APMA 1720 : Computational Project

1. stock prices

$$X_T = X_0 \, e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 \sqrt{T} Z_1}$$

$$Y_T = Y_0 \, e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 \sqrt{T} Z_2} ,$$

$$Z \sim N(0, \Sigma), \quad \underset{\sim}{Z} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

basket option price

$$V = E_{Z_1, Z_2} \left[e^{-rT} (c_1 X_T + c_2 Y_T - K)^+ \right]$$

method of conditioning

$$V = e^{-rT} E_{Z_2} \left[E_{Z_1} \left[(c_1 X_T + c_2 Y_T - K)^+ \mid Z_2 \right] \right]$$

$$B \sim N(0, \mathbb{1}), \quad \underset{\sim}{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

$$Z = RB, \quad RR^T = \Sigma$$

Cholesky decomposition

$$R = \begin{pmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} Z_1 = \sqrt{1-\rho^2}\, B_1 + \rho B_2 \\ Z_2 = B_2 \end{cases}$$

$$\bar{K} := K - C_2 Y_T = K - C_2 Y_0\, e^{(r-\frac{1}{2}\sigma_2^2)T + \sigma_2\sqrt{T} B_2}$$

$$S_0 := C_1 X_0\, e^{\sigma_1 \rho \sqrt{T} B_2 - \frac{1}{2}\sigma_1^2 \rho^2 T}$$

$$\bar{\sigma} := \sigma_1 \sqrt{1-\rho^2}$$

$$C_1 X_T = C_1 X_0\, e^{(r-\frac{1}{2}\sigma_1^2)T + \sigma_1\sqrt{T}(\sqrt{1-\rho^2}\, B_1 + \rho B_2)}$$

$$= C_1 X_0\, e^{\sigma_1\rho\sqrt{T}B_2 - \frac{1}{2}\sigma_1^2\rho^2 T}\, e^{[r-\frac{1}{2}\sigma_1^2(1-\rho^2)]T + \sigma_1\sqrt{1-\rho^2}\sqrt{T}B_1}$$

$$= S_0\, e^{(r-\frac{1}{2}\bar{\sigma}^2)T + \bar{\sigma}\sqrt{T}B_1}$$

for $\underline{\bar{K} > 0}$

$$V = e^{-rT} E_{B_2}\left[E_{B_1}\left[\left(S_0\, e^{(r-\frac{1}{2}\bar{\sigma}^2)T + \bar{\sigma}\sqrt{T}B_1} - \bar{K} \right)^+ \middle| B_2 \right]\right]$$

$$= E_{B_2}\left[\text{BLS\_Call}(S_0, \bar{K}, T, r, \bar{\sigma}) \mid B_2 \right]$$

$$\mathrm{BLS\_Call}(S_0, K, T, r, \sigma)$$

$$= S_0 \,\Phi(\sigma\sqrt{T} - \theta) - K e^{-rT} \Phi(-\theta)$$

$$\theta = \frac{1}{\sigma\sqrt{T}} \log\frac{K}{S_0} + \left(\frac{\sigma}{2} - \frac{r}{\sigma}\right)\sqrt{T}$$

for $\underline{\bar{K} < 0}$

$$V = e^{-rT} E_{B_2}\left[E_{B_1}\left[\left(S_0 e^{(r - \frac{1}{2}\bar{\sigma}^2)T + \bar{\sigma}\sqrt{T} B_1} - \bar{K} \right) \middle| B_2 \right]\right]$$

$$= E_{B_2}[S_0 - e^{-rT} \bar{K}]$$

2. mean-reverting stochastic volatility model

$$dS_t = r S_t \, dt + \theta_t S_t \, dW_t$$

$$d\theta_t = a(\Theta - \theta_t)dt + \beta dB_t$$

$r, a, \Theta, \beta > 0$

$$\underset{\smile}{\begin{matrix} \widetilde{W} \\ B \end{matrix}} \sim N(0, \Sigma), \quad \Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

vertical spread option price

$$V = e^{-rT} E[(S_T - K_1)^+ - (S_T - K_2)^+]$$

$$0 < K_1 < K_2$$

$$Y_t = \log S_t \Rightarrow dY_t = (r - \frac{1}{2}\theta_t^2)dt + \theta_t dW_t$$

$$\hat{Y}_{t_{i+1}} = \hat{Y}_{t_i} + (r - \frac{1}{2}\hat{\theta}_{t_i}^2)(t_{i+1} - t_i) + \hat{\theta}_{t_i}\sqrt{t_{i+1} - t_i}\; Z_{i+1}$$

$$\hat{Y}_{t_0} = Y_0$$

$$\hat{\theta}_{t_{i+1}} = \hat{\theta}_{t_i} + a(\Theta - \hat{\theta}_{t_i})(t_{i+1} - t_i) + \beta\sqrt{t_{i+1} - t_i}\; R_{i+1}$$

$$\hat{\theta}_{t_0} = \theta_0$$

$$R = \rho Z + \sqrt{1-\rho^2}\, X, \quad \text{iid } (Z, X) \sim N(0, \mathbb{1}_{2m})$$

control variates : artificial stochastic process

$$\bar{Y}_{t_{i+1}} = \bar{Y}_{t_i} + \left(r - \frac{1}{2}\theta_0^2\right)(t_{i+1} - t_i) + \theta_0 \sqrt{t_{i+1} - t_i}\, Z_{i+1}$$

$$\bar{Y}_{t_0} = Y_0$$

$$d\bar{Y}_t = \left(r - \frac{1}{2}\theta_0^2\right) dt + \theta_0 dW_t$$

$$\bar{V} = e^{-rT} \left(e^{\bar{Y}_T} - k\right)^+$$

$$V = e^{-rT} E\left[\left(e^{Y_T} - k_1\right)^+ - \left(e^{Y_T} - k_2\right)^+\right]$$

$$- b^* \left\{ e^{-rT} E\left[\left(e^{\bar{Y}_T} - \frac{k_1 + k_2}{2}\right)^+\right] \right.$$

$$\left. - BLS\_Call\left(e^{Y_0}, \frac{k_1 + k_2}{2}, T, r, \theta_0\right) \right\}$$

$$b^* = \frac{Cov\left[\left(e^{Y_T} - k_1\right)^+ - \left(e^{Y_T} - k_2\right)^+, \left(e^{\bar{Y}_T} - \frac{k_1 + k_2}{2}\right)^+\right]}{Var\left[\left(e^{\bar{Y}_T} - \frac{k_1 + k_2}{2}\right)^+\right]}$$

here we choose $b = 1$ for simplicity