- **Series Solution about an Ordinary Point** 常点

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_0(x)y(x) = f(x)$$

A point x_0 is called ordinary point of the given differential equation if each coefficients $p_0(x), p_1(x), \cdots, p_{n-1}(x)$ and $f(x)$ are analytic (可解析的) at $x=x_0$. that is $p_i(x)$ $i=1\cdots n-1$ and $f(x)$ can be expressed as power series about x_0 that are convergent for $|x-x_0|<r$, $r>0$:

$$p_i(x) = \sum_{n=0}^{\infty} p_{i,n}(x-x_0)^n, \quad f(x) = \sum_{n=0}^{\infty} f_n(x-x_0)^n.$$

Theo: If x_0 is an ordinary point of nth-order linear ordinary differential equation

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_0(x)y(x) = f(x)$$

then any solution can be expressed as a power series in $x-x_0$:

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n. \quad |x-x_0|<R$$

and the representation is unique. $R \geq r$ is the radius of convergence.

[eg] $(1-x^2)y'' - 2xy' + p(p+1)y = 0$.

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0. \quad p_1(x) = -\frac{2x}{1-x^2} \quad p_2(x) = \frac{p(p+1)}{1-x^2}$$

$$\therefore p_1(x) = -2x\sum_{n=0}^{\infty}(x^2)^n = -2\sum_{n=0}^{\infty}x^{2n+1}. \quad |x|<1$$

$$p_0(x) = p(p+1)\sum_{n=0}^{\infty}x^{2n}, \quad |x|<1$$

$\therefore$ $x=0$ is an ordinary point $\quad y(x) = \sum_{n=0}^{\infty}a_n x^n$, $|x|<1$ exists.

$$(1-x^2)\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty}na_n x^{n-1} + p(p+1)\sum_{n=0}^{\infty}a_n x^n = 0$$

since $\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$

$$\Rightarrow \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{n=2}^{\infty}n(n-1)a_n x^n - 2\sum_{n=1}^{\infty}na_n x^n + p(p+1)\sum_{n=0}^{\infty}a_n x^n = 0$$

x^0 : $2a_2 + p(p+1)a_0 = 0 \qquad a_2 = -\frac{p(p+1)}{2!}a_0$

x^1 : $3\cdot 2a_3 - 2a_1 + p(p+1)a_1 = 0 \qquad a_3 = -\frac{(p-1)(p+2)}{3!}a_1$

for $n \geq 2$. x^n 系数 $(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n = 0$.

$$\Rightarrow a_{n+2} = -\frac{(p-n)(p+(n+1))}{(n+2)(n+1)}$$

$$\Rightarrow a_{2k} = (-1)^k\frac{p(p+1)(p-2)(p+3)\cdots(p-2k+2)(p+2k-1)}{(2k)!}a_0$$

$$= \frac{(-1)^k}{(2k)!}\prod^{k}((p-2i+2)(p+2i-1))a_0$$

$$a_{2k+1} = (-1)^k \frac{(p-1)(p+2)(p-3)\cdots(p-2k+1)(p+2k)}{(2k+1)!} a_0$$

$$= \frac{(-1)^k}{(2k+1)!} \prod_{i=1}^{k} ((p-2i+1)(p+2i)) a_1$$

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \prod_{i=1}^{k} ((p-2i+2)(p+2i-1)) x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \prod_{i=1}^{k} ((p-2i+1)(p+2i)) x^{2k+1}$$

[eg2] $xy'' + y(1-x) = 0 \quad |x|<1.$

$p_1(x) = 0, \quad p_2(x) = \frac{\ln(1-x)}{x}$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ |x|<1.$ 两边积分

$\ln(1-x) = -\int \frac{1}{1-x} dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad |x|<1$

$p_0(x) = \frac{\ln(1-x)}{x} = -\sum_{n=0}^{\infty} \frac{x^n}{n+1}, \quad |x|<1.$ $\quad x=0$ ordinary point.

$\therefore \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \underline{\sum_{n=0}^{\infty} \frac{x^n}{n+1} \sum_{n=0}^{\infty} a_n x^n} = 0$ $\quad = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{x^m}{m+1} a_{n-m} x^{n-m} \right)$

$\Rightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - \sum_{m=0}^{n} \frac{a_{n-m}}{m+1} \right] x^n = 0.$ $\quad = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{a_{n-m}}{m+1} \right) x^n$

$\therefore a_{n+2} = \frac{1}{(n+2)(n+1)} \cdot \sum_{m=0}^{n} \frac{a_{n-m}}{m+1}$

• Series Solution about a Regular Singular Point 奇点

$y^{(n)}(x) + p_{n-1}(x) y^{(n-1)}(x) + \cdots + p_0(x) y(x) = f(x)$

✓ x_0 is a singular point (奇点) if it is not an ordinary point. that is, not all of coefficients $p_0(x), p_1(x), \cdots, p_{n-1}(x)$ are analytic at $x = x_0$.

✓ x_0 is a regular singular point (正则奇点) if it is not an ordinary point, but all of $(x-x_0)^{n-k} p_k(x)$ are analytic for $k = 0, 1, \cdots, n-1$

✓ x_0 is an irregular point (非正则点) if it is either ordinary point nor regular singular point.

Fuchs' Theo:

$$y''(x) + P(x) y'(x) + Q(x) y(x) = 0$$

if $x = 0$ is a regular singular point, then

$x P(x) = \sum_{n=0}^{\infty} P_n x^n, \quad x^2 Q(x) = \sum_{n=0}^{\infty} Q_n x^n. \quad |x| < r$

Let the indicial equation 指数方程 $\alpha(\alpha-1) + \alpha P_0 + Q_0 = 0$

has two real roots $\alpha_1 > \alpha_2$. Then DE has at least on Frobenius series solution

given by $y_1(x) = x^{\alpha_1}\sum_{n=0}^{\infty} a_n x^n$, $a_0 \neq 0$, $0<x<r$, a_n 由 $y_1(x)$ 代回原式确定

A second linearly independent solution is obtained by:

① $\alpha_1 - \alpha_2$ is not equal to an integer

$$y_2(x) = x^{\alpha_2}\sum_{n=0}^{\infty} b_n x^n, \quad 0<x<r$$ b_n 由 $y_2(x)$ 代回原式确定

② $\alpha_1 = \alpha_2 = \alpha$.

$$y_2(x) = y_1(x)\ln x + x^{\alpha}\sum_{n=0}^{\infty} b_n x^n, \quad 0<x<r$$

③ $\alpha_1 - \alpha_2$ is a positive integer

$$y_2(x) = a y_1(x)\ln x + x^{\alpha_2}\sum_{n=0}^{\infty} b_n x^n, \quad 0<x<r$$ parameter a might be zero

• Solution of Bessel's Equation

Bessel's equation: $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ $\quad x>0$. $\nu = \text{const} \geq 0$.

$\therefore$ in the form $y'' + P(x)y' + Q(x)y = 0$. $P(x) = \frac{1}{x}$, $Q(x) = \frac{x^2-\nu^2}{x^2}$

$x P(x) = 1 = 1 + 0\cdot x + 0\cdot x^2 + \cdots \Rightarrow P_0 = 1$

$x^2 Q(x) = x^2 - \nu^2 = -\nu^2 + 0\cdot x + 1\cdot x^2 + \cdots \Rightarrow Q_0 = -\nu^2$

$\therefore \alpha(\alpha-1) + \alpha\cdot 1 - \nu^2 = 0. \Rightarrow \alpha_1 = \nu, \ \alpha_2 = -\nu$

$y_1(x) = x^{\nu}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\nu}$, $a_0 \neq 0$, $0<x<\infty$ 求导得 y_1', y_1'' 代入

$$x^2\sum_{n=0}^{\infty} a_n(n+\nu)(n+\nu-1)x^{n+\nu-2} + x\sum_{n=0}^{\infty} a_n(n+\nu)x^{n+\nu-1} + (x^2-\nu^2)\sum_{n=0}^{\infty} a_n x^{n+\nu} = 0$$

末项 $\sum_{n=0}^{\infty} a_n x^{n+\nu+2} = \sum_{n=2}^{\infty} a_{n-2}x^{n+\nu}$

$$x^{\nu}\left(\sum_{n=0}^{\infty}[(n+\nu)(n+\nu-1) + (n+\nu) - \nu^2]a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n\right) = 0$$

$$\sum_{n=0}^{\infty} n(n+2\nu)a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$x^0 \Rightarrow a_0 \neq 0$ arbitrary $\qquad x^1$: $a_1 = 0$.

x^n: $n(n+2\nu)a_n + a_{n-2} = 0 \Rightarrow a_n = -\frac{a_{n-2}}{n(n+2\nu)}, \ n \geq 2$

$$a_{2n+1} = 0, \quad a_{2n} = (-1)^n \frac{a_0}{2^{2n}\cdot n!(1+\nu)(2+\nu)\cdots(n+\nu)}$$

$$y_1(x) = a_0 x^{\nu}\sum_{n=0}^{\infty}(-1)^n \frac{1}{n!(1+\nu)(2+\nu)\cdots(n+\nu)}\left(\frac{x}{2}\right)^{2n}, \quad 0<x<\infty$$

• Property of Gamma function

Gamma function:

$$\Gamma(\nu+1) = \int_0^\infty t^\nu e^{-t}\, dt \;,\; \nu>0$$

$$= -\int_0^\infty t^\nu\, d(e^{-t}) = -t^\nu e^{-t}\Big|_{t=0}^{\infty} + \int_0^\infty e^{-t}\,\nu t^{\nu-1}\, dt \quad \Leftarrow ?$$

$$= nu\int_0^\infty e^{-t} t^{\nu-1} dt = \nu\,\Gamma(\nu)$$

$$\therefore\ \Gamma(n+\nu+1) = (n+\nu)\Gamma(n+\nu) = (n+\nu)(n+\nu-1)\cdots(1+\nu)\Gamma(1+\nu)$$

if ν is an integer

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1 \;,\; \Gamma(2) = 1\cdot\Gamma(1) = 1 \;,\; \Gamma(3) = 2\Gamma(2) = 2!$$

$$\Rightarrow \Gamma(k+1) = k\,\Gamma(k) = k!$$

let $a_0 = \left[2^\nu\,\Gamma(1+\nu)\right]^{-1}$ $\Rightarrow$ first Frobenius solution is

$$y_1(x) = \underline{J_\nu(x)} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu} \;,\quad 0<x<\infty \quad \Leftarrow ?$$

↳ Bessel function of the first kind of order ν.

① $\alpha_1-\alpha_2 = 2\nu$ not an integer

$$y_2(x) = x^{-\nu}\sum_{n=0}^{\infty} b_n x^n \;,\quad 0<x<\infty$$

同理 $a_{2n-1} = 0$, $a_{2n} = (-1)^n \dfrac{1}{2^{2n}\, n!\,(1-\nu)(2-\nu)\cdots(n-\nu)}\left(\dfrac{x}{2}\right)^{2n}$,

Letting $b_0 = \left[2^{-\nu}\,\Gamma(1-\nu)\right]^{-1}$

$$y_2(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n!\,\Gamma(n-\nu+1)}\left(\frac{x}{2}\right)^{2n-\nu} = J_{-\nu}(x)$$

general sol : $y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x)$

also can be written as $y(x) = D_1 Y_\nu(x) + D_2 Y_{-\nu}(x)$

where $Y_\nu(x) = \dfrac{J_\nu\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$ Bessel function of second kind of order ν.

② $\alpha_1 = \alpha_2 \Rightarrow \nu = 0$.

$$y_1(x) = J_0(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2n} \;,\quad 0<x<\infty$$

$$y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^n \;.\quad 0<x<\infty$$

$$(x^2y_1'' + xy_1' + x^2y_1)\ln x + 2xy_1' + \sum_{n=2}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} n b_n x^n + \sum_{n=0}^{\infty} b_n x^{n+2} = 0$$

since $\tilde{y} = 0$, and

$$2xy_1' = 2x\sum_{n=1}^{\infty}(-1)^n \frac{1}{(n!)^2}\cdot\frac{2n\cdot x^{2n-1}}{2^{2n}} = \sum_{n=1}^{\infty}(-1)^n\frac{4n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}$$

由 x^n 系数为0.

$\Rightarrow b_1 = 0$, $\quad x^n$: $b_{2n+1} = 0$ and $b_{2n} = (-1)^{n+1}\frac{1}{n(n!)^2}\left(\frac{1}{2}\right)^{2n} - \frac{b_{2n-2}}{(2n)^2}$

若 $b_0 = 0$. $b_{2n} = (-1)^{n+1}\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{(n!)^2}\left(\frac{1}{2}\right)^{2n}$

$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{(n!)^2}\left(\frac{1}{2}\right)^{2n}$$

second order: $y_2(x) = \frac{\pi}{2}Y_0(x) + (\ln 2 - \gamma)J_0(x)$, $0<x,\infty$

后 ???

③ v is positive integer

$$y_1(x) = J_v(x) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!(n+v)!}\left(\frac{x}{2}\right)^{2n+v}. \quad 0<x<\infty$$

$$y_2(x) = ay_1(x)\ln x + x^{-v}\sum_{n=0}^{\infty}b_n x^n, \ 0<x<\infty$$

$$y_2'(x) = a\left(y_1'(x)\ln x + \frac{y_1}{x}\right) + \sum_{n=0}^{\infty}(n-v)b_n x^{n-v-1}$$

$$y_2''(x) = a\left(y_1''(x)\ln x + \frac{2y_1'}{x} - \frac{y_1}{x^2}\right) + \sum_{n=0}^{\infty}(n-v)(n-v-1)b_n x^{n-v-2}$$

代入