二重积分

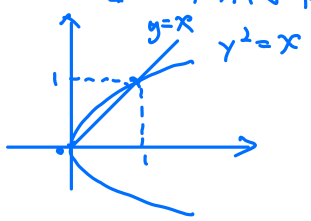
x型函数 $\iint_D f(x,y)\,d\sigma = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right] dx$ （平行y轴）

$= \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$

y型函数 $\iint_D f(x,y)\,d\sigma = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx$

eg: $I = \iint_D \frac{\sin y}{y}\,dx\,dy$. D是 $y^2=x$ 与 $y=x$ 围成的区域

★ 当一个方向积不出，则换个方向　换顺序称为 Fubini's Theorem



$I = \int_0^1 dy \int_y^{y^2} \frac{\sin y}{y}\,dx$　后略.

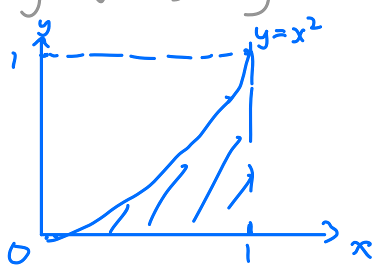
利用奇偶性：
当D关于x(y)轴对称，
f(x,y)是关于y(x)的奇函数，
$\iint_D f(x,y)\,d\sigma = 0$.

$\iint_R f(x,y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dx\,dy$
$= \int_a^b g(x)\,dx \int_c^d h(y)\,dy$

交换积分次序

(1) $\int_0^1 dx \int_0^{x^2} f(x,y)\,dy$

如图，



$\int_0^1 dy \int_{\sqrt{y}}^1 f(x,y)\,dy$.

当积分区域的边界曲线或被积函数用极坐标表示简单时，二重用极坐标　（如看到 x^2+y^2）

$\iint_D f(x,y)\,d\sigma$

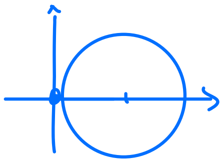
$\Delta\sigma = \frac{1}{2}\left[(r+\Delta r)^2\Delta\theta - r^2\Delta\theta\right] = \frac{1}{2}\left[2r\Delta r + (\Delta r)^2\right]\Delta\theta$

$\Rightarrow d\sigma = r\,dr\,d\theta$

若 $D = (r,\theta)\,|\,\{r_1(\theta) \le r \le r_2(\theta),\ \alpha \le \theta \le \beta\}$ 二重积分累次化为

$\iint_D (r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \int_\alpha^\beta d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr$

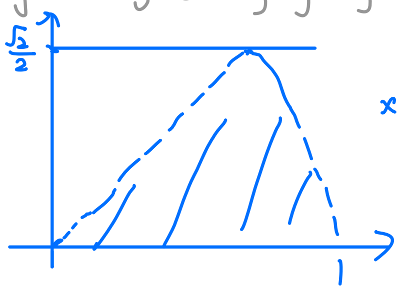
eg: $I = \iint x^2+y^2\,dx\,dy$. $D = \{x^2+y^2 \le 2x\}$



$I=\iint x^2+y^2\,dxdy$

$=2\int_0^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r^2\,rdr=2\int_0^{\frac{\pi}{2}}2^4\cos^4\theta\,d\theta=\frac{3}{2}\pi.$

eg: $\int_0^{\frac{\sqrt{2}}{2}}dy\int_y^{\sqrt{1-y^2}}\arctan\frac{y}{x}\,dx$



$x=y \sim x=1-y^2$

$=\int_0^{\frac{\pi}{4}}d\theta\int_0^1\theta\,rdr$

$=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\pi^2}{16}$

eg: $I=\int_0^{+\infty}e^{-x^2}dx$

$I^2=\int_0^{+\infty}e^{-x^2}dx\cdot\int_0^{+\infty}e^{-y^2}dy=\iint e^{-(x^2+y^2)}dxdy$

$=\int_0^{\frac{\pi}{2}}d\theta\int_0^{+\infty}e^{-r^2}\,rdr$ 想出现

$=\frac{\pi}{2}\left(\frac{1}{2}e^{-r^2}\Big|_{+\infty}^{0}\right)$

$=\frac{\pi}{4}\quad\rightarrow\quad I=\frac{\sqrt{\pi}}{2}$

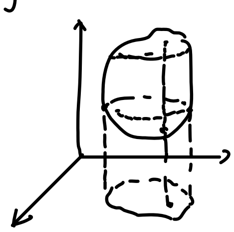
三重积分

$\iiint_\Omega f(x,y,z)\,dV$ 计算

1. 柱线法

Ω 以 曲面 $z=z_1(x,y)$ 为底，$z=z_2(x,y)$ 为顶．侧面是母线平行 z 轴的柱面所围成的区域．

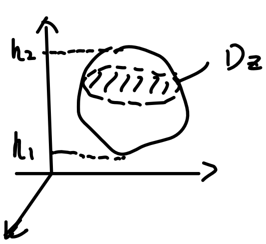
$=\iint_D dxdy\int_{z_1(x,y)}^{z_2(x,y)}f(x,y,z)\,dz$



2. 截面法

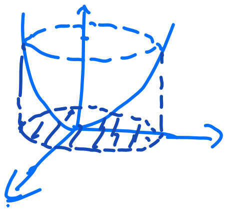
Ω 在 z 轴投影区间 $[h_1,h_2]$．即 Ω 介于 h_1,h_2 之间．垂直于 z 轴过 z 处平面截 Ω 所得截面为区域 D_z.

$\int_{z_1}^{z_2}\left(\iint_{D_z}f(x,y,z)\,dxdy\right)dz$



eg: $I=\iiint\limits_{\Omega}(x+z^2)\,dx\,dy\,dz$ Ω由 $z=x^2+y^2$ 与 $z=2$ 围成

奇函数 $\int x^{+}+\int x^{-}=0$ (方向)



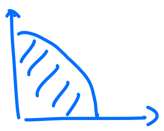
① 柱线

投区: $x^2+y^2\le 2$

$$I=\iint\limits_{x^2+y^2\le 2} dx\,dy\int_{x^2+y^2}^{2}(x+z^2)\,dz$$

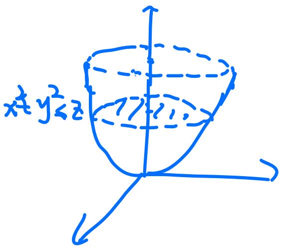
$$=4\iint\limits_{\substack{x^2+y^2\le 2\\ x\ge 0,y\ge 0}} dx\,dy\int_{x^2+y^2}^{2} z^2\,dz.$$

$$=\frac{4}{3}\iint\limits_{\substack{x^2+y^2\le 2\\ x\ge 0,\,y\ge 0}}\left[8-(x^2+y^2)^3\right]dx\,dy$$



极坐标 $=\frac{4}{3}\int_0^{\frac{\pi}{2}}d\theta\int_0^{\sqrt{2}}(8-r^6)\,r\,dr$

$$=\frac{2}{3}\pi\left((4\sqrt{2})^2-\frac{16}{8}\right)=4\pi$$



② 截面法 *

$$I=\int_0^2 dz\iint\limits_{x^2+y^2\le z} z^2\,dx\,dy=\int_0^2\pi z^3\,dz=4\pi$$

当被积函数与x,y无关，D_z面积易用z表示.

$$\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz=\iiint\limits_{\Omega} F(\rho,\theta,z)\,\rho\,d\rho\,d\theta\,dz.$$

其中 $F(\rho,\theta,z)=f(\rho\cos\theta,\rho\sin\theta,z)$

eg. $\iiint\limits_{\Omega} z\,dx\,dy\,dz$. Ω由 $z=x^2+y^2$ 与 $z=4$ 围成

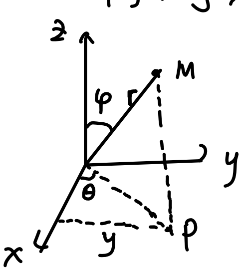
$D_{xy}=\{(\rho,\theta)\mid 0\le\rho\le 2,\ 0\le\theta\le 2\pi\}$

$\therefore\ \rho^2\le z\le 4,\quad 0\le\rho\le 2,\ 0\le\theta\le 2\pi$

$$=\int_0^{2\pi}d\theta\int_0^2\rho\,d\rho\int_{\rho^2}^{4}z\,dz$$

$$=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^2\rho(16-\rho^4)\,d\rho=\frac{64}{3}\pi$$

利用球面坐标求三重积分



$0\le r<+\infty$

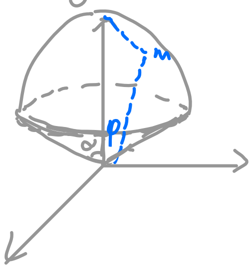
$0\le\varphi\le\pi$

$0\le\theta\le 2\pi$

$$\Rightarrow\begin{cases}x=r\sin\varphi\cos\theta\\ y=r\sin\varphi\sin\theta\\ z=r\cos\varphi\end{cases}$$

$$\iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \iiint_\Omega F(r,\varphi,\theta)\, r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

eg. 求半径a球面与半顶角 α 内接锥面所围成立体体积



$0 \le r \le 2a\cos\varphi$, $0 \le \varphi \le \alpha$, $0 \le \theta \le 2\pi$

$$V = \iiint_D r^2 \sin\varphi \, dr\, d\varphi\, d\theta = \int_0^{2\pi} d\theta \int_0^{\alpha} \sin\varphi \, d\varphi \int_0^{2a\cos\varphi} r^2\, dr$$

$$= \frac{4\pi a^3}{3}(1-\cos^4\alpha)$$

Moment of centers of Mass

moment and center of mass

of x-axis $\quad M_x = \iint_D y\,\rho(x,y)\,dA$

y-axis $\quad M_y = \iint_D x\,\rho(x,y)\,dA$

center of mass $\left(\frac{1}{m}M_y, \frac{1}{m}M_x\right)$

$$m = \iint_D \rho(x,y)\,dA$$

Moment of inertia

moment of inertia is mr^2 . r is distance from the particle to the axis

x-axis $\quad I_x = \iint_D y^2 \rho(x,y)\,dA$

y axis $\quad I_y = \iint_D x^2 \rho(x,y)\,dA$

moment of inertia about the origin. polar moment of inertia.

$$I_0 = \iint_D (x^2+y^2)\rho(x,y)\,dA . \qquad I_0 = I_x + I_y$$

Probability

$$P(a \le x \le b) = \int_a^b f(x)\,dx .$$

$$P((x,y)\in D) = \iint_D f(x,y)\,dA .$$

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

expected value

if X is a variable. $\mu = \int_{-\infty}^{\infty} x f(x)\, dx$.

if X, Y are variables with joint density function f.

X- mean $\quad \mu_1 = \iint_{\mathbb{R}^2} x f(x,y)\, dA$

Y - mean $\quad \mu_2 = \iint_{\mathbb{R}^2} y f(x,y)\, dA$

Surface area.

$z = f(x,y)$ $\qquad A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA$

反常二重积分

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

证: $\iint e^{-(x^2+y^2)}\, dx\, dy \qquad D_k = x^2+y^2 \le k^2.$

$$= \lim_{k\to\infty} \int_0^{2\pi} d\theta \int_0^k e^{-r^2}\, r\, dr$$

$$= \lim 2\pi \left(-\tfrac{1}{2}\right) \int_0^k d(e^{-r^2})$$

$$= \lim_{k\to\infty} \pi\left(1 - \frac{1}{e^{k^2}}\right) = \pi.$$

三重积分应用

$$m = \iiint_E \rho(x,y,z)\, dV$$

moments: $M_{yz} = \iiint_E x\rho(x,y,z)\, dV \qquad M_{xz} = \iiint_E y\rho(x,y,z)\, dV$

$M_{xy} = \iiint_E z\rho(x,y,z)\, dV$

center of mass $\quad \bar{x} = \dfrac{M_{yz}}{M} \quad \bar{y} = \dfrac{M_{xz}}{M} \quad \bar{z} = \dfrac{M_{xy}}{M}$

moments of inertia $\quad I_x = \iiint_E (y^2+z^2)\rho(x,y,z)\, dV$

$I_y = \iiint_E (x^2+z^2)\rho(x,y,z)\, dV$

$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\, dV$