Inequality Process Simulation

Modeling and Simulation in Science, Engineering, and Economics

Yao Xiao | Ziyue Wang

Courant Institute of Mathematical Sciences
December 22, 2022

Introduction

- *n* people in a room, each with *m* dollars initially
- In each step, a random person gives a dollar to another random person
- How will the money be distributed?

Introduction

- Uniform?
- Exponential-like
- The inequality process

Model

- *n* people in a society, with some distribution of initial wealth
- In each step, randomly select two people
- Exchange their wealth following some transaction function

The Fugivity of Surplus Wealth Principle

Surplus is the difference between subsistence and the total production of wealth; societal net product. At the level of the individual person, where people are able to produce a surplus, some of the surplus will be **fugitive** and **leave the possession of people who produce it**. Moreover, this implies encounters in which surplus wealth changes hands fairly readily.

Transaction Function #1

$$X'_A = X_A + dU \cdot X_B - (1-d)U \cdot X_A, \tag{1}$$
$$X'_B = X_B + (1-d)U \cdot X_A - dU \cdot X_B, \tag{2}$$

where

$X_A, X'_A =$ the surplus wealth of A before (respectively, after) an encounter with B,

$X_B, X'_B =$ the surplus wealth of B before (respectively, after) an encounter with A,

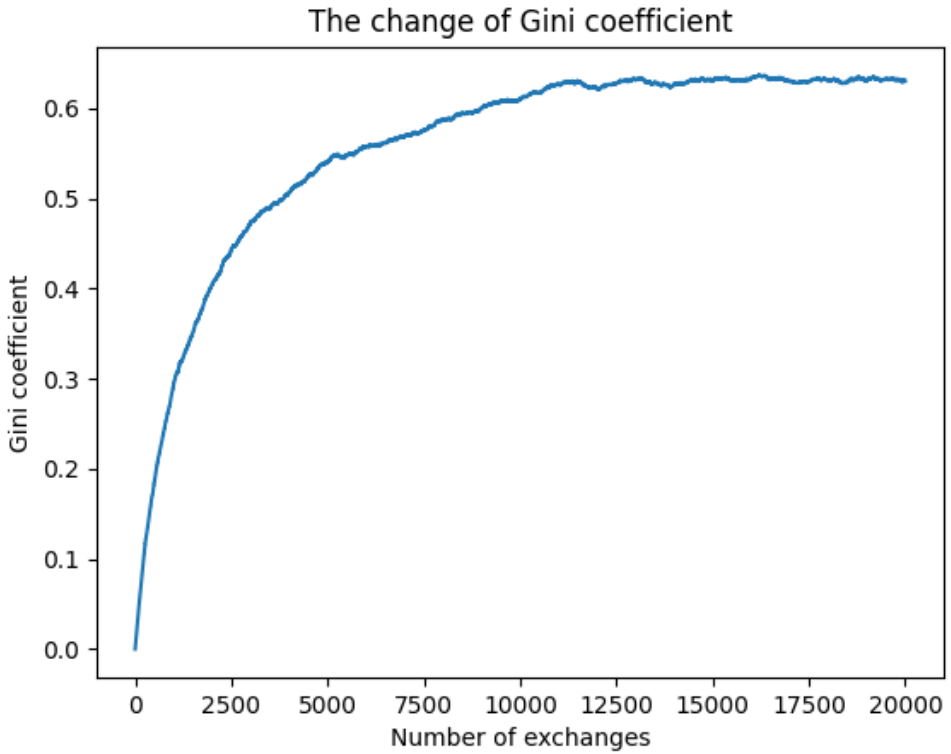
$$d = \begin{cases} 1, & \text{with probability } 0.5, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad U \sim \mathbf{Uniform}(0,1).$$

The Hunter-Gatherer Society

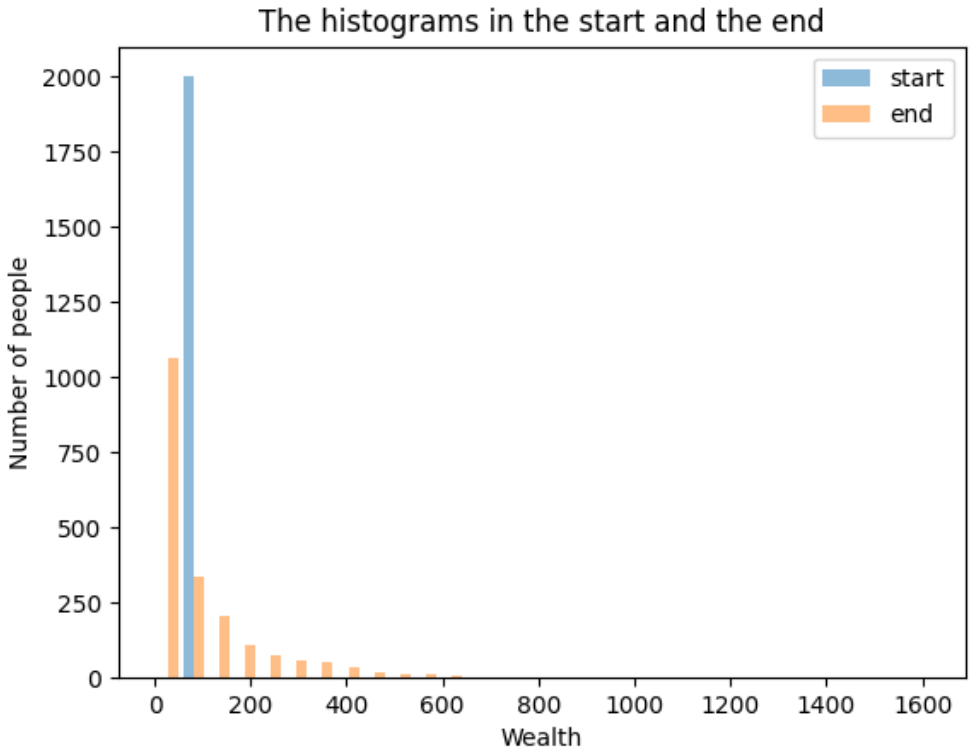
Equal population, size=2000, mean=100.0, simulating 20000 steps
Exchange strategy: winner takes random proportion of wealth from the loser

step	gini	std	1%	5%	25%	50%	75%	95%	99%
0	0.00	0.00	100	100	100	100	100	100	100
2000	0.41	74.62	0	3	41	100	139	237	320
4000	0.51	97.35	0	1	23	73	147	294	404
6000	0.56	113.12	0	1	17	62	140	323	490
8000	0.59	122.36	0	0	14	56	133	356	557
10000	0.61	133.63	0	0	13	51	132	357	592
12000	0.62	135.18	0	0	12	47	137	365	613
14000	0.63	134.91	0	0	11	45	140	382	597
16000	0.63	140.42	0	0	9	45	136	376	638
18000	0.63	138.91	0	0	9	46	136	372	609
20000	0.63	139.88	0	0	10	46	133	378	603

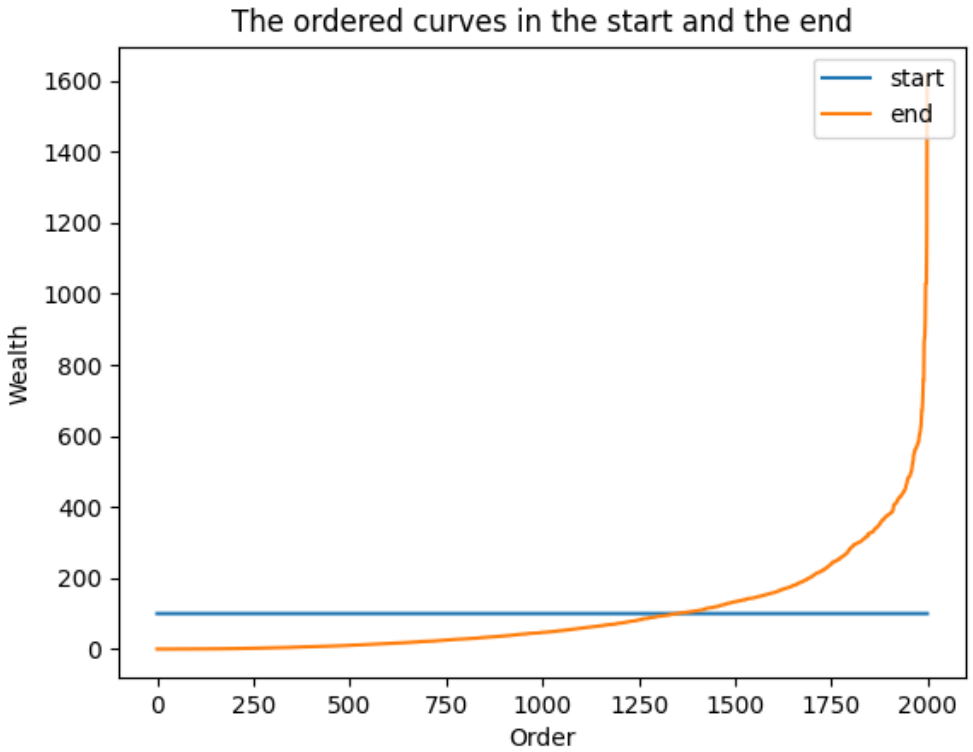
The Hunter-Gatherer Society



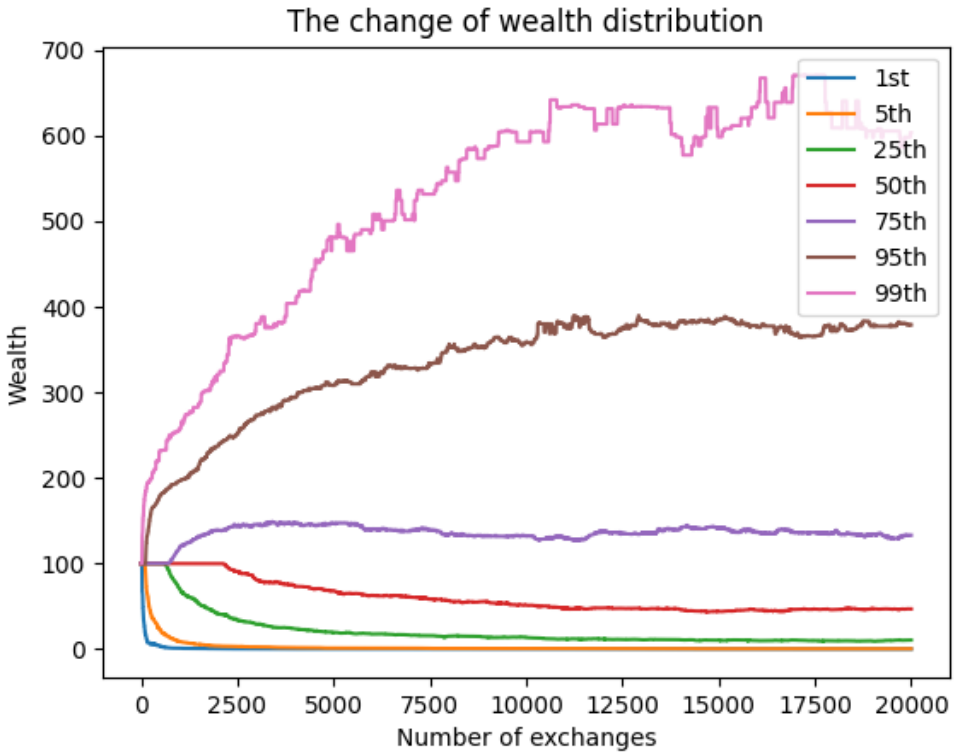
The Hunter-Gatherer Society



The Hunter-Gatherer Society



The Hunter-Gatherer Society



The Snowball Principle

Wealth confers on those who possess it the ability to extract wealth from others. So netting out each person's ability to do this in a general competition for surplus wealth, **the rich tend to take surplus away from the poor**.

Transaction Function #2

$$X'_A = X_A + dU \cdot X_B - (1-d)U \cdot X_A, \tag{3}$$
$$X'_B = X_B + (1-d)U \cdot X_A - dU \cdot X_B, \tag{4}$$

where

X_A, X'_A, X_B, and X'_B are as previously stated,

$$d = \begin{cases} 1, & \text{with probability } \delta \text{ if } X_A > X_B \text{ and } (1-\delta) \text{ otherwise,} \\ 0, & \text{otherwise,} \end{cases}$$

$U \sim \mathrm{Uniform}(0,1)$.

The Ranked Society

Equal population, size=2000, mean=100.0, simulating 20000 steps
Exchange strategy: winner takes random proportion of wealth from the loser
however, the richer party has 80% chance of winning

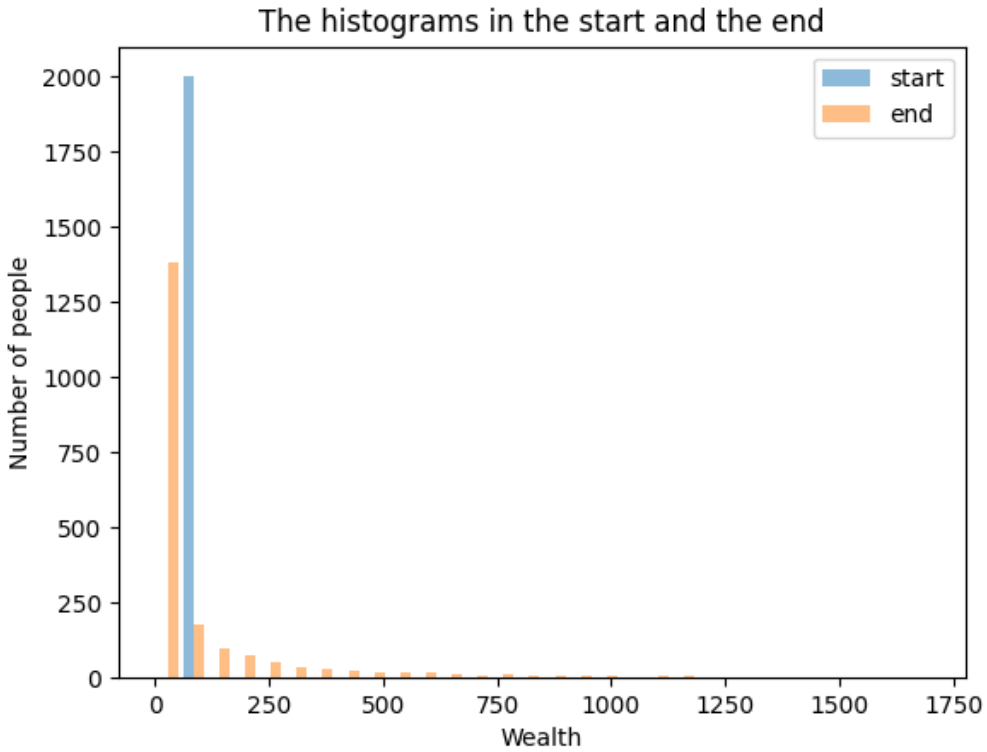
step	gini	std	1%	5%	25%	50%	75%	95%	99%
0	0.00	0.00	100	100	100	100	100	100	100
2000	0.43	80.77	0	2	31	100	139	257	356
4000	0.57	113.32	0	0	11	64	145	342	477
6000	0.64	136.25	0	0	5	42	145	363	632
8000	0.69	153.72	0	0	2	30	140	402	741
10000	0.72	166.94	0	0	1	22	125	449	776
12000	0.75	177.94	0	0	1	18	114	469	843
14000	0.76	185.20	0	0	0	14	104	502	831
16000	0.77	191.89	0	0	0	13	101	537	891
18000	0.78	201.22	0	0	0	12	95	550	957
20000	0.79	209.72	0	0	0	10	90	536	1065

The Ranked Society

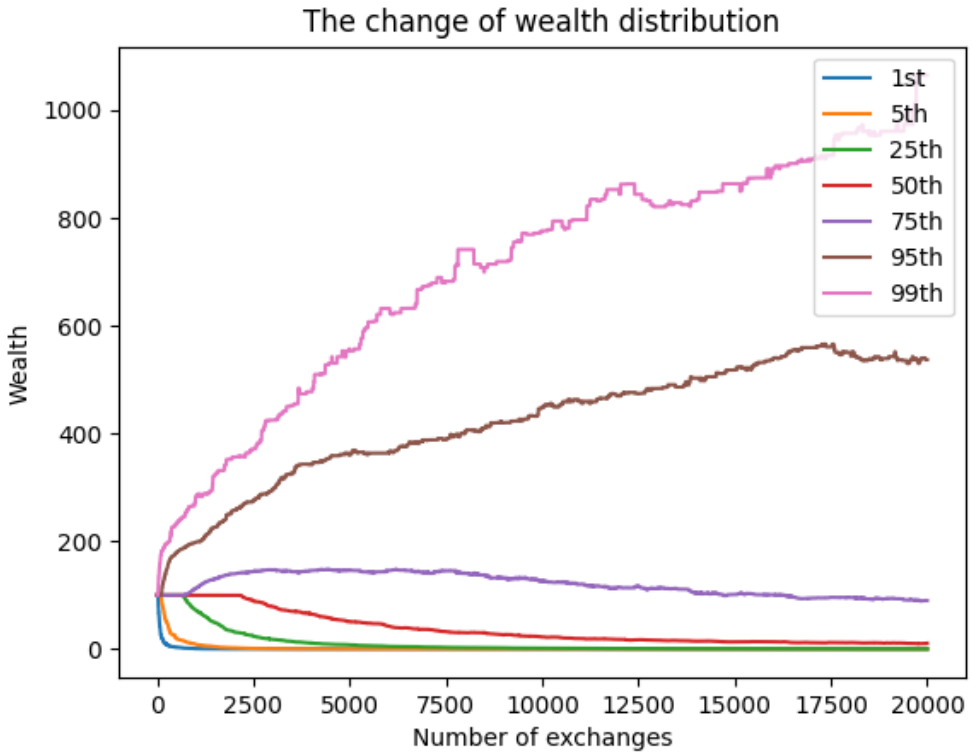
Equal population, size=2000, mean=100.0, simulating 20000 steps
Exchange strategy: winner takes random proportion of wealth from the loser
however, the richer party has 60% chance of winning

step	gini	std	1%	5%	25%	50%	75%	95%	99%
0	0.00	0.00	100	100	100	100	100	100	100
2000	0.42	78.41	0	3	39	100	136	249	363
4000	0.54	106.00	0	1	17	72	143	308	482
6000	0.60	122.39	0	0	11	56	144	346	565
8000	0.62	133.45	0	0	9	48	142	361	636
10000	0.64	136.92	0	0	7	42	138	389	619
12000	0.65	144.07	0	0	7	41	137	390	671
14000	0.67	148.14	0	0	6	39	131	416	681
16000	0.67	147.34	0	0	5	34	134	412	670
18000	0.68	153.62	0	0	5	35	125	416	711
20000	0.69	158.94	0	0	4	34	124	403	789

The Ranked Society (80% bias)



The Ranked Society (80% bias)



Problem

- Biased towards the poorer party?
- Equality or inequality?

The Resistance of Loss Principle

Surplus should be viewed as being **made up of layers** and that the top layers are **more fugitive**, more easily lost than the bottom layers, those close to the level of subsistence.

Transaction Function #3

$$X'_A = X_A + dZ \cdot X_B - (1-d)Z \cdot X_A, \tag{5}$$
$$X'_B = X_B + (1-d)Z \cdot X_A - dZ \cdot X_B, \tag{6}$$

where

X_A, X'_A, X_B, X'_B, and d are as previously stated,

$$Z = \sum_{k=1}^{l} \frac{U_k^k}{l}, \quad \text{with } U_k \sim \mathbf{Uniform}(0,1), k = 1, \cdots, n.$$

The Industrial Society

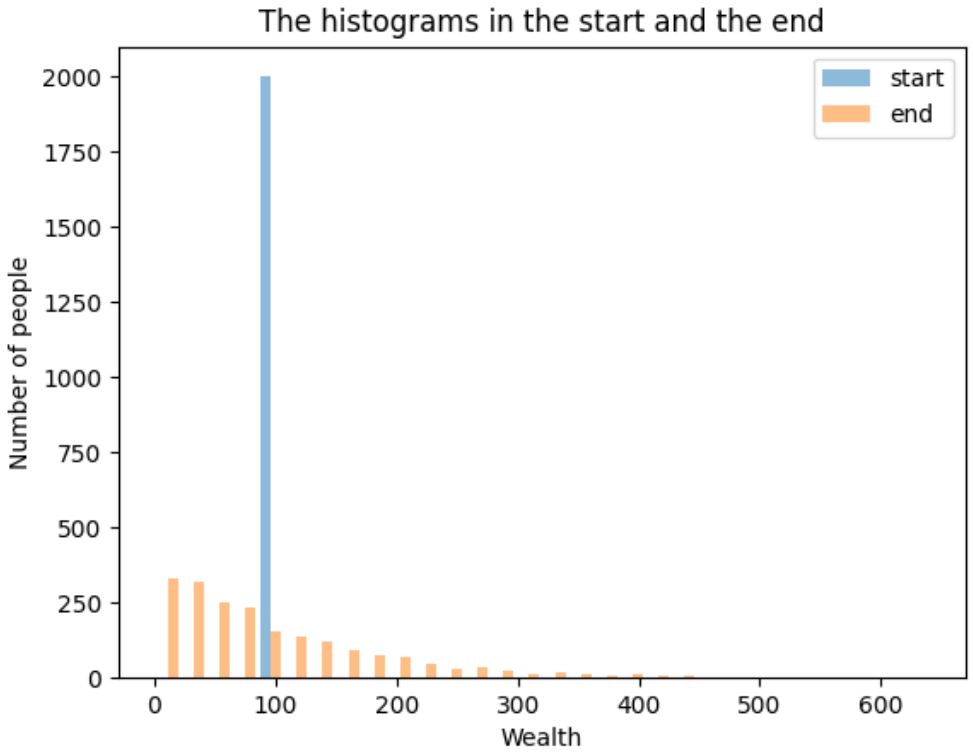
Lenski Hypothesis

With the evolution of the industrial society, there will be more layers with increasing resistance to loss, and the total expectation of loss should thus drop.

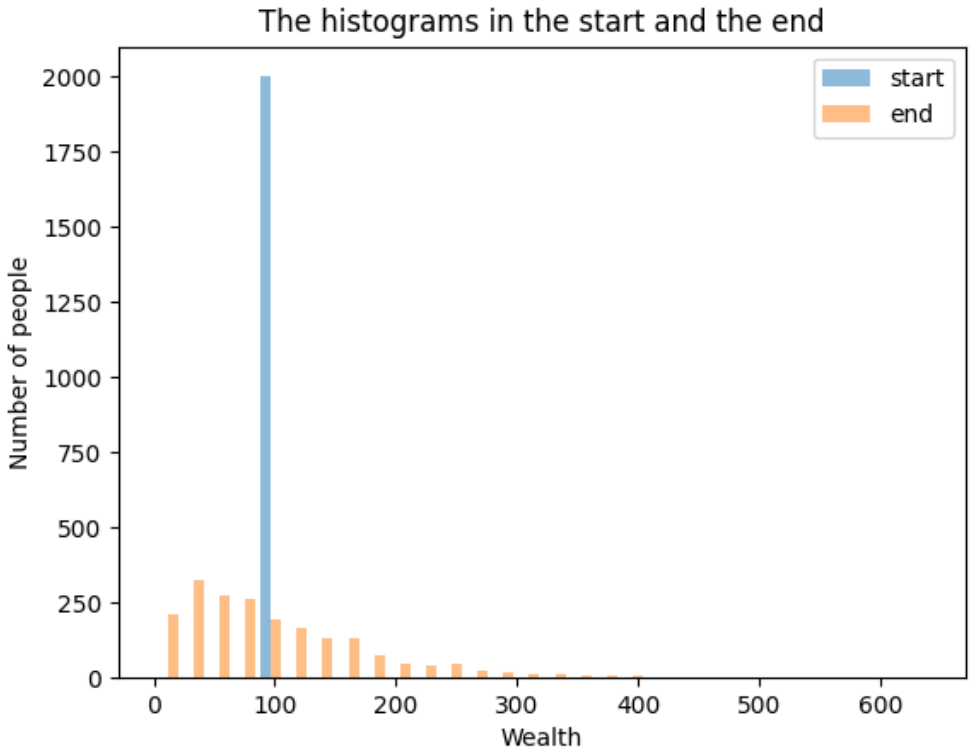
Indeed, we can compute that

$$\mathbb{E}(Z) = \frac{1}{l}\sum_{k=1}^{l}\mathbb{E}(U_k^k) = \frac{1}{l}\sum_{k=1}^{l}\int_0^1 x^k dx = \frac{1}{l}\sum_{k=1}^{l}\frac{1}{k+1}.$$

The Industrial Society (2 layers)



The Industrial Society (5 layers)



Transaction Function #4

$$X'_A = X_A + d \cdot (ZX_B - \texttt{tax}(ZX_B)) - (1 - d) \cdot ZX_A, \tag{7}$$
$$X'_B = X_B + (1 - d) \cdot (ZX_A - \texttt{tax}(ZX_A)) - d \cdot ZX_B, \tag{8}$$

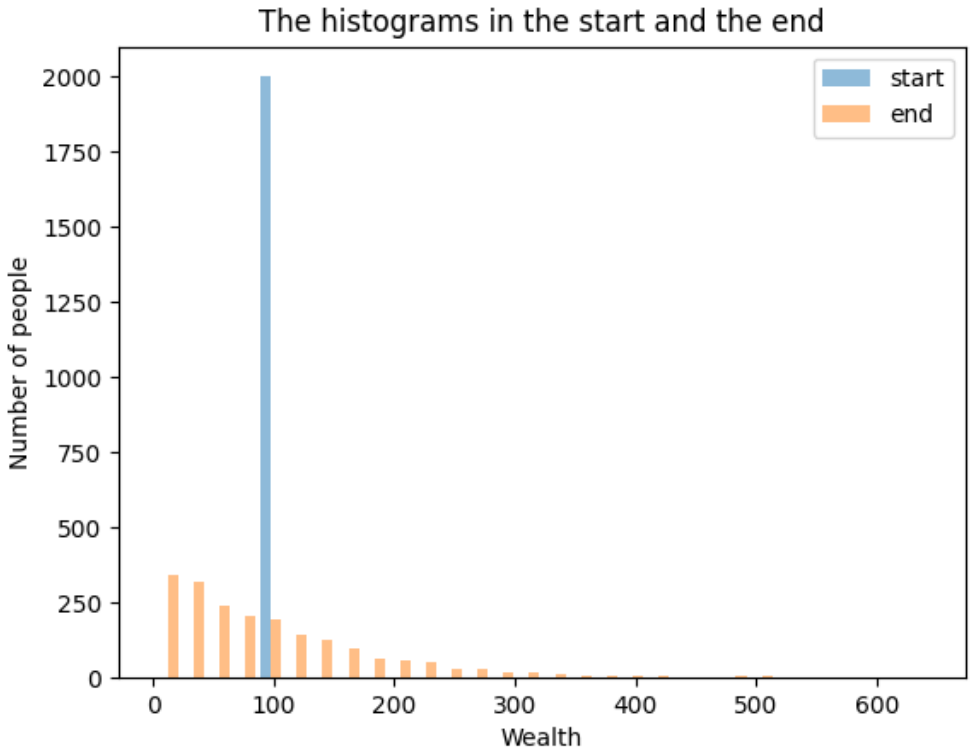
where

X_A, X'_A, X_B, X'_B, d, and Z are as previously stated,

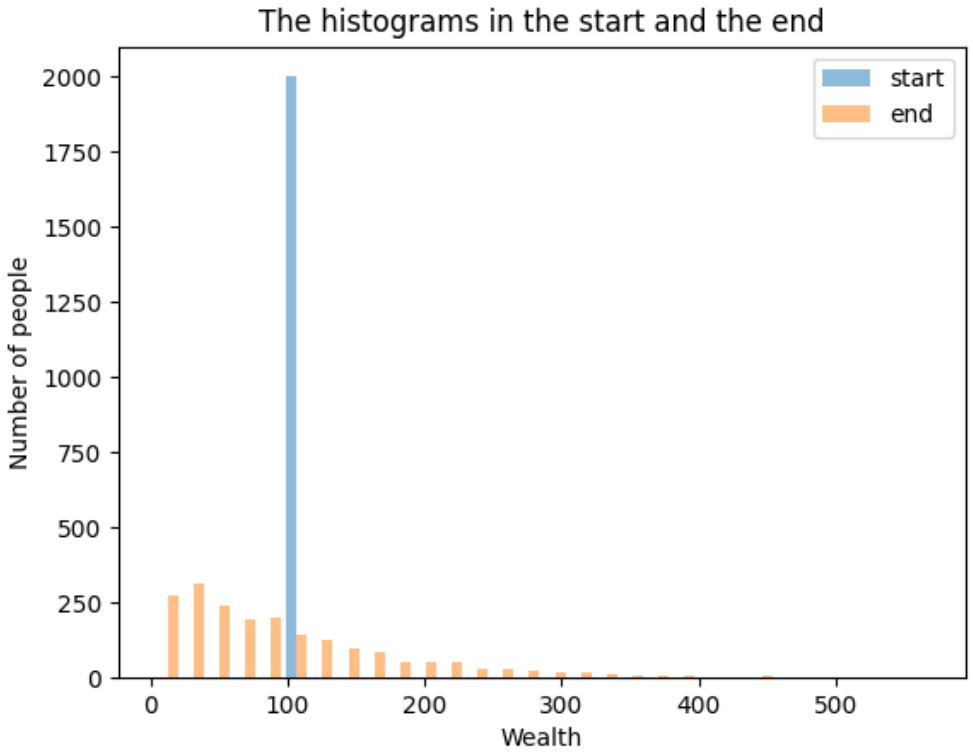
$\texttt{tax}$: transaction amount $\mapsto$ tax amount.

Note that the tax will be later **evenly distributed across the total population**.

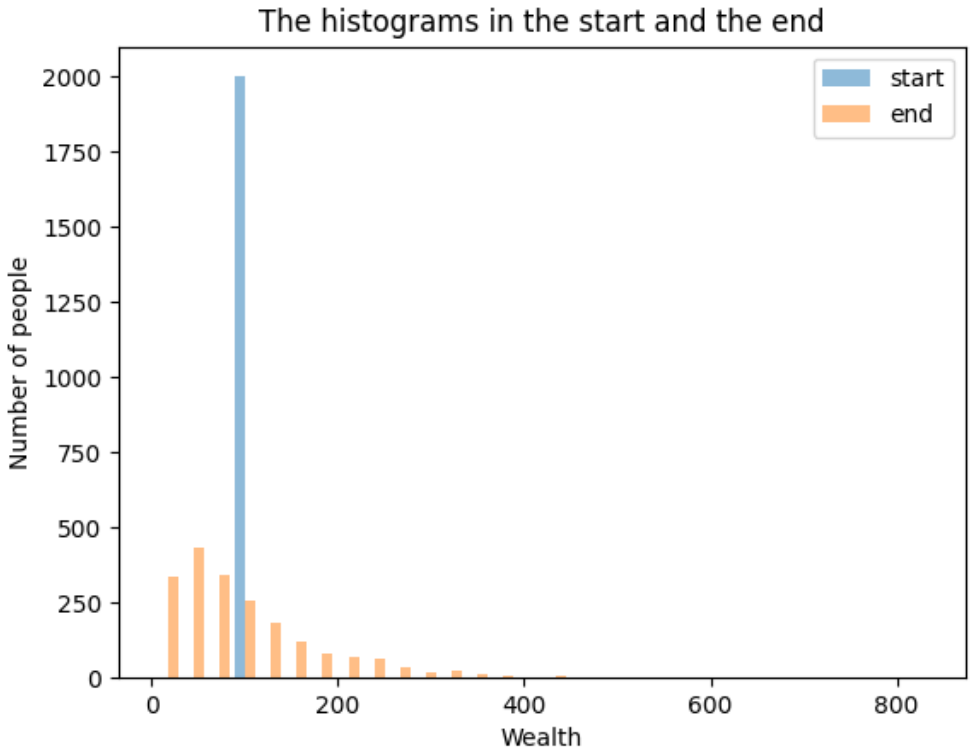
The Modern Society (3% tax)



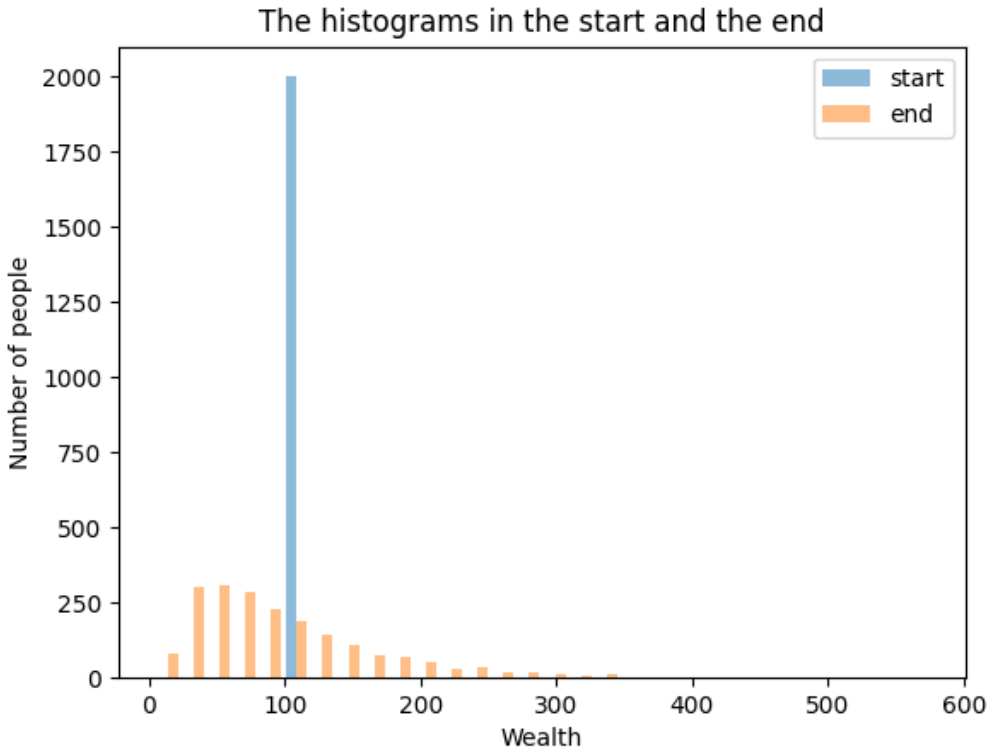
The Modern Society (10% tax)



The Modern Society (20% tax)



The Modern Society (45% tax)



Distribution Fitting

- Take the normalized histogram
- For each distribution, maximize its likelihood function $\mathcal{L}(\theta|x)$

Algorithm

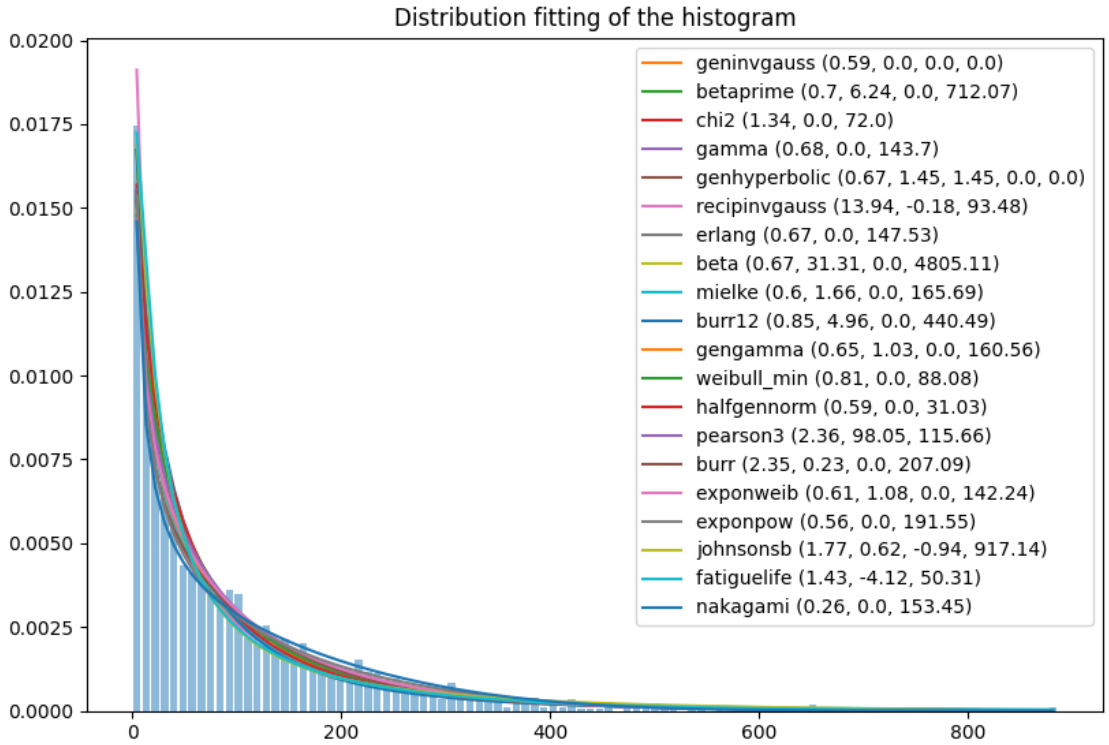
$$\hat{\theta} = \arg\max_{\theta} f(\mathbf{x}; \theta) = \arg\min_{\theta} \left(-\sum_{i=1}^{n} \log f(x_i, \theta) \right), \tag{9}$$

where f is the probability density function, and $\mathbf{x}$ is the random vector as a sample of n independent random variables with this probability density function.

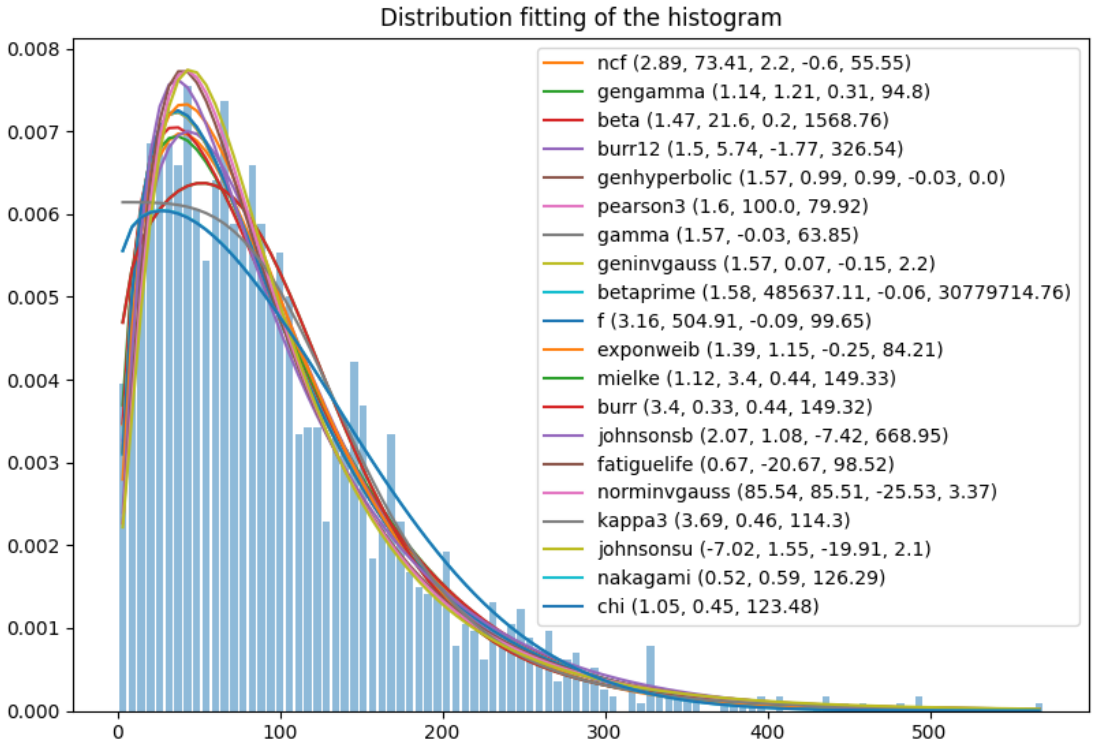
Distribution Fitting

- Take all distributions from the `scipy.stats` library
- Compute the mean-squared error and perform the Kolmogorov-Smirnov test
- Pick the top 20 fits according to MSE

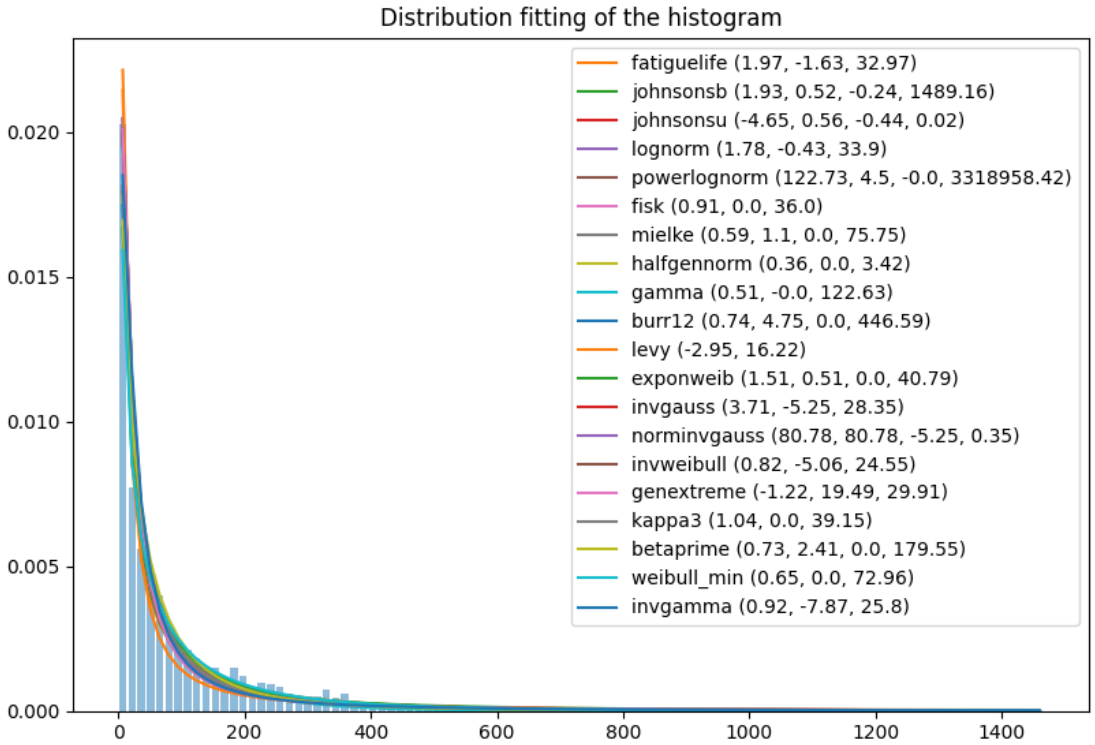
Distribution Fitting (unbiased, single layer)



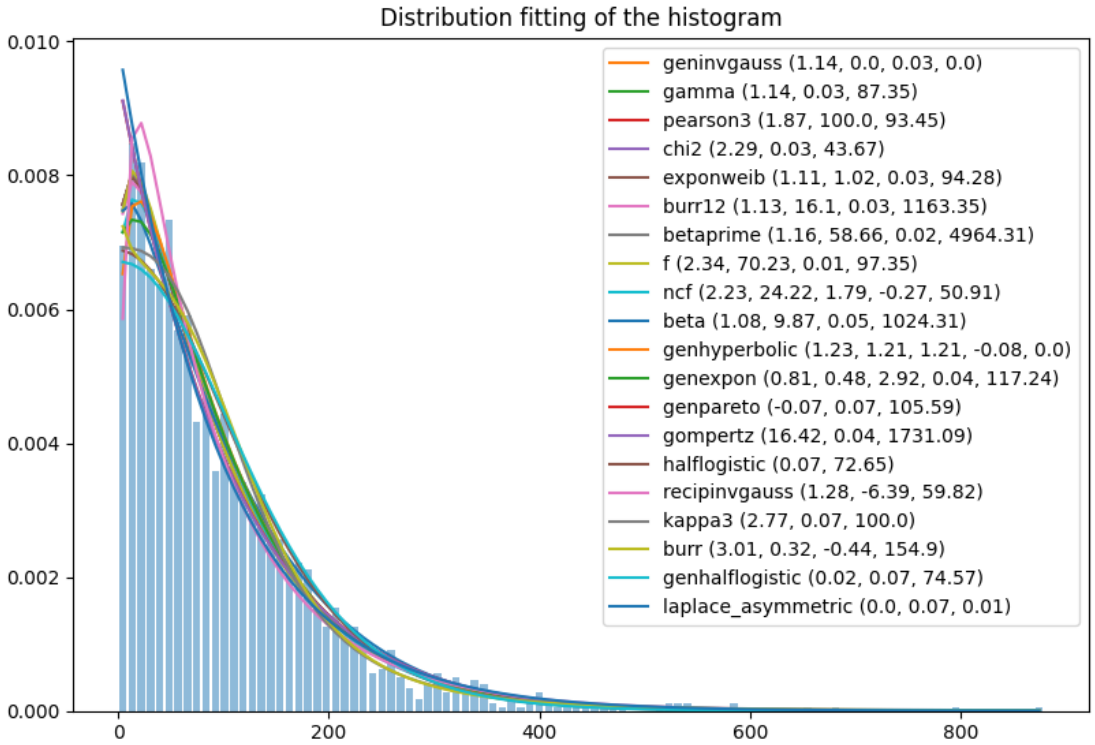
Distribution Fitting (unbiased, 5 layers)



Distribution Fitting (60% biased, single layer)



Distribution Fitting (60% biased, 5 layers)



Distribution Fitting (unbiased, single layer)

```
Fitting with transaction function win_take_layer, simulating 200000 steps
Testing on equal population of size=2000, mean=100.00, transaction bias=60%, layers=5
+-------------------+----------+----------+----------+
|       distr       |   MSE    | KS-stat  | KS-pval  |
+-------------------+----------+----------+----------+
|    geninvgauss    | 5.51E-06 | 1.16E-02 | 9.47E-01 |
|       gamma       | 5.51E-06 | 1.16E-02 | 9.47E-01 |
|      pearson3     | 5.51E-06 | 1.16E-02 | 9.47E-01 |
|        chi2       | 5.51E-06 | 1.16E-02 | 9.47E-01 |
|     exponweib     | 5.51E-06 | 1.14E-02 | 9.55E-01 |
|       burr12      | 5.55E-06 | 1.29E-02 | 8.88E-01 |
|     betaprime     | 5.56E-06 | 1.41E-02 | 8.18E-01 |
|         f         | 5.62E-06 | 1.52E-02 | 7.39E-01 |
|        ncf        | 5.94E-06 | 1.31E-02 | 8.80E-01 |
|        beta       | 6.24E-06 | 2.33E-02 | 2.24E-01 |
+-------------------+----------+----------+----------+
```

Validation

- Real-world economic system
  Stochastic simulation of economic exchange
- Gamma distribution
- Pareto distribution

Remark

The fitting results are algorithm-dependent, but should not affect the final conclusion. The maximum likelihood estimation may detect some nice fits while missing others, and the same can happen to other methods (*e.g.*, the method of moments estimation).

Taxes on Personal Income

People's Republic of China

Annual taxable income (CNY)	Tax rate (%)	Quick deduction (CNY)
0 to 36,000	3	0
Over 36,000 to 144,000	10	2,520
Over 144,000 to 300,000	20	16,920
Over 300,000 to 420,000	25	31,920
Over 420,000 to 660,000	30	52,920
Over 660,000 to 960,000	35	85,920
Over 960,000	45	181,920

Taxes on Personal Income

Design

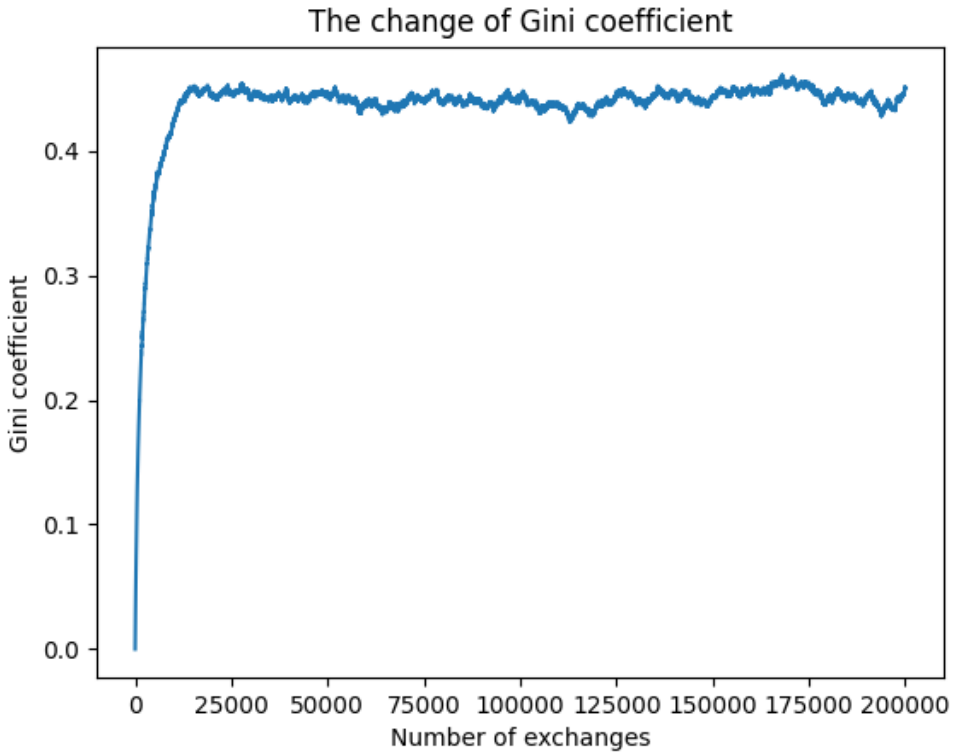
below 0.15 times initial mean	3%
0.15 to 0.50 times initial mean	10%
0.50 to 1.04 times initial mean	20%
1.04 to 1.96 times initial mean	25%
1.96 to 2.29 times initial mean	30%
2.29 to 3.33 times initial mean	35%
above 3.33 times initial mean	45%

Simulation of Real-World Economy

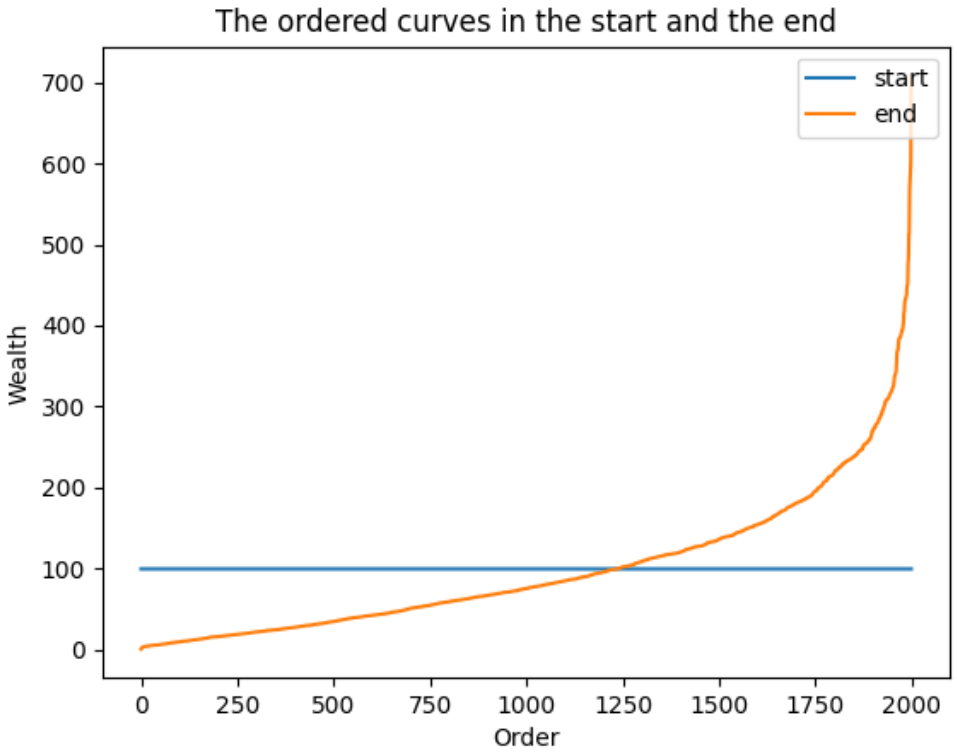
Equal population, size=2000, mean=100.00, simulating 200000 steps
Exchange strategy: winner takes some proportion of wealth from the loser (taxed)
with the loser resisting the loss of wealth at Lvl. 5
the richer party has 60% chance of winning

step	gini	std	1%	5%	25%	50%	75%	95%	99%
0	0.00	0.00	100	100	100	100	100	100	100
20000	0.45	85.33	5	11	35	77	136	271	390
40000	0.44	84.68	5	11	36	76	138	268	396
60000	0.44	85.21	5	12	37	75	137	260	400
80000	0.44	85.50	5	12	37	74	136	271	401
100000	0.44	85.67	5	11	37	77	134	276	387
120000	0.44	85.41	5	12	39	76	133	280	407
140000	0.45	87.31	4	10	36	74	136	269	414
160000	0.45	86.74	5	11	35	74	139	275	407
180000	0.44	86.12	5	11	37	74	137	270	399
200000	0.45	87.59	4	10	35	75	136	270	404

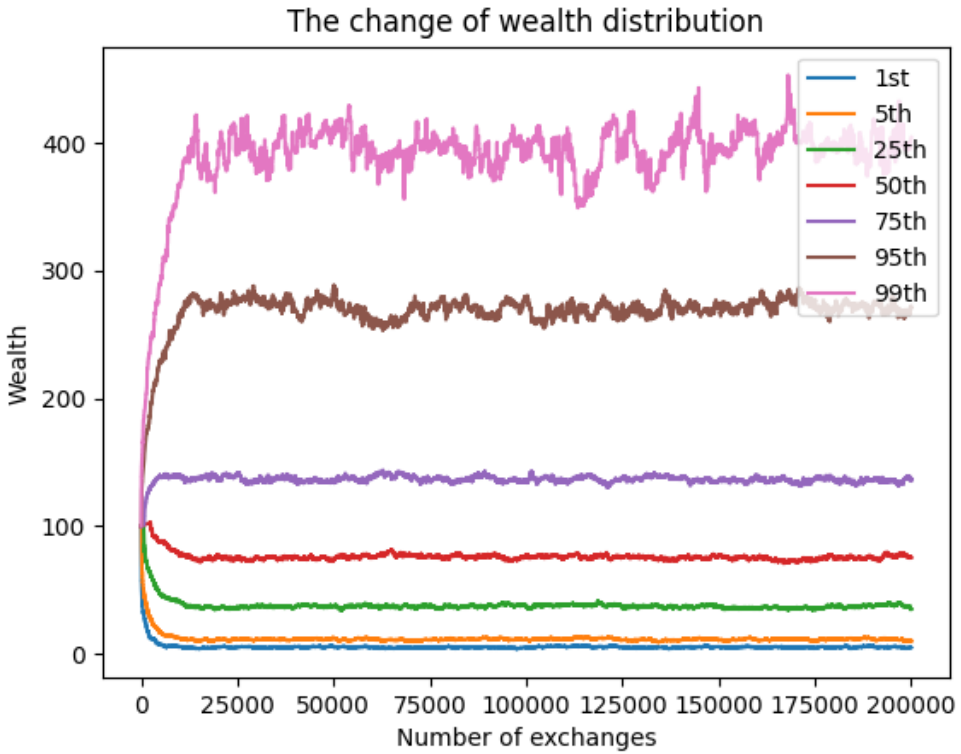
Simulation of Real-World Economy



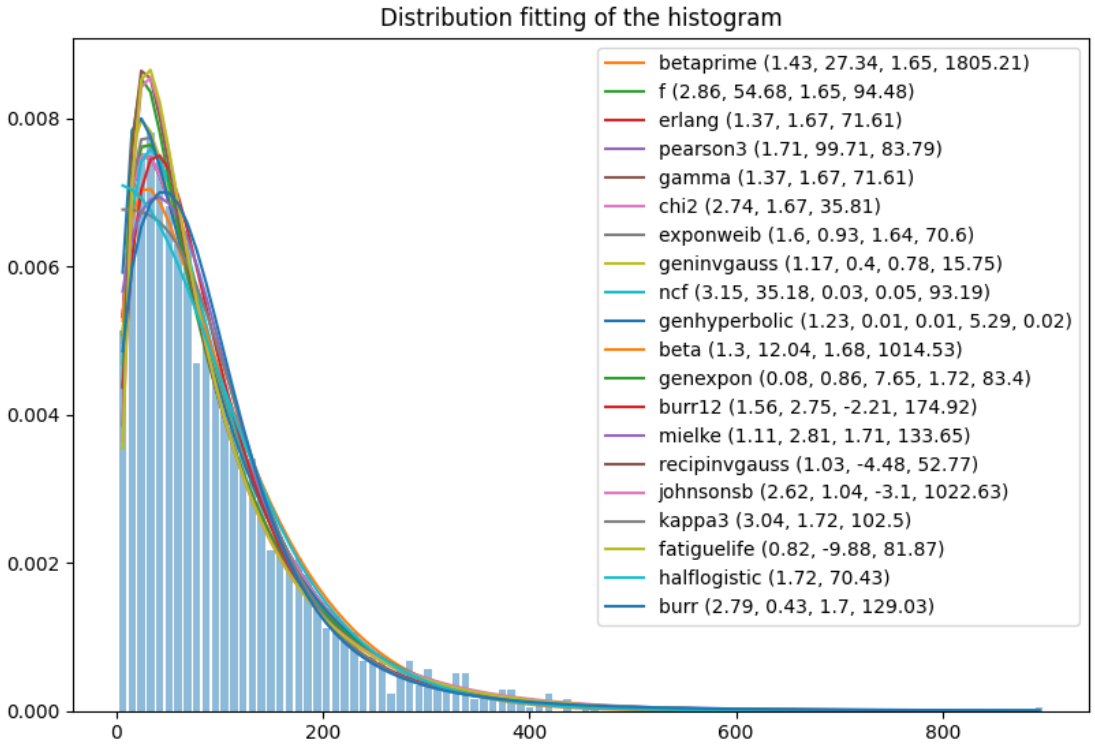
Simulation of Real-World Economy



Simulation of Real-World Economy



Simulation of Real-World Economy



Validation

- The Gini coefficient of China has been around 0.43—0.49 for years
- Recent research on economic exchange
- The beta prime distribution is suitable for describing real-world wealth distribution, given that it can mimic various behaviors (*e.g.*, exponential behavior for both large and small values).

distr	MSE	KS-stat	KS-pval
betaprime	2.04E-06	1.01E-02	9.85E-01

Concluding Remarks

Thanks for listening!