THE UNIVERSITY OF HONG KONG
DEPARTMENT OF PHYSICS

Assignment 2

Course: *Computational Physics (PHYS4150/8150)* – Professor: *Prof. Zi Yang Meng*
Tutor: *Mr. Min Long, Mr. Tim Lok Chau*
Due date: *Oct. 21st, 2024 (Monday)*

This assignment is also a project, you will need to form group of 2 to 4 people and present your work on these questions, during the classes on Oct. 21st and 24th. The presentation should contain your answer to the questions, as well as how you solve the problems and your understanding of them.

1. Soliton in pendulum
   Soliton is a special solution of the equation of motion initially observed in shallow water as shown in Fig.1. In mathematics and physics, a soliton is a nonlinear, self-reinforcing, localized wave packet that is strongly stable. It preserves its shape while propagating freely, at a constant velocity, and recovers it even after collisions with other such localized wave packets.

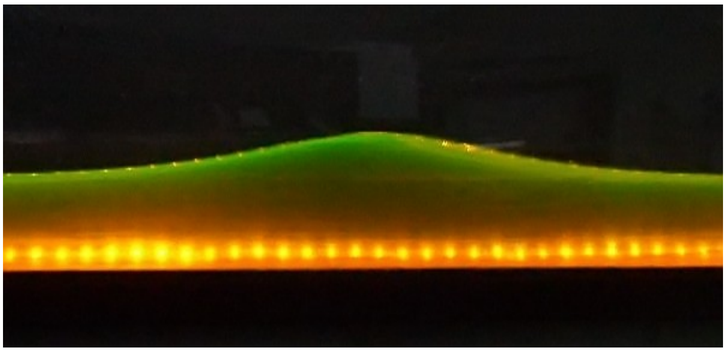

Figure 1: Soliton in shallow water, the wave packet will preserve its shape during the propagation. The figure comes from WiKi.

   The soliton has a quantum counterpart called instanton, which plays an important role in the modern context of quantum field theory. For more background of Soliton and Instanton, please refer to the links to Wikipedia.

   In this question, we study the soliton in pendulum.

   We define the potential felt by a pendulum as $-\cos\theta$, where $\theta \in [-\pi/2, \pi/2]$, and in this way, the equation of motion becomes:

$$\ddot{\theta} = k \sin\theta$$

   where k refers to the acceleration of gravity. Note the angle between the vertical axis and the pendulum has been shifted by π.

Solve the equation using **4th order Runge–Kutta method**. Numerically simulate the motion of a pendulum with 3 **different initial conditions** ($\theta_0 = 10^{-1}, 10^{-3}, 10^{-6}, \dot{\theta}_0 =$ 0 **and** $k = 4 \times 10^{-4}$**)**. The recommended **time step is** 0.1.. The reference for the motion of pendulum for one sets of parameters is shown in Fig. 2.

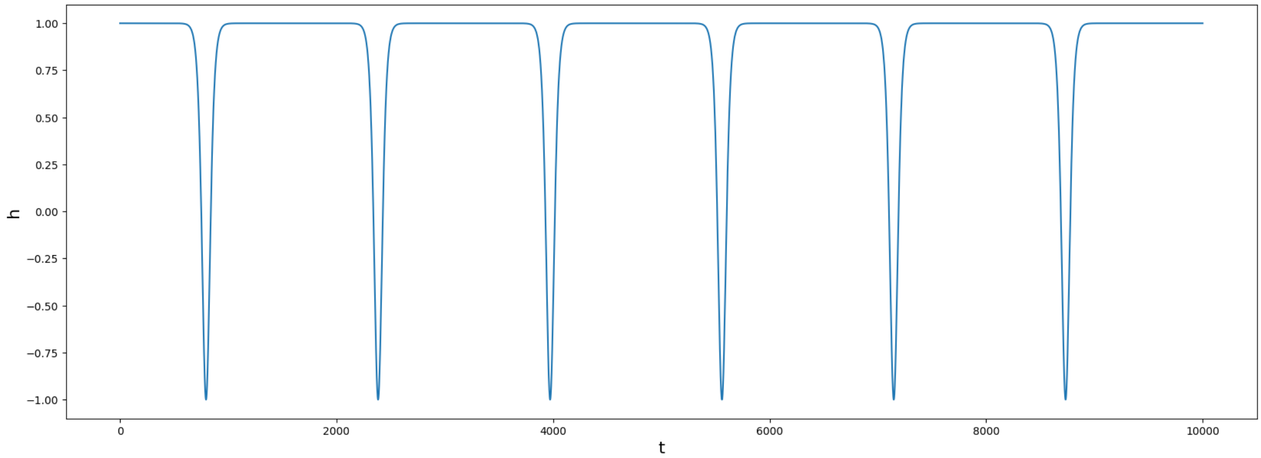

Figure 2: Soliton solution for $k = 1 \times 10^{-4}, \theta_0 = 1 \times 10^{-6}$, where $h = \cos\theta$ is the height of pendulum.

Explain the meaning of your results and how such behavior relates to the concept of soliton discussed at the beginning of this question.

2. Poisson Equation
In the lectures, we have learned to use three different relaxation methods to solve Poisson Equation $\Delta\phi(\vec{x}) = -\frac{1}{\epsilon_0}\rho(\vec{x})$ with Dirichlet boundary conditions.
In this question, the error bound is set to be 10^{-4}, h and ϵ_0 are set to be 1.
We consider the Dirichlet boundary condition, $\phi(\vec{x})|_{boundary} = 0$ on the 2D plane of two configurations. Using **Jacobi Relaxation, Gauss-Seidel Relaxation and Successive Overrelaxation scheme**, compare the numerical result on the numbers of iterations for each method to the theoretical prediction on computational complexity introduced in the lecture notes (Chap1_3). **Comment on the leading order, as well as its coefficient.**

   (a) We consider a **ring** charge distribution with **outer radius N/2 and inner radius N/4 centers on (N,N)** , as shown in Figure 3. The points **in the ring (i.e. we do not account for the points on the boundary) have charge 1** , while those outside have no charge. The plane is a square lattice with size (2N+1)×(2N+1), one clearly sees that this makes the circle lies on the center of the meshgrid. **Make a plot of its potential, as shown in Figure 3. Please consider N = 12, 32, 40 for this configuration.**

   (b) We consider a **Star of David** centers on a N×N square lattice, as shown in Figure 4. The **Star of David** can be view as a overlap of two equilateral triangles whose centers of gravity both lie on (N/2, N/2) and put on opposite direction, an example is shown in Figure 2. **The side length of the equilateral triangles is 0.6N. Make a plot of its potential, as shown in Figure 4. Please consider N = 20, 40, 60 for this configuration.** We have no specific requirement for the relative orientation of the star of David.

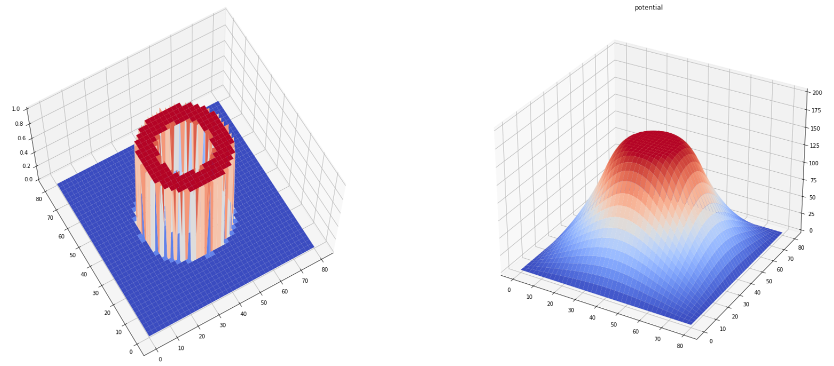

Figure 3: Ring charge distribution and the potential.

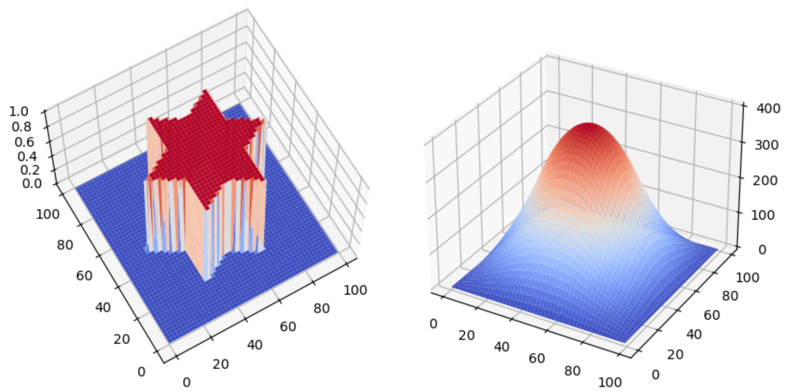

Figure 4: Star of David charge distribution and potential.

3. Perform the von Neumann stability analysis on FTCS (forward time derivative&centerd space derivative), FTFS, CTCS (which is the Leap-Frog method), and Lax-Wendroff schemes for advection equation $\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$ respectively. Discuss the stability for each scheme as shown in the lecture. You need to show the complete analysis process.

4. Consider again the problem of traffic flow. In this question, we simplify the flux as $F(x) = C * \rho(x)$ so we get an advection function $\frac{\partial \rho}{\partial t} = -C\frac{\partial \rho}{\partial x}$. We set $L = 400$ and $C = 1$. Use **Lax-Wendroff method** to see the different directions of flow for these initial conditions (the animation is needed for display). Figure 5 is the snapshot of $\rho(t = 0, x)$ of parts a) and b).

   (a) $\rho(t = 0, x) = \rho_0(x) = \begin{cases} (x - \frac{L}{2})(x + \frac{L}{2})\frac{4}{L^2}, & \frac{L}{2} < x < L \\ 0, & \text{else .} \end{cases}$,

   (b) $\rho(t = 0, x) = \rho_0(x) = \begin{cases} (\frac{L}{4} - |x - \frac{L}{2}|)\frac{4}{L}, & \frac{L}{4} < x < \frac{3L}{4} \\ 0, & \text{else .} \end{cases}$,

   (c) Set $C = -1$ and use the $\rho(t = 0, x)$ in part a), check the moving direction of the traffic flow

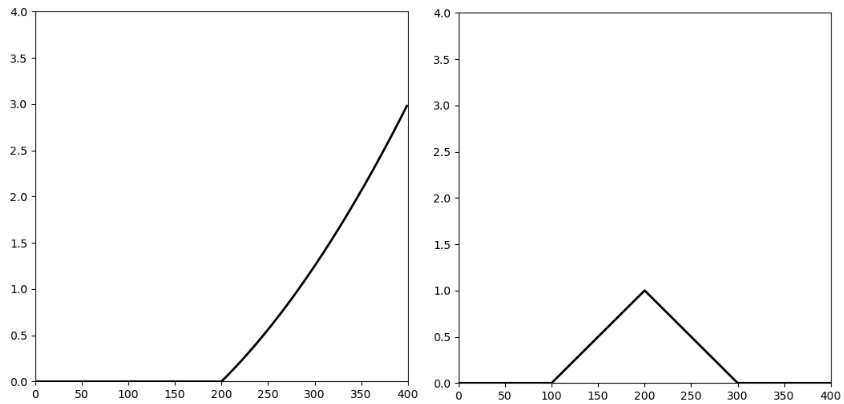

Figure 5: Initial condition of a) and b).