# 曲线积分

2D: $\int_C f(x,y)\,ds = \int_C f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

3D: $r(t) = x(t)i + y(t)j + z(t)k$

$$\int_C f(x,y,z)\,dS = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$
$$= \int_a^b f(r(t))\cdot|r'(t)|\,dt$$

起点到终点 $r(t) = (1-t)r_0 + t\,r_t$

第二类 $W = \int_a^b F(r(t))\cdot r'(t)\,dt.$

$\int_C F\cdot dr = \int_C F\cdot T\,ds = W \qquad T(t) = \frac{r'(t)}{|r'(t)|}$

$F = Pi + Qj + Rk$ 型.

$F$ 可写为 $\nabla f$ 则 $\int_C F\,dr = \int_C \nabla f\,dr = f(r(b)) - f(r(a))$ 从a到b

与路径无关 $\Leftrightarrow$ $F = \nabla f$, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\mathrm{curl}\,F = 0.$
conservative

格林 $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$
[ 二维区域 $\Leftrightarrow$ 边界曲线 ]

旋度 $\mathrm{curl}\,F = \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

散度 divergence $F = \nabla\cdot F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}$

定理 1. $f$ 二阶偏导 $\mathrm{curl}(\nabla f) = 0.$

2. $\mathrm{div}\,\mathrm{curl}\,F = 0.$

$\oint_C F\cdot dr = \iint_D \mathrm{curl}\,F\cdot k\,dA.$

$\oint_C F\cdot n\,ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iint_D \mathrm{div}\,F(x,y)\,dA.$

# 曲面积分

$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$

- Surfaces of Revolution

$y = f(x)$ rotate about $x$-axis ($f(x) \geq 0$. $(x,y,z)$ on $S$)

$x = x$, $y = f(x)\cos\theta$, $z = f(x)\sin\theta$

- tangent plane

① 得 $r_u = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k$

$r_v = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j + \frac{\partial z}{\partial v} k.$

② 得 normal vector $r_u \times r_v$ 表达式

③ 代入点 $(x_0, y_0, z_0) \Rightarrow$ 得 $u, v \Rightarrow$ 算 normal vector

④ 得 tangent plane.

曲面积分: $A(S) = \iint_D |r_u \times r_v| \, dA$

若给了 $z = g(x,y)$ $A(S) = \iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} \, dS$

然后有了 $\iint_S f(x,y,z) \, dS = \iint_D f(r(u,v)) |r_u \times r_v| \, dA$

若给了 $z = g(x,y)$ $= \iint_D f(x,y,g(x,y)) \sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1} \, dA$

第二类 $\iint_S F \cdot dS = \iint_S F \cdot n \, dS$ <u>flux</u> of $F$ across $S$

$\Rightarrow \iint_S F \cdot dS = \iint_D F \cdot (r_u \times r_v) \, dA.$

若给了 $z = g(x,y)$ $F \cdot (r_u \times r_v) = (Pi + Qj + Rk) \cdot (-\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k)$

$\therefore \iint_S F \cdot dS = \iint_D (-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R) \, dA$ (outward)

Stoke's $\int_C F \cdot dr = \iint_S \text{curl } F \cdot dS$

[曲线 $\Longleftrightarrow$ 曲面]

Divergence 高斯 $\iint_S F \, dS = \iiint_E \text{div} F \, dV$

[曲面 $\Longleftrightarrow$ 物体]