VE 281 Homework 1

Name: , ID No.:

- **Exercise 1.**

  Definition 1 (o-Notation)**.** *Let $f(n)$ and $g(n)$ be functions from the set of natural numbers to the set of nonnegative real numbers. $f(n)$ is said to be $o(g(n))$, written as $f(n) = o(g(n))$, if*

  $$\forall c > 0. \exists n_0. \forall n \geq n_0. f(n) < cg(n).$$

  An equivalence relation $\mathcal{R}$ on the set of complexity functions is defined as follows:

  $$f \mathcal{R} g \text{ if and only if } f(n) = \Theta(g(n)).$$

  A complexity class is an equivalence class of $\mathcal{R}$.

  The equivalence classes can be ordered by $\prec$ defined as: $f \prec g$ iff $f(n) = o(g(n))$.

  Example: $1 \prec \log\log n \prec \log n \prec \sqrt{n} \prec n^{\frac{3}{4}} \prec n \prec n \log n \prec n^2 \prec 2^n \prec n! \prec 2^{n^2}$.

  Please order the following functions by $\prec$ and give your (verbal or math) explanation:

  $$(\sqrt{2})^{\log n}, (n+1)!, ne^n, (\log n)!, n^3, n^{1/\log n}.$$

  f(n), g(n). bigger ?

  $$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 / \infty,$$

  $$n^{1/\log n} < (\sqrt{2})^{\log n} < n^3 < (\log n)! < ne^n < (n+1)!$$

- **Exercise 2**

  Prove that $\log(n!) = \Omega(n \log n)$. Notice that after you prove this, you will understand the lower bound of comparison sort is $\Theta(n \log n)$.

- **Exercise 3**

  Show how QUICKSORT can be made to run in $O(n \log n)$ time in the worst case.

- **Exercise 4**

  Let A be a list of n (not necessarily distinct) integers. Describe an $O(n)$ -algorithm to test whether any item occurs more than $\lceil n/2 \rceil$ times in A.

  find mediam

  check if it appears that much.

- **Exercise 5**

  Modify the $Merge(int *\ a, int\ left, int\ mid, int\ right)$ function taught in class and make it in-place (provide pseudo code). Explain why the new mergeSort might still run slower than quickSort in practice. Hint: Piazza.

  in-place → sacrifice time

- **Exercise 6**

  Let A be a ~~sorted~~ unsorted array of integers and S a target integer. Design algorithms for determining if there exist a pair of indices i, j(not necessarily distinct) such that A[i]+A[j]=S.

  1. Design an $O(n^2)$ algorithm (stating it in plain English is OK).
  2. Design an $O(nlogn)$ algorithm (stating it in plain English is OK).

O(1)

O(1) Hash

search S - A[i]

∴ n·O(1) = O(n)

1. double loop.

linear search (S-A[i])

n · O(n) = O(n²)

binary search

$\boxed{\text{sorted}}$

2. O(n) → O(logn).

linear search

- **Exercise 7**

  What is the most efficient sorting algorithm for each of the following situations and briefly explain:

  1. A small array of integers. (3~5 ele)
  2. A large array of integers that is already almost sorted.
  3. A large collection of integers that are drawn from a very small range.
  4. Stability is reqired, i.e., the relative order of two records that have the same sorthg key should not be changed.

1. any algorithm. (perform similar).

   most efficient: insertion sort. { easy implementation; O(n) best case.

2. insertion / bubble sort.

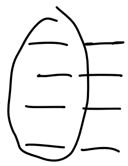
   insertion → O(n). bubble → O(1)·n = O(n)

3. counting sort O(n).

4. Merge { stability; O(nlogn).

- **Exercise 8**

  Suppose you are given a set of names and your job is to produce a set of unique first names. If you just remove the last name from all the names, you may have some duplicate first names. How would you create a set of first names that has each name occurring only once? Specifically, design an efficient algorithm for removing all the duplicates from an array.



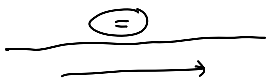
job: remove duplicates.

sort ↑

($\boxed{\text{sort strings}}$

fingerprint of string

Σ ASCII.)



- **Exercise 9**

  Suppose $\lim_{n\to\infty} f_1(n)/g_1(n)$ and $\lim_{n\to\infty} f_2(n)/g_2(n)$ exist, and we assume that all the functions are larger than 0 when $n > 0$. Judge whether the following statement is correct or not when $n \to \infty$. Justify your answers.

  1. $n \log n = O(n)$.
  2. $2^n = O(n!)$.
  3. If $f_1(n) = \Omega(g_1(n))$, $f_2(n) = \Omega(g_2(n))$, then $f_1(n)f_2(n) = \Omega(g_1(n)g_2(n))$.
  4. If $f_1(n) = \Theta(g_1(n))$, $f_2(n) = \Theta(g_2(n))$, then $f_1(n) + f_2(n) = \Theta(\max\{g_1(n), g_2(n)\})$.

1. $\lim_{\infty} \frac{n\log n}{n} = \lim_{\infty} \log n \to \infty$

def

limit

3. $f_1(n) = \Omega(g_1(n))$

if there exist C and n_0

$T(n) \geq c g_1(n)$ for all $n > n_0$

$f_1(n) \geq C_1 g_1(n)$, $f_2(n) \geq C_2 g_2(n)$

for all $n > \max\{n_{01}, n_{02}\}$

$f_1(n) \geq C_1 g_1(n)$

$f_2(n) \geq C_2 g_2(n)$

$\Rightarrow$ ..

- **Exercise 10**

We want to find the 6[th] largest element, which is 6, in the following array, Insertion sort is a simple and fast sorting algorithm when the length of array n is short. However, when n goes large, insertion sort may not be the best choice, as the worst case time complexity is $O(n^2)$. We can speed up insertion sort by combining it with `merge` in `mergeSort` we learnt in the lectures,

Alg. 1: `timSort`$(a[\cdot], x)$

Input : an array a of n elements, an integer $x > 0$ (you can assume that $x \ll n$)
Output: the sorted array of a

fixed length array sorting $O(n)$

for $i \leftarrow 0$; $i < n$; i += x **do**
  `insertionSort`$(a, i, \min(i + x - 1, n - 1))$;
for step $\leftarrow x$; step $< n$; step $*= 2$ **do** $O(\log n)$
  for left $\leftarrow 0$; left $< n$; left += $2 \times$ step **do** $\rightarrow O(n)$
    mid $\leftarrow$ left + step $- 1$;
    right $\leftarrow \min($left $+ 2 \times$ step $- 1,\ n - 1)$;
    `merge`$(a,$ left, mid, right$)$; $\rightarrow O($left $-$ right$)$

This algorithm is used as the default sorting algorithm in Java and Python. Here, we assume that $x \ll n$ is a known constant.

1. Suppose $n = 1000$ and $x = 32$. How many times will `insertionSort` be performed?
2. Suppose $x = 32$. Express in terms of n how many comparisons in the worst case will be performed in `insertionSort`.
3. Express the worst case running time of the whole algorithm in terms of the big-Oh notation.
4. Is this algorithm in-place? If not, express the additional space needed in terms of the big-Oh notation.

1. $1000/32 = 32$.

2. insertion sort worst case

$\lfloor n/32 \rfloor$

$x(x-1)/2$

??

$$\underset{32(32-1)/2}{496} \times \lfloor n/32 \rfloor + \frac{(n \bmod 32)(n \bmod 32 - 1)}{2} = O(n)$$

3. $O(n) + O(\log n) \cdot O(n)$
$= O(n \log n)$

4. in-place.