MTH101: Lecture 14

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Improper Real Integrals

By using Residue Theorem it is possible to Compute Real improper Integrals.

We will consider an example of the application of Residues Theory.

Example

Compute the following Integral

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx := \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4} dx$$

Step 1:
Consider the complex version of the function $f(x)$:

$$f(x) = \frac{1}{1+x^4} \quad \Longrightarrow \quad f(z) = \frac{1}{1+z^4}$$

and find its singular points, that is, the points such that $z^4 = -1$:

$$z_1 = e^{i\frac{\pi}{4}}, z_2 = e^{i\frac{3\pi}{4}}, z_3 = e^{-i\frac{3\pi}{4}}, z_4 = e^{-i\frac{\pi}{4}}.$$

We observe that z_1 and z_2 are in the upper half plane while z_3 and z_4 are in the lower half plane.

Step 2:

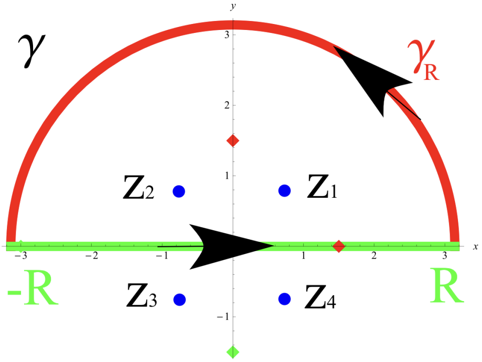
Consider the simple closed path $\gamma = [-R, R] \cup \gamma_R$ with counterclockwise orientation, where:

$[-R, R]$ is the segment on the real axis with extremes $-R$ and R. Its parametrization is

$$z(t) = t, \quad t \in [-R, R], \quad \text{and } z'(t) = 1, \quad t \in [-R, R].$$

γ_R is the upper Semicircle with center $z_0 = 0$ and Radius R. Its parametrization is

$$z(t) = R \cdot e^{it}, \quad t \in [0, \pi], \quad \text{and } z'(t) = iR \cdot e^{it}, \quad t \in [0, \pi].$$



Note: R must be chosen big enough such that z_1 and z_2 are inside γ. In this case $R > 1$ since $|z_1| = |z_2| = 1$.

Step 3:

Use the **Theorem of Residues**:

$$\oint_{\gamma} \frac{1}{1+z^4}dz = \int_{\gamma_R} \frac{1}{1+z^4}dz + \int_{[-R,R]} \frac{1}{1+z^4}dz = 2\pi i[\mathrm{Res}_{z_1}(f)+\mathrm{Res}_{z_2}(f)$$

Compute the Residues, we use that

$$f(z) = \frac{p(z)}{q(z)}, \quad \text{where } p(z) = 1,\ q(z) = z^4 + 1.$$

We observe that $p(z_1), p(z_2) \neq 0$ and $q(z)$ has a simple zero at z_1 and at z_2, then

$$\mathrm{Res}_{z_1} f(z) = \frac{p(z_1)}{q'(z_1)} = \frac{1}{4z_1^3} = \frac{e^{-i\frac{3\pi}{4}}}{4},$$

$$\mathrm{Res}_{z_2} f(z) = \frac{p(z_2)}{q'(z_2)} = \frac{1}{4z_2^3} = \frac{e^{-i\frac{\pi}{4}}}{4}.$$

Then

$$\begin{aligned}\oint_{\gamma} \frac{1}{1+z^4}\,dz &= \int_{\gamma_R} \frac{1}{1+z^4}\,dz + \int_{[-R,R]} \frac{1}{1+z^4} \\ &= 2\pi i[\mathrm{Res}_{z_1}(f) + \mathrm{Res}_{z_2}(f)] \\ &= 2\pi i\left[\frac{e^{-i\frac{3\pi}{4}}}{4} + \frac{e^{-i\frac{\pi}{4}}}{4}\right] \\ &= \frac{\pi}{\sqrt{2}}.\end{aligned}$$

Step 4: We observe that using the parametrization of $[-R, R]$ we have:

$$\int_{[-R,R]} \frac{1}{1+z^4} dz = \int_{-R}^{R} \frac{1}{1+t^4} \cdot 1 \; dt$$

Then

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{1+x^4} \; dx :&= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4} dx \\ &= \frac{\pi}{\sqrt{2}} - \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{1+z^4} dz \end{aligned}$$

Step 5:

Prove that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{1}{1+z^4} dz = 0.$$

We use the ML- inequality with $L = \pi R$ and $M(R)$ such that

$$\left| \frac{1}{1+z^4} \right| = \frac{1}{|z^4+1|} \leq M(R), \quad \text{on } \gamma_R,$$

and

$$\lim_{R\to\infty} \left| \oint_{\gamma_R} \frac{1}{1+z^4} dz \right| \leq \lim_{R\to\infty} M(R)L.$$

In order to find $M(R)$ we use the Triangle inequality:

$$|a| - |b| \leq |a + b| \leq |a| + |b|,$$

from which

$$|z^4| - 1 \leq |z^4 + 1|.$$

Then

$$\frac{1}{|1 + z^4|} \leq \frac{1}{|z|^4 - 1} = \frac{1}{R^4 - 1} = M(R) \quad \text{on } \gamma_R.$$

Finally, from ML-inequality we get

$$0 \leq \lim_{R\to\infty} \left| \int_{\gamma_R} \frac{1}{1 + z^4} dz \right| \leq \lim_{R\to\infty} \frac{R\pi}{R^4 - 1} = 0.$$

We conclude that

$$\lim_{R\to\infty}\int_{\gamma_R}\frac{1}{1+z^4}dz=0$$

from which

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{1+x^4}dx &:= \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{1+x^4}dx\\ &= \frac{\pi}{\sqrt{2}}-\lim_{R\to\infty}\int_{\gamma_R}\frac{1}{1+z^4}dz\\ &= \frac{\pi}{\sqrt{2}}.\end{aligned}$$

Introduction to Ordinary Differential Equations

Definition

Most of the natural phenomenon involving rates of change (derivatives), e.g. the motion of fluids, the flow of current in electric circuits, the dissipation of heat in solid objects, etc., can be expressed in mathematical terms. Equation containing derivatives are **differential equations**. A differential equation that describes some physical process is often called a **Mathematical Model** of the process.

Classification of Differential Equations

Definition

If only ordinary derivatives appear in the differential equation, it is called an **ordinary differential equation**, if the derivatives are partial derivatives, the equation is called a **partial differential equation**.

For example, the equation

$$L\frac{d^2\,Q(t)}{d\,t^2}+R\frac{d\,Q(t)}{d\,t}+\frac{1}{C}Q(t)=E(t)$$

is an **ODE**; and the equation

$$a^2\frac{\partial^2\,u(x,t)}{\partial\,x^2}=\frac{\partial^2\,u(x,t)}{\partial\,t^2}$$

is a **PDE**.

Classification of Differential Equations

Definition

The **order** *of a differential equation is the the order of the highest derivative that appears in the equation. The equation*

$$F(t, u(t), u'(t), ..., u^{(n)}(t)) = 0$$

is called an **ordinary differential equation of the n-the order**.

For example,

$$y''' + 2y'' + yy' = t^4$$

is a third order differential equation for $y = u(t)$.

Classification of Differential Equations

Definition

The Ordinary differential equation $F(t, y, y', ..., y^{(n)})$ is said to be **linear** *if F is a linear function of the variables $y, y', ..., y^{(n)}$, otherwise, the equation is* **nonlinear***; a similar definition applies to PDEs. The general* **linear ordinary equation of order n** *is*

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t).$$

For example, $y'' + y = t$ is **liner** and $y''' + 2y'' + yy' = t^4$ is **nonlinear**.

Solutions to First order ordinary differential equations

Case 1: **Homogeneous Linear ODEs**

$$y' = p(x)y \quad \Rightarrow \quad y = ce^{\int p(x)\ dx};$$

Case 2: **Nonhomogeneous Linear ODEs**:

$$y' + p(x)y = r(x) \quad \Rightarrow \quad y = e^{-\int p(x)\ dx}\left(\int e^{\int p(x)\ dx} r(x)\ dx + c\right)$$

Case 3: **Nonlinear ODEs** are relatively complicated, however, people have generalized several tricks to solve certain types of nonlinear first order ordinary differential equations.

- Separable ODEs:

$$g(y)y' = f(x) \quad \Rightarrow \quad \int g(y)\, dy = \int f(x)\, dx + C$$

  (See section 1.3 for details)

- Exact ODEs When $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$, we have

$$M(x,y)dx + N(x,y)dy = 0 \quad \Rightarrow \quad u(x,y) = c$$

  See section 1.4 for details.

- Nonlinear ODEs which can be reduced to linear ODEs(Bernoulli Equation)

$$y' + p(x)y = g(x)y^a \quad \underset{\text{let } u=y^{1-a}}{\Rightarrow} \quad u' + (1-a)pu = (1-a)g$$

Solutions to Second order Homogeneous Linear ODEs with Constant Coefficients

$$ay''+by'+cy=0 \quad \underset{\text{let } y=e^{rt}}{\Longrightarrow} \quad ar^2+br+c=0 \quad \text{(Characteristic equation)}$$

- If $\Delta = b^2 - 4ac > 0$, then $r = r_1,\ r_2$, and
$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
- If $\Delta = b^2 - 4ac = 0$, then it has one repeated root r and
$$y = c_1 e^{rt} + c_2 t e^{rt}$$
- If $\Delta = b^2 - 4ac < 0$, then $r = \alpha \pm \beta i$ and
$$y = c_1 e^{\alpha t} \sin \beta t + c_2 e^{\alpha t} \cos \beta t$$

This is frequently used to describe the oscillations of Mass-Spring system:

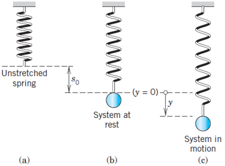
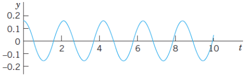


Fig. 33. Mechanical mass–spring system

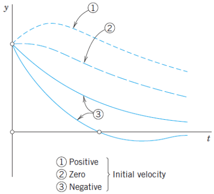
$$my''(\text{inertial}) + cy'(\text{damping}) + ky(\text{stiffness}) = 0$$

Assuming that both the mass m and the stiffness k are positve.

- $c = 0$ (Undamped Systems): $y = c_1 \cos \beta t + c_2 \sin \beta t$;

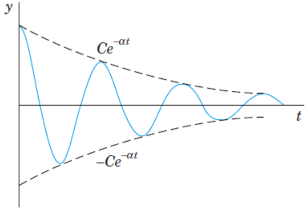


- $c^2 - 4mk > 0$ (over-damping): $y = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$;
- $c^2 - 4mk = 0$ (critical damping): $y = (c_1 + c_2 t)e^{-rt}$;



- $c^2 - 4mk < 0$ (under-damping):

$$y = c_1 e^{-\alpha t} \cos \beta t + c_2 e^{-\alpha t} \sin \beta t$$



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