- Elimination

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + b_1 & ① \\ y_2' = a_{21}y_1 + a_{22}y_2 + b_2 & ② \end{cases}$$

①式求导: $y_1'' = a_{11}y_1' + a_{12}y_2' \overset{②}{=} a_{11}y_1' + a_{12}(a_{21}y_1 + a_{22}y_2 + b_2)$

用①式消去y_2: $y_1'' = a_{11}y_1' + a_{12}a_{21}y_1 + a_{12}b_2 + a_{22}(y_1' - a_{11}y_1 - b_1)$

$$y_1'' - (a_{11}+a_{22})y_1' + (a_{11}a_{22} - a_{12}a_{21})y_1 = a_{12}b_2 - a_{22}b_1$$

$$\Rightarrow \quad \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0.$$

then $y_1(t) = y_{1c}(t) + y_{1p}(t)$. complementary solution is

$$y_{1c}(t) = \begin{cases} C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}, & \lambda_1 \neq \lambda_2 \\ (C_1 + C_2t)e^{\lambda t}, & \lambda_1 = \lambda_2 \end{cases}$$

y_{1p} 解法略.

$$y_2(t) = \frac{1}{a_{12}(y_1' - a_{11}y_1 - b_1)}$$

- Matrix Method

Consider normal homogeneous nth-dimensional system with constant coefficient

$y'(t) = Ay(t)$ sol: $y = e^{\lambda t}v$

代入得 $e^{\lambda t}(A - \lambda I)v = 0$. $\det(A - \lambda I) = 0$

λ_i: eigenvalue v_i: eigenvector of matrix A.

If all eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$ are distinct, then complementary sol is

$y(t) = C_1e^{\lambda_1 t}v_1 + C_2e^{\lambda_2 t}v_2 + \cdots + C_ne^{\lambda_n t}v_n$. That is, $y(t) = \phi(t)C$

ini condition $y(t_0) = y_0$, $y(t_0) = \phi(t_0)C = y_0$. $C = \phi^{-1}(t_0)y_0$

eg: Solve $\begin{cases} y_1' - y_2' - 6y_2 = 0 \\ y_1' + 2y_2' - 3y_1 = 0. \end{cases}$

$\begin{cases} y_1' = y_1 + 4y_2 \\ y_2' = y_1 - 2y_2 \end{cases}$ $A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$

characteristic equation: $\det(A - \lambda I) = \begin{vmatrix} -\lambda+1 & 4 \\ 1 & -2-\lambda \end{vmatrix}$

$= (\lambda+3)(\lambda-2)$

$\lambda_1 = -3$: $(A - \lambda_1 I)v_1 = \begin{vmatrix} 4 & 4 \\ 1 & 1 \end{vmatrix}\begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$ $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 2$: $(A - \lambda_2 I)v_2 = |-1 \quad 4 \quad |(v_2^1) = 0$ $v_2 = (4$

complementary sol: $y(t) = C_1 e^{-3t}\begin{pmatrix}1\\-1\end{pmatrix} + C_2 e^{2t}\begin{pmatrix}4\\1\end{pmatrix}$

or $\begin{cases} y_1(t) = C_1 e^{-3t} + 4C_2 e^{2t} \\ y_2(t) = -C_1 e^{-3t} + C_2 e^{2t} \end{cases}$

若 $y'(t) = Ay(t)$ is real matrix. $\lambda = \alpha + \beta i$ is its eigenvalue with corresponding eigenvector v. then $y_1(t) = \mathcal{R}(e^{\lambda t} v)$ and $y_2 = \mathcal{I}(e^{\lambda t} v)$.

complementary sol: $y(t) = C_1 \mathcal{R}(e^{\lambda t} v) + C_2 \mathcal{I}(e^{\lambda t} v)$

eg: $\begin{cases} y_1' + y_1 - 5y_2 = 0. \\ 4y_1 + y_2' + 5y_2 = 0 \end{cases}$

$A = \begin{vmatrix} 1 & 5 \\ -4 & -5 \end{vmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 5 \\ -4 & -5-\lambda \end{vmatrix} = 0.$ $\lambda_{1,2} = -3 \pm 4i$

For $\lambda = -3+4\lambda$

$\therefore (A - \lambda I)v = \begin{pmatrix} 2-4i & 5 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore (2-4i)v_1 + 5v_2 = 0.$ $v_2 = -\frac{1}{5}(2-4i)v_1$ take $v_1 = 5$ $\therefore v_2 = -2+4i$

$v = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + i\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$\therefore e^{\lambda t} v = e^{-3t}(\cos 4t + i\sin 4t)\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix} + i\begin{pmatrix} 0 \\ 4 \end{pmatrix}\right)$

$= e^{-3t}\left[\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\cos 4t - \begin{pmatrix} 0 \\ 4 \end{pmatrix}\sin 4t\right) + i\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\sin 4t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}\cos 4t\right)\right]$

$\therefore y(t) = C_1 e^{-3t}\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\cos 4t - \begin{pmatrix} 0 \\ 4 \end{pmatrix}\sin 4t\right) + C_2 e^{-3t}\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\sin 4t - \begin{pmatrix} 0 \\ 4 \end{pmatrix}\cos 4t\right)$

$\hookrightarrow$ $y_1(t) = 5e^{-3t}(C_1\cos 4t + B\sin 4t)$

$y_2(t) = 2e^{-3t}\left[(-C_1 + 2C_2)\cos 4t - (2C_1 + C_2)\sin 4t\right]$

The Matrix Method: Multiple Eigenvalues

- Recall, that if a matrix $A_{n\times n}$ has n distinct eigenvalues λ_i, $i = 1..n$, then the corresponding eigenvectors are linearly independent and form a complete basis of eigenvectors.
- What happens if $A_{n\times n}$ has repeated eigenvalues? In general case, the matrix $A_{n\times n}$ may not have n linearly independent eigenvectors!
- To obtain a FSS, we augment the eigenvectors with generalized eigenvectors. Let λ is an eigenvalue of multiplicity m, and there are only $k < m$ linearly independent eigenvectors corresponding to λ. A FSS is obtained by including $(m-k)$ generalized eigenvectors.

??

$$
\begin{aligned}
&(A-\lambda I)v_i = 0 && \Rightarrow v_i,\ i = 1..k \text{ are lin. independent}\\
&(A-\lambda I)v_{k+1} = v_k && \Rightarrow (A-\lambda I)^2 v_{k+1} = 0\\
&(A-\lambda I)v_{k+2} = v_{k+1} && \Rightarrow (A-\lambda I)^3 v_{k+2} = 0\\
&\dots\\
&(A-\lambda I)v_m = v_{m-1} && \Rightarrow (A-\lambda I)^{m-k+1} v_m = 0
\end{aligned}
$$

The Matrix Method: Multiple Eigenvalues

If the matrix A of the homogeneous system $y'(t) = Ay(t)$ has an eigenvalue λ of algebraic multiplicity $m > 1$, and a sequence of generalized eigenvectors corresponding to λ is $v_1, v_2, \dots, v_m$. Then the corresponding m linearly independent solutions of the homogeneous system are

$$y_i(t) = e^{\lambda t} v_i,\ i = 1, \dots, k,$$

$$y_{k+2} = e^{\lambda t}\left(v_k \frac{t^2}{2!} + v_{k+1}t + v_{k+2}\right),$$

$$\dots$$

$$y_m = e^{\lambda t}\left(v_k \frac{t^{m-k}}{(m-k)!} + v_{k+1}\frac{t^{m-k-1}}{(m-k-1)!} + \dots + \dots + v_{m-2}\frac{t^2}{2!} + v_{m-1}t + v_m\right).$$

eg: $\begin{cases} y_1' - 4y_1 + y_2 = 0 \\ 3y_1 - y_2' + y_2 - y_3 = 0 \\ y_1 - y_3' + y_3 = 0 \end{cases}$

$A = \begin{pmatrix} 4 & -1 & 0 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \cdots$

$\lambda_{1,2,3} = 2.$

$(A-\lambda I)v_1 = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{pmatrix} = 0.$

Take $v_1^1 = 1. \Rightarrow v_1^2 = 2,\ v_1^3 = 1$

$(A-\lambda I)v_2 = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{pmatrix} = \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{pmatrix}$

Take $v_2^1 = 2. \Rightarrow v_2^2 = 3,\ v_2^3 = 1$

$(A-\lambda I)v_3 = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_3^1 \\ v_3^2 \\ v_3^3 \end{pmatrix} = \begin{pmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{pmatrix}$

Take $v_3^1 = 1 \Rightarrow v_3^2 = 0,\ v_3^3 = 0.$

$\therefore y_1(t) = e^{2t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\quad y_2(t) = e^{2t}\left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}t + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right]$

??

$y_3(t) = e^{2t}\left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\frac{t^2}{2} + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]$

$\therefore y(t) = C_1 e^{2t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + C_2 e^{2t}\left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}t + \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}\right] +$

?? $\longrightarrow$ $2C_3 e^{2t}\left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\frac{t^2}{2} + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]$

◦ non-homogeneous System

Consider $y'(t) = Ay(t) + b(t)$

$y'(t) = Ay(t)$ has obtained $y(t) = \phi(t)C$

Therefore, $\phi'(t) = A\phi(t)$, C is n dimensional constant vector

by variation of parameters, $C = c(t)$. $\therefore y(t) = \phi(t)C(t)$

$\phi'(t)C(t) + \phi(t)C'(t) = Ay(t) + b(t)$

$A\phi(t)C(t) + \phi(t)C'(t) = A\phi(t)C(t) + b(t)$

$\therefore \phi(t)C'(t) = b(t)$

$C(t) = C + \int \phi^{-1}(t)b(t)\,dt. \Rightarrow y(t) = \phi(t)\left(C + \int \phi^{-1}(t)b(t)\,dt\right)$

eg: $\begin{cases} y_1' + 3y_1 + 4y_2 = 2e^{-t} \\ y_1 - y_2' + y_2 = 0. \end{cases}$

$A = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}$ $\quad b(t) = \begin{pmatrix} 2e^{-t} \\ 0 \end{pmatrix}$ $\qquad \det(A - \lambda I) = 0 \Rightarrow \lambda_{1,2} = -1$

$(A - \lambda I)V_1 = \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} V_1' \\ V_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

* a second linearly independent eigenvector doesn't exist.

generalized eigenvector: $(A - \lambda I)V_2 = V_1 \quad \Rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$y_1(t) = e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad y_2(t) = e^{-t}\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}t + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

$\phi(t) = [y_1(t)\ y_2(t)] = \begin{pmatrix} 2e^{-t} & (2t+1)e^{-t} \\ -e^{-t} & -(t+1)e^{-t} \end{pmatrix} \qquad \det\phi = -e^{-2t}$

$\phi^{-1}(t) = \begin{pmatrix} (t+1)e^{t} & (2t+1)e^{t} \\ -e^{t} & -2e^{t} \end{pmatrix}$

Evaluate $\int \phi^{-1}(t)b(t)\,dt = \int \begin{pmatrix} (t+1)e^{t} & (2t+1)e^{t} \\ -e^{t} & -2e^{t} \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 0 \end{pmatrix} dt$

$= \int \begin{pmatrix} 2(t+1) \\ -2 \end{pmatrix} dt = \begin{pmatrix} t^2 + 2t \\ -2t \end{pmatrix}$

$y(t) = \phi(t)\left(C + \int \phi^{-1}(t)b(t)\,dt\right) = \begin{pmatrix} 2e^{-t} & (2t+1)e^{-t} \\ -e^{-t} & -(t+1)e^{-t} \end{pmatrix}\begin{pmatrix} C_1 + t^2 + 2t \\ C_2 - 2t \end{pmatrix}$

$A = \begin{pmatrix} a, b \\ c, d \end{pmatrix}$

$\therefore\ y_1(t) = e^{-t}(-2t^2 + 2(C_2+1)t + (2C_1 + C_2))$ $\quad \Leftarrow$??

$y_2(t) = e^{-t}(t^2 - C_2 t - (C_1 + C_2))$

$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$A^{-1} = \frac{1}{|A|}A^*$，$A^*$伴随矩阵：第k列元素是A第k行元素的代数余子式

• Matrix Exponent Method

Consider $y'(t) = A(t)\,y(t) + b(t) \quad y(t_0) = y_0.$

Let A be $n \times n$ matrix. We define exponent of A by

$$\exp A = I + A + \frac{A^2}{2} + \cdots + \frac{A^k}{k!} + \cdots = \sum_{k=0}^{\infty}\frac{A^k}{k!}$$

where $A^2 = A \cdot A$

??, Properties of $\exp A$.

① $\exp 0 = I$.

② For a constant matrix of A.

$$\frac{d\exp(At)}{dt} = \frac{d}{dt}\left(\sum_{k=0}^{\infty}\frac{A^k t^k}{k!}\right) = \sum_{k=0}^{\infty}\frac{A^k t^{k-1}}{(k-1)!} = A\exp(At)$$

③ $\exp A$ commutes with any power of A.

④ If B commutes with A, that is $AB=BA$, then B commutes with $\exp A$.

⑤ A, B are $n\times n$ matricies, then

$$\exp A \exp B = \sum_{k=0}^{\infty} \frac{A^k}{k!} \sum_{j=0}^{\infty} \frac{A^j}{j!} = \sum_{m=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \right)$$

$\therefore \exp A \exp B = \exp(A+B) = \exp B \exp A.$

⑥ Since A and $-A$ commute then

$$\exp A \exp(-A) = \exp 0 = I = \exp(-A) \exp A.$$

Thus, $\exp A$ has inverse $\exp(-A)$ for any matrix A.

· $y'(t) = A(t)y(t) + b(t)$ let $M(t)$ be the solution of matrix equation.

? $\frac{d(M(t))}{dt} = -M(t)A(t)$, $M(t_0) = I$

Then $\frac{d(My)}{dt} = M\frac{dy}{dt} + \frac{dM}{dt} y = M(Ay+b) - (MA)y = Mb.$

$\therefore M(t)y(t) = \int_{t_0}^{t} M(u)b(u)\,du + M(t_0)y(t_0) = \int_{t_0}^{t} M(u)b(u)\,du + y(t_0)$

$y(t) = M^{-1}(t)\int_{t_0}^{t} M(u)b(u)\,du + M^{-1}(t)\,y(t_0)$

$M(t) = \sum_{k=0}^{\infty} (-1)^{k+1} M_k(t)$ that can be used $M(t)$ satisfies the corresponding matrix equation.

If A is a constant matrix, $M(t) = \exp(-A(t-t_0))$

hense, $y(t) = \int_{t_0}^{t} [\exp(A(t-u))]\, b(u)\,du + \exp(A(t-t_0))\,y(t_0)$

eg: Consider $y'(t) = Ay(t)$ with $A = \begin{pmatrix} -5 & 4 \\ -9 & 7 \end{pmatrix}$,

g.s: $y(t) = \exp(At)\,C$, C is a vector of 2 arbitrary constants.

$y(t) = \exp[(A-I)t + It]C = \exp[(A-I)t]\exp(It)\,C$

$\exp(It) = e^t I$, $\exp[(A-I)t] = \sum_{k=0}^{\infty} \frac{t^k (A-I)^k}{k!} = I + t(A-I)$

$\therefore y(t) = e^t C + te^t (A-I) C$

with component form: $y_1(t) = C_1 e^t + 2(2C_2 - 3C_1)te^t$

$y_2(t) = 3(2C_2 - 3C_1)te^t + C_2 e^t$

The Matrix Exponent Method

- Let D be a diagonal matrix, $D = diag(d_1, d_2, \dots, d_n)$. What is D^2, D^3, $\dots$, $D^n, \dots$? A direct verification shows that $\exp(Dt) = diag(\exp(d_1 t), \exp(d_2 t), \dots, \exp(d_n t))$, where

$$\exp(d_i t) = 1 + \sum_{k=0}^{\infty} \frac{d_i^k t^k}{k!}, \quad i = 1 \dots n$$

- Recall, that if a matrix A is diagonalizable then A is similar to a diagonal matrix D: $D = T^{-1}AT$, T is the transformation matrix (its columns are eigenvectors of A!)
- A and D have the same eigenvalues.Moreover, the elements of D are eigenvalues of A!!!
- Introduce a new function $y = Tx \Rightarrow Tx' = ATx$ and

$$x' = T^{-1}ATx = Dx$$

The Matrix Exponent Method

- The solution of this equation is

$$x = \exp(Dt)C = \begin{pmatrix} e^{d_1 t} & 0 & \dots & 0 \\ 0 & e^{d_2 t} & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & e^{d_n t} \end{pmatrix} C$$

- $$y = Tx = \begin{pmatrix} \varphi_1^1 e^{d_1 t} & \dots & \varphi_n^1 e^{d_n t} \\ \dots & & \\ \varphi_1^n e^{d_1 t} & \dots & \varphi_n^n e^{d_n t} \end{pmatrix} C$$

- For example, the solution of the DE $y' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} y$ is $y = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ with eigenvalues $\lambda_1 = 2\, \lambda_2 = -1$ of A and the corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$