Charlie Bradford

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1. **Deriving a competitive LOB.** Consider the model developed in section 6.2.3. We make the following parametric assumptions:

   (i) The trader who arrives in period 1 knows the final value of the security v with probability π. Otherwise, he is uninformed.

   (ii) If the trader who arrives in period 1 is uninformed, he buys or sells (with equal probability) a number of shares x that has an exponential distribution with parameter θ. That is, the size distribution of the market order submitted by an uninformed trader arriving in period 2 is $f(x) = \frac{1}{2}\theta e^{-\theta|x|}$

   (iii) The final value of the security in period 2 has the following probability distribution:

   $$g(v) = \frac{1}{2\sigma}\exp\left(-\frac{|v-\mu|}{\sigma}\right).$$

   (iv) The tick size is nil ($\Delta = 0$).

   Assumption 3 implies that σ is the mean absolute deviation of v and $E[v|v \geq z] = z + \sigma$.

   (a) Let $Y(A)$ be the cumulative depth up to ask price A in the book and $A*$ be the lowest ask price in the LOB. Show that when $v \geq A*$ the optimal strategy of the informed trader is to buy $Y(v)$ shares.

   (b) Using this observation and the zero profit condition (6.13), show that in equilibrium:

   $$Y(A) = \frac{1}{\theta}\left[\ln\left(\frac{1-\pi}{\pi}\right) + \ln\left(\frac{A-\mu}{\sigma}\right) + \frac{A-\mu}{\sigma}\right] \text{ if } A > A*$$

   (c) Show that the book becomes thinner on the ask side when (i) π increases or (ii) σ increases. What is the economic intuition for this result?

   (a) The optimal strategy is to buy any shares at or below the true value v. To do this one must buy $Y(v)$ shares.

   (b) Consider the inverse of Y, a function that maps quote sizes to prices

   (c)

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2. **Time priority vs. random tie-breaking rule.** Consider example 2 in section 6.2.3 but assume that the tick size Δ is strictly positive, such that

   $$A_1 = \mu + \Delta < \mu + \sigma < \mu + 2\Delta$$

   Time priority is enforced as in the baseline model of section 6.2.3.

   (a) Explain why the LOB will feature at least q_L shares offered at price A_2.

   (b) Let Y_1 be the number of shares offered at price A_1. Define $r = \frac{\sigma}{\Delta}$. Observe that $r \in [1, 2]$. Using the assumptions regarding the order flow at time 1, show that:

      i. $Y_1 = 0$ iff $\frac{(r-1)\pi}{1-\pi} \geq 1$.

ii. $Y_1 = q_S$ iff $\frac{(r-1)\pi}{1-\pi} \in [1-\psi, 1)$.
iii. $Y_1 = q_L$ iff $\frac{(r-1)\pi}{1-\pi} \in [0, 1-\psi)$.

(c) Why does Y_1 decrease with π?

(d) Now assume that $\frac{(r-1)\pi}{1-\pi}$ and suppose that time priority is not enforced any more. Instead, if two traders post a limit order at price A_1, then the offer that is executed first is determined randomly. Specifically, the limit order posted by trader $j \in \{1, 2\}$ is executed first with probability 0.5. Let Y_1^j be the number of shares offered by trader $j \in \{1, 2\}$ at price A_1. Explain why, in equilibrium, Y_1^1 and Y_1^2 must satisfy the following conditions for $j = 1$ and $j = 2$:

$$\left(A_1 - E\left[v|q \geq Y_1^j\right]\right) P(q \geq Y_1^j) + (A_1 - E[v|q \geq Y_1^1 + Y_1^2])P(q \geq Y_1^1 + Y_1^2)$$

with a strict inequality if $Y_1^1 = Y_1^2 = 0$.

(e) Suppose that $q_L = 2q_S$. Deduce that $Y_1^1 = Y_1^2 = q_S$ form an equilibrium if:

$$\frac{(r-1)\pi}{1-\pi} \in \left[(1-\psi), \frac{1+(1-\psi)}{2}\right],$$

when the "random" tie-breaking rule is used.

(f) Why is cumulative depth greater when the random tie-breaking rule is used for $\frac{(r-1)\pi}{1-\pi} \in \left[1-\psi, \frac{1+(1-\psi)}{2}\right]$

3. **Time priority vs. pro-rata allocation.** Consider the model developed in section 6.2.2 and suppose $C < A_1 - v_0$. The size of the incoming market order (in absolute value) has a uniform distribution on $[0, Q]$, that is,

$$F(q) = \frac{q}{Q}.$$

(a) Show that in this case the cumulative depth at price Ak is:

$$Y_k = Q\left(1 - \frac{C}{A_k - \mu}\right), \forall k.$$

(b) Now suppose that instead of time priority, a pro-rata allocation rule is used, as described in section 6.3.2. Further assume that $A_1 - \mu > 2C$. Then show that the cumulative depth at price A_1 is $Y_1^r = \frac{(A_1-\mu)Q}{2C}$.

(c) Why does the pro-rata allocation rule yield greater cumulative depth at all ask prices?

4. Competition among specialists and liquidity. Consider the model of section 6.3.3 and assume that two specialists can stop out a market order. When a market order arrives, they post a stop-out price at which they are ready to fill. The specialist with the more competitive price executes. If there is a tie, the order is split equally between the two specialists.

(a) Show that if

$$\frac{q_L}{q_S} > 1 + \frac{\pi}{(1-\pi)(1-\psi)},$$

then, in equilibrium, the offers in the LOB are as described by equations (6.25) and (6.26), and the specialists stop out the small orders at a price (bid or ask) equal to μ.

(b) In this case, do the specialists improve liquidity?

5. **Make/take fees and bid-ask spreads.** Consider the model of section 6.4.1 with $\sigma = 0$ and $\tau = 1$. As Chapter 7 explains, trading platforms often charge different fees for market and limit orders. Let f_{mo} be the fee per share paid by a market order placer and f_{lo} the fee per share for a limit order placer when the limit order executes (there is no entry fee for limit orders). Finally let f be the total fee earned by the platform on each trade, $f = f_{mo} + f_{lo}$.

(a) Compute bid and ask quotes in equilibrium.

(b) Show that the bid-ask spread decreases in f_{mo} and increases in f_{lo}. Explain.

(c) Trading platforms often subsidize traders who submit limit orders. That is, they set $f_{lo} < 0$ and $f_{mo} > 0$, maintaining that this practice ultimately helps to narrow the spread and benefits traders submitting market orders. Holding the total trading fee fixed, is this argument correct?