Direction fields.

$y' = f(t,y)$. $f(x,y)$ is the slope of the tangent to the solution curve $y = y(t)$ at a generic point (t,y)

Choose a lattice / grid in the (t,y) plane and draw short segments of the line with the slope $f(t,y)$ at each node.

Interval of existence

Theo: Let J be an open interval of a form $a<t<b$ and t_0 be a point in J. Consider IVP: $y' + p(t)y = q(t)$. $y(t_0) = y_0$. where y_0 is given initial value. If p and q are continuous on J, then the IVP has a unique solution on J for any y_0.

the largest open interval J is called <u>maximal interval of existence</u>.

Theo: Consider IVP: $y' = f(t,y)$, $y(t_0) = y_0$. where f, f_y are continuous in an open rectangle R functions, $R = \{(t,y) : a<t<b, c<y<d\}$ If $(t_0, y_0) \in R$ then the IVP has a unique solution in some open interval J of the form $t_0 - h < t < t_0 + h$ contained in the internal $a<t<b$.

PPT P62 ??

Autonomous Equation

形如 $y' = f(y)$

- 一种模型: $y' - u(y)y = 0$ $y(0) = y(0)$ Population with logistic growth

要求. $\cdot\, u(y) \approx r$ for small y

$\cdot\, u(y) \downarrow$ when $y \uparrow$

$\cdot\, u(y) < 0$ for large y. $\Rightarrow$ simplest function $u(y) = r - ay$

$\therefore$ $y' = r(1 - \frac{y}{B})y$. $B = r/a$

critical points : $y = 0$, $y = B$.

solution: $\dfrac{y}{B-y} = Ce^{rt}$

if $y(0)=y_0>0$. $\Rightarrow y(t) = \dfrac{y_0 B}{y_0+(B-y_0)e^{-rt}}$

定义

stable: if any other solution starting close to y_0 remains close to y_0 for all time.

asymptotically stable: if it is stable and any solution starting close to y_0 becomes arbitrarily close to y_0 as t increases.

- 一种模型: $y' = r(1-\frac{y}{B})y - \alpha$ *Logistic population with harvesting*

eg: $y' = 28(1-\frac{y}{7})y - 40 = 4(y-2)(y-5)$

$$\frac{dy}{dt} = 4(y-2)(y-5).$$

$$\frac{1}{4(y-2)(y-5)}dy = dt.$$

$$\frac{1}{4}\cdot\frac{1}{3}\int\left(\frac{1}{y-5} - \frac{1}{y-2}\right)dy = dt \quad ??$$

$$\frac{1}{12}\ln\frac{y-5}{y-2} = t$$

general solution: $\dfrac{y-2}{y-5} = Ce^{12t}$

$$\Rightarrow y(t) = \frac{5(y_0-2)-2(y_0-5)e^{-12t}}{y_0-2-(y_0-5)e^{-12t}}$$

- 一种模型 $y' = -r(1-\frac{y}{B})y$ *Population with a critical threshold*

solution: $y(t) = \dfrac{y_0 B}{y_0+(B-y_0)e^{rt}}$ for $y(0)=y_0$ B is critical threshold

Gronwall - Bellman

Let $u(t)\geq 0$, $f(t)\geq 0$ for all $t\geq t_0$. $u(t), f(t)\in C[t_0,+\infty)$, and $\forall t\geq t_0$, $u(t)\leq C+\int_{t_0}^{t} f(t_1)u(t_1)dt_1$, where C is positive constant.

Then $\forall t\geq t_0$, $u(t)\leq C\exp\left(\int_{t_0}^{t} f(t_1)dt_1\right)$

Bihari - LaSalle

Let $u(t)\geq 0$, $f(t)\geq 0$ for all $t\geq t_0$. $u(t), f(t)\in C[t_0,+\infty)$ and $u(t)\leq C+\int_{t_0}^{t} f(t_1)\phi(u(t_1))dt_1$

where C is a positive constant, $\phi(u)$ is a positive non-decreasing continuous function for all $0<u<\bar{u}$ ($\bar{u}\leq\infty$). Define $\psi(u) = \int_{C}^{u}\frac{du_1}{\phi(u_1)}$, $0<u<\bar{u}$.

If $\int_{t_0}^{t} f(t_1)dt_1 < \psi(\bar{u}-0)$, $t_0<t<\infty$, then

$$u(t)\leq\psi^{-1}\left[\int_{t_0}^{t} f(t_1)dt_1\right] \quad \forall t_0\leq t<\infty.$$

Implicit first-order ODEs

$F(t,y,y')=0$, F is a known smooth function.

解法: 令 $y'=p$. ① $t=\varphi(y,y')=\varphi(y,p)$

$$\frac{1}{p} = \frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial p}\cdot\frac{\partial p}{\partial y} \Rightarrow \frac{dp}{dy} = \frac{p\frac{\partial\varphi}{\partial p}}{1-p\frac{\partial\varphi}{\partial y}}$$

if general solution : $y=\Theta(p,c)$, where Θ is known and C is a constant

then, $\begin{cases} t=\varphi(\Theta(p,c),p) \\ y=\Theta(p,c) \end{cases}$ is general solution of equation $t=\varphi(y,y')$ in parametric form.

② $y=\psi(t,y')=\psi(t,p)$

$$p=\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial p}\cdot\frac{\partial p}{\partial t} \Rightarrow \frac{dt}{dp}=\frac{\frac{\partial\psi}{\partial p}}{p-\frac{\partial\psi}{\partial t}}$$

if general solution $t=\Theta(p,c)$ exists, then

$\begin{cases} t=\Theta(p,c) \\ y=\psi(\Theta(p,c),p) \end{cases}$ is the general solution $y=\psi(t,y')$ in parametric form.

Lagrange Equation

$\rightarrow$ $y=t\varphi(y')+\psi(y')$

$$(\varphi(p)-p)\frac{dt}{dp}+\varphi'(p)\,t+\psi'(p)=0.$$

sol: $t=\phi(p,c)$

general sol: $\begin{cases} t=\phi(p,c) \\ y=\phi(p,c)\,\varphi(p)+\psi(p) \end{cases}$

if $\varphi(y')=y'$, call **Clairaut equation**

$\underline{y=ty'+\psi(y')}$

P35 ???

Letting $y'=p$, obtain $p=C$. $t=-\psi'(p)$

general sol : $y=Ct+\psi(C)$

eliminate p and get $\begin{cases} t=-\psi'(p) \\ y=pt+\psi(p) \end{cases}$ (the singular equation)

solve $y=y'+(y')^2e^{y'}$

$y'=p$. $y=p+p^2e^p$. Therefore, $p \simeq \frac{d(p+p^2e^p)}{dp}\cdot\frac{dp}{dt}$

$$dt=\frac{1+(p^2+2p)e^p}{}\,dp$$

$$\Rightarrow \text{general sol} \begin{cases} t = \ln|p| + (p+1)e^p + C \\ y = p + p^2 e^p \end{cases}$$

complement with $y=0$.

Singular solution

$F(t, y, y') = 0$. $\quad \dfrac{\partial F(t,y,y')}{\partial y'} = 0$

$\psi_p(t, y) = 0 \quad$ p- discriminant