range of radii.

Average the results for each r over the dataset

Identifying a range of scale three groups of singular values

(intrinsic, extrinsic, noise) can be distinguished

Estimation of the dimensionality

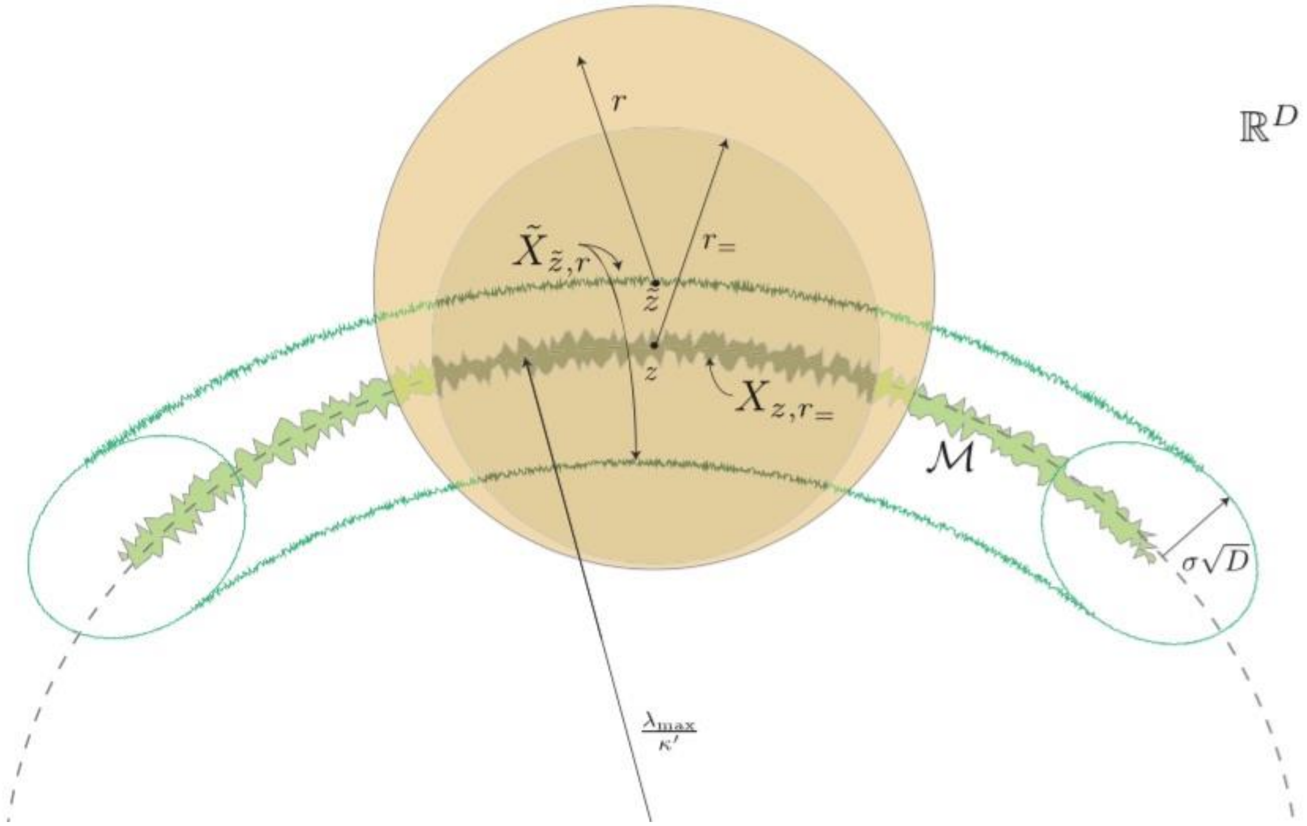


Figure taken from : *Little, A. V., Maggioni, M., & Rosasco, L. (2011). Multiscale geometric methods for data sets I: Intrinsic dimension.*

The paper's case study

- Sphere is defined as $S^k = \left\{ x \in \mathbb{R}^{k+1} \middle| \|x\|_2 = 1 \right\}$

- extrinsic dimension : k+1

- intrinsic dimension : k
  - since the equation which describes it is of k+1 parameters with one constraint, hence it has k degrees of freedom which are the intrinsic dimension.

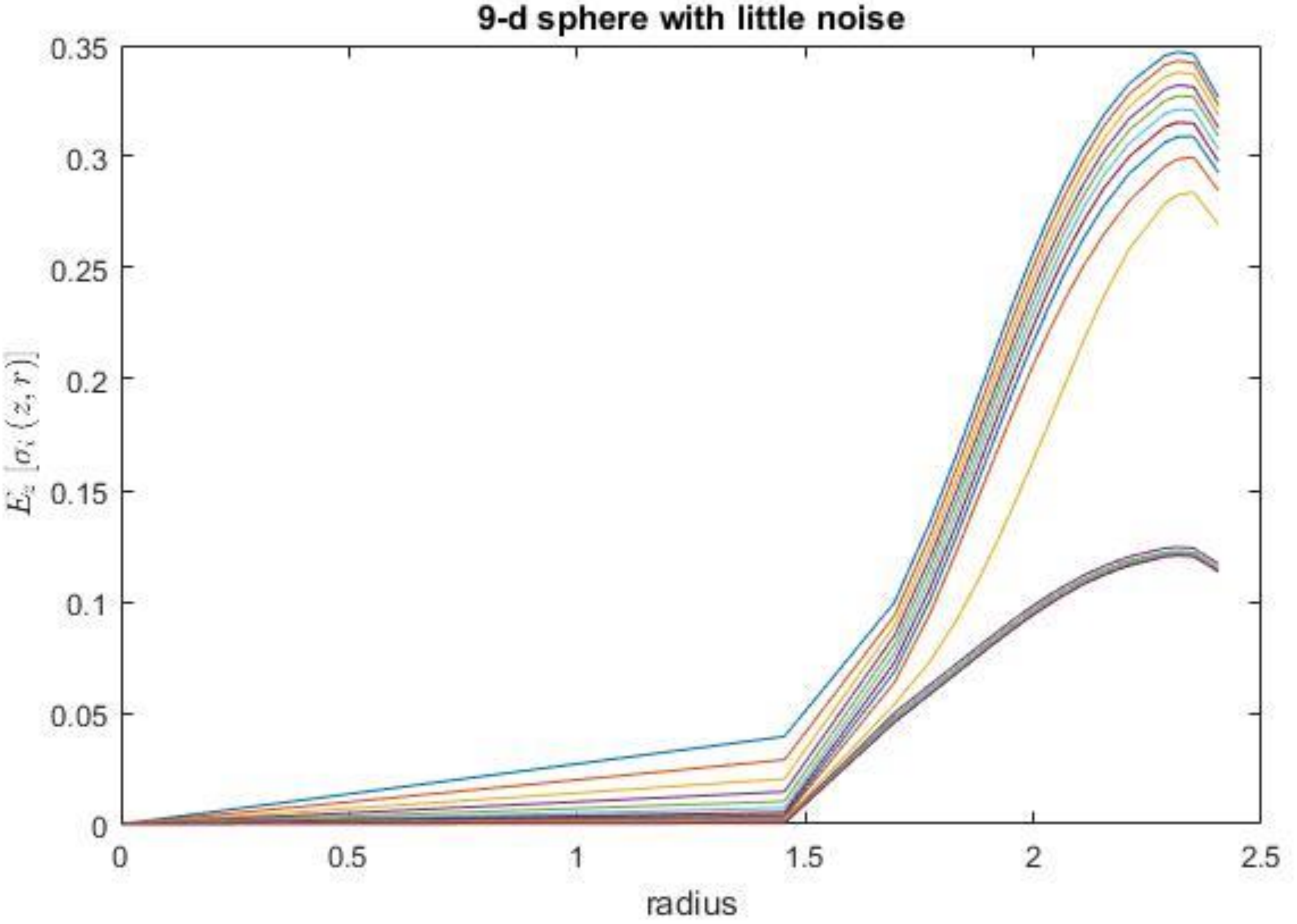
- The sphere is embedded in R^D

BEHAVIOR OF THE S.V.

The noise S.V. converge to the std of the white noise (here N(0,0.1))

Intrinsic S.V. are the top 9 S.V.

Extrinsic S.V. is the 10^{th} S.V. recognizable by the gap



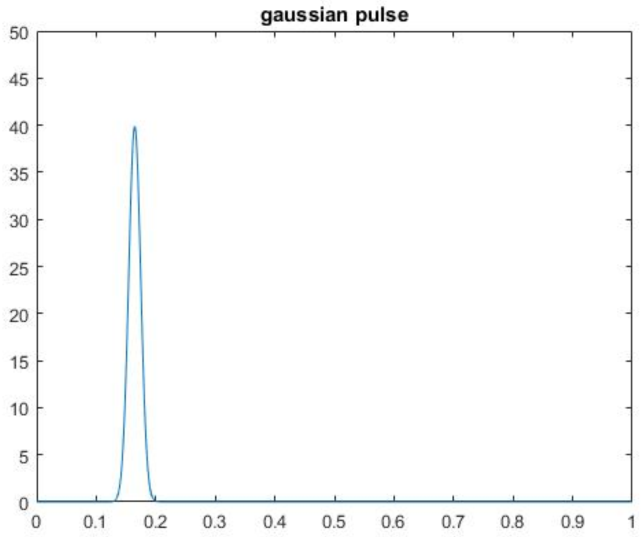
Analysis on the pulse dataset

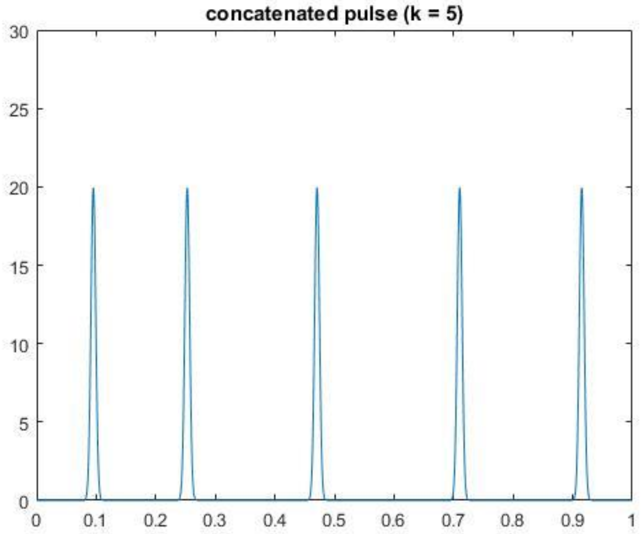
THE PULSE DATASET

This dataset is more related to signal processing : heartbeats, musical tempo, breathing …

Only one parameter : mu (fixed width)

Intrinsic dimension is scalable (concatenation)





MSVD WORKS ON PULSES

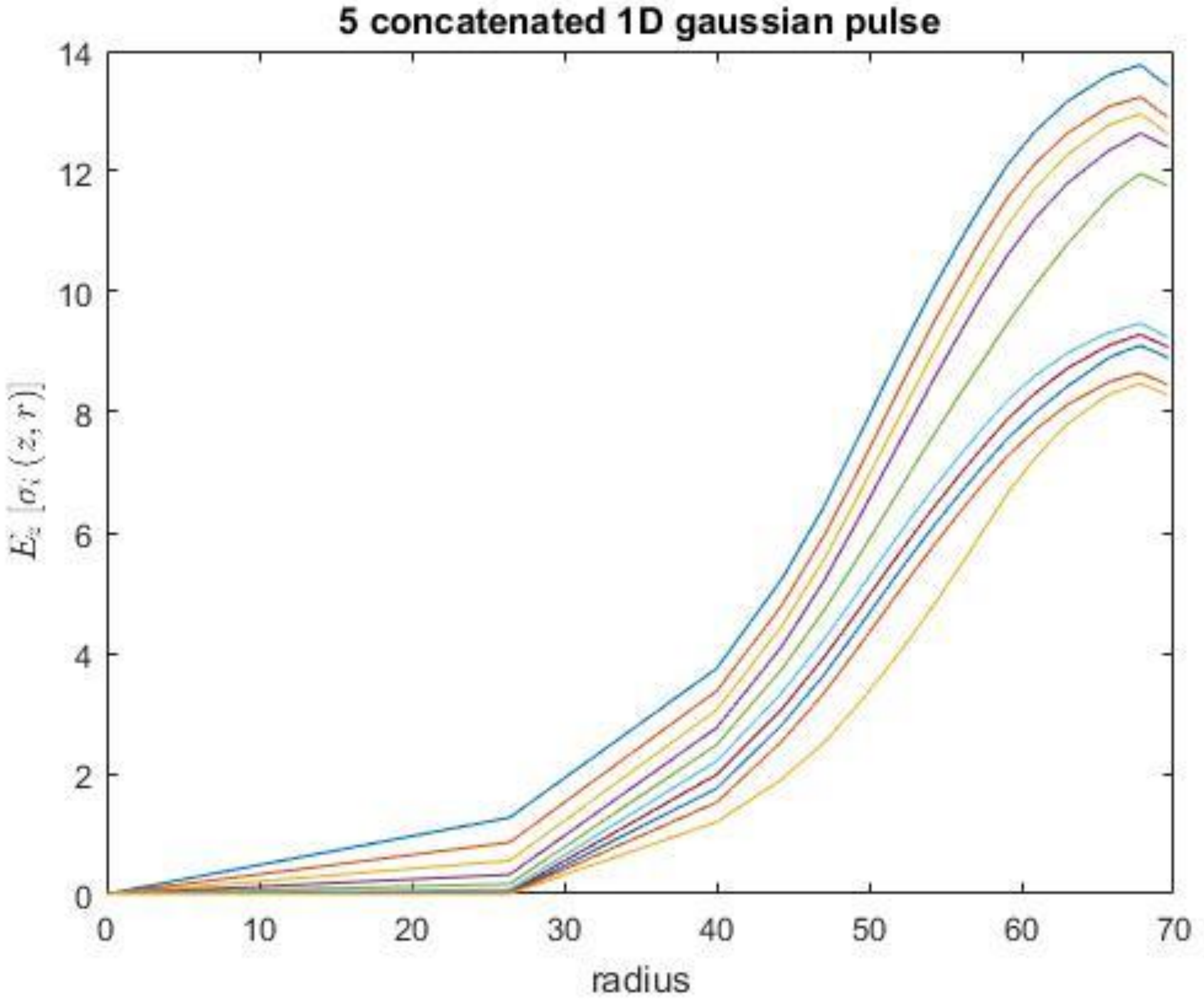
MSVD on the 5 concatenated pulses (k=5)

There is a gap between the fifth and the sixth curve

-> The MSVD estimation is correct

MSVD is much more accurate than SVD

-> Global SVD estimation is exceeds 50 in this case



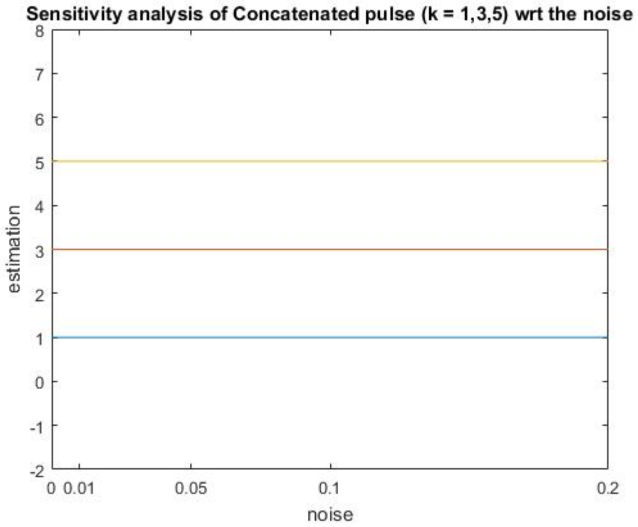
SENSITIVITY ANALYSIS

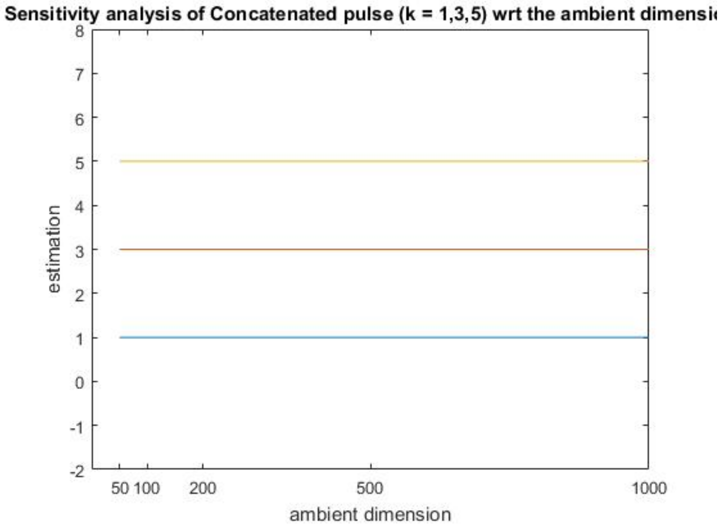
MSVD is robust :

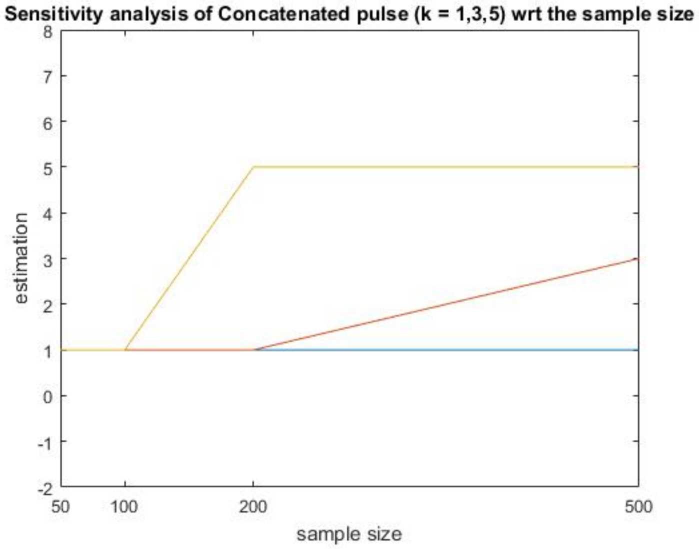
-> noise, up to a limit

-> ambient dimension

-> sample size when it isn't too low







Conclusion

- Accurate technique to estimate the intrinsic dimensionality of high dimensional datasets

- Generalizes well on the pulse dataset
  - Robust to main parameters

- Future improvements and research direction :
  - Measure improvements on real data
  - Computational : MSVD is natively a distributed technique

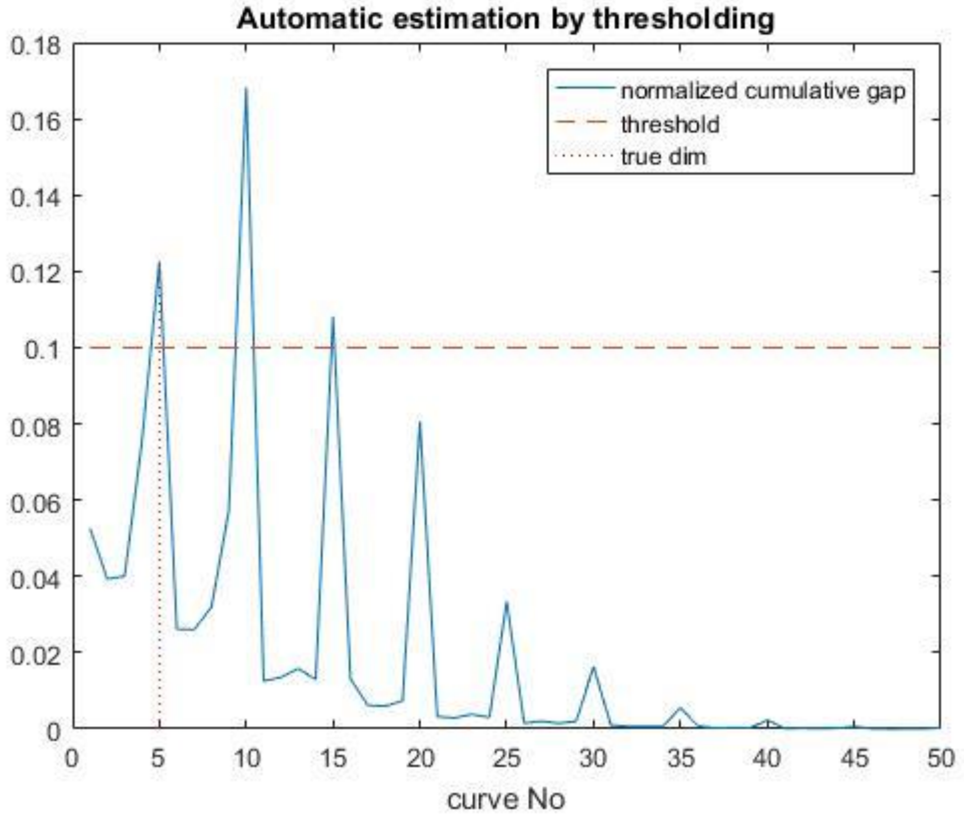
Questions

BONUS 1 : AUTOMATIC ESTIMATION

We compute the average gap between the curves given of the MSVD

We estimate the intrinsic dimensionality by thresholding on the normalized gap

Best threshold found : 0.1



Bonus 2 : Uniform sampling on a sphere

USING THE HYPERCUBE

- Sampling points in the hypercube, keep only point that belong to the ball (z,r) and normalize
- Problem : the volume of the sphere decreases too fast with regard to the dimensionality
- Very inefficient in high dimension

USING NORMAL DISTRIBUTION

- Gaussian distribution is symmetric, therefore it is uniformly distributed over all the directions
- Only normalize points drawn from the gaussian distribution