II) Perfect dipole – special : $d \ll \frac{c}{\omega} = \frac{\lambda}{2\pi}$ $\omega \to 0$ (radio waves)

$$\lim_{\omega\to 0} \Phi(r,\Theta,t) = \lim_{\omega\to 0} \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r}\left[\frac{d}{r}\cos\Theta * \cos\left(\omega\left(t+\tfrac{r}{c}\right)\right) * \underbrace{\cos\left(\tfrac{\omega}{c}\tfrac{d}{2}\cos\Theta\right)}_{\to 1} - 2\underbrace{\sin\left(\omega\left(t+\tfrac{r}{c}\right)\right)}_{\to 0}\underbrace{\sin\left(\tfrac{\omega}{c}\tfrac{d}{2}\cos\Theta\right)}_{\to \frac{\omega}{c}\frac{d}{2}\cos\Theta}\right]$$

$$= \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r}\left[\frac{d}{r}\cos\Theta * \cos\left(\omega\left(t+\tfrac{r}{c}\right)\right) * 1 + \cancel{2}\underbrace{\sin\left(\omega\left(t-\tfrac{r}{c}\right)\right)}_{\to 0}\frac{\omega}{c}\frac{d}{\cancel{2}}\cos\Theta\right]$$

$$= \frac{q_0}{4\pi\varepsilon_0}\frac{d}{r^2}\cos\Theta * \cos\left(\omega\left(t+\tfrac{r}{c}\right)\right) \quad \text{decays rapidly!}$$

III perfect dipole – radiation zone: $d \ll \frac{c}{\omega} = \frac{\lambda}{2\pi} \ll r$

$$\lim_{r\to\infty} \Phi(r,\Theta,t) = \lim_{r\to\infty} \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r}\left[\frac{d}{r}\cos\Theta * \cos\left(\omega\left(t+\tfrac{r}{c}\right)\right) * \underbrace{\cos\left(\tfrac{\omega}{c}\tfrac{d}{2}\cos\Theta\right)}_{d\,\text{small} \Rightarrow \to 1} - 2\sin\left(\omega\left(t+\tfrac{r}{c}\right)\right)\underbrace{\sin\left(\tfrac{\omega}{c}\tfrac{d}{2}\cos\Theta\right)}_{d\,\text{small} \Rightarrow \to \frac{\omega}{c}\frac{d}{2}\cos\Theta}\right]$$

$$= \lim_{r\to\infty} \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r}\left[\frac{d}{r}\cos\Theta * \cos\left(\omega\left(t+\tfrac{r}{c}\right)\right) + \cancel{2}\sin\left(\omega\left(t+\tfrac{r}{c}\right)\right)\frac{\omega}{c}\frac{d}{\cancel{2}}\cos\Theta\right]$$

$$= \lim_{r\to\infty} \frac{q_0}{4\pi\varepsilon_0}\frac{d}{r}\cos\Theta\left[\frac{1}{r}\cos\left(\omega\left(t+\tfrac{r}{c}\right)\right) + \frac{\omega}{c}\sin\left(\omega\left(t+\tfrac{r}{c}\right)\right)\right]$$

$$= \lim_{r\to\infty} \frac{q_0}{4\pi\varepsilon_0}\left[\frac{d}{r^2}\cos\Theta\cos\left(\omega\left(t+\tfrac{r}{c}\right)\right) + \frac{d}{r}\cos\Theta\frac{\omega}{c}\sin\left(\omega\left(t+\tfrac{r}{c}\right)\right)\right]$$

$$= + \frac{p_0}{4\pi\varepsilon_0}\frac{\cos\Theta}{r}\frac{\omega}{c}\sin\left(\omega\left(t+\tfrac{r}{c}\right)\right) = \Phi(r,\Theta,t)$$