AE353 (Spring 2021)

Day 11 – Stability

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nonlinear ODEs

$\downarrow$ put in standard form

$$\dot{m} = f(m,n)$$
$$l = g(m,n)$$

$\downarrow$ linearize

$$x = m - m_e$$
$$u = n - n_e$$
$$y = l - g(m_e, n_e)$$

$$0 = f(m_e, n_e)$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$\downarrow$ choose input

$$u = -Kx$$

$\downarrow$ find closed-loop system

$$\dot{x} = (A-BK)x = Fx$$
$$y = (C-DK)x = Gx$$

$\downarrow$ solve by matrix exponential

$$\left.\begin{aligned} x(t) &= e^{F(t-t_0)} x(t_0) \\ y(t) &= C e^{F(t-t_0)} x(t_0) \end{aligned}\right\}$$

for which choices of K does $x(t) \to 0$ as $t \to \infty$???

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STRATEGY

1. Answer this question in the special case when F is diagonal
2. Show how to rewrite (almost) any F as diagonal
3. Answer this question for (almost) any F

suppose F is diagonal

$$F = \begin{bmatrix} & & \end{bmatrix}$$

then:

$$e^{Ft} = I + Ft + \frac{1}{2!}(Ft)^2 + \cdots$$

$$= \begin{bmatrix} & & \end{bmatrix} + \begin{bmatrix} & & \end{bmatrix} + \begin{bmatrix} & & \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} & & \end{bmatrix}$$

$$= \begin{bmatrix} & & \end{bmatrix}$$

← when F is diagonal, e^{Ft} is easy to find

coordinate invariance

$\dot{x} = Fx$

$\downarrow$ plug in $x = Vz$ for some invertible V

$\downarrow$ solve for $\dot{z}$

$\downarrow$ solve for $z(t)$ with matrix exponential

$\downarrow$ plug in $z = V^{-1}x$ and solve for x

solve for $x(t)$ with matrix exponential who cares?

$$e^{Ft} = V \underbrace{e^{V^{-1}FVt}}_{} V^{-1}$$

this is easy to find if $V^{-1}FV$ is diagonal
so let's choose V so this is true

our goal - find invertible V such that

$$V^{-1}FV = \text{diag}(s_1, \dots, s_n)$$

← this is the same as
$FV = V\,\text{diag}(s_1, \dots, s_n)$

for example, suppose F is 2×2:

$$V = [v_1 \quad v_2] \qquad \text{diag}(s_1, s_2) = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

columns of V (both are 2×1)

then:

$$FV = [\qquad\qquad] \qquad V\,\text{diag}(s_1, s_2) = [\qquad\qquad]$$

if

then

$$x(t) = e^{F(t-t_0)} x(t_0) = V \underbrace{e^{\mathrm{diag}(s_1, \dots, s_n)(t-t_0)}}_{\mathrm{diag}\left(e^{s_1(t-t_0)}, \dots, e^{s_n(t-t_0)}\right)} V^{-1} x(t_0)$$

what if $s = a + jb$???

$$e^{(a+jb)(t-t_0)} = e \qquad e$$

$$= e \qquad (\qquad)$$

The system

$$\dot{x} = Fx$$

is asymptotically stable if and only if all of F have

$x(t) \to 0$ as $t \to \infty$