Chapter 6

Limit Order Book Markets

6.7 Exercises

1. Deriving a competitive LOB. Consider the model developed in section 6.2.3. We make the following parametric assumptions:

1. The trader who arrives in period 1 knows the final value of the security v with probability π. Otherwise, he is uninformed.

2. If the trader who arrives in period 1 is uninformed, he buys or sells (with equal probability) a number of shares x that has an exponential distribution with parameter θ. That is, the size distribution of the market order submitted by an uninformed trader arriving in period 2 is $f(x) = \frac{1}{2}\theta e^{-\theta|x|}$.

3. The final value of the security in period 2 has the following probability distribution:
$$g(v) = \frac{1}{2\sigma}\exp\left(-\frac{\mid v-\mu\mid}{\sigma}\right).$$

4. The tick size is nil ($\Delta = 0$).

Assumption 3 implies that σ is the mean absolute deviation of v and $\mathrm{E}(v \mid v \geq z) = z + \sigma$.

a. Let $Y(A)$ be the cumulative depth up to ask price A in the book and A^* be the lowest ask price in the LOB. Show that when $v \geq A^*$, the optimal strategy of the informed trader is to buy $Y(v)$ shares.

b. Using this observation and the zero profit condition (6.13), show that in equilibrium:

$$Y(A) = \frac{1}{\theta}\left[\ln\left(\frac{1-\pi}{\pi}\right) + \ln\left(\frac{A-\mu}{\sigma}\right) + \frac{A-\mu}{\sigma}\right] \text{ if } A > A^*.$$

c. Show that the book becomes thinner on the ask side when (i) π increases or (ii) σ increases. What is the economic intuition for this result?

2. Time priority vs. random tie-breaking rule. Consider example 2 in section 6.2.3 but assume that the tick size Δ is strictly positive, such that

$$A_1 = \mu + \Delta < \mu + \sigma < \mu + 2\Delta.$$

Time priority is enforced as in the baseline model of section 6.2.3.

a. Explain why the LOB will feature at least q_L shares offered at price A_2.

b. Let Y_1 be the number of shares offered at price A_1. Define $r = \sigma/\Delta$. Observe that $r \in [1,2]$. Using the assumptions regarding the order flow at time 1, show that:

1. $Y_1 = 0$ iff $\frac{(r-1)\pi}{1-\pi} \geq 1$.
2. $Y_1 = q_S$ iff $\frac{(r-1)\pi}{1-\pi} \in [1-\phi, 1)$.
3. $Y_1 = q_L$ iff $\frac{(r-1)\pi}{1-\pi} \in [0, 1-\phi)$.

c. Why does Y_1 decrease with π?

d. Now assume that $\frac{(r-1)\pi}{1-\pi} \in [1-\Phi, 1)$ and suppose that time priority is not enforced any more. Instead, if two traders post a limit order at price A_1, then the offer that is executed first is determined randomly. Specifically, the limit order posted by trader $j \in \{1,2\}$ is executed first with probability 0.5. Let Y_1^j be the number of shares offered by trader $j \in \{1,2\}$ at price A_1. Explain why, in equilibrium, Y_1^1 and Y_1^2 must satisfy the following conditions for $j = 1$ and$j = 2$:

$$(A_1 - \mathrm{E}(v \mid q \geq Y_1^j)) \Pr(q \geq Y_1^j) + (A_1 - \mathrm{E}(v \mid q \geq Y_1^1 + Y_1^2)) \Pr(q \geq (Y_1^1 + Y_1^2)) \leq 0$$

with a strict inequality if $Y_1^1 = Y_1^2 = 0$.

e. Suppose that $q_L = 2q_S$. Deduce that $Y_1^I = Y_1^2 = q_S$ form an equilibrium if:

$$\frac{(r-1)\pi}{1-\pi} \in \left[(1-\phi), \frac{1+(1-\phi)}{2}\right],$$

when the "random" tie-breaking rule is used.

f. Why is cumulative depth greater when the random tie-breaking rule is used for $\frac{(r-1)\pi}{1-\pi} \in \left[1-\phi, \frac{1+(1-\phi)}{2}\right]$?

3. Time priority vs. pro-rata allocation. Consider the model developed in section 6.2.2 and suppose $C < A_1 - v_0$. The size of the incoming market order (in absolute value) has a uniform distribution on $[0, Q]$, that is,

$$F(q) = \frac{q}{Q}.$$

a. Show that in this case the cumulative depth at price A_k is:

$$Y_k = Q\left(1 - \frac{C}{A_k - \mu}\right), \quad \forall k.$$

b. Now suppose that instead of time priority, a pro-rata allocation rule is used, as described in section 6.3.2. Further assume that $A_1 - \mu > 2C$. Then show that the cumulative depth at price A_1 is $Y_1^r = \frac{(A_1-\mu)Q}{2C}$.

c. Why does the pro-rata allocation rule yield greater cumulative depth at all ask prices?

4. Competition among specialists and liquidity. Consider the model of section 6.3.3 and assume that two specialists can stop out a market order. When a market order arrives, they post a stop-out price at which they are ready to fill. The specialist with the more competitive price executes. If there is a tie, the order is split equally between the two specialists.

a. Show that if

$$\frac{q_L}{q_S} > 1 + \frac{\pi}{(1-\pi)(1-\phi)},$$

then, in equilibrium, the offers in the LOB are as described by equations (6.25) and (6.26), and the specialists stop out the small orders at a price (bid or ask) equal to μ.

b. In this case, do the specialists improve liquidity?

5. Make/take fees and bid-ask spreads. Consider the model of section 6.4.1 with $\sigma = 0$ and $\tau = 1$. As Chapter 7 explains, trading platforms often charge different fees for market and limit orders. Let f_{mo} be the fee per share paid by a market order placer and f_{lo} the fee per share for a limit order placer when the limit order executes (there is no entry fee for limit orders). Finally let f be the total fee earned by the platform on each trade, $f = f_{mo} + f_{lo}$.

a. Compute bid and ask quotes in equilibrium.

b. Show that the bid-ask spread decreases in f_{mo} and increases in f_{lo}. Explain.

c. Trading platforms often subsidize traders who submit limit orders. That is, they set $f_{lo} < 0$ and $f_{mo} > 0$, maintaining that this practice ultimately helps to narrow the spread and benefits traders submitting market orders. Holding the total trading fee fixed, is this argument correct?

6.8 Solutions

Exercise 1:

a. If $A > A^*$ then $Y(A) > 0$ by definition of A^*. Moreover, if an informed trader buys $Y(A)$ shares, he obtains an expected profit equal to $(v - A)$ on the marginal share purchased (and a greater profit on inframarginal shares since he buys them at a lower price). Thus, the optimal strategy of the informed investor is to purchase all shares up to $A = v$, i.e., $Y(v)$ shares.

b. Let q be the size of the buy market order submitted in period 1. As the tick size is zero, the limit order book in equilibrium must be such that

$$A(y) = E(v\,|q > y) \text{ for } A > A^*.$$

Let H be an indicator variable equal to one if the trader arriving at date 1 is informed and zero otherwise and let $p(y)$ be the probability that the next trader is uninformed conditional on the market order size being greater than y. For $A > A^*$, we have

$$\begin{aligned} A(y) &= p(y)\cdot\mu + (1 - p(y))\cdot E(v\,|Y(v) > y) \\ &= \mu + (1 - p(y))(E(v\,|v > A) - \mu) \\ &= \mu + (1 - p(y))(A + \sigma - \mu) \end{aligned}$$

Furthermore

$$\begin{aligned} p(y) &= \Pr(H = 0\,|q > y) \\ &= \frac{(1-\pi)\exp(-\theta y)}{(1-\pi)\exp(-\theta y) + \pi\exp(-(A-\mu)/\sigma)}. \end{aligned}$$

We deduce that

$$A(y) - \mu = \frac{(\pi\exp(-(A-\mu)/\sigma))}{(1-\pi)\exp(-\theta y) + \pi\exp(-(A-\mu)/\sigma)}(A(y) - \mu + \sigma)$$

That is,

$$\exp(\theta y) = \frac{(1-\pi)}{\pi} \times \frac{A(y)-\mu}{\sigma} \times \exp(\frac{A(y)-\mu}{\sigma})$$

It is then immediate that $Y(A)$ (the inverse of $A(y)$) must be such that

$$Y(A) = \frac{1}{\theta}[\ln(\frac{1-\pi}{\pi}) + \ln(\frac{A-\mu}{\sigma}) + \frac{A-\mu}{\sigma}] \text{ if } A > A^*.$$

c. It is easily checked that $\frac{\partial Y(A)}{\partial \pi} < 0$ and $\frac{\partial Y(A)}{\partial \sigma} < 0$. Thus, the cumulative depth at a given price becomes smaller when it becomes more likely that the investor arriving at date 1 is informed or when the dispersion of the asset payoff becomes greater. Intuitively, the reason is that, in these two cases, liquidity suppliers become more exposed to the risk of informed trading: execution of a sell limit order at a given price is more likely to happen when the limit order undervalues the security. Hence, at each possible price, limit order traders are less willing to supply liquidity.

Exercise 2:

a. Consider an informed trader arriving at date 1 and assume that the payoff of the security is $v_H = \mu + \sigma$. As $A_1 < v_0 + \sigma < A_2$, the informed investor should send a buy market order for a quantity just equal to that offered at price A_1 but no more (otherwise part of her order will execute at a price larger than her valuation for the security). Hence, only sell limit orders at price A_1 are exposed to the risk of being hit by informed orders and sell limit orders at price A_2 will only execute (if that happens) against market orders submitted by uninformed orders. Hence, limit order traders should fill the book so that at least q_L shares (the maximal liquidity demand) are offered at this price.

b. Let q be the size of the market order submitted at date 1. The expected profit on the marginal sell limit order offered at price $A_1$is given by

$$(A_1 - E(V \mid q \geq Y_1) \Pr(q \geq Y_1)$$

Now, as shown in Chapter 6 (see Example 2), we have:

$$E(\widetilde{v} \mid q \geq Y_1) = \begin{cases} \mu + \pi\sigma & \text{for } Y_1 \leq q_S \\ \mu + \frac{\pi}{\pi+(1-\pi)(1-\phi)}\sigma & \text{for } q_S < Y_1 \leq q_L \\ \mu + \sigma & \text{for } Y_1 > q_L \end{cases}$$

Remember that $A_1 = \mu + \Delta$. We deduce that $E(V \mid q \geq q_L) \leq A_1$ iff

$$\frac{(r-1)\pi}{1-\pi} \leq (1-\phi) \tag{6.45}$$

Under this condition, the number of shares offered at price A_1 should be q_L. Offering less would attract competition while offering more would be suboptimal as additional shares would execute only if the trader arriving at date 2 is informed (since the maximal order size for a liquidity trader is q_L).

When Condition (6.45) is not satisfied then $Y_1 = q_S$ if $E(V \mid q \geq q_S) < A_1$ and $Y_1 = 0$ otherwise. We have $E(V \mid q \geq q_S) < A_1$ iff:

$$\pi\sigma < \Delta,$$

which rewrites:

$$\frac{(r-1)\pi}{1-\pi} < 1.$$

c. Observe that the number of shares offered in equilibrium at price A_1 is weakly decreasing in π. Intuitively, as π increases, the risk of being hit by an informed order for limit orders at price A_1 gets larger and accordingly the number of shares offered at this price is reduced.

d. Consider trader $j = 1$. With probability $\frac{1}{2}$, trader 1's limit order at price A_1 executes first. In this case, the trader obtains an expected profit on the marginal share of his limit order equal to:

$$(A_1 - E(v \mid q \geq Y_1^j))\Pr(q \geq Y_1^1)$$

With probability $\frac{1}{2}$, trader j's limit order at price A_1 executes second. In this case, the trader obtains an expected profit on the marginal share of his limit order equal to:

$$(A_1 - E(v \mid q \geq Y_1^j))\Pr(q \geq Y_1^1 + Y_1^2)$$

Hence, the expected profit on the marginal share offered by trader $j = 1$ is:

$$0.5(A_1 - E(v \mid q \geq Y_1^j))\Pr(q \geq Y_1^1) + 0.5(A_1 - E(v \mid q \geq Y_1^j))\Pr(q \geq Y_1^1 + Y_1^2)$$

The reasoning is similar for trader $j = 2$. In equilibrium, either this expected profit is less than zero for $Y_1^1 = Y_1^2 = 0$ or not. If the former case is realized then no investor finds optimal to submit a limit order at price A_1 as he would make a loss. If not then in equilibrium Y_1^1 and Y_1^2 must be such that the expected profit on the marginal share offered by trader j is just equal to zero as otherwise one trader would find it profitable to supply more shares.

e. If $Y_1^j = q_S$ then $Y_1^1 + Y_1^2 = q_L$. Thus, using 2.2 and the fact that $\Pr(q \geq q_S) = 1$ and $\Pr(q \geq q_L) = (\alpha + (1-\phi)(1-\alpha))$, we obtain:

$$\begin{aligned}0.5(A_1 - E(\tilde{V} \mid q \geq q_S))Pr(q \geq q_S) + 0.5(A_1 - E(\tilde{V} \mid q \geq q_L))Pr(q \geq q_L) = \\ 0.5(\Delta - \alpha\sigma) + 0.5((\alpha + (1-\phi)(1-\alpha))\Delta - \alpha\sigma).\end{aligned}$$

We deduce that :

$$0.5(A_1 - E(\tilde{V} \mid q \geq q_S))Pr(q \geq q_S) + 0.5(A_1 - E(\tilde{V} \mid q \geq q_L))Pr(q \geq q_L) \geq 0$$

iff

$$(1-\phi) \leq \frac{(r-1)\alpha}{1-\alpha} \leq 0.5 + 0.5(1-\phi).$$

It is also easily checked that a trader offering more shares than q_S shares at price A_1 expects a negative expected profit on the marginal share if he expects the other trader to offer q_S shares at price A_1.

f. The previous finding implies that there exists a range of parameters for which the cumulative depth at price A_1 is equal to q_S when time priority is used but $q_L = 2q_S$ when the random tie-breaking rule is used.To understand why, suppose first that time priority is enforced. If trader $j = 1$ is first to offer q_S shares at price A_1 then trader $j = 2$ cannot post a limit order without making loss (see question 2.2). Yet, trader $j = 2$ would like to bypass trader 1's time priority since shares offered at the top of the book yields a strictly positive expected profit. This is exactly what he can do when time priority is replaced by the random tie-breaking rule. In this case the first mover advantage of trader $j = 1$ does not exist any more. Even if trader $j = 2$ moves second, he can be selected as the trader who will be first executed and enjoys the expected profit that trader $j = 1$ was expecting with time priority. Thus, the random tie breaking rule intensifies competition among liquidity providers and results in a greater depth at price A_1, at least for some parameter values.

Exercise 3:

a. As explained in Section 6.2.2, the cumulative depth at price A_k in this case solves:

$$\Pr(q \geq Y_k)(A_k - v_0) = C.$$

As $F(q) = \frac{q}{Q}$, we have $\Pr(q \geq Y_k)$=$\frac{Q-Y_k}{Q}$. Hence

$$Y_k = Q(1 - \frac{C}{A_k - \mu}), \quad \forall k.$$

b. As explained in Section 6.3.2, if the pro-rata allocation rule is used, the cumulative depth at price A_k solves:

$$\int_{Y_{k-1}^r}^{Y_k^r} (P^{ex}(y)(A_k - E(\widetilde{v} \mid q \geq y)) - C)dy \leq 0, \tag{6.46}$$

with a strict equality if $Y_k^r = Y_{k-1}^r$ (i.e., if it is not profitable to post a limit order at price A_k).

As the order flow does not contain information, we have $E(\widetilde{v} \mid q \geq y) = \mu$. Moreover

$$P^{ex}(y) = \Pr(q \geq y) = \frac{Q - y}{Q} \text{ if } y \leq Q \text{ and } P^{ex}(y) = 0 \text{ for } y > Q.$$

Now consider the cumulative depth offered at price A_1. Using (6.46), it solves:

$$\int_0^{Y_1^r} (\left(\frac{Q-y}{Q}\right)(A_1 - \mu) - C)dy = 0.$$

That is:

$$Y_1^r = \begin{array}{lll} 2Q(1 - \frac{C}{A_1 - \mu}) & \text{if} & Y_1^r \leq Q, \text{ i.e., } A_1 - \mu \leq 2C, \\ \frac{(A_1-\mu)Q}{2C} & \text{if} & Y_1^r > Q, \text{ i.e., } A_1 - \mu > 2C. \end{array}$$

In the exercise we assume that $A_1 - \mu > 2C$. It follows that the cumulative depth at price A_1 is $\frac{(A_1-\mu)Q}{2C}$, which is strictly greater than the maximum order size for the market order, Q. Hence the execution probability of sell limit orders placed at prices greater than A_1 is zero. We deduce that $Y_k^r = Y_1^r = \frac{(A_1-\mu)Q}{2C}$ for all $k \geq 2$.

c. Clearly the cumulative depth offered at each price in the book is greater with the pro-rata allocation rule. The reason is as follows. With time priority, it is not profitable to expand the queue of limit orders at a given price if the expected profit on the marginal share offered at this price is zero. In contrast, with the pro-rata allocation rule, in such a situation, it is optimal for a trader to expand the queue with a limit order, for say, one share. Indeed, in this way, the trader can reap part of the profits enjoyed by infra-marginal limit orders queuing at the same price. In other words, the pro-rata allocation rule encourages traders to compete away the average expected profit on all limit orders queuing at a given price rather than the expected profit on the marginal limit order. This intuition is similar to that given for the effect of a random tie breaking rule in Exercise 2.

Exercise 4:

a. Suppose first that the two specialists stop out small buy market orders at price μ. If the informed investor behaves as when there is a single specialist (i.e., always submit a large buy market order when she observes that the value of the security is v_H) then the offers in the limit order book must be as described in equations (6.25), and (6.26). Thus, we just need to check whether the informed investor is better submitting a large buy market order rather than a small buy market order when the value of the security is v_H. The expected profit of the informed investor with a large buy market order when the value of the security is v_H is:

$$q_S(v_H - A^h(q_S)) + (q_L - q_S)(v_H - A^h(q_L)).$$

Indeed the order walks up the book and executes partially at $A^h(q_S)$ and then at $A^h(q_L)$. In contrast he expected profit of the informed investor with a small buy market order when the value of the security is v_H is:

$$q_S(v_H - \mu),$$

since the order will be stopped by the specialists (who cannot tell whether the order comes from an informed or an uninformed investor). Thus, the informed investor is better submitting a large buy market order rather than a small buy market order iff

$$q_S(v_H - A^h(q_S)) + (q_L - q_S)(v_H - A^h(q_L)) > q_S(v_H - \mu)$$

That is,

$$q_L(v_H - A^h(q_L)) > q_S(v_H - \mu + A^h(q_S) - A^h(q_L)).$$

Replacing $A^h(q_L)$ and $A^h(q_S)$ by their expressions in in equations (6.25), we obtain that this condition is equivalent to:

$$\frac{q_L}{q_S} > 1 + \frac{\pi}{(1-\pi)(1-\phi)}. \tag{6.47}$$

Now suppose that the informed investor submits only large buy market order when the value of the security is v_H. Then when a small order arrives, the specialists know that the order comes from an uninformed investor. Thus, stopping out the order by offering to execute it at a price below $A^h(q_S)$ but greater than μ is profitable. Price competition among specialists drives their offers to the lowest possible price, i.e., μ.

b. We observed in Section 6.3.3 that both large and small investors are worse off when a single specialist can stop out market orders. With two specialists the answer is not so clear cut. Indeed, when a small order arrives and Condition (6.47) is satisfied, the order will execute at μ. Hence, there is no bid-ask spread for small orders. For these orders the possibility of multiple specialists stopping out incoming market orders is a good thing. In contrast investors submitting large orders are still worse off in the hybrid market structure relative to the pure limit order market.

Exercise 5:

a. We can solve for the quotes chosen by limit order traders as we do in Section 6.4.1. Let $\overline{A}$ and $\overline{B}$ be, respectively, buyers and sellers' cut-off prices. As explained in Section 6.4.1, if a buyer submits a limitv order, she will optimally price it at $B^* = \overline{B}$ and if a seller submits a sell limit order he will optimally price it at $A^* = \overline{A}$. Thus, accounting for the trading fee, the expected payoff with a buy limit order is:

$$\frac{\tau}{2}(\mu_0 + L - \overline{B} - f_{lo})$$

while the expected payoff with a buy market order is

$$\mu_0 + L - \overline{A} - f_{mo}.$$

By definition of buyers' cut-off prices, we must have:

$$\mu_0 + L - \overline{A} - f_{mo} = \frac{\tau}{2}(\mu_0 + L - \overline{B} - f_{lo}) \tag{6.48}$$

Similarly, for the sellers, we obtain:

$$\overline{B} - (\mu_0 - L) - f_{mo} = \frac{\tau}{2}(\overline{A} - (\mu_0 - L) - f_{lo}). \tag{6.49}$$

Solving the previous system of equations for $\overline{A}$ and $\overline{B}$, we obtain:

$$A^* = \overline{A} = \mu_0 - L - f_{mo} + \frac{2(2L - \tau f)}{(2+\tau)}, \tag{6.50}$$

$$B^* = \overline{B} = \mu_0 + L + f_{mo} - \frac{2(2L - \tau f)}{(2+\tau)}. \tag{6.51}$$

b. Thus, the bid-ask spread is:

$$A^* - B^* = (2L - \tau f)\frac{(2-\tau)}{(2+\tau)} - 2f_{mo} = (2L - \tau f_{mo} - \tau f_{lo})\frac{(2-\tau)}{(2+\tau)} - 2f_{mo}$$

Thus, the bid-ask spread decreases with the fee charged on market orders and it increases with the fee charged on limit orders. To understand why, consider an increase in f_{mo}. Other things equal, this increase reduces traders' incentives to submit a market order since the cost of these orders increases. Thus, buyers' cut-off price fall and sellers' cut-off price increases. As a result, traders submitting limit orders must post more attractive offers. The reasoning is symmetric for an increase in the make fee.

c. Consider a buyer submitting a market order. In equilibrium he pays:

$$A^* + f_{mo} = \mu_0 - L + \frac{2(2L - \tau f)}{(2+\tau)}.$$

Hence, the payment of the buyer just depends on the total fee, not the division of the fee between the market order submitters and the limit order submitters. Thus, the platfortms' argument is incorrect if one looks at the quotes cum fees, which, in the model, adjusts so as to neutralize the effects of any change in the allocation of the total fee between those submitting limit orders and those submitting market orders.