- **Inner product**

A complex-valued function $(\cdot,\cdot): X \times X \to C$ satisfying

1. $(x,x) \geq 0 \quad \forall x \in X$
2. $(\alpha x+\beta y, z) = \alpha(x,z) + \beta(y,z) \quad \forall x,y,z \in X, \ \forall \alpha,\beta \in K$
3. $\overline{(x,y)} = (y,x) \quad \forall x,y \in X$

is called an inner product. linear space X is called inner product space

- Cauchy - Schwartz inequality:

$$|(x,y)| \leq \sqrt{(x,x)}\sqrt{(y,y)} \quad \forall x,y \in X$$

- Any inner product space is also a normed linear space with the natural norm induced by the inner product. $\|x\| = \sqrt{(x,x)}$

- An inner product is continuous function w.r.t both arguments:

$$x_n \to x, \ y_n \to y \Rightarrow (x_n, y_n) \to (x,y) \quad n \to \infty$$

- Parallelogram identity-

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

- Orthogonality

Elements $\{e_i\} \subset X$, $i=1,2\cdots$ are said to be 正交的 orthogonal if (e_i, e_j), $i \neq j$.
If $\|e_i\| = 1$, $i=1,2\cdots$ then the orthogonal system is said to be 标准正交 orthonormal.

* Any orthonormal system is linearly independent.

- **Fourier series**

A functional series of the form

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, where coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

is called a Fourier series of function $f(x)$. [real form of Fourier series]

Theo 1: A Fourier series of periodic $(\omega = 2\pi)$, piecewise continuous bounded function

$f(x)$ converges at all points $x \in \mathbb{R}$ and its sum equals

$$S(x) = \frac{f(x-0)+f(x+0)}{2}.$$ 两个x是左右使$f(x)$不连续的极限点,

eg. Find a Fourier series expansion of periodic ($T=2\pi$) function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 \le x \le \pi . \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{1}{\pi n^2}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = -\frac{\pi}{n\pi}\cos n\pi = \frac{(-1)^{n-1}}{n}$$

$$\therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(-\frac{2}{\pi(2n-1)^2}\cos(2n-1)x + \frac{(-1)^{n-1}}{n}\sin nx\right)$$

Series converges to $f(x)$ at all $x \ne (2n-1)\pi$

sum of Fourier series equal to $\frac{\pi+0}{2} = \frac{\pi}{2}$ at points $x = (2n-1)\pi$

- A Fourier series of a periodic function $y=f(x)$ with $T=2l$ can be represented

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{\pi n}{l}x + b_n\sin\frac{\pi n}{l}x\right)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\,dx, \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x\,dx$$

A Fourier series of periodic $T=2l$ piecewise continuous bounded on $[-l, l]$ function $f(x)$ converges at all points $x \in \mathbb{R}$ and sum equals

$$S(x) = \frac{f(x-0)+f(x+0)}{2}$$

eg. Find Fourier series expansion of the periodic ($T=4$) function

$$f(x) = \begin{cases} -1 & -2 < x < 0 \\ 2 & 0 \le x \le 2 \end{cases}$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = 1$$

$$a_n = \frac{1}{2}\left(\int_{-2}^{0}(-1)\cos\frac{\pi n}{2}x\,dx + \int_0^2 2\cos\frac{\pi n}{2}x\,dx\right) = 0,$$

$$b_n = \frac{1}{2}\left(\int_{-2}^{0}(-1)\sin\frac{\pi n}{2}x\,dx + \int_0^2 2\sin\frac{\pi n}{2}x\,dx\right) = \frac{3}{\pi n}\left(1-(-1)^n\right)$$

$$\therefore f(x) = \frac{1}{2} + \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{2}$$

- If periodic function $y=f(x)$ is even 偶函数, then its Fourier series is a Fourier cosine series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n}{I} x \quad \text{with} \quad a_n = \frac{2}{I}\int_0^I f(x)\cos\frac{\pi n}{I} x\, dx$$

• If periodic function $y=f(x)$ is odd (奇函数), then its Fourier series is a Fourier sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{\pi n}{I} x \quad \text{with} \quad b_n = \frac{2}{I}\int_0^I f(x)\sin\frac{\pi n}{I} x\, dx$$

- A function $f(x)$: piecewise continuous, bounded on $[a,b] \subset (-I, I)$

a function $f(x)$ defined on $[a,b] \subset (-I, I)$ with extension

$$f(x) = \begin{cases} 0 & -I < x < -b \\ -f(x) & -b \le x \le -a \\ 0 & -a < x < a \\ f(x) & a \le x \le b \\ 0 & b < x < I \end{cases}$$

The sum $S(x)$ is $f(x)$ in (a,b), $S(a) = f(a)/a$, $S(b) = f(b)/2$

eg. $f(x) = 2-x$ on $[0,2]$.

odd extension on $[-2,0)$. $f(x) = \begin{cases} -2-x & -2 \le x < 0 \\ 2-x & 0 \le x \le 2. \end{cases}$

Then $a_n = 0$, $n = 0, 1, \cdots$

$$b_n = \frac{2}{I}\int_0^I f(x)\sin\frac{\pi n}{I}x\, dx$$

$$= \int_0^2 (2-x)\sin\frac{\pi n}{2}x\, dx = \frac{4}{\pi n} \quad \Rightarrow f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi n}{2}x$$

* We use Fourier series to find sums of series.

令 $x = \cdots$ 左右 除·除.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

Complex Exponential Form

$T = 2\pi$ 时,

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}, \quad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{inx} + \frac{a_n + ib_n}{2} e^{-inx} \quad \Leftarrow \quad ?$$

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)\, dx$$

$$C_0 = \frac{a_0}{2}, \quad \frac{a_n + ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i\sin nx)\, dx = C_{-n}$$

$\therefore\ f(x) = \sum_{-\infty}^{+\infty} C_n e^{inx}$. $\quad C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$

$\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}$ is orthonormal. $\quad f(x) = \sum_{i=1}^{\infty} (f(x), e_i)\, e_i$. $\{e_i\}$ is orthonormal and complete

有 $a_n - ib_n = 2C_n$ $\quad\Rightarrow\quad$ $a_n = C_n + C_{-n}$

$a_n + ib_n = 2C_{-n}$ $\quad\quad\quad$ $ib_n = C_{-n} - C_n$

if $f(x)$ is real, then $C_{-n} = \overline{C_n}$ $\Rightarrow$

$C_n = |C_n| e^{i\varphi_n}$. $\quad C_{-n} = |C_n| e^{-i\varphi_n}$

real form becomes $f(x) = C_0 + \sum_{n=1}^{\infty} |C_n| \left(e^{i(nx+\varphi_n)} + e^{-i(nx+\varphi_n)}\right)$

$f(x) = C_0 + \sum_{n=1}^{\infty} 2|C_n| \cos(nx + \varphi_n)$

derive that $2|C_n| = \sqrt{a_n^2 + b_n^2}$. $\quad \varphi_n = -\tan^{-1}(b_n/a_n)$

- Multiplication

$f(x) = \sum_{n=-\infty}^{+\infty} f_n e^{inx}$, $\quad g(x) = \sum_{n=-\infty}^{+\infty} g_n e^{inx}$

$h(x) = f(x)\, g(x) = \sum_{n=-\infty}^{+\infty} h_n e^{inx}$ $\quad\quad$???

Fourier coefficients $h_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) g(x) e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{+\infty} f_k e^{ikx} g(x) e^{-inx}\, dx$

$= \sum_{k=-\infty}^{+\infty} f_k \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n-k)x} g(x)\, dx = \sum_{k=-\infty}^{+\infty} f_k g_{n-k}$

- Parceval's Indentity

$\overline{f_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x) e^{-inx}}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\, e^{-i(-n)x}\, dx = g_{-n}$

Let $g(x) = \overline{f(x)}$ and $n=0$. Then $h_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{+\infty} f_k g_{-k}$

The first form of Parseval Identity

$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_{k=-\infty}^{+\infty} |f_k|^2$

- Mean-square Error Approximation

$f(x) - \sum_{n=-\infty}^{+\infty} f_n e^{inx}$ by a finite sum $f_N(x) = \sum_{n=-N}^{N} \alpha_n e^{inx}$

$\mathcal{E}_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - f_N(x)|^2\, dx = \underline{\sum_{n=-N}^{N} |f_n - \alpha_n|^2 + \sum_{|n|>N} |f_n|^2}$

let $\alpha_n = f_n$ $\quad\Rightarrow\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx - \sum_{n=-N}^{N} |f_n|^2 \to 0$ as $N \to \infty$

- BVP Problem

$F''(x) + \alpha F(x) = 0 \quad F(0)=0, \; F(1)=0. \quad (1)$

general sol of equation $\alpha>0$ is $F(x) = A\sin\sqrt{\alpha}x + B\cos\sqrt{\alpha}x$

$\Rightarrow 0=F(0)=B. \quad 0=F(1) = A\sin\sqrt{\alpha}1 + B\cos\sqrt{\alpha}1 \Rightarrow A\sin\sqrt{\alpha}1 = 0.$

$A\neq 0 \Rightarrow$ non-trivial solution 非平凡解 $\Rightarrow \sin\sqrt{\alpha}1 = 0,$

requires $\alpha = \alpha_n = \dfrac{n^2\pi^2}{1^2}. \quad n=1,2,3\cdots$

Each value of α for which (1) has non-trivial sol is called an eigenvalue, corresponding expression $F(x) = \sin\frac{n\pi x}{I}$ is called eigenfunction.

general sol of (1) is $F(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{I}$. with arbitrary constants a_n, $n=1,2\cdots\infty$

[eg] $y'' - 2y' + (1+\lambda)y = 0. \quad y(0)=0, \; y(1)=0.$

$y = e^{\alpha x}. \quad \alpha^2 - 2\alpha + (1+\lambda) = 0 \Rightarrow \alpha = 1\pm\sqrt{1-(1+\lambda)} = 1\pm i\sqrt{\lambda}$

$\therefore y = e^x(C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x)$

$y(0) = 0 = C. \quad y(1)=0 \Rightarrow$ non-trivial sol $\sqrt{\lambda} = n\pi. \quad n=1,2,3\cdots$

$y_n(x) = k_n e^x \sin\pi n x, \; n=1,2\cdots \quad k_n$ arbitrary constants.

example 2 ??

Sturm-Liouville boundary value problem

$(p(x)y')' - q(x)y + \lambda r(x)y = 0. \; (1)$ on interval $0<x<1$. with boundary condition on

$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \; \beta_1 y(1) + \beta_2 y'(1) = 0 \quad (2)$ at end points.

introduce linear homogeneous differential operator L defined by

$L[y] = -(p(x)y')' + q(x)y \quad (3)$

then (1) 写成 $L[y] = \lambda r(x) y$

Lagrange's identity: $\int_0^1 (L(u)v - uL(v))\,dx = -p(x)(u'(x)v(x) - u(x)v'(x))\Big|_0^1 \quad (5)$

if (5) statisfy (2), then $\int_0^1 (L[u]v - uL[v])\,dx = 0 \quad (6)$

let $(u,v) = \int_0^1 u(x)v(x)\,dx$

(6) becomes $(L[u], v) - (u, L[v]) = 0.$

$(u,v)=\int_0^1 u(x)\bar{v}(x)\,dx.$

unbounded
simple

Theo: All eigenvalues of S-L problem (1)(2) are real-valued

Theo: Eigenfunctions of S-L boundary value problem (1)(2) are orthogonal with respect to weight function r. if ϕ_m, ϕ_n two eigenfunctions of S-L problem, (1)(2) corresponding to λ_m, λ_n. if $\lambda_m \neq \lambda_n$,

$$\int_0^1 r(x)\phi_m(x)\phi_n(x)\,dx=0.$$

· normalization condition $\int_0^1 r(x)\phi_n^2(x)\,dx=1.$ $n=1,2\cdots$ (20) → 确定常数 k_n

$$\delta_{mn}=\begin{cases}0 & m\neq n\\ 1 & m=n\end{cases}\qquad \int_0^1 r(x)\phi_m(x)\phi_n(x)\,dx=\delta_{mn}$$

$$f(x)=\sum_{n=1}^{\infty}C_n\phi_n(x)$$

$$\int_0^1 r(x)f(x)\phi_m(x)\,dx=\sum_{n=1}^{\infty}C_n\int_0^1 r(x)\phi_m(x)\phi_n(x)\,dx=\sum_{n=1}^{\infty}C_n\delta_{mn}$$

$$C_m=\int_0^1 r(x)f(x)\phi_m(x)\,dx=(f,\ r\phi_m)\qquad m=1,2,\cdots$$

[eg] expand $f(x)=x$ $0\leq x\leq 1$ in terms of normalized eigenfunctions $\phi_n(x)$

$\phi_n=k_n\sin\sqrt{\lambda_n}x.$

$$f(x)=\sum_{n=1}^{\infty}C_n\phi_n(x)$$

$$C_n=k_n\int_0^1 x\sin(\sqrt{\lambda_n}x)\,dx$$

$$C_n=k_n\frac{2\sin\sqrt{\lambda_n}}{\lambda_n}$$

代入 $f(x)$ ..

[eg] $y''+\lambda y=0.$ $y(0)=0.$ $y'(1)+y(1)=0.$

$\phi=k_n\sin\sqrt{\lambda_n}x$ $\qquad\sin\sqrt{\lambda_n}+\sqrt{\lambda_n}\cos\sqrt{\lambda_n}=0.$

determine k_n from normalization condition

$$\int_0^1 r(x)\phi_n^2(x)\,dx=k_n^2\int_0^1\sin^2(\sqrt{\lambda}x)\,dx=k_n^2\frac{1+\cos^2\lambda_n}{2}$$

$$k_n=\left(\frac{2}{1+\cos^2\sqrt{\lambda_n}}\right)^{1/2}$$

$$\therefore\ \phi_n=\frac{\sqrt{2}\sin\sqrt{\lambda}x}{(1+\cos^2\sqrt{\lambda_n})^{1/2}}\qquad n=1,2,\cdots$$

non-homogeneous Boundary Value Problem

$L[y] = \mu r(x) y + f(x)$ boundary condition $\alpha_1 y(0) + \alpha_2 y'(0) = 0.$ $\beta_1 y(1) + \beta_2 y'(1) = 0$

$\phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$

$b_n = \int_0^1 r(x) \phi(x) \phi_n(x)\, dx$

case 1. $\mu \neq \lambda_n$. $b_n = \frac{c_n}{\lambda_n - \mu}$

$$y = \phi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} \phi_n(x)$$

[eg] $y'' + 2y = -x.$ $y(0) = 0$ $y'(1) + y(1) = 0$

homo: $y'' + \lambda y = 0.$ $y(0) = 0,$ $y'(1) + y(1) = 0.$

$\lambda = 0.$ $y(x) = c_1 x + c_2$

$\lambda > 0.$ $y(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x$ ...

$\lambda < 0$ $\mu = -\lambda$ $y(x) = c_1 \sinh\sqrt{\mu}x + c_2 \cosh\sqrt{\mu}x.$