controlled queueing model

state space

2 classes

$$S = \{ \{x\},\ x \in \mathbb{N},\ \sum_{i=1}^{2} x_i \le B \in \mathbb{N} \}$$

maximum buffer size

# of customers of class i $\quad i = 1, 2$

arrival rate $\quad \lambda_i \in (0, \infty)$

service rate $\quad \mu_i \in (0, \infty)$

control $\quad u \in U = \{1, 2\}$

1. running cost : $a \cdot x$
2. serving cost : C_u, $u \in U$
3. ~~punishment~~ Penalty : $C_i \mid x_1 + x_2 = B$

discounting rate : $\rho > 0$ (infinite horizon)

$$V(x) = \inf_{\pi} E_x^{\pi} \Big[\int_0^{\infty} dt\, e^{-\rho t} \left(a \cdot X(t) + C_{U(t)} \right) + \sum_{i=1}^{2} \sum_{j=1}^{\infty} e^{-\rho t_j^i} \Big]$$

time that j-th customer of class i was lost

$$E_x^\pi \left[\sum_{i=1}^{2} c_i \sum_{j=1}^{\infty} e^{-\rho t_j^i} \right] \qquad X_1(t_j^i) + X_2(t_j^i) = B$$

$$t_j^i(X)$$

$$= E_x^\pi \left[\sum_{i=1}^{2} c_i \int_0^\infty dt\, e^{-\rho t} \sum_{j=1}^{\infty} \delta(t_j^i - t) \right]$$

$$= E_x^\pi \left[\sum_{i=1}^{2} c_i \int_0^\infty dt\, e^{-\rho t} \underbrace{\sum_{j=1}^{\infty} \frac{\partial X_i}{\partial t_j^i}}_{\lambda_i} \delta(X_1(t) + X_2(t) - B) \right]$$

$$= E_x^\pi \left[\int_0^\infty dt\, e^{-\rho t} \, \lambda \cdot C \, \delta\left(\sum_{i=1}^{2} X_i(t) - B\right) \right]$$

$$V(x) = \inf_\pi E_x^\pi \left[\int_0^\infty dt\, e^{-\rho t} c(x(t), u(t)) \right]$$

$$c(x(t), u(t)) = \bar{c}(X(t)) + C_{u(t)}$$

$$+ \lambda \cdot C \, \delta\left(\sum_{i=1}^{2} X_i(t) - B\right)$$

Hamilton-Jacobi-Bellman equation

$$0 = \inf_u \left\{ c(x, u) + \sum_y r(x, y | u) [V(y) - V(x)] - \rho V(x) \right\}$$

transition probability

$$p(x, y | u) = \frac{r(x, y | u)}{r(x | u)}, \quad r(x | u) = \sum_y r(x, y | u)$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad e_{-1} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad e_{-2} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

for $x_1 \wedge x_2 > 0$

$$r(x, x+e_1 \mid i) = \lambda_1, \quad r(x, x+e_2 \mid i) = \lambda_2$$

$$r(x, x+e_{-1} \mid 1) = \mu_1, \quad r(x, x+e_{-2} \mid 2) = \mu_2$$

$$y = \{x + e_k\}, \quad k = \pm 1, \pm 2$$

$$V(x) = \inf_u \left\{ \overbrace{\frac{C(x,u)}{\rho + r(x|u)}}^{\tilde{c}(x,u)} + \overbrace{\frac{r(x|u)}{\rho + r(x|u)}}^{\tilde{\alpha}(x,u) \leq 1} \times \sum_y p(x,y|u) V(y) \right\}$$

discretization

$$V_n(x) = \inf_\pi E_x^\pi \left[\sum_{l=0}^{n-1} \tilde{\alpha}^l(x,u) \, \tilde{c}(x,u) \right]$$

iterative solution

$$V_{n+1}(x) = \tilde{c}(x,u) + \tilde{\alpha}(x,u) \sum_y p(x,y|u) V_n(y)$$

iteration in value space

$$V_0(x) = 0$$

$$V_{n+1}(x) = \inf_{u \in U} \left[\tilde{c}(x,u) + \tilde{\alpha}(x,u) \sum_{y \in S} p(x,y|u) V_n(y) \right]$$

$$g(x) = \underset{u \in U}{\arg\min} \left[\tilde{c}(x,u) + \tilde{\alpha}(x,u) \sum_{y \in S} p(x,y|u) V_n(y) \right]$$

iteraction in policy space

$$\pi_0 = \{ g_0(x) = 1 \}$$

$$W(x,\pi_0) = \tilde{c}(x, g_0(x)) + \tilde{\alpha}(x, g_0(x)) \times \sum_{y \in S} p(x,y|g_0(x)) W(y,\pi_0)$$

$$W(x,\pi_n) = \tilde{c}(x, g_n(x)) + \tilde{\alpha}(x, g_n(x)) \times \sum_{y \in S} p(x,y|g_n(x)) W(y,\pi_n)$$

$$g_{n+1}(x) = \underset{u \in U}{\arg\min} \left\{ \tilde{c}(x,u) + \tilde{\alpha}(x,u) \times \sum_{y \in S} p(x,y|u) W(y,\pi_n) \right\}$$