APPLIED MATH 2812A
Computational Projects

In these projects all parameters should be open to choice by the user of the program. It is often useful for deciding on convergence to use an iteration that is monotone. The first one is a discounted problem of queueing control in continuous time, the second is control till exit in discrete time. One is not limited to the two problems described below. However, if you are interested in doing something else please see me so we can make sure the problem is not too complicated.

Project 1. We consider the control of a two class queueing model. The state space is

$$S = \{(x_1, x_2) : x_i \in \mathbb{Z}, x_i \geq 0, i = 1, 2, x_1 + x_2 \leq B\}$$

where $B \in \mathbb{N}$ is some maximum buffer size for the two queues. Each x_i denotes the number of customers of class i waiting to complete service, and service is granted to the customer at the head of each queue, depending on the control. The arrival rates are $\lambda_i \in (0, \infty)$ and the service rates are $\mu_i \in (0, \infty)$. Only one class is given service at any time, which is the control, so $U = \{1, 2\}$.

We have the following non-negative costs.

1. There is a cost proportional to queue length per unit time: $a_1 x_1 + a_2 x_2$.
2. There is a cost for the class being served, also assessed per unit time, and depending on the class served: $c_u, u \in U$.
3. If a customer of class i arrives to find the buffer full (i.e., $x_1 + x_2 = B$), then that customer is lost, and there is a fixed (i..e., not per unit time) cost of C_i. These events will occur at rates $\lambda_i, i = 1, 2$ when the buffer is full.

With a discounting rate of $\rho > 0$, the objective is to minimize the expected discounted sum of these costs. Note that there is a mix of costs charged per unit time and fixed costs. The charge for lost customers is discounted according to the moment it is lost, i.e., if at time t then $C_i e^{-\rho t}$. Hence the overall cost to be minimized is

$$E_x^\pi \left[\int_0^\infty e^{-\rho t} \left(a_1 X_1(t) + a_2 X_2(t) + c_{U(t)} \right) dt + \sum_{i=1}^{2} C_i \sum_{j=1}^{\infty} e^{-\rho t_j^i} \right],$$

where t_j^i is the time that the jth customer of class i was lost.

Write down an equivalent cost is the standard form for continuous time problems by expressing

$$E_x^\pi \left[\sum_{i=1}^{2} C_i \sum_{j=1}^{\infty} e^{-\rho t_j^i} \right]$$

in the form

$$E_x^\pi \left[\int_0^\infty e^{-\rho t} \left(C_1 r_1 + C_2 r_2 \right) 1_{\{X_1(t)+X_2(t)=B\}} dt \right].$$

(*Hint:* write the sum as an integral with respect to a Poisson random measure and use some martingales.) Identify the Bellman equation for this problem, as well as the Bellman equation for a discrete time problem with the same solution. Using the discrete time Bellman equation, write codes that solve for the value function by iteration in value space and iteration in policy space, and present the solutions for two sets of problem data (probably assymetric costs would be more interesting). Display the optimal control as a function of the state.

Project 2. We consider the state space

$$S = \{(x_1, x_2) : x_i \in \mathbb{Z}, x_i \geq 0, i = 1, 2, x_1 \leq N, x_2 \leq M\}.$$

We let B be a collection of *boundary* states, which include the boundary of the domain

$$\{(x_1, x_2) \in S : x_1 \in \{0, N\}, x_2 \in \{0, M\}\},$$

as well as barriers or obstacles in the domain. Given a state $x \in S \backslash B$, the controlled dynamics are as follows. The controls allowed at state x are

$$U_x = \{v = (\pm 1, \pm 1) : x + v \in S \backslash B\}.$$

Thus the controls are identified with vectors that do not move the state into the boundary. There is also a noise parameter $\delta \in [0, 1/4)$ such that if at state x control v is used, then the actual state reached has the distribution

$$y + v \text{ with probability } 1 - 3\delta$$

$$y + w \text{ with probability } \delta$$

for $w \in U_y$, $w \neq v$, and

$$y \text{ with probability } 1 - \delta |U_y|.$$

If $\delta = 0$ then the process can wander deterministically so long as it avoids the boundary, but when $\delta > 0$ noise is added, though the noise also cannot move the process into the boundary. Finally there is a set of *target states* $T \subset S \backslash B$. The problem of interest is to solve for

$$V(x) = \inf W(x, \pi), \quad x \in S \backslash B$$

$$W(x, \pi) = E_x^{\pi} \tau,$$

$$\tau = \inf\{i \geq 0 : X_i \in T\}.$$

Hence we want to minimize the expected time for the noisy process to reach the target states, for any starting position.

For this problem the inputs are M, N, the additional states in B, δ and the targets T. Identify the Bellman equation for this problem, and identify conditions for which you think there is a unique finite solution. Write codes that solve for the value function by iteration in value space and iteration in policy space, and present the solutions (i.e., if possible plot the value function) for two sets of problem data.

Note. To keep the problem simple this problem reflects a lattice structure, but one can also formulate a version of the problem in which the control and noise are not confined in this way to the lattice directions.