Set X is called linear space over the scalar field K is there are two binary operations of addition and scalar multiplication defined on X.

a) $\forall x, y \in X \Rightarrow x+y \in X$

b) $\forall x \in X,\ \forall \alpha \in K \Rightarrow \alpha x \in X$

satisfying the follow properties:

1. $x+y = y+x$ commutativity
2. $x+(y+z) = (x+y)+z$ associativity
3. $\exists\, 0 \in X : 0+x = x+0 = x \quad \forall x \in X$
4. $\forall x \in X \ \exists\, (-x) \in X : x+(-x) = 0.$?
5. $1 \cdot x = x \quad \forall x \in X$
6. $(\alpha\beta)x = \alpha(\beta x) \quad \forall x \in X \ \forall \alpha, \beta \in K$
7. $\alpha(x+y) = \alpha x + \alpha y$
8. $(\alpha+\beta)x = \alpha x + \beta x \quad \forall x, y \in X,\ \forall \alpha, \beta \in K$

$R^n = \{x = (x_1, x_2, \cdots, x_n) : x_i \in R,\ i = 1, 2, \cdots, n\}$, C^n

$l_\infty = \{x = (x_1, x_2, \cdots) : x_i \in K,\ i = \overline{1,\infty},\ \sup_{i=\overline{1,\infty}} |x_i| < \infty\}$

$l_1 = \{x = (x_1, x_2, \cdots) : x_i \in K,\ i = \overline{1,\infty},\ \sum_{i=1}^{\infty} |x_i| < \infty\}$

$l_p = \{x = (x_1, x_2, \cdots) : x_i \in K,\ i = \overline{1,\infty},\ \sum_{i=1}^{\infty} |x_i|^p < \infty\}$

?

$C[a,b]$ = the set of all continuous functions defined on $[a,b]$

- elements $x_1, x_2, \cdots, x_n$ of a linear space X over K are linearly independent if equality $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$, $\alpha_i \in K$, $i = \overline{1,n}$ implies that $\alpha_1 = \alpha_2 = \cdots \alpha_n = 0$. If there is at least one $\alpha_i \neq 0$ then elements $x_1, x_2 \cdots x_n$ linearly dependent.

linearly dependent $\Leftrightarrow$ those elements can be expressed as a linear combination of others

- Dimension of a linear space X is the max number of linearly independent elements in X.

If $\dim X = n$, a system $e^1, e^2, \cdots, e^n$ of n linearly independent elements is said to be a basis in X, provided that $\forall x \in X$ can be expressed a linear combination of basis elements $x = \sum_{i=1}^{n} x_i e^i$. Then $x = (x_1, x_2, \cdots, x_n)$ in the basis $\{e^i\}$, $i = \overline{1,n}$

$[\dim R^n = n, \ \dim C[a,b] = \infty]$

• Distance in Linear Space

1) Metric Space: A space with a metric $d: X \times X \to R$

1. $d(x,y) \geq 0$, $d(x,y) = 0$ iff $x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y \in X$

2) Normed linear spaces: A linear space with a norm

$\|\cdot\|: X \to R$

1. $\|x\| \geq 0$, $\|x\| = 0$ iff $x = 0$.
2. $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall x \in X, \forall \alpha \in K$
3. $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

3) Inner product Spaces: A linear space with an inner product

$(\cdot,\cdot): X \times X \to C$

1. $(x,x) \geq 0$, $(x,x) = 0$ iff $x = 0$.
2. $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$
3. $(y,x) = \overline{(x,y)}$

The Wronskian

a continuously differentiable function

• The Wronskian of <u>n smooth enough functions</u> is defined by

$$W[f_1, f_2, \cdots f_n](t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}$$

if $W[f_1, \cdots, f_n] \neq 0 \Rightarrow$ Functions are linearly independent.

- The Wronskian of n elements $x^1(t), x^2(t), \cdots, x^n(t)$ of n components each is

$$W[x^1, x^2, \cdots, x^n](t) = \det([x^1, x^2, \cdots, x^n]) = \begin{vmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{vmatrix}$$

If $W[x^1, x^2, \cdots, x^n](t_0) \neq 0$ at some t_0, then system $x^1(t), x^2(t), \cdots, x^n(t)$ is linearly independent.

Systems of linear Algebraic Equations

$Ax = b$. where $A = (a_{ij})$ is the $n \times n$ matrix. $x = (x_1, x_2, \cdots, x_n)$ is unkown,

$b = (b_1, b_2, \cdots, b_n)$ is given.

- if $\det A \neq 0$, exists A^{-1} $\quad A^{-1} = \frac{1}{|A|} A^*$, A^*伴随矩阵：第k列元素是A第k行元素的代数余子式
- $\det A \neq 0$, $b = 0 \Rightarrow x = 0$. (trivial sol)
- $\det A = 0$, $b = 0 \Rightarrow$ infinitely many nonzero solutions
- $\det A = 0$, $b \neq 0 \Rightarrow$ no sol, but if b satisfies condition $\sum b_i y_i = 0$ for all $y = (y_1, y_2, \cdots, y_n)$ s.t. $\bar{A}^T y = 0$.

1) Cramer's rule,

$$x_j = \frac{D_j}{D}$$

2) Gaussian elimination

对方程进行三种变换：

(1) 乘一个非零系数

(2) 将一个方程若干倍加到另一个方程上

(3) 交换两方程位置

- **Eigenvalues and Eigenvectors**

$y = \lambda x$ where $\lambda \in K$ and obtain $Ax = \lambda x$

The value of λ for which there are nonzero vectors x satisfying the eq. is called the <u>eigenvalue</u> of A, those nonzero vectors x are called the <u>eigenvectors</u> of A associated with λ.

$A = \begin{pmatrix} 5 & -4 \\ 8 & -7 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -4 \\ 8 & -7-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 3$

$\lambda_1 = 1$ and $\lambda_2 = -3$

1° $\lambda = 1$: $(A - I)x = 0$ and

$\begin{pmatrix} 4 & -4 \\ 8 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 - x_2 = 0. \Rightarrow x = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ a \neq 0.$

2° $\lambda = -3$ $(A + 3I)x = 0.$

$\Rightarrow 2x_1 - x_2 = 0.$

$x = b\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad b \neq 0.$

$A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ 2 & 0 & 3 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & -1 \\ -1 & 1-\lambda & -1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$

$\lambda_1 = 2, \lambda_2 = \lambda_3 = 1$

when $\lambda = 2$ $\quad x = \begin{pmatrix} a \\ a \\ -2a \end{pmatrix}, \ a \neq 0.$

when $\lambda = 1$ $\quad x = b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ b, c \neq 0$

性质: 1. A real matrix A has a complex eigenvalue λ and x is a corresponding eigenvector, then the complex conjugate $\bar{\lambda}$ is also an eigenvalue with y, the conjugate vector of x, as a corresponding eigenvector.

2. A matrix has an inverse matrix A^{-1} if and only if it does not have zero as an eigenvalue. If $\lambda_1, \lambda_2, \cdots, \lambda_n \neq 0$ are the eigenvalues of A, then the eigenvalues of A^{-1} are $1/\lambda_1, \cdots, 1/\lambda_n$.

3. The eigenvalues of A are the same as the eigenvalues of A^T.

4. If λ is an eigenvalue of A with an eigenvector x, then λ^k is an

eigenvalue of A^k with a corresponding eigenvector x, $c\lambda$ is an eigenvalue of CA with a corresponding eigenvector x. and $C_m\lambda^m + C_{m-1}\lambda^{m-1} + \cdots + C_1\lambda + C_0$ is an eigenvalue of $C_mA^m + C_{m-1}A^{m-1} + \cdots + C_1A + C_0I$ with a corresponding eigenvector of x.

Theorem: Eigenvectors associated with distinct eigenvalues are linearly independent.

Diagonalizable matrices

$n \times n$ matrices A and B are said to be similar if exist matrix T, s.t. $B = T^{-1}AT$. in this case $\det(B-\lambda I) = \det(A-\lambda I)$

if λ is an eigenvalue of B with corresponding eigenvector x, then $A(Tx) = \lambda(Tx)$

g

Def: A matrix A is said to be diagonalizable if it is similar to a diagonal matrix B.

Theo: 1. $n \times n$ matrix A is digonalizable if and only if it has n linearly independent eigenvectors.

2. If A has n linearly independent eigenvector $x^1, x^2, x^3, \cdots, x^n$ $Ax^i = \lambda_i x^i$. Denote by T the matrix whose columns are the vectors $x^1, \cdots, x^n$. Then the rank of T is n, and T^{-1} exists.

$(x^1, \cdots, x^n)\,\mathrm{diag}(\lambda_1, \cdots, \lambda_n) = AT \Rightarrow \mathrm{diag}(\lambda_1, \cdots, \lambda_n) = T^{-1}AT$