- Autonomous system

n-dimensional vector field where solutions of the system are interpreted in the form of trajectories is called phase space of the system. The trajectories are called phase trajectories. (portraits)

- phase plane of linear homogeneous system with constant coefficient

$$y' = Ay \quad \text{or} \quad \begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 \\ y_2' = a_{21}y_1 + a_{22}y_2 \end{cases}$$

Origin (0,0) always in equilibrium state. let λ_1, λ_2 of A be real, distinct, nonzero.

expand $y(t) = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t}$ in terms of the basis eigen vectors v_1, v_2.

$y = \varepsilon_1 V_1 + \varepsilon_2 V_2$. $\varepsilon_1 = C_1 e^{\lambda_1 t}$, $\varepsilon_2 = C_2 e^{\lambda_2 t}$

$\varepsilon_1, \varepsilon_2$ on a phase plane P of the system are not rectangular $\Rightarrow$ make affine (仿射图) mapping of phase plane P onto an auxiliary plane P^* st. $V_1, V_2 \to e_1, e_2$

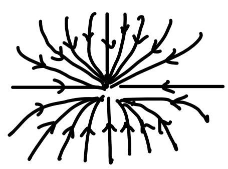
$C_1 = C_2 = 0$. (0, 0)

$C_1 = 0, C_2 > 0$. motion positive semi axis of ordinates (纵坐标). Case $\lambda_2 > 0$ / $\lambda_2 < 0$. motion away / toward origin

$C_2 = 0, C_1 > 0$. 〃 〃 — — — abscissas.

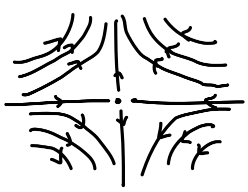
Theo: eigenvalue of A real. If a line l lies along an eigenvector of A, then in phase plane any solution of $y' = Ay$ starts at a point (y_1, y_2) on line l remains on l for all t; as $t \to \infty$ it approaches the origin if $\lambda_1 < 0$, move away from origin if $\lambda_1 > 0$.

- $\lambda_2 < \lambda_1 < 0$. Stable node
  1. positive semi axes go toward origin
  2. 第一象限 渐近原点
  3. $t \to -\infty$ motion goes in the direction of axis of ordinates



$0 < \lambda_1 < \lambda_2$ Unstable node.

$\lambda_1 < 0 < \lambda_2$ Saddle point
1. motion along positive semi axis of abscissas directed toward origin.
2. motion along positive semi axis of ordinates is directed away from the origin.
3. forms of trajectories 在一象限 resemble 双曲线



- Classification of states of equilibrium

$\lambda_{1,2} = \mu + i\nu$ ⇒ corresponding eigenvectors can be chosen to be complex conjugate $v, \bar{v}$

Any solution $y = Cve^{\lambda t} + \bar{C}\bar{v}e^{\bar{\lambda}t}$ $\quad C \in \mathbb{C}$

Denote $v = \frac{1}{2}(v_1 - iv_2)$, v_1, v_2 : real vectors, then v_1、v_2 forms basis in phase plane P

$\xi = \xi_1 + i\xi_2 = Ce^{\lambda t} \Rightarrow y = \xi_1 v_1 + \xi_2 v_2$

The trajectory $y = \xi_1 v_1 + \xi_2 v_2$ will be mapped into a phase trajectory described by $\xi = \xi_1 + i\xi_2 = Ce^{\lambda t}$.

polar coordinates : $\xi = \rho e^{i\varphi}$, $C = Re^{i\alpha}$

$\therefore \rho = Re^{\mu t}$, equ of motion of a point in phase plane P^*

$\varphi = \alpha + \nu t$

$\mu \neq 0$. ⇒ every trajectory is a logarithmic spiral. image on phase plane call focus.

$\mu < 0$. point approach origin at $t \to +\infty$ ⇒ stable focus

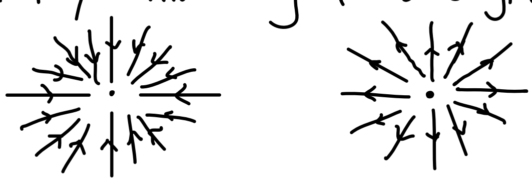
$\mu > 0$ ⇒ unstable focus

$\mu = 0$. ⇒ every phase trajectory except state of equilibrium $(0,0)$ is closed ⇒ center.

- Degenerated cases : $\lambda_1 = \lambda_2 = \lambda \neq 0$.
  - two independent eigenvectors v_1, v_2 :

    $y = C_1 v_1 e^{\lambda t} + C_2 v_2 e^{\lambda t} = y_0 e^{\lambda t}$

    A ray emanating from origin : $\lambda < 0$ toward, $\lambda > 0$ away

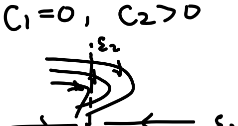
  - one independent eigenvector v_1.

    $y = \xi_1 v_1 + \xi_2 v_2$ $\quad \xi_1 = e^{\lambda t}(C_1 + C_2 t)$, $\xi_2 = C_2 e^{\lambda t}$

    $\lambda < 0$: C_1, C_2 符号变, consider trajectories in upper half-plane.

    $C_2 = 0$. $C_1 \neq 0$. positive $(C_1 > 0)$ and negative $(C_1 < 0)$ semi axes

    $C_1 = 0$, $C_2 > 0$ $\quad \xi_1 = C_2 e^{\lambda t} t$, $\xi_2 = C_2 e^{\lambda t}$ as $t \uparrow$ from 0, point 先右再左. 向origin降

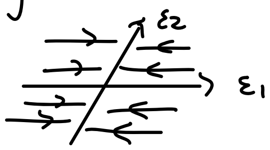
$\lambda_1 = \lambda_2 = 0.$

Case 1: $y = y_0 \Rightarrow$ every point of P state of equilibrium

Case 2: $\varepsilon_1 = C_1 + C_2 t$, $\varepsilon_2 = C_2 \Rightarrow$ straight lines; all points of line $\varepsilon_2 = 0$ are equilibrium

$\lambda_1 \neq 0,\ \lambda_2 = 0.$

$y = \varepsilon V_1 + \varepsilon_2 V_2$, $\varepsilon_2 = \text{const}$. $\varepsilon_1 = C_1 e^{\lambda_1 t}$.



motion along straight line $\varepsilon_2 = \text{const}$ in the direction of line $\varepsilon_1 = 0$. All points of $\varepsilon_1 = 0$. ~~states~~ of equilibrium.

· Stability X 不考.

Lyapunov stable:

1. $\exists \rho > 0$, for $|\varepsilon - a| < \rho$ $\varphi(\varepsilon, t)$ defined for $t > 0$.

2. $\forall \varepsilon > 0, \exists\, 0 < \delta < \rho$: $|\varepsilon - a| < \delta \Rightarrow |\varphi(\varepsilon, t) - a| < \varepsilon$ for all $t > 0$.

Asymptotically stable:

stable and $\exists\, \sigma < \rho$: $|\varepsilon - a| < \sigma$, $\lim_{t \to +\infty} |\varphi(\varepsilon, t) - a| = 0$

∘ A function $W(y) = \sum_{i,j=1}^{n} w_{ij} y_i y_j$, $w_{ij} = w_{ji} \in \mathbb{R}$.

is called a quadratic form of vector $y = (y_1, \cdots, y_n)$

The quadratic form $W(y)$ is positive definite if $W(y) > 0$ for $y \neq 0$.

Define $y_i = a_i + \Delta y_i$

$$(\Delta y_i)' = \sum_{j=1}^{n} a_{ij} \Delta y_j + R_i, \quad i = 1, \cdots n.$$

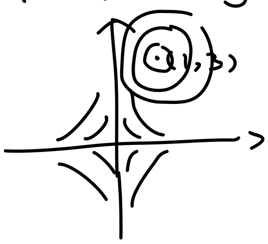
例. $$\begin{cases} y_1' = \frac{3}{2} y_1 - \frac{1}{2} y_1 y_2 = \phi_1 \\ y_2' = -\frac{1}{2} y_2 + \frac{1}{2} y_1 y_2 = \phi_2 \end{cases}$$

$$\begin{cases} \frac{3}{2} y_1 - \frac{1}{2} y_1 y_2 = 0 \\ -\frac{1}{2} y_2 + \frac{1}{2} y_1 y_2 = 0. \end{cases} \Rightarrow \begin{cases} y_1 = 0 \\ y_2 = 0 \end{cases} \quad \begin{cases} y_1 = 1 \\ y_2 = 3 \end{cases}$$

Jacobian $J = \begin{vmatrix} \frac{\partial \phi_1}{\partial y_1} & \frac{\partial \phi_1}{\partial y_2} \\ \frac{\partial \phi_2}{\partial y_1} & \frac{\partial \phi_2}{\partial y_2} \end{vmatrix} = \begin{pmatrix} \frac{3}{2} - \frac{1}{2}y_2 & -\frac{1}{2}y_1 \\ \frac{1}{2}y_2 & -\frac{1}{2}+\frac{1}{2}y_1 \end{pmatrix}$

$(0,0)$ 代入 $J \Rightarrow J = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \Rightarrow \lambda = -\frac{1}{2}, \frac{3}{2}$ saddle.

$(1,3)$ 代入 $J = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{3}{2} & 0 \end{pmatrix} \Rightarrow \lambda = \pm\frac{\sqrt{3}}{2}i$ center



书

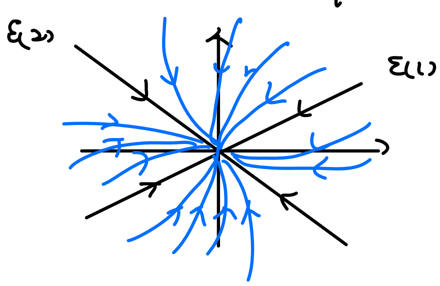
Case 1. 实, 异, eigenvalues. 同号.

$x = c_1 \xi^{(1)} e^{r_1 t} + c_2 \xi^{(2)} e^{r_2 t}$

suppose $r_1 < r_2 < 0$.

$t \to \infty$ 所有点近 critical point. 渐近 $\xi^{(2)}$

这类 critical point 叫 node, nodal sink.



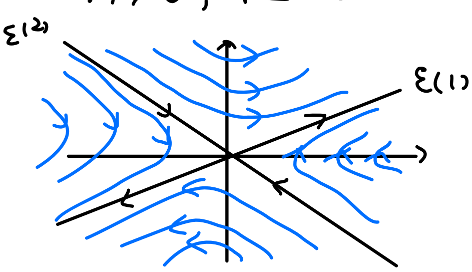
当 $0 < r_2 < r_1$ 时 方向相反, 图不变

Case 2. 实, 异号

$X = c_1 \xi^{(1)} e^{r_1 t} + c_2 \xi^{(2)} e^{r_2 t}$

小于0 向内
大于0 向外.

$r_1 > 0, r_2 < 0$.



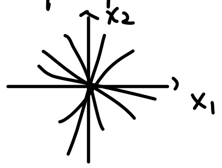
saddle

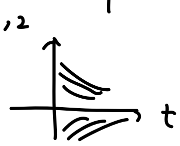
Case 3. 等 eigenvalue.

(a) two independent eigenvectors.

$X = c_1 \xi^{(1)} e^{rt} + c_2 \xi^{(2)} e^{rt}$

proper node, star point. 过原点直线



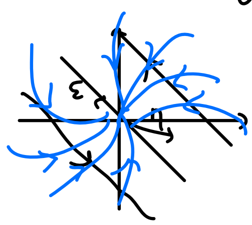


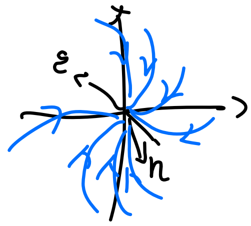
(b) one independent eigenvector

$$X = c_1 \xi e^{rt} + c_2(\xi t e^{rt} + \eta e^{rt})$$

$$X = y e^{rt}, \quad y = (c_1 \xi + c_2 \eta) + c_2 \xi t$$

improper / degenerate node.





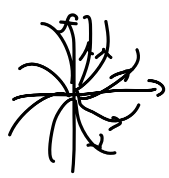
case 4. complex eigen nonzero Real part

$$X = \lambda \pm i\mu$$

$$X' = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} X.$$

$$X_1' = \lambda x_1 + \mu x_2 \qquad X_2' = -\mu x_1 + \lambda x_2.$$

- spiral sink / source
  $\lambda < 0$ $\quad$ $\lambda > 0$.



case 5. Pure img Eigen.

$$X' = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} x.$$

- center.

$\mu > 0$ 顺时 $\qquad$ $\mu < 0$ 逆时 $\qquad$ solution $T = \frac{2\pi}{\mu}$

总结.

$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically stable.

$r_2 < 0 < r_1$	Saddle	Unstable.
$r_1 = r_2 > 0.$	Proper/improper node	Unstable
$r_1 = r_2 < 0$	同上	Asymptotically stable

$r_{1,2} = \lambda \pm i\mu$

$\lambda > 0$	Spiral point	Untable
$\lambda < 0$	Spiral point	Asymptotically stable
$\lambda = 0$	Center	Stable

Stable:

critical point If

given $\varepsilon > 0$, there is $\delta > 0$ s.t. every solution $X = X(t)$ of system(1) which $t=0$ satisfies $\|X(0) - X^0\| < \delta$, $\|X(t) - X^0\| < \varepsilon$.

asymptotically stable: $\|X(0) - X^0\| < \delta_0$. then $\lim_{t \to \infty} X(t) = X^0$.

$$\frac{dx}{dt} = 4 - 2y, \quad \frac{dy}{dt} = 12 - 3x^2.$$

$4 - 2y = 0$, $12 - 3x^2 = 0$ ⇒ critical point. ✓

$$\frac{dy}{dx} = \frac{12 - 3x^2}{4 - 2y}.$$

$$H(x, y) = 4y - y^2 - 12x + x^3 = c.$$