vv256: Week4-5.
Normal systems and higher-order equations.

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Outline

Lecture 9: Normal Systems of ODEs. Proof of the Existence Theorem.

Lecture10: Higher-order ODEs

Lecture 11: Linear homogeneous equations with constant coefficients

Lecture 12: Vibrations.

Definition

A system

$$\begin{cases} y_1' = f_1(t, y_1, y_2, \dots, y_n), \\ y_2' = f_2(t, y_1, y_2, \dots, y_n), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ y_n' = f_n(t, y_1, y_2, \dots, y_n). \end{cases} \tag{1}$$

of ordinary differential equations is called a normal system. In this system,

- t is an independent variable,
- $y_1, \dots, y_n$ are unknown functions, and
- $f_1, \dots, f_n$ are functions of $n + 1$ independent variable defined in some open domain Γ.

Definition

A normal system (1) can be also written in the form

$$y' = f(t, y), \tag{2}$$

- $y = (y_1, \dots, y_n)$ is a vector, and
- $f(t, y) = (f_1(t, y), \dots, f_n(t, y))$ is a vector-function.
- We assume that $f_i(t, y_1, \dots, y_n)$, $i = \overline{1, n}$ and $\frac{\partial f_j(t, y_1, \dots, y_n)}{\partial y_i}$, $i, j = \overline{1, n}$ are continuous on Γ.

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Definition: A vector-function

$$y = \varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))$$

is called a solution of the normal system (1) provided $\varphi(t)$ is defined on some interval $r_1 < t < r_2$ and it satisfies (1) $\Rightarrow$ a point with coordinates $(t, \varphi_1(t), \dots, \varphi_n(t)) \in \Gamma$ for all $r_1 < t < r_2$. The interval $r_1 < t < r_2$ is called the interval of definition of the solution.

Definition

Theorem 1: Let the right-hand sides of (1) be defined on the open set Γ, and $f_i(t, y_1, \ldots, y_n)$, $i = \overline{1, n}$, $\frac{\partial f_j(t, y_1, \ldots, y_n)}{\partial y_i}$, $i, j = \overline{1, n}$ be continuous on Γ.
Then for each point $(t_0, y_1^0, \ldots, y_n^0) \in \Gamma$ there exists a solution $y_i = \varphi_i(t)$, $i = \overline{1, n}$ of the normal system (1) which satisfies $\varphi_i(t_0) = y_i^0$, $i = \overline{1, n}$.
If there exist two solutions $y_i = \varphi_i(t)$ and $y_i = \psi_i(t)$, $i = \overline{1, n}$ of the normal system (1) which satisfy the same conditions $\psi_i(t_0) = \varphi_i(t_0) = y_i^0$, $i = \overline{1, n}$ then those solutions coincide.

Existence of a maximum interval

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1. To each solution of (1) with the i.v. $(t_0, y_1^0, y_2^0, \ldots, y_n^0)$ there corresponds its own interval of definition. Denote by R_2 the set of all right-hand points and by R_1 the set of all left-hand points of those intervals.

Let m_1 and m_2 be the lower and the upper bounds of R_1 and R_2.

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2. Construct a solution $\hat{\varphi}(t) = (\hat{\varphi}_1(t), \ldots, \hat{\varphi}_n(t))$ with the initial values $(t_0, y_1^0, y_2^0, \ldots, y_n^0)$ defined on $m_1 < t < m_2$.

Let t^* be an arbitrary point of that interval: $m_1 < t^* < m_2$ and $t^* \geq t_0$. m_2 is an upper bound of R_2 $\Rightarrow$

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Let t^* be an arbitrary point of that interval: $m_1 < t^* < m_2$ and $t^* \geq t_0$. m_2 is an upper bound of R_2 $\Rightarrow$there exists a solution $\psi(t) = (\psi_1(t), \dots, \psi_n(t))$ of (1) with the i.v. $(t_0, y_1^0, y_2^0, \dots, y_n^0)$, and its interval of definition (r_1, r_2) is such that $r_1 < t^* < r_2$. We set $\hat{\varphi}(t^*) = \psi(t^*)$.

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No. If we choose another solution $\chi(t)$ with the same i.v. and the interval of definition $s_1 < t < s_2$ such that $t^* \in (s_1, s_2)$ then by the uniqueness property $\chi(t^*) = \psi(t^*)$. $\Rightarrow$

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4. If (r_1, r_2) is an arbitrary interval of definition which corresponds to the solution $\varphi(t)$ of (1) with the i.v. $(t_0, y_1^0, y_2^0, \ldots, y_n^0)$, then $r_1 \in R_1$, $r_2 \in R_2$ and hence, $m_1 \leq r_1 < r_2 \leq m_2 \Rightarrow (m_1, m_2)$ is maximal.

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Remark: Note that any nth-order system of ODEs solved for highest derivatives may be reduced to a normal system.

Existence Theorem 2

Theorem 2: Consider a normal system of *linear* ODEs

$$\begin{cases} y_1' = a_{11}(t)y_1 + a_{12}(t)y_2 + \ldots + a_{1n}(t)y_n + b_1(t), \\ y_2' = a_{21}(t)y_1 + a_{22}(t)y_2 + \ldots + a_{2n}(t)y_n + b_2(t), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n' = a_{n1}(t)y_1 + a_{n2}(t)y_2 + \ldots + a_{nn}(t)y_n + b_n(t), \end{cases} \tag{3}$$

where $a_{ij}(t)$, $b_i(t)$, $i, j = 1, \ldots n$, are continuous functions defined on some interval $q_1 < t < q_2$.
Then for arbitrary initial values $(t_0, y_1^0, y_2^0, \ldots, y_n^0)$, $q_1 < t_0 < q_2$ there exists a unique solution of (3) defined on the entire interval (q_1, q_2).

Existence Theorem 2: Lemma 1

Lemma 1. Let $A = (a_{ij})$ be a $n \times n$ matrix and

$$\exists K > 0\colon |a_{ij}| \leq K, \forall i, j = 1, \ldots, n.$$

If a vector $v \in \mathbb{K}^n$ is defined by

$$v_i = \sum_{j=1}^{n} a_{ij} u_j, \; i = 1, \ldots, n$$

with an arbitrary $u \in \mathbb{K}^n$, then

$$|v| \leq n^2 K |u|, \quad \text{where} \quad |v| = ||v||_2 = \left(\sum_{i=1}^{n} |v_i|^2 \right)^{1/2}$$

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$$\Rightarrow \ |v_i| = \left| \sum_{j=1}^{n} a_{ij} u_j \right| \leq \sum_{j=1}^{n} |a_{ij}||u_j| \leq \sum_{j=1}^{n} K|u| \leq nK|u|$$

Therefore, $|v| \leq \sqrt{n(nK)^2|u|^2} \leq n^2 K|u|$.

Existence Theorem 2: Lemma 2

Lemma 2: Let $z(t)$ be an n-dimensional vector continuously depending on the parameter t. Then for $t_0 \leq t_1$

$$\left| \int_{t_0}^{t_1} z(\tau)\, d\tau \right| \leq \int_{t_0}^{t_1} |z(\tau)|\, d\tau.$$

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Proof:

Recall, that for n-dimensional vectors u_1, u_2 $|u_1 + u_2| \leq |u_1| + |u_2|$ and hence, $|u_1 + \ldots + u_k| \leq |u_1| + \ldots + |u_k|$.

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Recall, that for n-dimensional vectors u_1, u_2 $|u_1 + u_2| \leq |u_1| + |u_2|$ and hence, $|u_1 + \ldots + u_k| \leq |u_1| + \ldots + |u_k|$.
Use the definition of the integral

$$\int_{t_0}^{t_1} z(\tau)\, d\tau = \lim \sum_{i=0}^{k-1} z(t_i)(t_{i+1} - t_i)$$

and the last inequality to obtain

$$\left| \int_{t_0}^{t_1} z(\tau)\, d\tau \right| \leq \lim \sum_{i=0}^{k-1} |z(t_i)|(t_{i+1} - t_i) = \int_{t_0}^{t_1} |z(\tau)|\, d\tau.$$

Existence Theorem 2: Proof

Rewrite the normal linear system (3) in the vector form

$$y' = A(t)y + b(t),$$

where $y = (y_1, \dots, y_n)$, $b(t) = (b_1(t), \dots, b_n(t))$, $A(t) = (a_{ij}(t))$.

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   $\Rightarrow$ the operator equation $L\varphi(t) = \varphi(t)$ is equivalent to the system $y' = A(t)y + b(t)$ with the i.c. $y(t_0) = y_0 \Rightarrow$ it is enough to solve the operator equation $L\varphi(t) = \varphi(t)$ and prove the uniqueness of its solution.

Existence Theorem 2: Proof

2. Note that

$$L\varphi(t) - L\psi(t) = \int_{t_0}^{t} A(\tau)(\varphi(t) - \psi(\tau))\, d\tau$$

for all continuous v.-functions $\varphi(t)$, $\psi(t)$ defined on (q_1, q_2).

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   To this aim, choose $[r_1, r_2] \subset (q_1, q_2)$, $t_0 \in [r_1, r_2]$ and obtain an estimate on $[r_1, r_2]$ as follows.

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4. Take an arbitrary v.-function $\varphi_0(t)$ defined on (q_1, q_2) and define the first iteration $\varphi_1(t) = L\varphi_0(t)$. What is the property of φ_0, φ_1? continuity

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$$\Rightarrow |\varphi_{i+1}(t) - \varphi_i(t)| \leq n^2 K \int_{t_0}^{t} |\varphi_i(\tau) - \varphi_{i-1}(\tau)|\, d\tau$$

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$$i = 1: \quad |\varphi_2(t) - \varphi_1(t)| \leq n^2 KC|t - t_0|,$$

$$i = 2: \quad |\varphi_3(t) - \varphi_2(t)| \leq \frac{(n^2K)^2 C|t - t_0|^2}{2!} \text{ and so on.}$$

By induction, $|\varphi_{i+1}(t) - \varphi_i(t)| \leq \dfrac{(n^2K)^i C|t - t_0|^i}{i!} \leq \dfrac{(n^2K)^i C(r_2 - r_1)^i}{i!}$

Existence Theorem 2: Proof

Since the series $\sum_{i=1}^{\infty} \frac{(n^2K)^i(r_2 - r_1)^i}{i!} < +\infty$, Check it

so the sequence of continuous functions $\{\varphi_i(t)\}$ converges uniformly to a continuous function $\varphi(t)$.

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Letting $i \to \infty$ in $\varphi_{i+1}(t) = L\varphi_i(t)$, we have $\varphi(t) = L\varphi(t) \Rightarrow$ the solution of the operator equation exists for all $t \in [r_1, r_2]$.

Explain why we are allowed to let $i \to 0$ in $\varphi_{i+1}(t) = L\varphi_i(t)$.

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Since $[r_1, r_2]$ is an arbitrary subinterval of (q_1, q_2) containing t_0, so the solution of the operator equation exists on the entire interval (q_1, q_2).

Existence Theorem 2: Proof

5. Uniqueness
   If there exist two solutions $\varphi(t)$, $\psi(t)$ of the operator equation then

$$\varphi(t) = L\varphi(t), \quad \psi(t) = L\psi(t)$$

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$$\Rightarrow |\varphi(t) - \psi(t)| \leq k|\varphi(t) - \psi(t)| \quad \forall t \in [r_1, r_2]$$

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Normal Systems: Properties

We shall consider a normal system of linear ODEs in the vector form

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where y, b are n-dimensional vectors and $A(t)$ is a $n \times n$ matrix.

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P1. If $y = \varphi(t)$ and $y = \psi(t)$ are two solutions of the linear system (4) then their difference $\chi(t) = \varphi(t) - \psi(t)$ is a solution of the homogeneous system (5). Prove it.

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P2. Any solution $y = \varphi(t)$ of the non-homogeneous system (4) can be represented as $\varphi(t) = \chi(t) + \psi(t)$, where $\psi(t)$ is a particular solution of (4) and $\chi(t)$ is a solution of (5). Prove it.

Normal Systems: Properties

P3. Let the free term $b(t) = \alpha b_1(t) + \beta b_2(t)$ in the system (4). Consider the two systems of ODEs

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P4. A) If $y = \varphi(t)$ is a solution of the homogeneous system (5) and $\exists t_0 \in (q_1, q_2)$: $\varphi(t_0) = 0$ then $\varphi(t) = 0 \, \forall t \in (q_1, q_2)$.

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P4. B) If $\varphi_1(t), \ldots, \varphi_r(t)$ are solutions of the homogeneous system (5) then $\varphi(t) = C_1\varphi_1(t) + \ldots + C_r\varphi_r(t)$, $C_i = const, i = \overline{1, r}$ is also a solution of (5). Verify it directly.

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Let $a_1, \ldots, a_n$ be an arbitrary system of linearly independent constant vectors. Give an example.

We define solutions $\varphi_1(t), \ldots, \varphi_n(t)$ of (5) by the initial conditions $\varphi_i(t_0) = a_i, i = 1..n$. Therefore, $\varphi_i(t_0)$ are linearly independent $\Rightarrow \varphi_i(t)$ are linearly independent.

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P5. A system $\varphi_1(t), \ldots, \varphi_n(t)$ of solutions of the homogeneous nth-order system (5) is called a fundamental system of solutions (FSS) if it is linearly independent.
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Normal Systems: Properties

B) Any solution $y = \varphi(t)$ of the homogeneous system (5) can be written in the form $\varphi(t) = C_1\varphi_1(t) + \ldots + C_r\varphi_r(t)$, where $\{\varphi_i\}$ is a FSS of (5).

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Indeed, let t_0 be a certain value of t. Since $\{\varphi_i(t)\}$ is linearly independent, so and $\{\varphi_i(t_0)\}$, and the number of the elements in the system equals to the dimension of the considered linear space $\Rightarrow \{\varphi_i(t_0)\}$ is a basis and hence, any solution $\varphi(t_0) = \sum_{i=1}^{n} c_i\varphi_i(t_0)$, $c_i = const$, $i = 1..n$. Thus, the solutions $\varphi(t)$ and $\sum_{i=1}^{n} c_i\varphi_i(t)$ have the same i.c. $\Rightarrow$ they coincide Explain it. $\Rightarrow$ an arbitrary solution $\varphi(t)$ of the homogeneous system (5) has the representation $\varphi(t) = \sum_{i=1}^{n} c_i\varphi_i(t)$.

Normal Systems: Properties

P6. The matrix of the FSS $\{\varphi_i(t)\}$,
$\varphi_i(t) = (\varphi_i^1(t), \varphi_i^2(t), \dots, \varphi_i^n(t))$ of the homogeneous system (5)

$$\Phi(t) = \begin{pmatrix} \varphi_1^1(t) & \varphi_2^1(t) & \dots & \varphi_n^1(t) \\ \varphi_1^2(t) & \varphi_2^2(t) & \dots & \varphi_n^2(t) \\ \dots & \dots & \dots & \dots \\ \varphi_1^n(t) & \varphi_2^n(t) & \dots & \varphi_n^n(t) \end{pmatrix} \tag{8}$$

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P7. Let the matrix (8) be an arbitrary matrix of continuously differentiable on (q_1, q_2) functions and it's determinant be nonzero on (q_1, q_2). Then (8) is a fundamental matrix for a unique homogeneous system defined on (q_1, q_2). Prove it.
Hint: Show that the v.-function $\varphi_k(t)$ (kth colomn of $\Phi(t)$) is a solution of the homogeneous system (5).

Normal Systems: Properties

P8. If $(\varphi^i_j(t))$ is an nth -order square matrix with differentiable entries, and $W(t) = \det(\varphi^i_j(t))$, then

$$W'(t) = W_1(t) + \ldots + W_n(t),$$

where $W_i(t)$ is the determinant of a matrix obtained by differentiating all elements in the ith row of the matrix $(\varphi^i_j(t))$.

Normal Systems: Properties

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P9. Liouville Formula

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} S(\tau)\,d\tau\right),$$

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Since $\{\varphi_i(t)\}$ are linearly independent, so $C_i'(t)$ are uniquely determined. Explain why so.

Outline

Higher-order ODEs: Properties

We shall onsider the nth-order linear ODE in the form

$$y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = b(t) \tag{9}$$

with continuous on (q_1, q_2) coefficients $a_i(t)$ and the free term $b(t)$.

P1. Reduce (9) to the normal system of ODEs by introducing new functions $y_1 = y$, $y_2 = y', \ldots, y_n = y^{(n-1)}$ to obtain

$$\begin{cases} y_1' = y_2, \\ y_2' = y_3, \\ \ldots \\ y_{n-1}' = y_n, \\ y_n' = b(t) - a_n(t)y_1 - a_{n-1}(t)y_2 - \ldots - a_1(t)y_n \end{cases}$$

OR

$$\bar{y}' = A(t)\bar{y} + \bar{b}(t) \tag{10}$$

Higher-order ODEs: Properties

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{pmatrix}, \ \bar{b}(t) = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ b(t) \end{pmatrix}$$

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Equations (9) and (10) are equivalent $\Rightarrow$ to each solution $y = \psi(t)$ of (9) there corresponds the solution

$$\bar{y} = \varphi(t) = (\psi(t), \psi'(t), \dots, \psi^{(n-1)}(t))$$

of (10) and conversely, to every solution

$$\bar{y} = \varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))$$

of (10) there corresponds the solution $y = \varphi_1(t)$ of (9).

Higher-order ODEs: Properties

Therefore, the existence theorem can be re-formulated as

Theorem. If the functions $a_1(t), a_2(t), \ldots, a_n(t)$, and $b(t)$ are continuous on an open interval (q_1, q_2) and $t_0 \in (q_1, q_2)$, then the IVP for the equation (9) with the initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ y''(t_0) = y_0'', \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)}$$

has a unique solution on (q_1, q_2).

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has a unique solution on (q_1, q_2).

Example: Consider the IVP

$$(t+2)y''' - 2ty'' + y \sin t = te^t,\ y(0) = -1,\ y'(0) = 2,\ y''(0) = 0.$$

The functions

$$a_1(t) = \frac{2t}{t+2},\ a_2(t) = 0,\ a_3(t) = \frac{\sin t}{t+2},\ b(t) = te^t t + 2$$

are continuous on the intervals $-\infty < t < -2$, $-2 < t < +\infty$.

What is the existence interval? $t_0 = 0 \Rightarrow -2 < t < +\infty$

Higher-order ODEs: Properties

P2. Let $\psi_1(t), \ldots, \psi_r(t)$ be the solutions of the homogeneous equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = 0, \qquad (11)$$

and $\varphi_1(t), \ldots, \varphi_r(t)$ be solutions of

$$\bar{y}' = A(t)\bar{y} + \bar{b}(t), \qquad (12)$$

corresponding to the solutions $\psi_i(t)$, $i = 1..r$

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The solutions $\psi_i(t)$, $i = 1..r$ are linearly dependent if and only if $\varphi_i(t)$, $i = 1..r$ are linearly dependent. Prove it.

Higher-order ODEs: Properties

P3. A system of solutions of the homogeneous equation (11) is called fundamental if it is linearly independent.

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A fundamental system of solutions $\{\psi_i(t)\}$, $i = 1..n$ of the homogeneous equation (11) exists and any solution $\psi(t)$ of (11) can be written in the form Prove it.

$$\psi(t) = C_1\psi_1(t) + \ldots + C_n\psi_n(t)$$

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P4. If $\{\psi_i(t)\}$, $i = 1..n$ is a FSS of the homogeneous equation (11), the determinant

$$W(t) = \begin{vmatrix} \psi_1(t) & \psi_2(t) & \ldots & \psi_n(t) \\ \psi_1'(t) & \psi_2'(t) & \ldots & \psi_n'(t) \\ \ldots & \ldots & \ldots & \ldots \\ \psi_1^{(n-1)}(t) & \psi_2^{(n-1)}(t) & \ldots & \psi_n^{(n-1)}(t) \end{vmatrix} \neq 0$$

Prove it.

Higher-order ODEs: Properties

P5. The Liouville formula for $W(t)$ is

$$W(t) = W(t_0) \exp\left(- \int_{t_0}^{t} a_1(\tau)\, d\tau \right)$$

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Remark: Consider two solutions $y_1(t)$ and $y_2(t)$ of the second-order linear homogeneous ODE

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What is the Wronskian of y_1, y_2?

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$$\exp\left(-\int \frac{4t}{2t+1}\,dt\right) = e^{-2t}(2t+1) \Rightarrow y_2(t) = C_1 t + C_2 e^{-2t}$$

Higher-order ODEs: Properties

P6. If $\chi_0(t)$ is a particular solution of (9), then an arbitrary solution of (9) is represented in the form

$$y(t) = \psi(t) + \chi_0(t),$$

where $\psi(t)$ is a solution of (11).

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where $\psi(t)$ is a solution of (11).
P7. Method of variation of parameters. If $\{\psi_i(t)\}$, $i = 1..n$ is a FSS of the homogeneous equation (11), then a solution of (9) can be obtained in the form $y(t) = C_1(t)\psi_1(t) + \ldots + C_n(t)\psi_n(t)$,

$$\begin{cases} C_1'(t)\psi_1(t) + \ldots + C_n'(t)\psi_n(t) = 0, \\ C_1'(t)\psi_1'(t) + \ldots + C_n'(t)\psi_n'(t) = 0, \\ \ldots \\ C_1'(t)\psi_1^{(n-1)}(t) + \ldots + C_n'(t)\psi_n^{(n-1)}(t) = b(t). \end{cases}$$

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Therefore, $\{t,\ e^t\}$ is linearly independent.

$$W[t, e^t, y](t) = \begin{vmatrix} t & e^t & y \\ 1 & e^t & y' \\ 0 & e^t & y'' \end{vmatrix} = 0 \Rightarrow (t-1)y'' - ty' + y = 0$$

Higher-order ODEs: Problems

Check if the following systems of functions are linearly dependent. Functions are considered in the domains where they are well-defined.

$$1.\, t+2,\, t-2;\ 2. t^2+2,\, 3t^2-1,\, t+4;\ 3.\ \ln(t^2), \ln(3t), 7.$$

Find a homogeneous ODE with the following particular solutions

$$1.\, 3t,\, t-2,\, e^t+1;\ 2.\, 1, \cos t.$$

Find general solutions of the following ODEs. A particular solution is given in some questions.

$1.\, (2t+1)y''+4ty'-4y=0$; $2.\ t^2(t+1)y''-2y=0,\, y_1=1+1/t$;

$3.\ ty''+2y'-ty=0,\, y_1=e^t/t$; $4.\ ty''-(2t+1)y'+(t+1)y=0$;

$5.\ y''-2(1+\tan^2 t)y=0,\, y_1=\tan t$, $6.\ t(t-1)y''-ty'+y=0$

Higher-order ODEs: Problems

Solve the following ODEs reducing the order

$$1.\ t^2y'' = y''^2,\ 2.\ 2ty'y'' = y'^2 - 1,\ 3.\ y^3y'' = 1,\ 4.\ y'''y'^2 = y''^3,$$

$$5.\ y'^2 + 2yy'' = 0,\ 6.\ yy'' + y = y'^2,\ 7.\ y'' = 2yy',\ 8.\ y''' = y''^2.$$

Reduce the order of the following ODEs using homogeneity and then solve them

$$1.\ tyy'' - ty'^2 = yy',\ 2.\ yy'' = y'^2 + 15y^2\sqrt{t},$$

$$3.\ (t^2+1)(y'^2 - yy'') = tyy',\ 4.\ tyy'' + ty'^2 = 2yy'.$$

Complex Differential Equations

So far, we have considered only real DE and their real solutions. However, in some cases it is easier to first find complex solutions of the real equation and then to select from them the real solutions.
A complex function $\chi(t)$ of a real variable t is said to be defined if $\forall t \in (r_1, r_2)$ there corresponds a complex number

$$\chi(t) = \varphi(t) + i\psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are real functions of the real variable t.
A complex function $\chi(t)$ is continuous if both $\varphi(t)$ and $\psi(t)$ are continuous functions. The derivative of the complex function is defined by $\chi'(t) = \varphi'(t) + i\psi'(t)$.

Complex Differential Equations

Theorem. Let

$$z^{(n)} = f(t, z, z', \dots, z^{(n-1)})$$

be an nth-order equation and the function f be a polynomial in the variables z, z' $\dots, z^{(n-1)}$ with continuous real or complex coefficients defined on $q_1 < t < q_2$. There exists a unique solution $z = \varphi(t)$ of the equation which satisfies the initial conditions

$$\varphi(t_0) = z_0,\ \varphi'(t_0) = z_0', \dots, \varphi^{(n-1)}(t_0) = z_0^{(n-1)}.$$

Any two solutions with identical initial conditions coincide on the common part of their intervals of definition.
If the equation is linear (polynomial is of the first degree), then for arbitrary initial values there exists a solution defined on the entire interval $q_1 < t < q_2$. Moreover, if $z = \varphi(t) = u(t) + iv(t)$ is a solution of the homogeneous linear equation then both real $u(t)$ and imaginary $v(t)$ part are solutions of the equation.

Complex Differential Equations

Let $w = u + iv$ be an arbitrary complex number. Define

$$e^w = e^u(\cos v + i \sin v)$$

(it can be proved using the expansion
$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots + \frac{w^n}{n!} + \ldots)$
Then the **Euler formulas** follow directly

$$\cos v = \frac{e^{iv} + e^{-iv}}{2}, \quad \sin v = \frac{e^{iv} - e^{-iv}}{2}.$$

Exercise: Prove that

$$e^{w_1} e^{w_2} = e^{w_1 + w_2}, \quad \frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$$

Outline

Linear homogeneous equations with constant coefficients

Consider a linear nth-order homogeneous equation with constant coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_{n-1} y' + a_n y = 0. \qquad (13)$$

First, we shall find all complex solutions of (13), and then separate them from the real solutions.

Denote $\dfrac{d^k y}{dt^k} = p^k y$

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = a_0 p^n y + a_1 p^{n-1} y + \ldots + a_{n-1} p y + a_n y$$

Let $L(p) = a_0 p^n + a_1 p^{n-1} + \ldots + a_{n-1} p + a_n$ and rewrite the equation (13) in the form $L(p)y = 0$. Then $L(p)e^{\lambda t} = L(\lambda)e^{\lambda t}$. Indeed,

$$L(p)e^{\lambda t} = a_0 (e^{\lambda t})^{(n)} + a_1 (e^{\lambda t})^{(n-1)} + \ldots + a_{n-1} (e^{\lambda t})' + a_n (e^{\lambda t}) =$$

$$= (a_0 \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1} \lambda + a_n) e^{\lambda t} = L(\lambda) e^{\lambda t}$$

Linear homogeneous equations with constant coefficients

The polynomial $L(p)$ is called the characteristic polynomial of the equation (13).

Theorem. If the characteristic polynomial $L(p)$ of the equation $L(p)y = 0$ has distinct roots $\lambda_1, \ldots, \lambda_n$ (no multiple roots) then the function

$$y = C_1 y_1 + \ldots + C_n y_n,\ y_i = e^{\lambda_i t},\ C_i \in \mathbb{K},\ i = 1..n,$$

is the solution of $L(p)y = 0$. It is a general solution $\Rightarrow$ any solution can be obtained from y by the proper choice of C_i, $i = 1..n$.

Example. The characteristic equation for the third-order ODE

$$y^{(4)} - 5y'' + 4y = 0$$

is $\lambda^4 - 5\lambda^2 + 4 = 0$ with the characteristic roots $1, -1, 2, -2$. All roots are distinct $\Rightarrow$ the general solution is

$$y(t) = C_1 e^{t} + C_2 e^{-t} + C_3 e^{2t} + C_4 e^{-2t}$$

Linear homogeneous equations with constant coefficients

Remark. Assume that the FSS satisfies

$$\bar{y}_1 = y_2, \dots, \bar{y}_{2k-1} = y_{2k}; \bar{y}_{2k+1}, \dots, \bar{y}_n = y_n$$

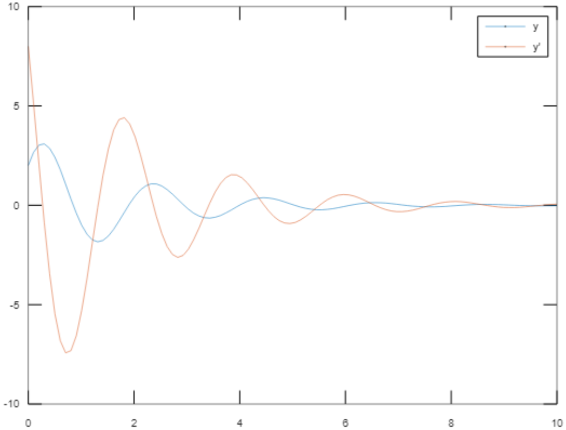
and $y_1 = a_1 + ib_1, y_3 = a_2 + ib_2, \dots, y_{2k-1} = a_k + ib_k$.
Let $C_1 = \bar{C}_2, \dots, C_{2k-1} = \bar{C}_{2k}$; $C_{2k+1} = \bar{C}_{2k+1}, \dots, \bar{C}_n = C_n$.
Define $C_1 = \frac{1}{2}(x^1 - iz^1), \dots, C_{2k-1} = \frac{1}{2}(x^k - iz^k)$, $x^i, z^i \in \mathbb{R}$.
Then the general solution is

$$y = x^1 a_1 + z^1 b_1 + \ldots + x^k a_k + z^k b_k + C_{2k+1} y_{2k+1} + \ldots + C_n y_n$$

$\Rightarrow$ for every pair of conjugate complex solutions we substitute the real and imaginary parts and obtain a FS of real solutions.
Example. $2y''' - 9y'' + 14y' - 5y = 0 \Rightarrow 2\lambda^3 - 9\lambda^2 + 14\lambda - 5 = 0 \Rightarrow \lambda_1 = 1/2$, $\lambda_2 = \lambda_3 = 2 \pm i$. Therefore, the general solution is $y(t) = C_1 e^{t/2} + e^{2t}(C_2 \cos t + C_3 \sin t)$

Linear homogeneous equations with constant coefficients

Example. The characteristic equation of $y'' + y' + 9.25y = 0$ has two complex roots $\lambda_{1,2} = -1/2 \pm 6i$.
Therefore, the solution is $y(t) = e^{-t/2}(C_1 \cos 6t + C_2 \sin 6t)$.
What is the solution satisfying $y(0) = 2$, $y'(0) = 8$?



Linear homogeneous equations with constant coefficients

What happens if the caracteristic polynomial

$$L(p) = a_0 p^n + a_1 p^{n-1} + \ldots + a_{n-1} p + a_n$$

of the equation $L(p)y = 0$ has multiple roots?

The idea is: if λ_1, λ_2 are two distinct roots of $L(p)$ then the function $\dfrac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}$ is also a solution of $L(p)y = 0$ Verify it

If $\lambda_2 \to \lambda_1$ then $\dfrac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \to te^{\lambda_1 t} \Rightarrow$ assume that $te^{\lambda_1 t}$ is a solution of $L(p)y = 0$ whenever λ_1 is a double root of the polynomial $L(p)$. Similarly, if λ is a k-fold root of $L(p)$ then $e^{\lambda t}$, $te^{\lambda t}, t^2 e^{\lambda t}, \ldots, t^{k-1} e^{\lambda t}$ are solutions of $L(p)y = 0$.

Linear homogeneous equations with constant coeff.

Theorem. Let $L(p) = 0$ be a linear nth-order homogeneous equation with constant coefficients. If $\lambda_1, \ldots, \lambda_m$ are distinct roots of the characteristic polynomial and the root λ_j has the multiplicity k_j : $k_1 + k_2 + \ldots + k_m = n$ then the functions

$$y_1 = e^{\lambda_1 t},\ y_2 = te^{\lambda_1 t},\ \ldots,\ y_{k_1} = t^{k_1 - 1} e^{\lambda_1 t};$$

$$y_{k_1+1} = e^{\lambda_2 t},\ y_{k_1+2} = te^{\lambda_2 t},\ \ldots,\ y_{k_2} = t^{k_2 - 1} e^{\lambda_2 t};$$

$$\ldots$$

$$\ldots\ldots\ldots\ldots\ldots, y_n = t^{k_m - 1} e^{\lambda_1 t}$$

are solutions of $L(p)y = 0$ and $y = C_1 y_1 + \ldots + C_n y_n$, $C_i \in \mathbb{K}$ is the general solution of of $L(p)y = 0$.

Multiple roots: Example

Example.

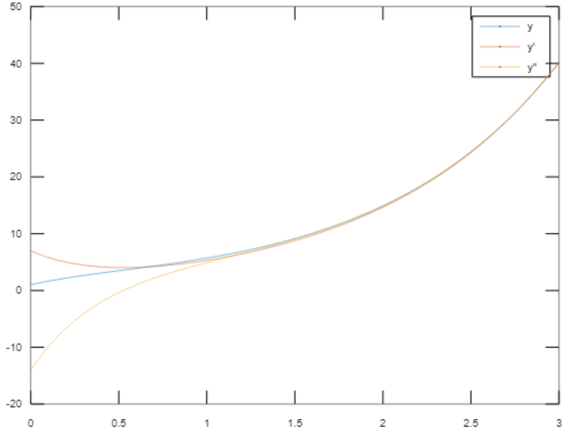
$$y''' + 3y'' - 4y = 0, y(0) = 1, y'(0) = 7, y''(0) = -14$$

$\lambda^3 + 3\lambda^2 - 4 = 0$ $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -2$
The general solution is $y(t) = C_1 e^t + (C_2 + C_3 t)e^{-2t}$.
Differentiate the solution $y' = C_1 e^t + C_3 e^{-2t} - 2(C_2 + C_3 t)e^{-2t}$,
$y'' = C_1 e^t - 4C_3 te^{-2t} + 4(C_2 + C_3 t)e^{-2t}$.
Use the initial conditions to determine C_1, C_2, C_3 from the system

$$\begin{aligned} C_1 + C_2 &= 1, \\ C_1 - 2C_2 + C_3 &= 7, \\ C_1 + 4C_2 - 4C_3 &= -14 \end{aligned}$$

$C_1 = 2,\ C_2 = -1,\ C_3 = 3 \Rightarrow\ y(t) = 2e^t + (3t - 1)e^{-2t}$

Multiple roots: Example



Multiple roots: Example

Example. Let the characteristic equations have the following roots

1. $-3 \pm 2i,\ -1,\ 0,\ 2;$
2. $3,\ 3, 3,\ 0,\ 0,\ 1 \pm 3i;$
3. $2 \pm 5i,\ 2 \pm 5i, 1,\ 1,\ 1,\ 1,\ 3.$

Write the solutions of hom. equations (complementary solutions).

Answer:

1. $y_C(t) = e^{-3t}(C_1 \cos 2t + C_2 \sin 2t) + C_3 e^{-t} + C_4 + C_5 e^{2t}$

2. $$\begin{aligned} y_C(t) = \ & (C_0 + C_1 t + C_2 t^2) e^{3t} + & \lambda = 3,\ 3,\ 3 \\ & + D_0 + D_1 t + & \lambda = 0,\ 0 \\ & + e^t (A \cos 3t + B \sin 3t) & \lambda = 1 \pm 3i \end{aligned}$$

3. $$\begin{aligned} y_C(t) = \ & e^{2t}[(A_0 + A_1 t)\cos 5t + (B_0 + B_1 t)\sin 5t] & \lambda = 2 \pm 5i,\ 2 \pm 5i \\ & +(C_0 + C_1 t + C_2 t^2 + C_3 t^3) e^t + & \lambda = 1, 1, 1, 1 \\ & + D e^{3t} & \lambda = 3 \end{aligned}$$

Linear non-homogeneous equations with constant coeff.

Define a **quasipolynomial** as any function $F(t)$ written in the form

$$F(t) = f_1(t)e^{\lambda_1 t} + f_2(t)e^{\lambda_2 t} + \dots f_m(t)e^{\lambda_m t},$$

where $\lambda_i \in \mathbb{C}$ and $f_i(t)$ are polynomials. Any solution of a linear homogeneous equation with constant coefficients is a quasipolynomial.

Theorem. Consider the homogeneous equation

$$L(p)y = f(t)e^{\lambda t}, \tag{14}$$

where $f(t)$ is a polynomial of degree r and $\lambda \in \mathbb{C}$. Define k be the multiplicity of the root λ if $L(\lambda) = 0$ and $k = 0$ if $L(\lambda) \neq 0$. Then there exists a particular solution of the equation (14) of the form

$$y = t^k g(t)e^{\lambda t},$$

where $g(t)$ is an rth degree polynomial.

Linear non-homogeneous equations with constant coeff.

$f(t)e^{\lambda t}$	$y_p(t)$
$f(t)$	$t^k g(t)$
$f(t)e^{\alpha t}$	$t^k g(t)e^{\alpha t}$
$f(t)e^{\alpha t}\sin\beta t$	$t^k e^{\alpha t}[g(t)\sin\beta t + h(t)\cos\beta t]$
$f(t)e^{\alpha t}\cos\beta t$	$t^k e^{\alpha t}[g(t)\sin\beta t + h(t)\cos\beta t]$

Example. Consider the equation $y'' + 9y = 3e^{2t}$.
What is the type of the equation? $F = 3e^{2t} \Rightarrow$ non-homogeneous

$$y(t) = y_C + y_P$$

The complementary solution y_C is a general solution of the corresponding homogeneous equation. The roots of the characteristic equation $\lambda^2 + 9 = 0$ are $\lambda = \pm 3i$ and hence,

$$y_C = A\cos 3t + B\sin 3t.$$

Linear non-homogeneous equations: Examples

To find a particular solution y_P, notice that $f(t) = 3 \Rightarrow r = 0$ and $\lambda = 2 \pm 0 \cdot i \Rightarrow \lambda$ is not a root of the characteristic polynomial $\Rightarrow k = 0$. Therefore, we shall look for the particular solution of the form

$$y_P(t) = Ce^{2t}.$$

Substitute it into the equation

$$4Ce^{2t} + 9Ce^{2t} = 3e^{2t} \Rightarrow 13Ce^{2t} = 3e^{2t} \Rightarrow C = 3/13$$

Therefore, the general solution of the non-homogeneous equation is

$$y(t) = y_C + y_P = A\cos 3t + B\sin 3t + \frac{3}{13}e^{2t}$$

Linear non-homogeneous equations: Examples

Example. Consider the equation

$$y'' + 3y' + 2y = 42e^{5t} + 390\sin 3t + 8t^2 - 2.$$

The characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$ and hence, $\lambda_1 = -2$, $\lambda_2 = -1 \Rightarrow y_C = C_1 e^{-2t} + C_2 e^{-t}$. We shall apply the method of undetermined coefficients to find y_P.

Term in $F(t)$	Assumed form for y_P
$42e^{5t}$	Ce^{5t}
$390\sin 3t$	$A\cos 3t + B\sin 3t$
$8t^2 - 2$	$D_0 + D_1 t + D_2 t^2$

Therefore, $y_P(t) = Ce^{5t} + A\cos 3t + B\sin 3t + D_0 + D_1 t + D_2 t^2$
You can represent y_P as a sum of particular solutions. Each particular solution satisfies the corresponding non-homogeneous equation!

Linear non-homogeneous equations: Examples

$\Rightarrow$ plug y_P into the equation to determine the coefficients.

$$y_P'' + 3y_P' + 2y_P = (25Ce^{5t} - 9A\cos 3t - 9B\sin 3t + 2D_2) +$$

$$+3(5Ce^{5t} - 3A\sin 3t + 3B\cos 3t + 2D_2 t + D_1) +$$

$$+2(Ce^{5t} + A\cos 3t + B\sin 3t + D_2 t^2 + D_1 t + D_0) =$$

$$= 42Ce^{5t} + (-7A + 9B)\cos 3t + (-9A - 7B)\sin 3t +$$

$$+2D_2 t^2 + (2D_1 + 6D_2)t + (2D_0 + 3D_1 + 2D_2)$$

$$= 42e^{5t} + 390\sin 3t + 8t^2 - 2.$$

Linear non-homogeneous equations: Examples

We equate the coefficients:

$$\begin{aligned}
e^{5t}: &\quad 42C = 42 \Rightarrow C = 1 \\
\cos 3t: &\quad -7A + 9B = 0 \\
\sin 3t: &\quad -9A - 7B = 390 \\
t^2: &\quad 2D_2 = 8 \Rightarrow D_2 = 4 \\
t: &\quad 2D_1 + 6D_2 = 0 \Rightarrow D_1 = -3D_2 = -12 \\
1: &\quad 2D_0 + 3D_1 + 2D_2 = -2 \Rightarrow D_0 = 13
\end{aligned}$$

Therefore, the particular solution is

$$y_P(t) = e^{5t} - 27\cos 3t - 21\sin 3t + 4t^2 - 12t + 13$$

Linear non-homogeneous equations: Examples

Example. Consider the equation

$$y'' + 2y' = 4t^2 + 2t + 3.$$

The roots of the characteristic equation $\lambda^2 + 2\lambda = 0$ are $\lambda = 0, -2$ and $y_C = C_1 + C_2 e^{-2t}$.
Determine the form of the particular solution using the RHS.
$F(t) = 4t^2 + 2t = f(t)e^{\lambda t}$ What is λ? $\lambda = 0$$\Rightarrow$ coincides with one of the roots of the characteristic polynomial $\Rightarrow k = 1$. Also, $r = 2$.
Therefore, the particular solusion has the form

$$y_P = t(D_2 t^2 + D_1 t + D_0)$$

Substitute y_P into the differential equation

$$y_P'' + 2y_P' = 6D_2 t + 2D_1 + 2(3D_2 t^2 + 2D_1 t + D_0) =$$

$$= 6D_2 t^2 + (4D_1 + 6D^2)t + (2D_0 + 2D_1) = 4t^2 + 2t + 3$$

Linear non-homogeneous equations: Examples

and equate the coefficients

$$\begin{aligned}
t^2: &\quad 6D_2 = 8 \Rightarrow D_2 = \tfrac{2}{3} \\
t: &\quad 4D_1 + 6D_2 = 0 \Rightarrow D_1 = \tfrac{1}{4}(2 - 6D_2) = -1/2 \\
1: &\quad 2D_0 + 2D_1 = 3 \Rightarrow D_0 = 2
\end{aligned}$$

The general solution of the equation is

$$y = y_C + y_P = C_1 + C_2 e^{-2t} + \frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t.$$

Exercise: Given the complementary solution y_C and the right-hand side $F(t)$ of the differential equation, specify the form of y_P

1. $y_C = C_1 e^{-t} + C_2 e^{3t} + (D_0 + D_1 t + D_2 t^2)e^{5t}$,
$F(t) = 3e^{-t} + 6e^{2t} - 4e^{5t}$;
2. $y_C = e^{2t}(A\cos 3t + B\sin 3t) + C_0 + C_1 t + C_2 t^2$,
$F(t) = 5te^{2t}\cos 3t + 3t + e^{2t}$.

Method variation of parameters: Examples

Consider the equation $y''' - y' = \frac{e^t}{1+e^t}$.
What is its complementary solution? $y_C = C_1 e^{-t} + C_2 e^t + C_3$ with constant C_1, C_2, C_3.
Assume that $C_1 = C_1(t)$, $C_2 = C_2(t)$ and $C_3 = C_3(t)$ and plug y_C into the equation to obtain Recall P7!!!

$$C_1' e^{-t} + C_2' e^t + C_3' = 0$$
$$-C_1' e^{-t} + C_2' e^t + C_3' \cdot 0 = 0$$
$$C_1' e^{-t} + C_2' e^t + C_3' \cdot 0 = e^t/(1+e^t)$$

Therefore, $C_1' = \frac{e^{2t}}{2(1+e^t)}$, $C_2' = \frac{1}{2(1+e^t)}$ and $C_3' = -\frac{e^t}{1+e^t}$.
Integrating, we obtain $C_1(t) = \frac{1}{2}(e^t - \ln(1+e^t))$, $C_2(t) = \frac{1}{2}(t - \ln(1+e^t))$ and $C_3(t) = -\ln(1+e^t)$.

Euler differential equations

An **Euler** differential equation is a linear differential equation with variable coefficients of the form

$$a_n t^n \frac{d^n y}{dt^n} + a_{n-1} t^{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 t \frac{dy}{dt} + a_0 y = f(t),$$

where a_i, $i = 0..n$ are constants.
Consider a second-order Euler equation

$$at^2 y''(t) + bty'(t) + cy(t) = 0.$$

It has at least one solution of type $y(t) = t^\lambda \Rightarrow$ substitute it into the equation and collect terms $\Rightarrow \lambda$ must satisfy the equation

$$a\lambda(\lambda - 1) + b\lambda + c = 0.$$

Euler differential equations

Case 1. The roots λ_1, λ_2 are real and distinct. The general solution is

$$y(t) = C_1 t_1^{\lambda} + C_2 t_2^{\lambda}.$$

Case 2. $\lambda_{1,2} = \alpha \pm \beta i$

$$y(t) = t^{\alpha}[C_1 \cos(\beta \ln t) + C_2 \sin(\beta \ln t)]$$

Case 3. There is a double root $\lambda_{1,2} = (a-b)/2a$. The procedure gives only one solution. How can we find a second solution? Apply variation of constants.

$$y(t) = A(t)t^{\lambda},\ \lambda = (a-b)/2 \to a\lambda(\lambda-1) + b\lambda + c = 0$$

After simplifications we obtain

$$A''(t) + \frac{A'(t)}{t} = 0 \Rightarrow A'(t) = \frac{1}{t} \Rightarrow A(t) = \ln t$$

Euler differential equations

In this case, the general solution is

$$y(t) = t^{\lambda}(C_1 + C_2 \ln t).$$

Example. Find the general solution of the equation

$$ty''(t) - \beta y'(t) + \frac{\beta}{t} y(t) = 0, \; \beta > 0.$$

It is the equation of Euler's type $\Rightarrow$ use the substitution $y(t) = t^{\lambda}$ to obtain

$$\lambda^2 - (\beta + 1)\lambda + \beta = 0.$$

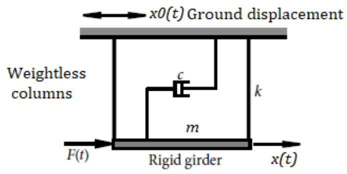
The roots are $\lambda_1 = 1$ and $\lambda_2 = \beta$. The general solution is given by

$$y(t) = \begin{cases} At + Bt^{\beta}, & \beta \neq 1, \\ (A + B \ln t)t, & \beta = 1. \end{cases}$$

Outline

Equation of motion

The number of degrees-of-freedom (DOF) is the total number of variables required to describe the motion of a system.
We shall consider the following model of a single DOF system.

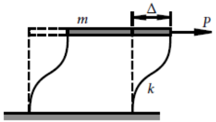


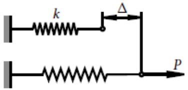
- A single story shear building consists of a rigid girder with mass m supported by
- columns with combined stiffness k. weightless, inextensible in the vertical direction $\Rightarrow$ they can only take shear forces but not bending moments.

Equation of motion

- In the horizontal direction, the columns act as a spring of stiffness $k \Rightarrow$
- the girder can only move in the horizontal direction, and
- its motion can be described by a single variable $x(t)$; hence the system is called a single degree-of-freedom(DOF) system.

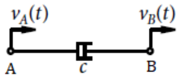
How can we determine the combined stiffness k of the columns?
Apply a horizontal static force P on the girder. If the displacement of the girder is Δ then the combined stiffness of the columns is $k = P/\Delta$.





Equation of motion

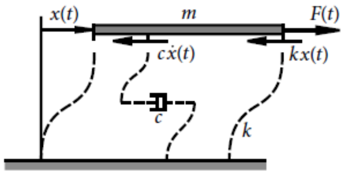
- The internal friction between the girder and the columns is described by a viscous dashpot damper with damping coefficient c.
- A dashpot damper is shown schematically in



- it provides a damping force relative velocity between points B and A.
- The damping force is opposite to the direction of the relative velocity.

Vibration of a shear building under externally applied force

Consider the vibration of the girder in case when it is subjected to an externally applied force $F(t)$ (for ex., a model of wind load).



What are the forces acting on the grinder?

- the shear force (elastic force) $kx(t)$,
- the viscous damping force $c\dot{x}(t)$,
- the externally applied load $F(t)$.

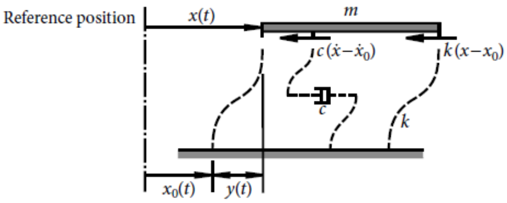
$$ma = \sum F$$

$$m\ddot{x} = -kx - c\dot{x} + F(t)$$

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

Vibration of a shear building under base excitation

Consider the vibration in the case when the base (foundation) of the building is subjected to a dynamic displacement $x_0(t)$ (for ex. a model of an earthquake) and is a known function.



The shear (elastic) force and the damping force applied on the girder are given by

- Shear force and
- Damping force

Vibration of a shear building under base excitation

- Shear force $= k\cdot$ (Relative displacement between girder and base) $= k(x - x_0)$,
- Damping force $= c\cdot$ (Relative velocity between girder and base) $= c(\dot{x} - \dot{x}_0)$

Apply Newton's Second Law to obtain

$$m\ddot{x}(t) = -k(x(t) - x_0(t)) - c(\dot{x}(t) - \dot{x}_0(t)).$$

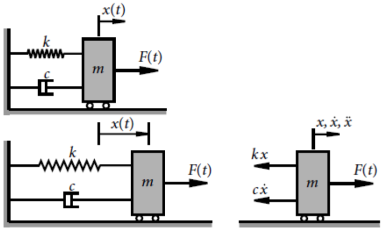
Denote the relative displacement between the girder and the base $y(t) = x(t) - x_0(t)$. Then the equation of motion is

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = -m\ddot{x}_0(t),$$

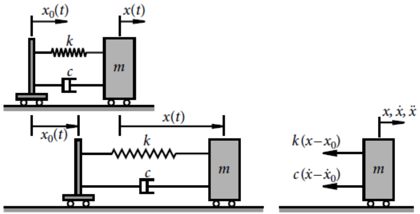
where $\ddot{x}_0(t)$ is the ground acceleration.
The loading on the girder created from ground excitation (earthquake) is $F(t) = -m\ddot{x}_0(t)$ (proportional to the mass of the girder and the ground acceleration).

Vibration of a shear building

In general, any linear single degree-of-freedom system can be modeled by a mechanical mass-damper-spring system.



A mass-damper-spring system under externally applied force.



A mass-damper-spring system under base excitation

Response of a single DOF system

The equation of motion of a single degree-of-freedom system is

$$m\ddot{x} + c\dot{x} + kx = F(t),$$

where $F(t)$ is a known function.
Denote

- $\dfrac{k}{m} = \omega_0^2, \quad \omega_0$ is a **natural circular frequency**,
- $\dfrac{c}{m} = 2\xi\omega_0, \quad \xi$ is a **nondimensional damping coefficient**

and result at the equation

$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2 x = \frac{F(t)}{m}.$$

Second-order, non-homogeneous $\Rightarrow x(t) = x_C(t) + x_P(t)$
The characteristic equation $\lambda^2 + 2\xi\omega_0\lambda + \omega_0^2 = 0$ has the roots
$\lambda_{1,2} = \omega_0(-\xi \pm \sqrt{\xi^2 - 1})$

Underdamped System $0 \leq \xi < 1$

Most engineering structures fall in this category with damping coefficient ξ usually less than 10 percent. The roots of the characteristic equation are

$$\lambda_{1,2} = \omega_0(-\xi \pm \sqrt{\xi^2 - 1}) = -\xi\omega_0 \pm i\omega_d,$$

where $\omega_d = \omega_0\sqrt{1 - \xi^2}$ is the **damped natural circular frequency**. The complementary solution is

$$x_C(t) = e^{-\xi\omega_0 t}(A\cos\omega_d t + B\sin\omega_d t).$$

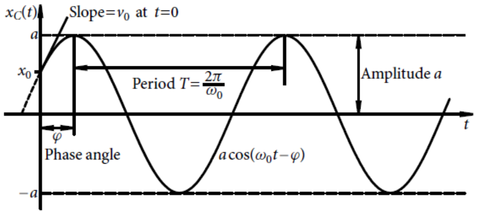
Determine A, B from the i.c. $x(0) - x_0$, $\dot{x}(0) = v_0$ (verify it) to obtain the following the response of free vibration

$$x_C(t) = e^{-\xi\omega_0 t}\left(x_0\cos\omega_d t + \frac{v_0 + \xi\omega_0 x_0}{\omega_d}\sin\omega_d t\right),\ 0 \leq \xi < 1.$$

Special Case: Undamped System $\xi = 0$, $\omega_d = \omega_0$

If $\xi = 0$ then $x_C(t) = x_0 \cos \omega_0 t + \dfrac{v_0}{\omega_0} \sin \omega_0 t = a \cos(\omega_0 t - \varphi)$

with $a = \sqrt{x_0^2 + \left(\dfrac{v_0}{\omega_0}\right)^2}$, $\quad \varphi = \tan^{-1}\left(\dfrac{v_0}{\omega_0 x_0}\right)$



The solution is harmonic with period $T = 2\pi/\omega_0 \Rightarrow \omega_0$ is called the natural circular frequency of the system. The natural frequency is $f = \omega_0/2\pi = 1/T$. The maximum displacement is a.

Underdamped free vibration

The response of the damped free vibration can be written as

$$x_C(t) = ae^{-\xi\omega_0 t}\cos(\omega_d t - \varphi),$$

where

$$a = \sqrt{x_0^2 + \left(\frac{v_0 + \xi\omega_0 x_0}{\omega_d}\right)^2}, \quad \varphi = \tan^{-1}\left(\frac{v_0 + \xi\omega_0 x_0}{\omega_d x_0}\right).$$

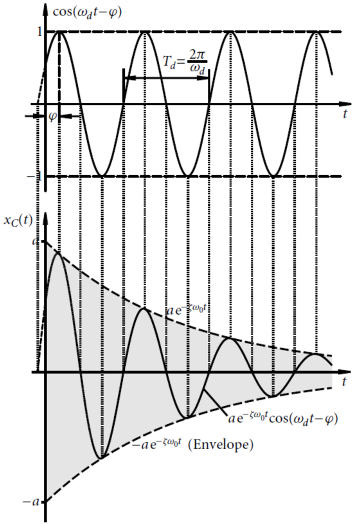
What is the main difference between responses of undamped and damped vibrations? Amplitude: instead of a constant amplitude a, $ae^{-\xi\omega_0 t}$ decays exponentially with time.

Underdamped free vibration

The response of the damped free vibration can be shown using the following steps.

- Sketch the sinusoidal function $\cos(\omega_d t - \varphi)$, ω_d is the damped natural circular frequency and φ is the phase angle. The period is $T_d = 2\pi/\omega_d$ and $f_d = \omega_d/(2\pi) = 1/T_d$ is the damped natural frequency.
- Sketch the amplitude $ae^{-\xi\omega_0 t}$ and its mirror image $-ae^{-\xi\omega_0 t}$. These two lines form the envelope of the response.
- Fit the sinusoidal function $\cos(\omega_d t - \varphi)$ inside the envelope to obtain the response of damped free vibration.

Underdamped free vibration



Underdamped free vibration

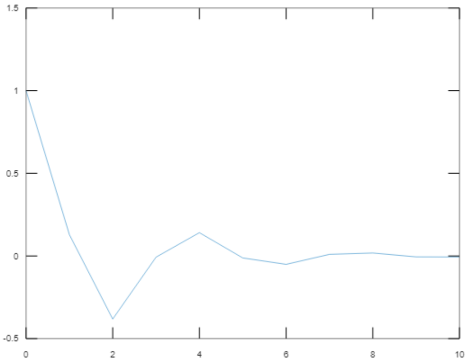
Show that the system $\ddot{x} + \dot{x} + 3x = 0$ is underdamped, find its damped circular frequency and graph the solution with initial conditions $x(0) = 1, x(0) = 0$.
The characteristic equation $\lambda^2 + \lambda + 3 = 0$ has two complex conjugate roots $\lambda_{1,2} = -1/2 \pm \sqrt{11}/2i$. The general solution is

$$x(t) = e^{-t/2}(A\cos(\sqrt{11}t/2) + B\sin(\sqrt{11}t/2))$$

Represent in the form $x(t) = Ce^{-t/2}\cos(\sqrt{11}t/2 - \varphi)$.

$$\varphi = \tan^{-1}(1/\sqrt{11}),\ C = \sqrt{12}/\sqrt{11}$$

Underdamped free vibration



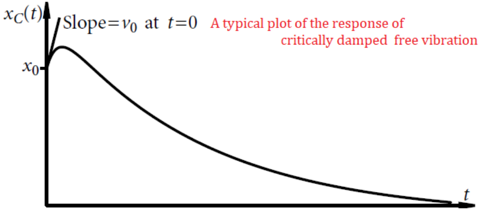
Critically Damped System $\xi = 1$

When $\xi = 1$, the system is critically damped. The characteristic equation has double root $\lambda_{1,2} = -\omega_0$ and the complementary solution is

$$x_C(t) = e^{-\omega_0 t}(C_0 + C_1 t).$$

Determine C_0, C_1 from the i.c. $x(0) = x_0$, $\dot{x}(0) = v_0$ to obtain

$$x_C(t) = e^{-\omega_0 t}\left(x_0 + (v_0 + \omega_0 x_0)t\right), \; \xi = 1.$$

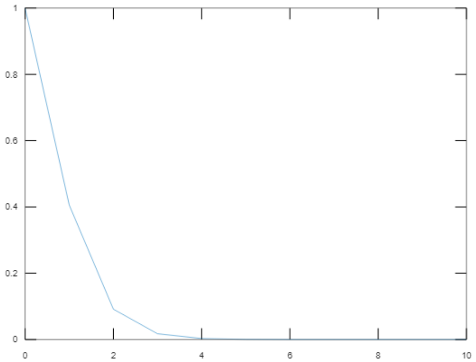


A typical plot of the response of critically damped free vibration

Critically damped free vibration

Show that the system $\ddot{x} + 4\dot{x} + 4x = 0$ is critically damped and graph the solution with initial conditions $x(0) = 1, x(0) = 0.$

$$x(t) = e^{-2t}(1 + 2t)$$



Overdamped System $\xi > 1$

When $\xi > 1$, the characteristic equation has two distinct roots $\lambda_{1,2} = \omega_0(-\xi \pm \sqrt{\xi^2 - 1})$ The complementary solution is

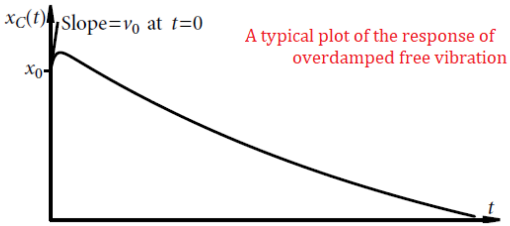
$$x_C(t) = C_1 e^{-\omega_0(\xi - \sqrt{\xi^2-1})t} + C_2 e^{-\omega_0(\xi + \sqrt{\xi^2-1})t}$$

Determine C_1, C_2 from the i.c. $x(0) = x_0$, $\dot{x}(0) = v_0$ to obtain

$$x_C(t) = \frac{1}{2\omega_0\sqrt{\xi^2 - 1}} \Big[(v_0 + (\xi + \sqrt{\xi^2 - 1})\omega_0 x_0) e^{-\omega_0(\xi - \sqrt{\xi^2-1})t} -$$

$$-(v_0 + (\xi - \sqrt{\xi^2 - 1})\omega_0 x_0) e^{-\omega_0(\xi + \sqrt{\xi^2-1})t} \Big]$$

Overdamped System $\xi > 1$



A typical plot of the response of overdamped free vibration

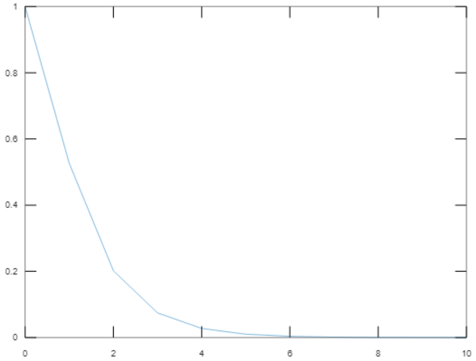
Overdamped free vibration

Show that the system $\ddot{x} + 4\dot{x} + 3x = 0$ is overdamped and graph the solution with initial conditions $x(0) = 1, x(0) = 0$. Which root controls how fast the solution returns to equilibrium?

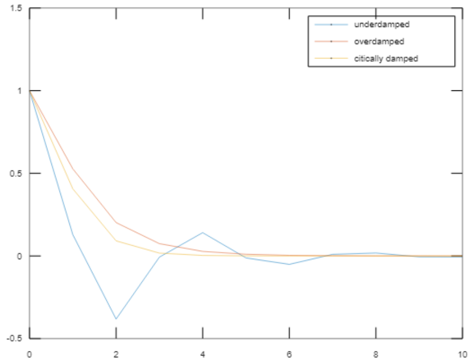
$$x(t) = 3e^{-t}/2 - e^{-3t}/2$$

Since e^{-t} goes to 0 more slowly than $e^{-3t/2}$, so it controls the rate at which x goes to 0.

Overdamped free vibration



Free vibrations



Forced Vibration: Particular Solution

For underdamped systems What is ξ?, the complementary solution rapidly decays (exponentially) $\Rightarrow$ its effect is small and not important in practice. It is called a **transient** solution.
The particular solution $x_P(t)$ is associated with the right-hand side of the differential equation and hence corresponds to forced vibration. The particular solution is called the **steady-state** solution, because it is the solution that persists when time is large.

Forced Vibration: Particular Solution

Consider a particular solution $x_P(t)$ in the case when $F(t)$ is periodic, $F(t) = F_0 \sin \Omega t$.

$$x_p(t) = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\xi\omega_0\Omega)^2}} \sin(\Omega t - \varphi),$$

where angle φ is defined by

$$\cos\varphi = \frac{\omega_0^2 - \Omega^2}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\xi\omega_0\Omega)^2}}, \ \sin\varphi = \frac{2\xi\omega_0\Omega}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\xi\omega_0\Omega)^2}}$$

Forced Vibration: Particular Solution

At what points do the applied force and the response have their maximum? $\Rightarrow$ the response $x_P(t)$lags behind the forcing by a time φ/Ω. The angle φ is called a phase angle or phase lag.
The amplitude of the response of forced vibration is

$$|x_P(t)| = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\xi\omega_0\Omega)^2}}.$$

Denote $r = \dfrac{\Omega}{\omega_0} = \dfrac{\text{excitation frequency}}{\text{undamped natural frequency}}$ = frequency ratio.

$$|x_P(t)| = \frac{F_0}{k}\frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}, \; k = m\omega_0^2.$$

What happens if the dynamic effect is not considered?

Resonance

Let $\xi = 0$ and $\Omega = \omega_0$. Then the equation of motion is

$$\ddot{x}_P + \omega_0^2 x_P = \frac{F_0}{m} \sin \omega_0 t.$$

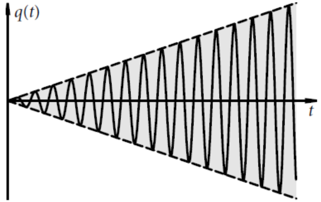
Show that the particular solution $x_P(t)$ has the representation

$$x_P(t) = -\frac{F_0 t}{2m\omega_0} \cos \omega_0 t$$

.

- to sketch the response, find its amplitude, $\dfrac{F_0 t}{2m\omega_0}$
- this straight line and its mirror image $-\dfrac{F_0 t}{2m\omega_0}$ form the **envelope** of $-\cos \omega_0 t$,
- fit $-\cos \omega_0 t$ inside of the envelope.

Resonance



If $\Omega = \omega_0 \Rightarrow$ the system is undamped and the excitation frequency is equal to the natural frequency, then the system is in **resonance** and the amplitude of the response of the system grows linearly with time.