- General Integral Transform

Laplace transform solves IVPs for ODE and PDEs by converting differential equation to algebraic equation.

A general integral transform of $f(t)$ into $\bar{f}(s)$ is $\bar{f}(s) = \int_\alpha^\beta K(s,t) f(t)\, dt$

function $K(s,t)$ is called the kernel of transform.

Laplace transform is a special case with $\alpha = 0$, $\beta = \infty$. $K(s,t) = e^{-st}$ ⇒ improper integral

assume $f(t)$ as a real-valued function defined for $t > 0$.

The Laplace transform of $f(t)$ is $\bar{f}(s) = \mathcal{L}[f(t): t \to s] = \int_0^\infty e^{-st} f(t)\, dt$

Inverse Laplace transform : if $\mathcal{L}[f(t): t\to s]$ then $\mathcal{L}^{-1}[\bar{f}(s): s \to t] = f(t)$

* The existence and uniqueness of Laplace transform of $f(t)$ is guaranteed if there exist real $K, M$ and $a$ s.t. ① $f(t)$ is piecewise continuous for $t > 0$.

② $|f(t)| \leq K e^{at}$ for $t \geq M$.

eg: Laplace transform of $f(t) = 1$. $\mathcal{L}[1: t\to s] = \int_0^\infty e^{-st} = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$

$f(t) = t$ $\mathcal{L}[t: t \to s] = \frac{1}{s^2}$

$f(t) = e^{at}$ $\mathcal{L}[e^{at}: t\to s] = \frac{1}{s-a}$, $s > a$

$f(t) = \sin at$ and $f(t) = \cos at$.

$\mathcal{L}[\cos at + i \sin at : t\to s] = \mathcal{L}[e^{iat}: t \to s] = \frac{1}{s - ia} = \frac{s+ia}{s^2+a^2}$, $s > 0$.

$\therefore \mathcal{L}[\cos at: t\to s] = \frac{s}{s^2+a^2}$, $\mathcal{L}[\sin at: t \to s] = \frac{a}{s^2+a^2}$

Some general properties

- Linearity : $\bar{f}(s) = \mathcal{L}[f(t): t\to s]$, $\bar{g}(s) = \mathcal{L}[g(t): t \to s]$ exist. then $\mathcal{L}[af(t)+bg(t): t\to s]$ exists for all constants $a$ and $b$, $\mathcal{L}[a f(t) + b g(t): t\to s] = a\bar{f}(s) + b\bar{g}(s)$

- First shifting property : Suppose $\bar{f}(s) = \mathcal{L}[f(t): t\to s]$ exists and $a$ is a constant, then

位移性质: $\mathcal{L}[e^{at} f(t): t \to s]$ exists and $\mathcal{L}[e^{at} f(t): t\to s]$ exists and $\mathcal{L}[e^{at} f(t): t \to s] = \bar{f}(s-a)$

or $\mathcal{L}^{-1}[\bar{f}(s-a): s \to t] = e^{at} f(t)$

小试:

| $f(t) = L^{-1}[\bar{f}(s): s\to t]$ | $\bar{f}(s) = L[f(t): t\to s]$ | |
|---|---|---|
| $1$ | $\frac{1}{s}$ | $Re(s)>0$ |
| $e^{at}$ | $\frac{1}{s-a}$ | |
| $t^n,\ n\in Z_+$ | $\frac{n!}{s^{n+1}}$ | |
| $t^p,\ p>-1$ | $\frac{\Gamma(p+1)}{s^{p+1}}$ | $\leftarrow$ ?? |
| $\cos at$ | $\frac{s}{s^2+a^2}$ | |
| $\sin at$ | $\frac{a}{s^2+a^2}$ | |
| $\cosh at$ | $\frac{s}{s^2-a^2}$ | $Re(s)>\lvert a\rvert$ |
| $\sinh at$ | $\frac{a}{s^2-a^2}$ | |
| $e^{at}\cos bt$ | $\frac{s-a}{(s-a)^2+b^2}$ | $Re(s)>a$ |
| $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2}$ | |
| $t^n e^{at},\ n\in Z_+$ | $\frac{n!}{(s-a)^{n+1}}$ | |

| $f(t) = L^{-1}[\bar{f}(s): s\to t]$ | $\bar{f}(s) = L[f(t): t\to s]$ |
|---|---|
| $H(t-a)$ | $\frac{e^{-as}}{s}$ |
| $H(t-a)f(t-a)$ | $e^{-as}\bar{f}(s)$ |
| $e^{at}f(t)$ | $\bar{f}(s-a)$ |
| $f(at)$ | $\frac{1}{a}\bar{f}(\frac{s}{a})$ |
| $\int_0^t f(t-w)g(w)\,dw$ | $\bar{f}(s)\bar{g}(s)$ |
| $\delta(t-a)$ | $e^{-as}$ |
| $(-t)^n f(t)$ | $\bar{f}^{(n)}(s)$ |
| $f^{(n)}(t)$ | $s^n\bar{f}(s) - \sum_{j=0}^{n-1} s^{n-j-1}f^{(j)}(0)$ |

## · Convolution 卷积

Let $\bar{f}(s) = L[f(t): t\to s]$, $\bar{g}(s) = L[g(t): t\to s]$ be Laplace transforms of $f(t)$ and $g(t)$.

$\therefore L^{-1}[\bar{f}(s)+\bar{g}(s): s\to t] = L^{-1}[\bar{f}(s): s\to t] + L^{-1}[\bar{g}(s): s\to t] = f(t)+g(t)$

✩ But $L^{-1}[\bar{f}(s)\bar{g}(s): s\to t] \neq f(t)\,g(t)$

Given two functions $f(t)$ and $g(t)$, convolution $f * g$ of $f$ and $g$ is defined by the integral $(f*g)(t) = \int_0^t f(t-u)g(u)\,du$. whenever this integral exists.

有 $(f*g)(t) = (g*f)(t)$

$L[(f*g)(t): t\to s] = \bar{f}(s)\bar{g}(s) = \int_0^\infty\left(\int_0^t f(t-u)g(u)\,du\right)e^{-st}\,dt = \int_0^\infty g(u)\left(\int_u^\infty f(t-u)e^{-st}dt\right)du$

eg: Compute $L^{-1}[s/(s^2+1)^2 : s\to t]$

since $L^{-1}[\bar{f}(s)\bar{g}(s): s\to t] = (f*g)(t)$

$L^{-1}\left[\frac{s}{(s^2+1)^2}: s\to t\right] = L^{-1}\left[\frac{s}{s^2+1}\cdot\frac{1}{s^2+1} : s\to t\right] = (\sin * \cos)(t)$

$(\sin*\cos)(t) = \int_0^t \sin(t-u)\cos u\,du = \sin t\int_0^t \cos^2 u\,du - \cos t\int_0^t \sin u\cos u\,du$.

$= \frac{t\sin t}{2}$

show $f(t)+\int_0^t e^u f(t-u)\,du = g(t)$ can be written as

$$f(t) = g(t) - \int_0^t g(u)\,du.$$

$f(t) \to \bar{f}(s)$ , $g(t) \to \bar{g}(s)$

$\therefore\ \bar{f} + L[(f * e)(t): t \to s] = \bar{g}$

$\bar{f} + \bar{f}\, L[e^t : t \to s] = \bar{g} \quad \Rightarrow \quad \bar{f} = \bar{g} - \frac{\bar{g}}{s}$

$f(t) = g(t) - L^{-1}\left[\frac{\bar{g}(s)}{s} : s \to t\right] = g(t) + \int_0^t g(x)\,dx$

- **Application to initial-Value Problems**

assume $f$ has a Laplace transform, $\dot{f}$ is piecewise continuous in interval $(0, \infty)$. Then, $L[\dot{f}(t): t \to s] = \int_0^\infty \dot{f}(t) e^{-st} dt$

$= [f(t)e^{-st}]_0^\infty + s\int_0^\infty f(t) e^{-st} dt$

$= s\bar{f}(s) - f(0) \qquad s > 0.$

Similarly, $L[\ddot{f}(t): t \to s] = \int_0^\infty \ddot{f}(t) e^{-st} dt = [\dot{f}(t) e^{-st}]_0^\infty + s\int_0^\infty \dot{f}(t) e^{-st} dt$

$= -\dot{f}(0) + sL[\dot{f}(t): t \to s]$

$= s^2 \bar{f}(s) - sf(0) - \dot{f}(0) \qquad s > 0.$

eg: $\ddot{y} + 4y = 0$, $y(0) = 1$, $\dot{y}(0) = 2$.

$\bar{y}(s) = L[y(t): t \to s]$, $y(s)$ satisfies $s^2\bar{y}(s) - sy(0) - \dot{y}(0) + 4\bar{y}(s) = 0$

$\bar{y}(s) = \frac{s+2}{s^2+4} = \frac{s}{s^2+4} + \frac{2}{s^2+4}$

$y(t) = \cos 2t + \sin 2t.$

给出 $Q(s) = \frac{s}{4(s^2+1)(s^2+6s+9)}$

$= \frac{1}{200}\left(\frac{4s}{s^2+1} + \frac{3}{s^2+1} - \frac{4}{s+3} - \frac{15}{(s+3)^2}\right)$

$\therefore\ Q(t) = \frac{1}{200}(4\cos t + 3\sin t - 4e^{-3t} - 15te^{-3t})$

- **The unit step function** $H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a. \end{cases}$

The Laplace transform of unit step function is

$L[H(t-a): t \to s] = \int_0^\infty e^{-st} H(t-a)\,dt = \int_a^\infty e^{-st} dt = \frac{e^{-as}}{s}$, $s > 0$.

second shifting property: $\mathcal{L}[f(t-a)H(t-a): t\to s] = e^{-as}\bar{f}(s)$

$$L^{-1}\left[\frac{e^{-\pi s/2}}{1+s^2}; s\to t\right] = H(t-\tfrac{\pi}{2})\sin(t-\tfrac{\pi}{2})$$

$$\bar{y}(s) = \frac{se^{-s}}{s^2+1} \Rightarrow y(t) = H(t-1)\cos(t-1)$$

- The Unit impulse Function

Unit impulse function $\delta(t)$ belongs to generalized functions or distribution.

$\int_{-\infty}^{+\infty} \delta(t-a) f(t)\, dt = f(a)$ for any integrable function $f(t)$.

Sequence 1. Let $\delta_\varepsilon(t-a) = \begin{cases} \frac{1}{2\varepsilon} & |t-a|<\varepsilon \\ 0 & |t-a|\geqslant\varepsilon \end{cases}$

$$\int_{-\infty}^{+\infty} \delta_\varepsilon(t-a)f(t)\,dt = \frac{1}{2\varepsilon}\int_{a-\varepsilon}^{a+\varepsilon} f(t)\,dt = f(a+\theta\varepsilon),\ |\theta|<1$$

$\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} \delta_\varepsilon(t-a) f(t)\,dt = f(a)$ for any real integrable function $f$. Therefore,

$\delta(t-a) = \lim_{\varepsilon\to 0}\delta_\varepsilon(t-a)$

Sequence 2. $\delta_n(t) = \lim_{n\to\infty}\frac{n}{\sqrt{\pi}}e^{-n^2t^2}$

Let $f(t)=1$.. $\int_{-\infty}^{+\infty}\delta(t-a)\,dt = 1$. Thus, $\delta(t-a)$ : unit impulse function

unit step function $H(t-a) = \int_{-\infty}^{t}\delta(u-a)\,du \Rightarrow \frac{d}{dt}H(t-a) = \delta(t-a)$

$\therefore \mathcal{L}[\delta(t-a): t\to s] = \int_0^\infty \delta(t-a)e^{-st}\,dt = \int_{-\infty}^{\infty}\delta(t-a)e^{-st}\,dt = e^{-as},\ a>0$

公式

| $f(t) = L^{-1}[\bar{f}(s): s\to t]$ | $\bar{f}(s) = \mathcal{L}[f(t): t\to s]$ |
|---|---|
| $H(t-a)$ | $\frac{e^{-as}}{s}$ |
| $H(t-a)f(t-a)$ | $e^{-as}\bar{f}(s)$ |
| $e^{at}f(t)$ | $\bar{f}(s-a)$ |
| $f(at)$ | $\frac{1}{a}\bar{f}(\frac{s}{a})$ |
| $\int_0^t f(t-w)g(w)\,dw$ | $\bar{f}(s)\bar{g}(s)$ |
| $\delta(t-a)$ | $e^{-as}$ |
| $(-t)^n f(t)$ | $\bar{f}^{(n)}(s)$ |

$f^{(n)}(t)$ $\qquad$ $s^n \bar{f}(s) - \sum_{j=0}^{n-1} s^{n-j-1} f^{(j)}(0)$

- Periodic Function

$f(t) = f(t+kT)$ for $0 \le t < \infty$, $k = 1, 2 \cdots$

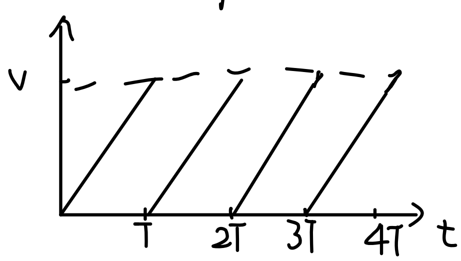
Laplace transform of periodic function f :

$$\mathcal{L}[f(t) : t \to s] = \bar{f}(s) = \int_0^\infty f(t) e^{-st} dt = \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} f(t) e^{-st} dt$$

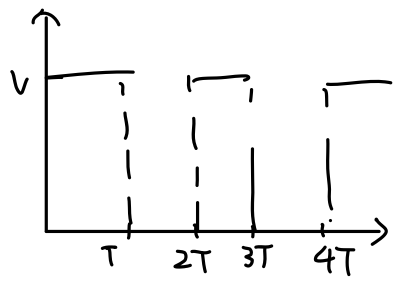
$\because t = u + kT \therefore$ $\bar{f}(s) = \sum_{k=0}^{\infty} \int_0^T f(u+kT) e^{-s(u+kT)} du$

$$= \frac{1}{1-e^{-sT}} \int_0^T f(t) e^{-st} dt$$

Common periodic waveforms




Saw tooth




square wave

saw tooth waveform of amplitude V and period T :

$$\bar{f}(s) = \frac{V}{Ts^2}\left(1 - \frac{sTe^{-sT}}{1-e^{-sT}}\right)$$

square wave of V and T :

$$\bar{f}(s) = \frac{V}{1-e^{-2sT}} \int_0^T t e^{-st} dt = \frac{V}{s} \frac{1}{1+e^{-sT}}$$

[eg] inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{a}{s(s+a)} : s \to t\right] = 1 - e^{-at}$$

$$\mathcal{L}^{-1}\left[\frac{e^{-skT}}{(s+a)} : s \to t\right] = e^{-a(t-kT)} H(t-kT)$$