EMS412U - MATHEMATICAL AND COMPUTATIONAL MODELLING 1

Forces and Moments

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Learning outcome

By the end of this lecture, you should be able to:

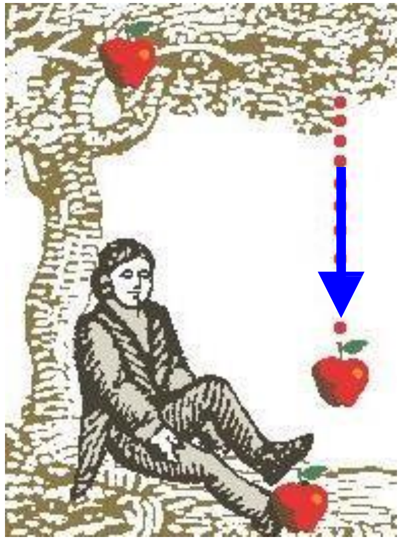
- Understand the physical meaning of **scalar** and **vectors**.
- Understand the physical meaning and definition of **force and moment.**
- Construct free body diagram.
- Calculate the **resultant** forces and moments.

What is a Force?

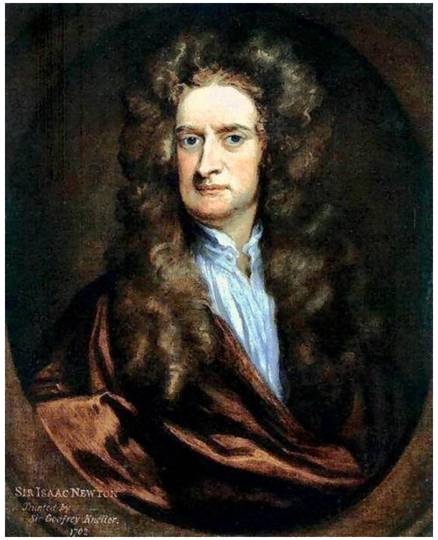
- You cannot see a force but feel it;
- A force is characterized by its effect;
- Force $\boldsymbol{F}$ is associated with a change in motion.
- A force accelerates an object
  Example: **weight** $\boldsymbol{W}$ is defined by
  1. **mass** m and
  2. **Earth gravitational acceleration,** g = 9.81 m/s²:

$$\boxed{\vec{W} = m \cdot \vec{g}\,[\mathrm{N}]} \qquad (1)$$

- The SI unit of force is Newton [N];
- g is different on other planets; on Earth it varies slightly, but can be assumed to be constant.



The legend about Newton's apple shows that a force (gravitation) changes the velocity of an object



Isaac Newton (1642-1727)
Portrait by Godfrey Kneller
National Portrait Gallery, London

Attention: mass is not equal to weight

Example:
The mass m of an apple was measured as m = 0.120 [kg].

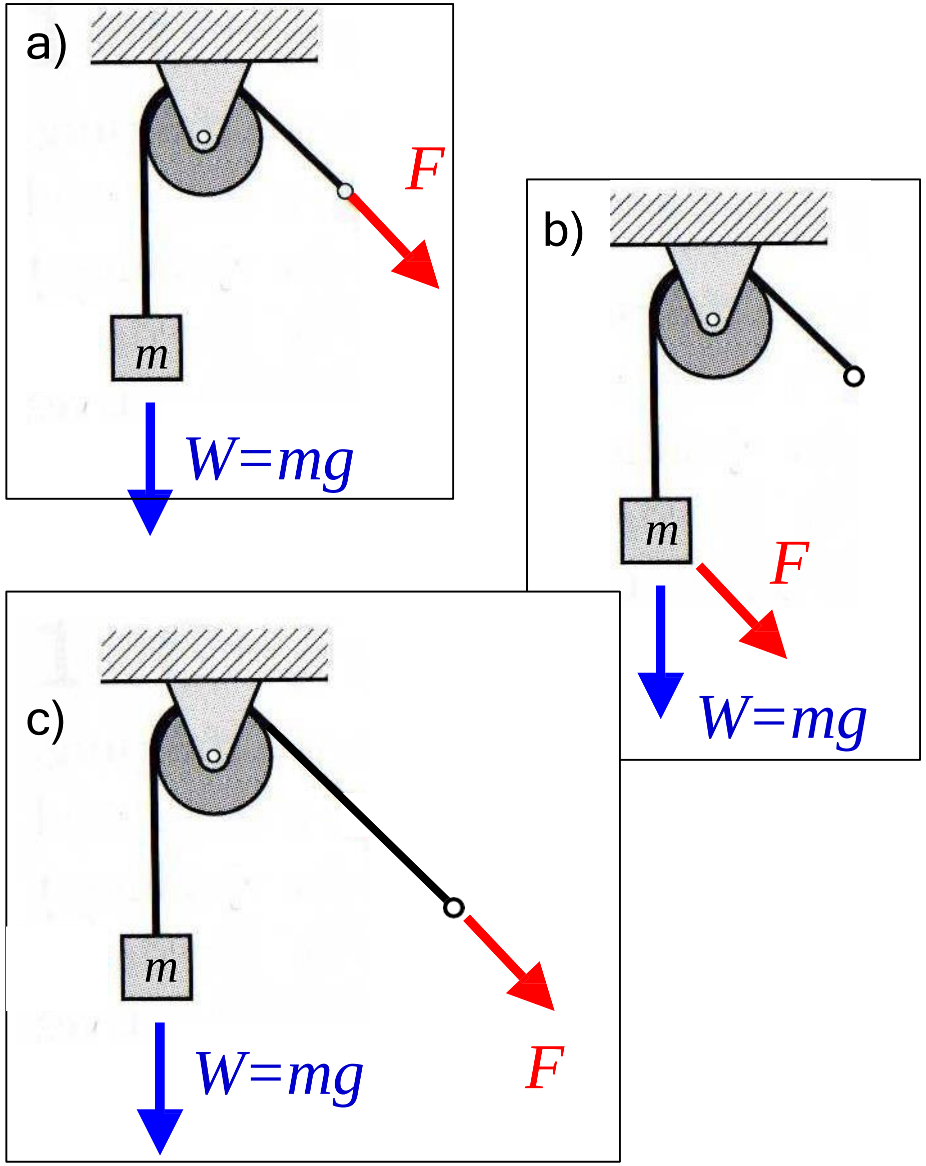
Its weight on Earth is hence:

$$\vec{W} = m \cdot \vec{g} = 0.12 \cdot 9.81 = 1.18\,[\mathrm{N}]$$

Magnitude, Direction and Point of Action

- A force is determined by
    1. **Magnitude;**
    2. **Direction;**
    3. **Point of action**.

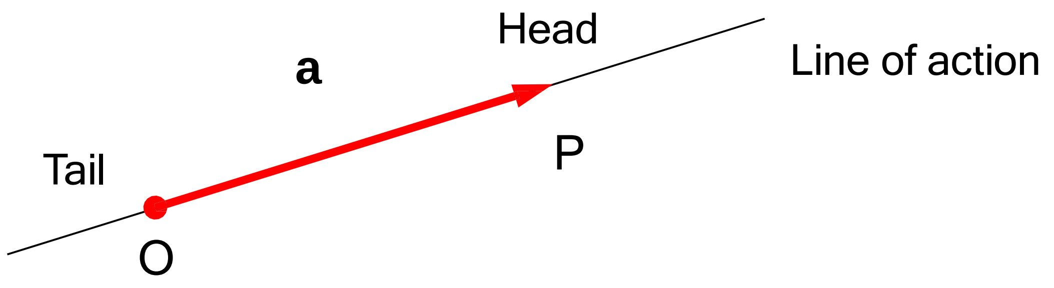
- In **Figure a)**, the mass m moves downwards if $F < W$ and upwards when $F > W$ (friction is neglected);
- The acceleration of the mass depends on the force magnitude.
- It would be different, when F is directly applied to the mass m as shown in the **Fig. b)**.
- If the force F is moved in its direction, cf. **Fig. c)**, the result is similar to case a), i.e.:
- Moving a force along its line of action does not change the effect.



Scalar and Vector

- A **scalar** is a positive or negative number; physical quantities described by scalars are for example: mass, volume, energy and temperature.
- A **vector** is defined by direction, magnitude, direction and sense; it can be depicted by an arrow where the length of the arrow indicates the magnitude. Forces, moments, displacements, velocity, acceleration are described via vectors.

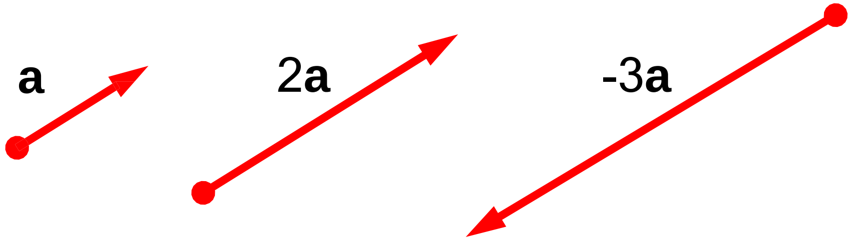
Remark: **bold** letters represent vectors or matrices while normal letters stand for scalar quantities. Vectors are also represented by letters with an arrow written over it.



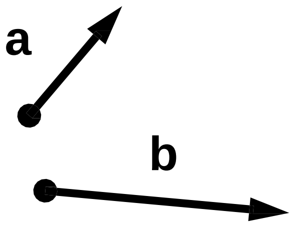
Graphic representation of a vector

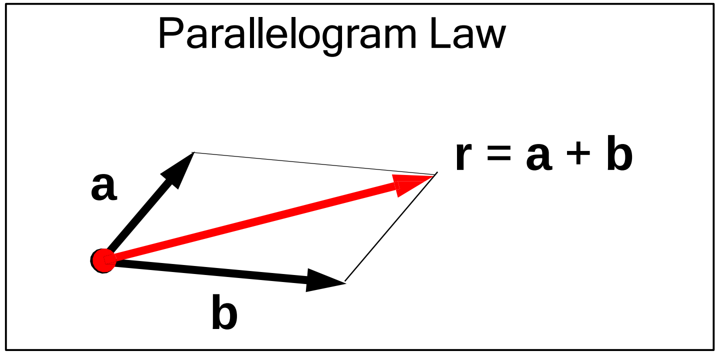
Operations on Vectors

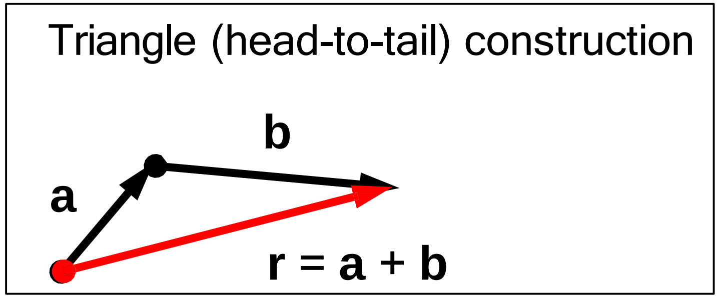
- **Multiplication** & **Division** of a vector by a scalar.



- **Addition** of two vectors.



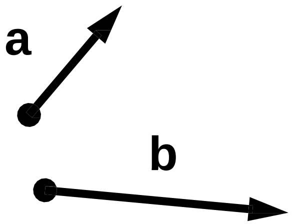


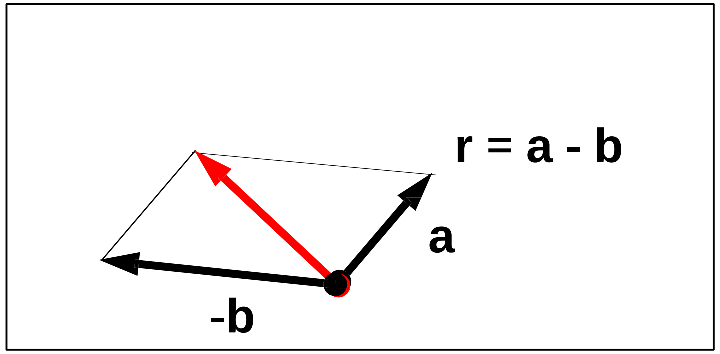


- **Subtraction** of two vectors. It's a special case of addition (First multiply the vector **B** by the scalar (-1) and then add to vector **A**)

a - **b** = **a** + (-**b**) =

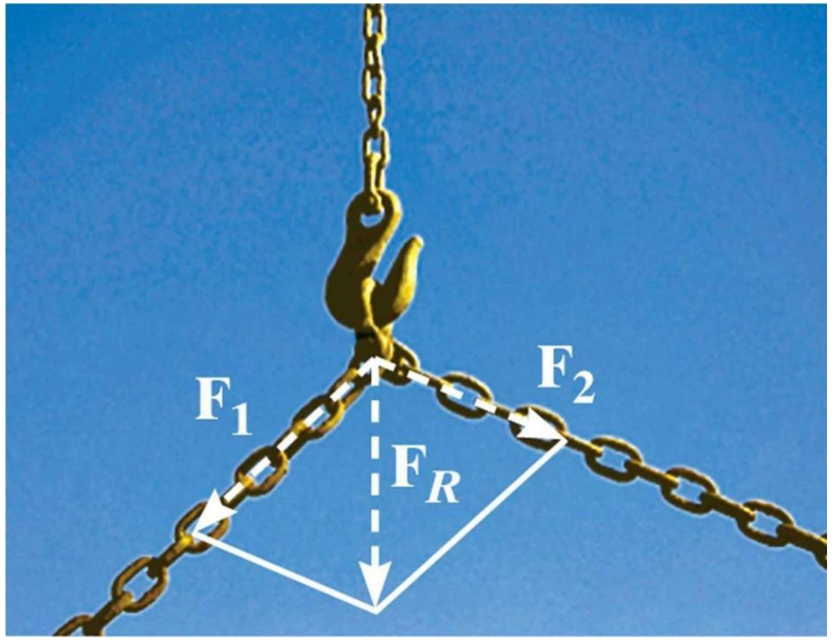
= **a** + (-1)·**b**





Operations on Vectors

If I need to calculate the resultant of two forces acting on the hook:



Components of a Vector (2D)

- The coordinate system is described by unit vectors with length 1:

$$\boxed{\begin{array}{ll} \text{Two dimensions}: & \mathbf{e}_x, \mathbf{e}_y \\ \text{Three dimensions}: & \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z \end{array}}$$

- A vector can be decomposed into its components in direction of the coordinate axis:

$$\mathbf{a} = \mathbf{a}_x + \mathbf{a}_y$$

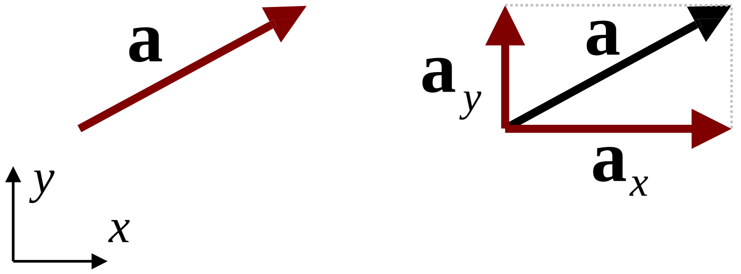
- These components are mathematically expressed by the product of a scalar and the unit vector:

$$\mathbf{a}_x = a_x \cdot \mathbf{e}_x ; \mathbf{a}_y = a_y \cdot \mathbf{e}_y$$

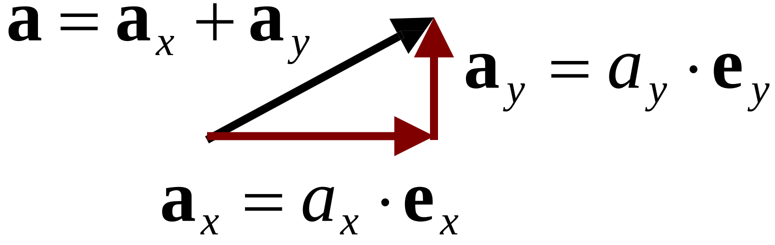
- Thus the vector is given by

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = a_x \cdot \mathbf{e}_x + a_y \cdot \mathbf{e}_y$$

Unit vectors for the coordinate system (2D and 3D)



Decomposition of a vector into components



Components of a vector

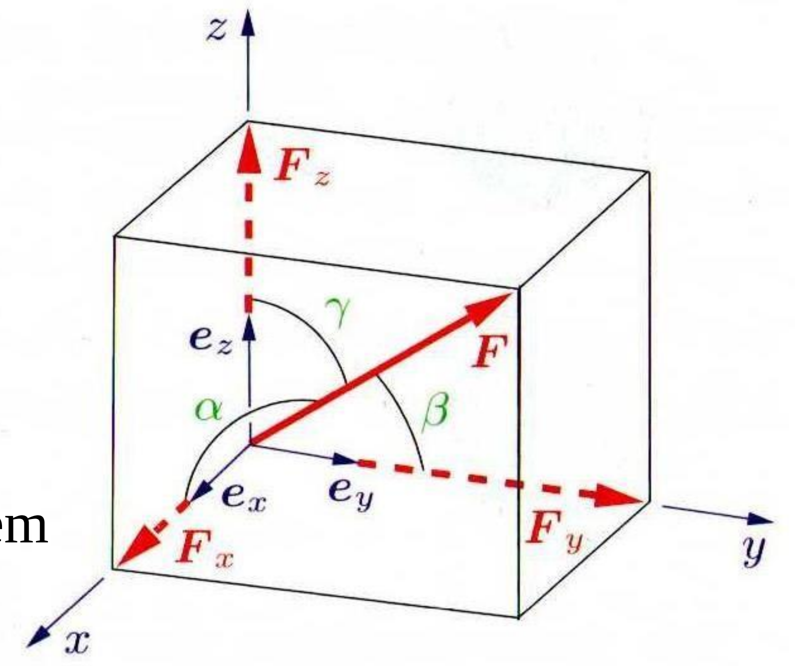
Components of a Vector (3D)

- In three dimensions, the vector ***F*** is composed of three components:

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y + \mathbf{F}_z$$
$$= F_x\mathbf{e}_x + F_y\mathbf{e}_y + F_z\mathbf{e}_z$$

- The magnitude F of the force is obtained via:

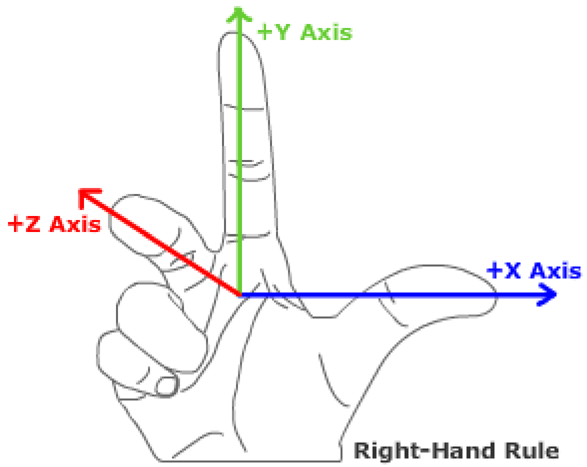
$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \quad \text{Pythagorean theorem}$$



- The angles (see Figure) are:

$$\cos\alpha = \frac{F_x}{F}, \cos\beta = \frac{F_y}{F}, \cos\gamma = \frac{F_z}{F}.$$

- The order of the coordinates x-y-z is given by the "right-hand rule"



Addition of Vectors via their Components (2D)

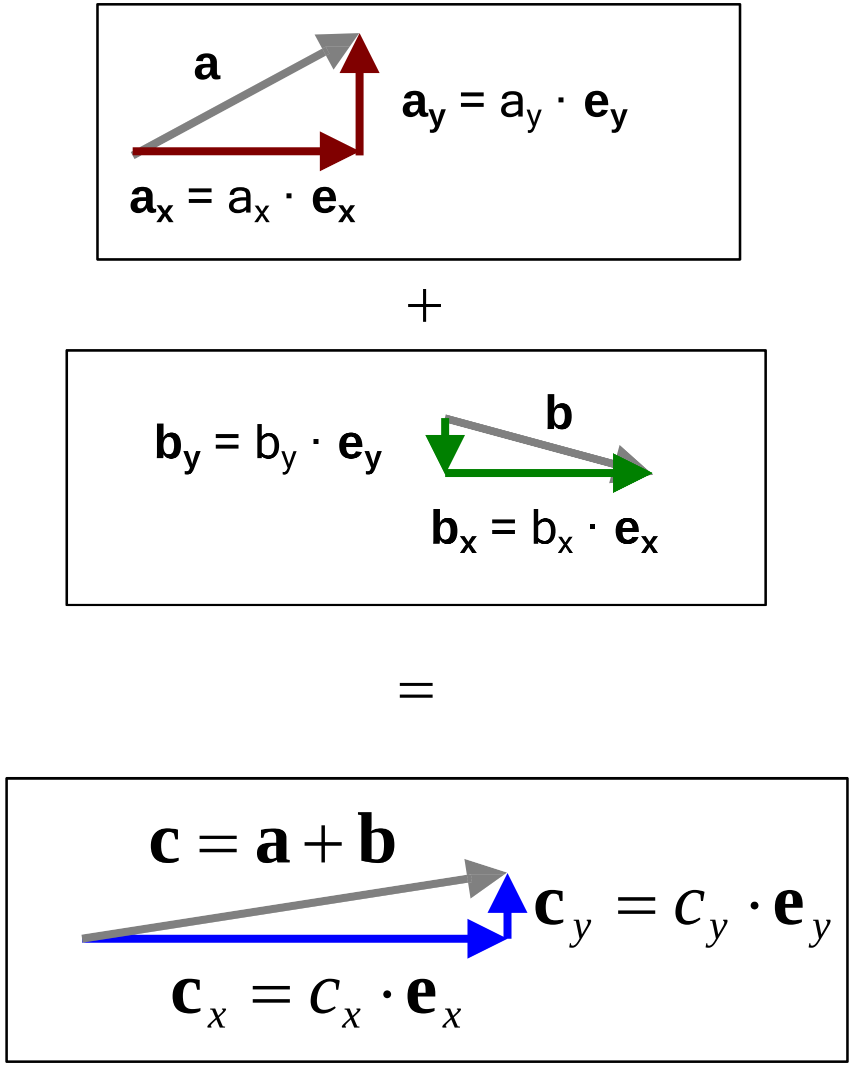
- Expressing the vectors **a** and **b** by their components a_x, a_y and b_x, b_y, the addition of two vectors is given by:

$$\mathbf{a}+\mathbf{b}=\begin{pmatrix} a_x \\ a_y \end{pmatrix}+\begin{pmatrix} b_x \\ b_y \end{pmatrix}=\mathbf{c}=\begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

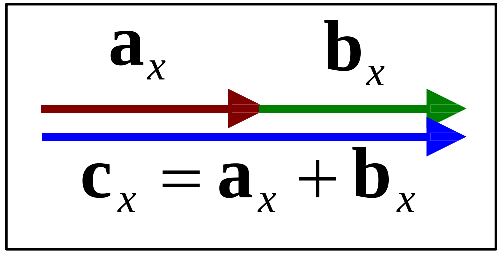
$$=(a_x+b_x)\cdot\mathbf{e}_x+(a_y+b_y)\cdot\mathbf{e}_y$$

$$\mathbf{c}=\mathbf{c}_x+\mathbf{c}_y=c_x\cdot\mathbf{e}_x+c_y\cdot\mathbf{e}_y$$

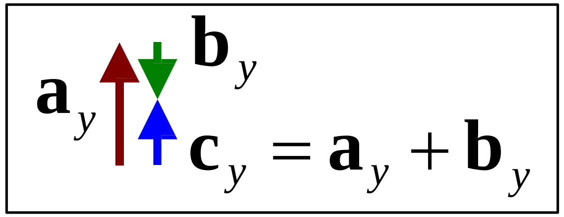
with: $c_x=(a_x+b_x)$; $c_y=(a_y+b_y)$



Along x-axis:



Along y-axis:

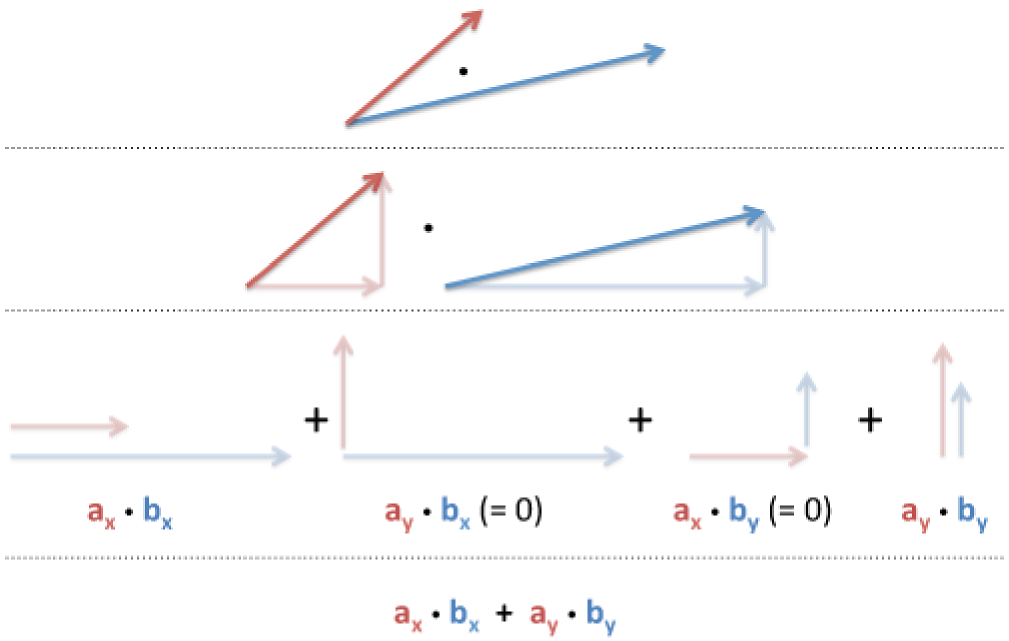


Dot product of vectors

Dot product calculates the sum of the two vectors' multiplied elements. Dot Product returns a **scalar** number as a result. The dot product is useful in calculating the projection of vectors. Dot product in Python also determines orthogonality and vector decompositions.

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Dot Product: Piece by Piece



$$\vec{a} \cdot \vec{b} = a_x \cdot b_x + a_y \cdot b_y = |\vec{a}||\vec{b}| \cos(\theta)$$

For example, the dot product of vectors [1, 3, −5] and [4, −2, −1] is

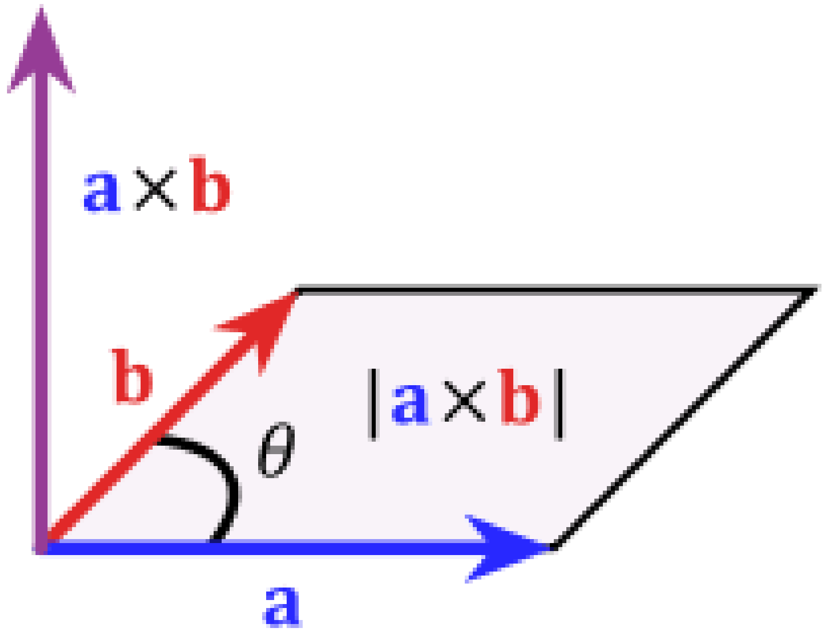
$$[1, 3, -5] \cdot [4, -2, -1] = (1 \times 4) + (3 \times -2) + (-5 \times -1)$$

$$= 4 - 6 + 5 = 3$$

Cross product of vectors

In mathematics, the cross product or vector product (occasionally directed area product, to emphasize its geometric significance) is a binary operation on two vectors in three-dimensional space. The magnitude of the product equals the area of a parallelogram with the vectors for sides.

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta)\, \mathbf{n}$$

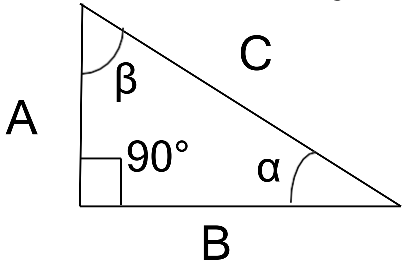


$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 \mathbf{i} + a_3 b_1 \mathbf{j} + a_1 b_2 \mathbf{k}) - (a_3 b_2 \mathbf{i} + a_1 b_3 \mathbf{j} + a_2 b_1 \mathbf{k})$$
$$= (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

https://betterexplained.com/articles/vector-calculus-understanding-the-dot-product/

Let's put some numbers in…

To Remember from trigonometry…

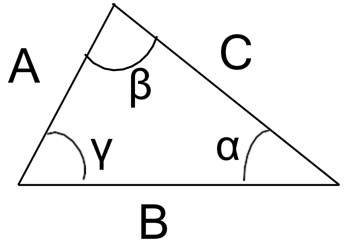


$A = C \cdot \sin\alpha = C \cdot \cos\beta$

$B = C \cdot \sin\beta = C \cdot \cos\alpha$

$C^2 = A^2 + B^2$

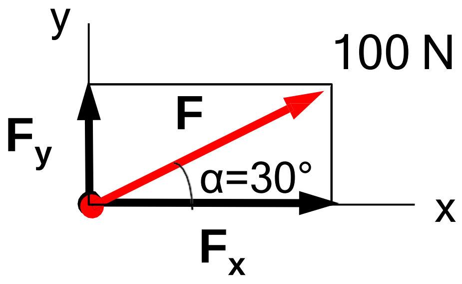
$\tan\alpha = A/B$



$A / \sin\alpha = B / \sin\beta = C / \sin\gamma$ (sine law)

$C^2 = A^2 + B^2 - 2AB\cos\gamma$ (cosine law)

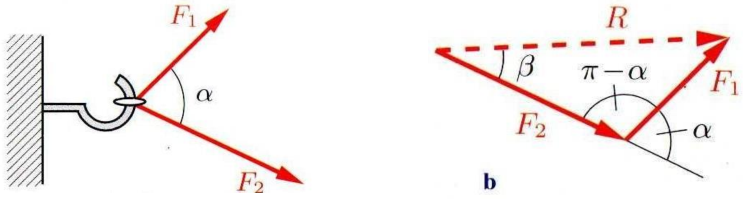
- Find the components of vector force F (magnitude 100 N) in the x and y directions:



$F_x = F \cdot \cos\alpha = 100 \cdot \cos 30° = 100 \cdot 0.866 = 86.6$ N

$F_y = F \cdot \sin\alpha = 100 \cdot \sin 30° = 100 \cdot 0.5 = 50$ N

- Find the force resultant of F_1=10 N and F_2=5 N, separated by an angle of 120°.



$$R = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos(\pi - \alpha)} \longrightarrow \mathbf{R} = \sqrt{(100 + 25 - 2 \cdot 10 \cdot 5 \cdot \cos 60°)} = 8.66 \text{ N}$$

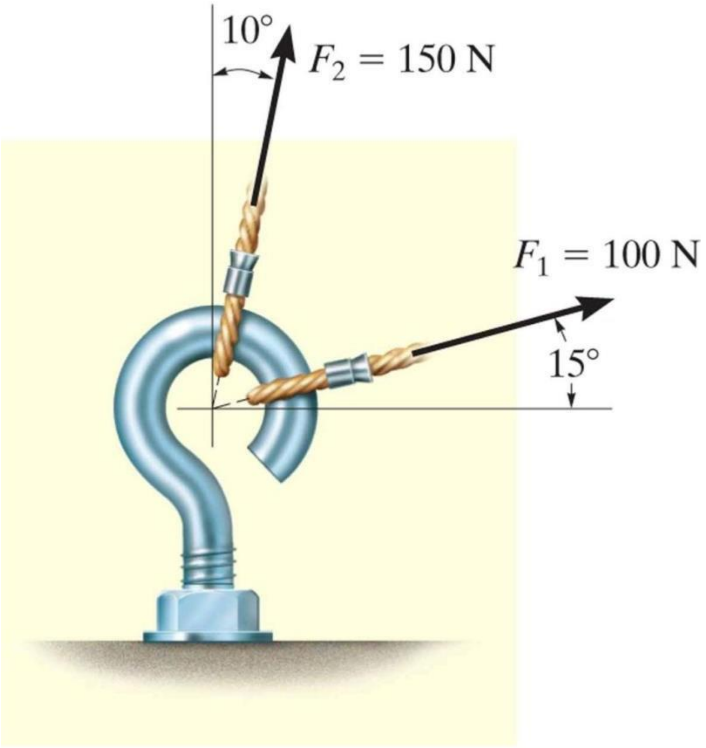
law of cosines

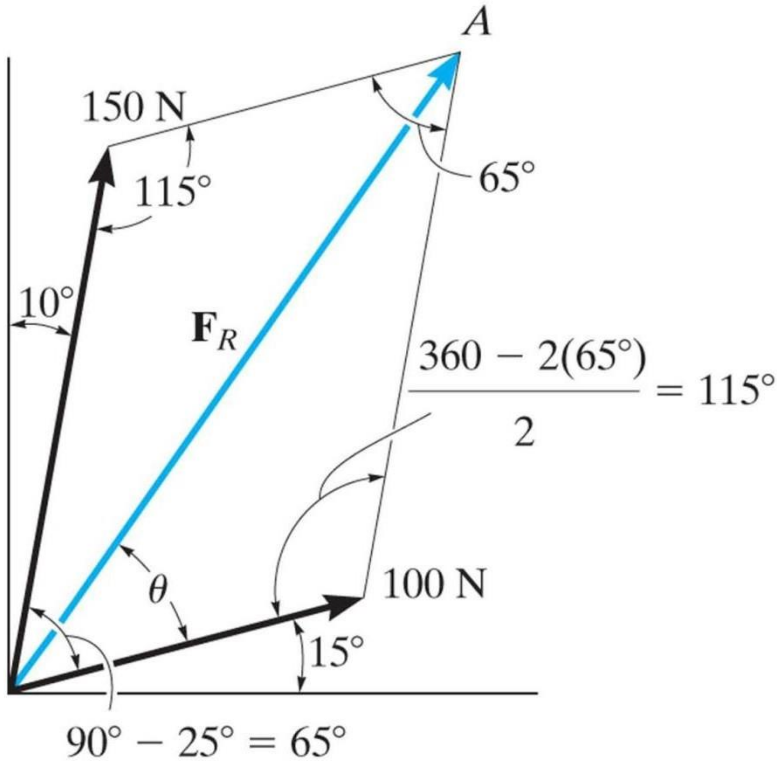
$$\sin\beta = \sin(\pi - \alpha) F_1 / R \longrightarrow \sin\beta = 0.866 \cdot 10 / 8.66 = 1 \longrightarrow \beta = 90°$$

law of sines

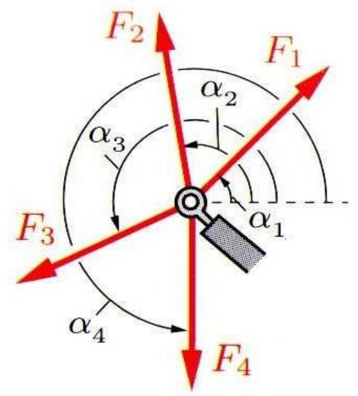
Try it yourself…

Calculate the resultant of two forces acting on the hook:



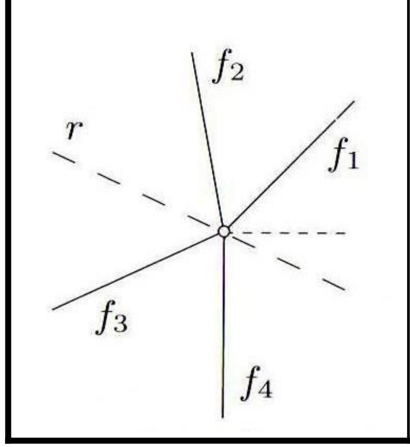
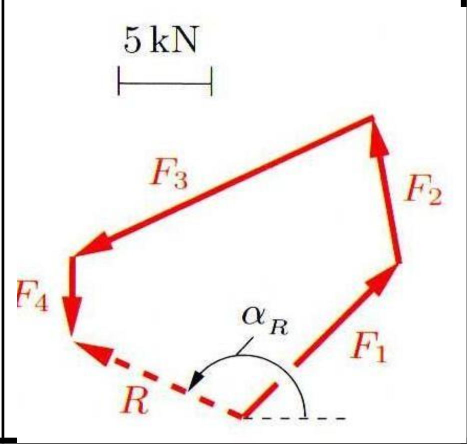


Another example: Resultant of more than 2 vectors



- Four forces are acting on a hook; determine the resultant force ***R***

$$F_1 = 12kN;\ F_2 = 8kN;\ F_3 = 18kN;\ F_1 = 4kN$$
$$\alpha_1 = 45°;\ \alpha_2 = 100°;\ \alpha_3 = 205°;\ \alpha_4 = 270°$$

Geometry	Forces
	

Note: Find the components of each individual force in the horizontal (x) and vertical (y) direction and then sum them up. The resultants can be easily be found by Pythagoras' rule.

Solution:

$$R_x = F_1\cos\alpha_1 + F_2\cos\alpha_2 + F_3\cos\alpha_3 + F_4\cos\alpha_4$$
$$= 12\cos 45° + 8\cos 100° + 18\cos 205° + 4\cos 270 = -9.22\,[\mathrm{kN}]$$

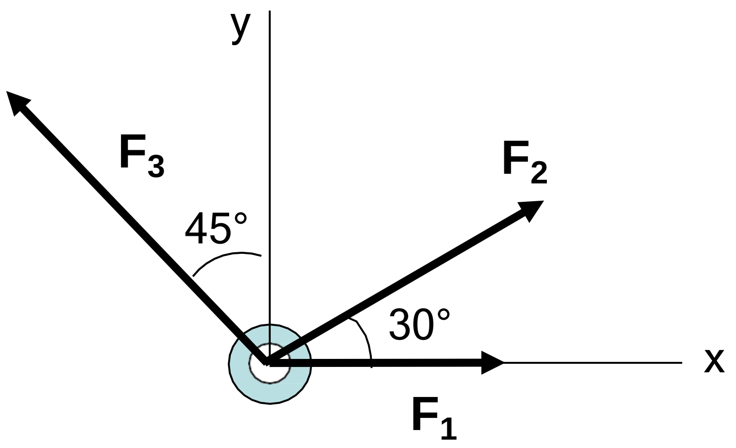
$$R_y = F_1\sin\alpha_1 + F_2\sin\alpha_2 + F_3\sin\alpha_3 + F_4\sin\alpha_4$$
$$= 12\sin 45° + 8\sin 100° + 18\sin 205° + 4\sin 270 = +4.76\,[\mathrm{kN}]$$

$$R = \sqrt{R_x^2 + R_y^2} = 10.4\,[\mathrm{kN}]$$

$$\tan\alpha_R = \frac{R_y}{R_x} = -0.52 \quad \rightarrow \quad \alpha_R = 152.5°$$

Try it yourself…

- Find the components of the forces $\mathbf{F}_1$ (5 N), $\mathbf{F}_2$ (10 N) and $\mathbf{F}_3$ (20 N) along the x and y direction. Also find the resultant forces $\mathbf{R_1}=\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3$ and $\mathbf{R}_2=F_1-F_2-F_3$

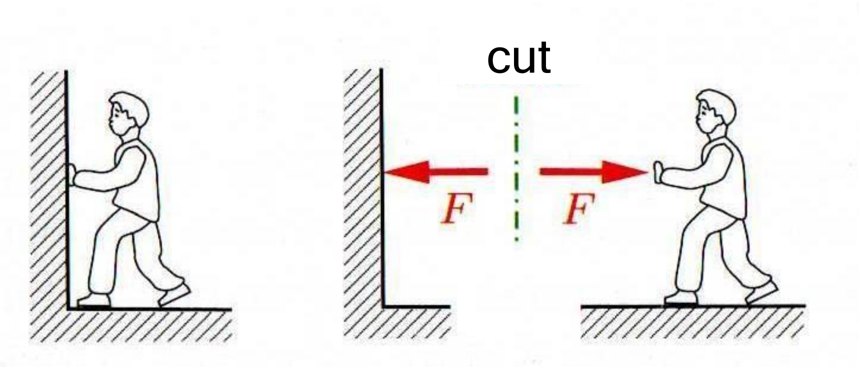


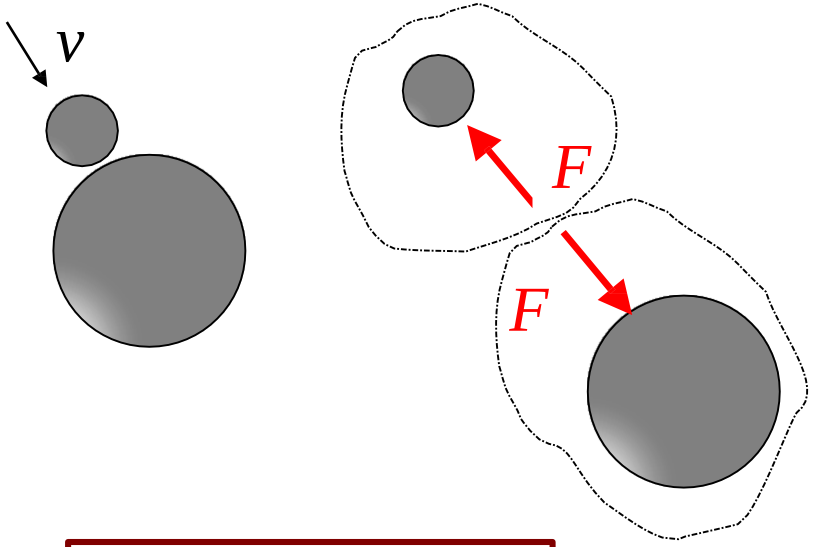
Newton's Laws of Motion (3)

- Newton's 3rd law states that: ***"To every action (force exerted) there is an equal and opposite reaction"***, which is:

  action = reaction (3)

- This principle is realized by **cutting** the objects from one another; the so-called **free-body diagram** is obtained.
- A free body diagram means:
  1. A closed cut around the object
  2. At each point where the object was separated, reaction forces (and eventually moments) have to be inserted
  3. The object should be totally isolated.





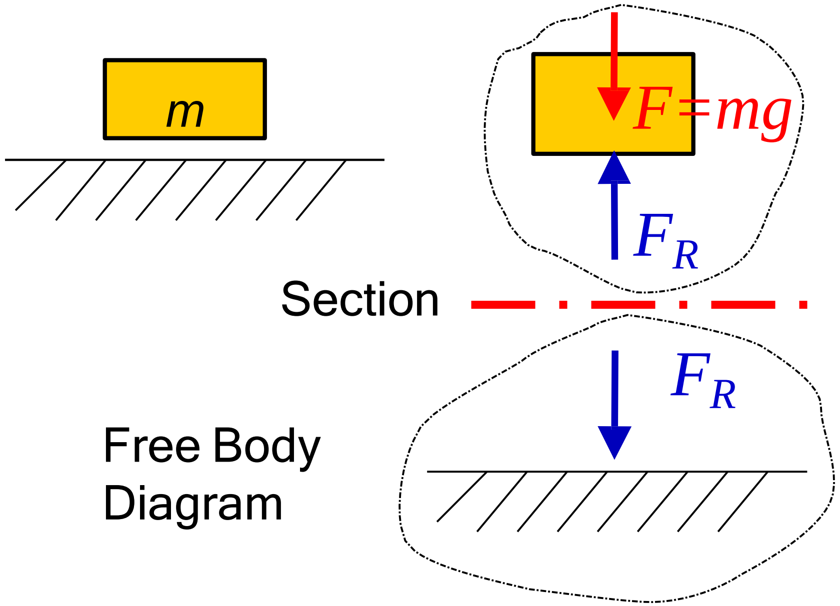
Free Body Diagram

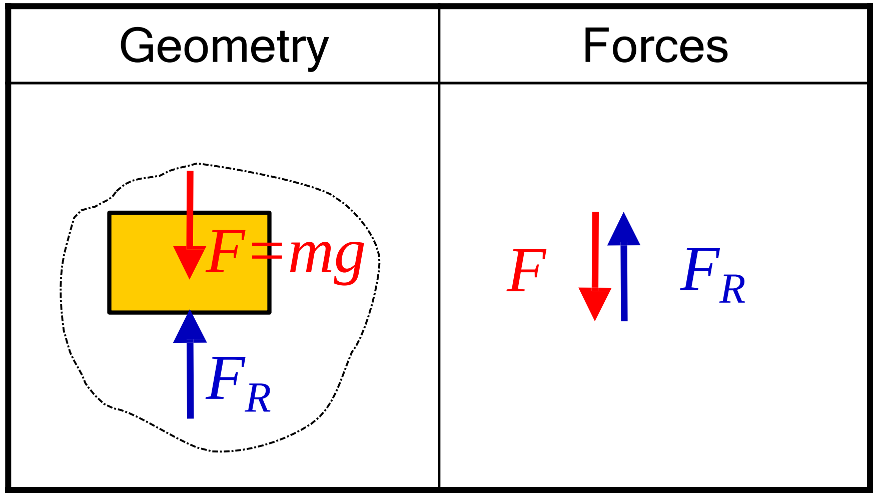
Free Body Diagram

- Consider a mass m on a horizontal surface. The force of gravity (weight) on the mass is $\boldsymbol{F} = m\boldsymbol{g}$.
- The mass is stationary hence there can be no net force on it; both forces have the same magnitude:

$$F = mg = F_R$$

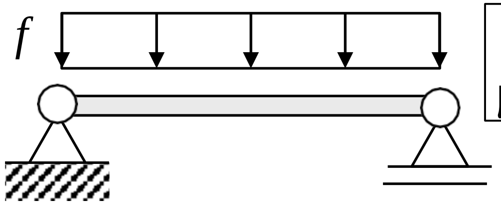
- The floor "reacts" to the weight and produces an equal and opposite force, $\boldsymbol{F}_R$, the force of reaction.
- This cancels out the force of gravity and hence the mass can be stationary. This will be later meant by "equilibrium".
- To analyze the problem, a plan of location and a plan of forces has to be drawn.





Distributed Forces f [N/m]

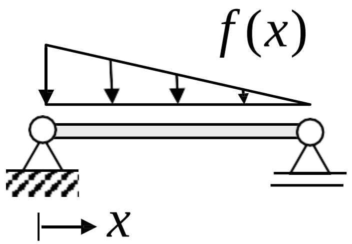
- Loads often occur distributed along a line (or over a surface in 3D):

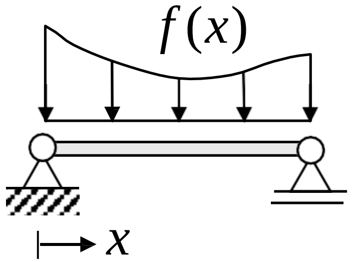


$$f = \mu \cdot g \,[\mathrm{N/m}]$$
$$\mu = \rho \cdot A \,[\mathrm{kg/m}]$$

ρ = density; A = cross-sectional area

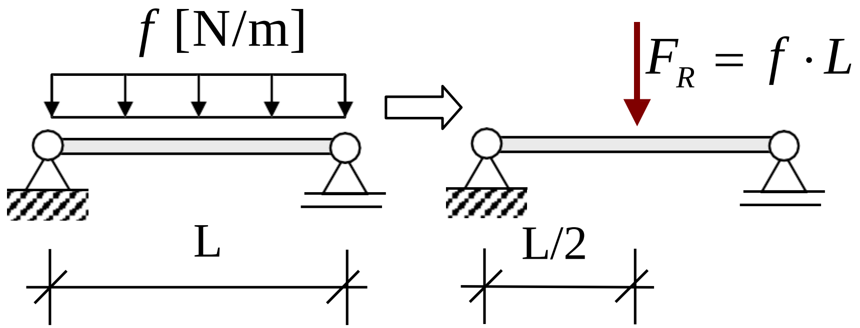
- Distributed loads may be constant, linear or more complex.



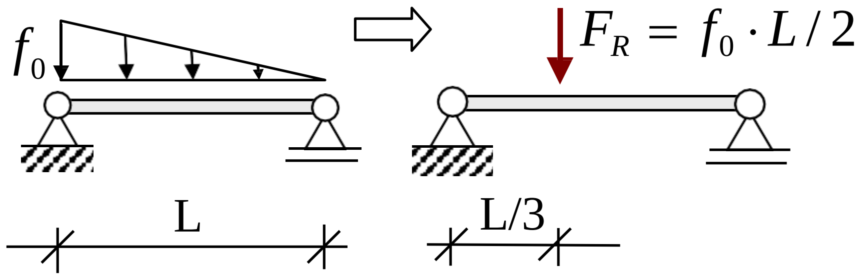


- Single forces are normally denoted by capital letters, distributed forces by small letters.

- To study the global reaction of the structure, distributed loads can be summarized in resultant forces:



or



- The resultant force is applied at the centre of gravity of the distributed load.

Example: Distributed Forces

- A material should be selected for a beam bridging from Point A to Point B made from an uniform material; the total load should not exceed

$$\max \mathbf{F}_R = 5\,[\text{kN}]$$

to prevent failure.

- The density of the material is:

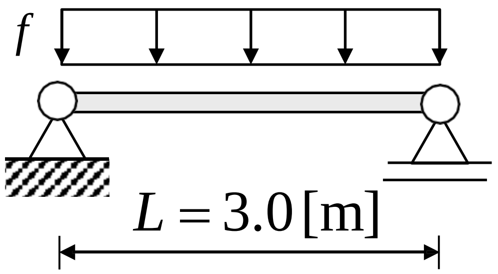
steel $\rho_S = 7.85 \cdot 10^3 [\text{kg/m}^3]$

beech $\rho_B = (0.4 \div 0.7) \cdot 10^3 [\text{kg/m}^3]$

- The cross-sectional area is:

$$A = 0.16\,[\text{m}^2]$$

- Determine the load f due to the weight of the beam and verify that it does not exceed the failure limit even if a person of m = 80 kg stands on it.



- Solution:

$$F_{\text{person}} = 80 \cdot 9.81 = 784.8\,[\text{N}] = 0.785\,[\text{kN}]$$

Steel :

$$f = \rho \cdot A \cdot g = 7.85 \cdot 10^3 \cdot 0.16 \cdot 9.81 = 12.32\,[\text{kN/m}]$$

$$F_{\text{R}} = f \cdot L = 12.32 \cdot 3.0 = 36.96\,[\text{kN}]$$

$$F_{\text{R}} + F_{\text{person}} = 37.7\,[\text{kN}] > \max F_{\text{R}} = 5.0\,[\text{kN}]$$

Beech :

$$f = \rho \cdot A \cdot g = 0.7 \cdot 10^3 \cdot 0.16 \cdot 9.81 = 1.01\,[\text{kN/m}]$$

$$F_{\text{R}} = f \cdot L = 12.32 \cdot 3.0 = 3.3\,[\text{kN}]$$

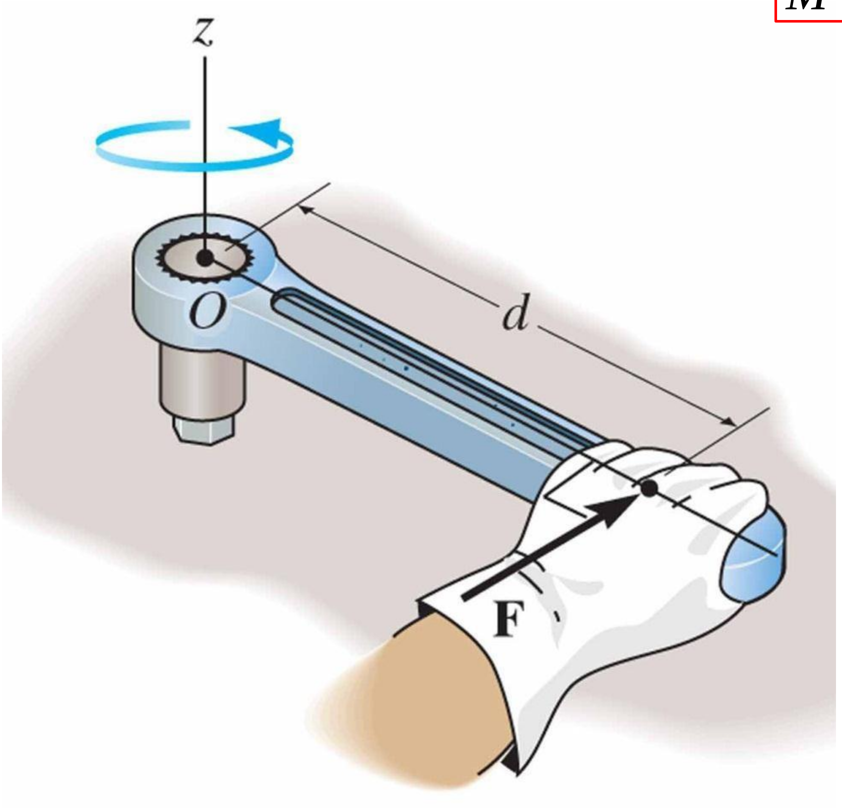
$$F_{\text{R}} + F_{\text{person}} = 4.08\,[\text{kN}] < \max F_{\text{R}} = 5.0\,[\text{kN}]$$

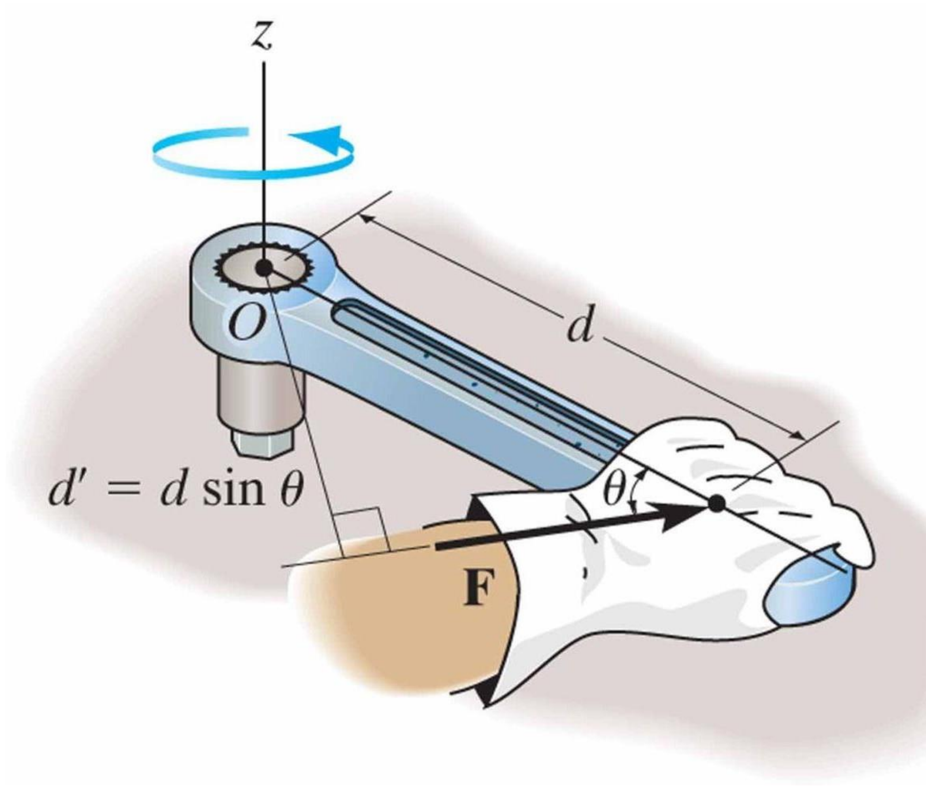
Break

Moment of a force

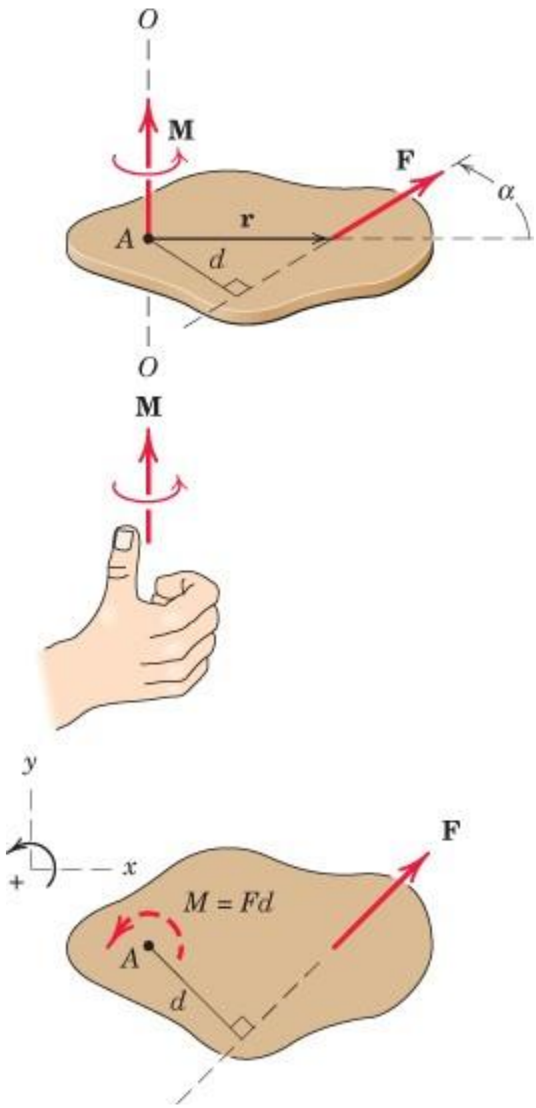
The Moment of a force is a measure of its tendency to cause a body to rotate about a specific point or axis. In order for a moment to develop, the force must act upon the body in such a manner that the body would begin to twist.

$$\boxed{M = Fd}$$





Moment M about a Point (2D) / Axis (3D)



Magnitude of the moment

$$\boxed{M = Fd}$$

where *d* is the moment arm.
SI unit of a moment is [Nm].

The cross product

$$\boxed{\mathbf{M} = \mathbf{r} \times \mathbf{F} = (Fr\sin\alpha)\mathbf{u_M}}$$

$$\boxed{\mathbf{r} \times \mathbf{F} = (r_y F_z - r_z F_y)\mathbf{i} - (r_x F_z - r_z F_x)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k}}$$

where **r** is the position vector, $Fr\sin\alpha$ is the magnitude and $\mathbf{u_M}$ is a unit vector in the direction of moment axis.

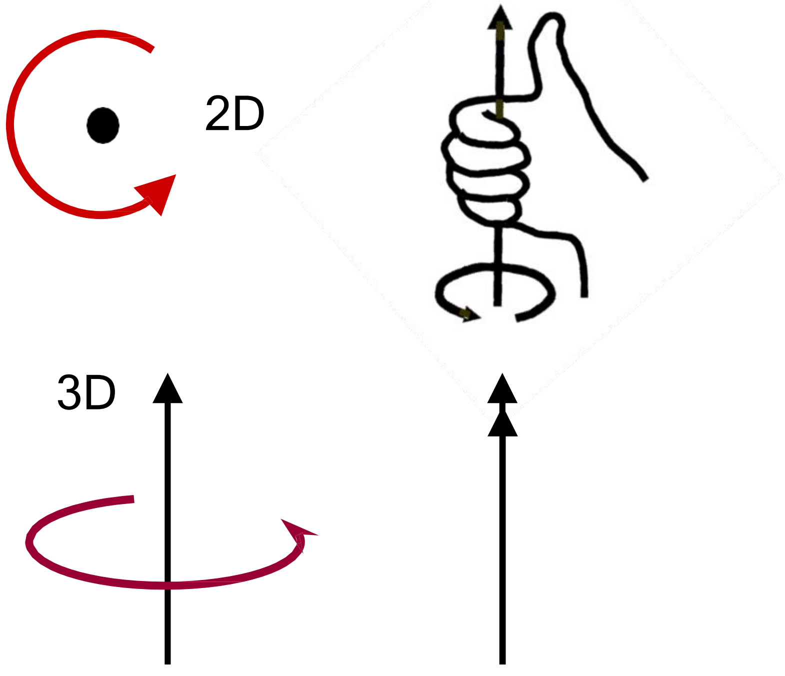
Varignon's Theorem

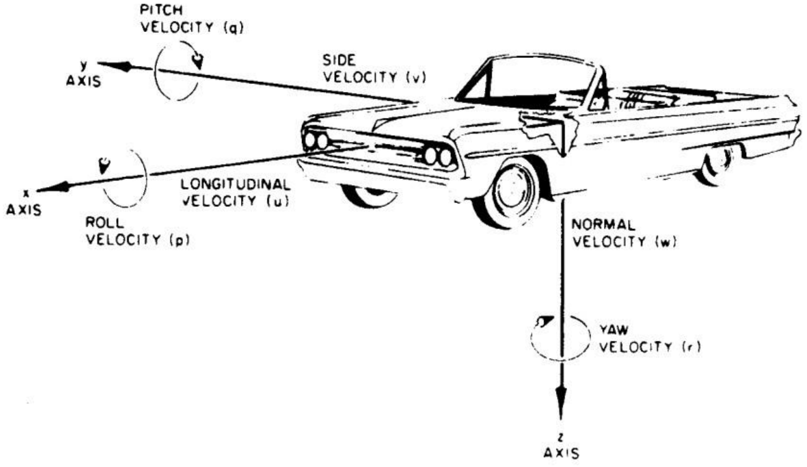
Moment of a force about any point is equal to the sum of the moments of the components of the force about the same point.

Direction of the Moment

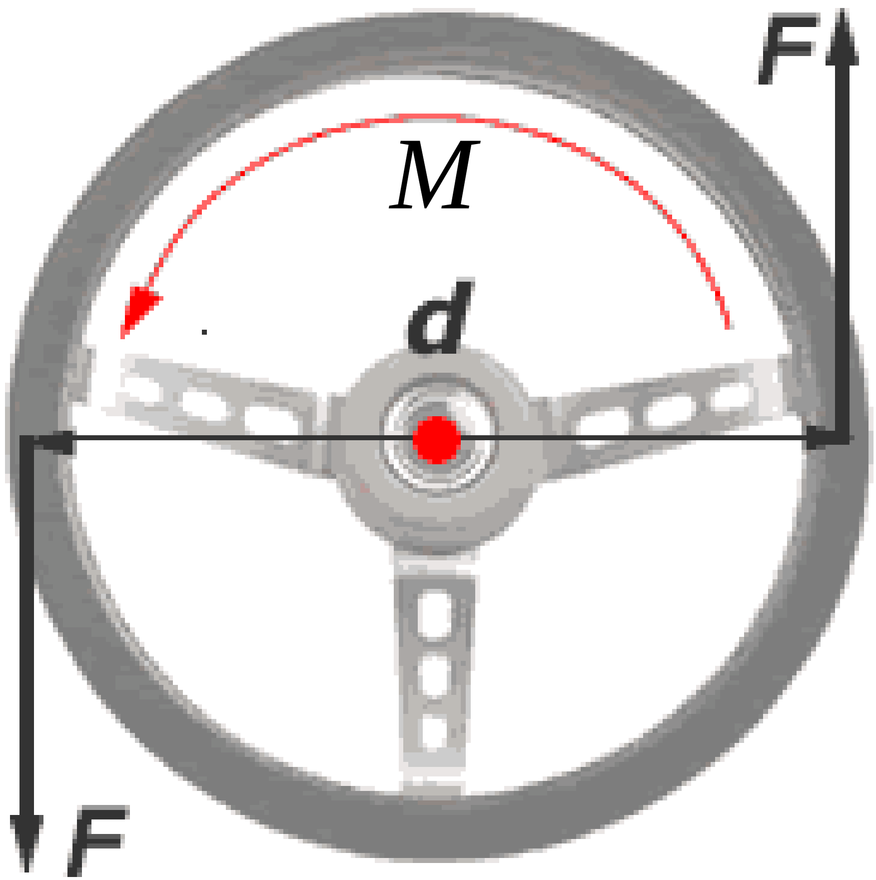
- In 2-dimensional problems, the moment is related to a point about which rotation occurs (the axis of rotation is out of the page, in this case).
- In 3D, the orientation of the axis can be obtained by the "right-hand rule" as shown in the Figure. The moment is as well drawn as a "double arrow".
- A positive moment turns always anti-clockwise.
- Thus in 3D, there are three principal directions for the moments each rotating about one of the coordinate axis.

$$\boxed{\mathbf{M}_x;\mathbf{M}_y;\mathbf{M}_z}$$





Moment *M* of a force couple



Moment M of a Force Couple

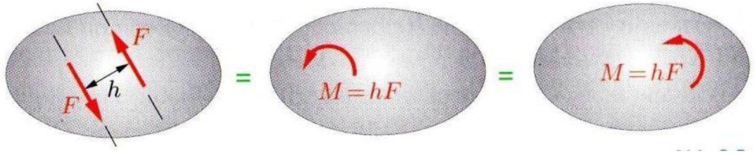
- Two equal and opposite forces separated by a distance *d* produce **moment** M of magnitude

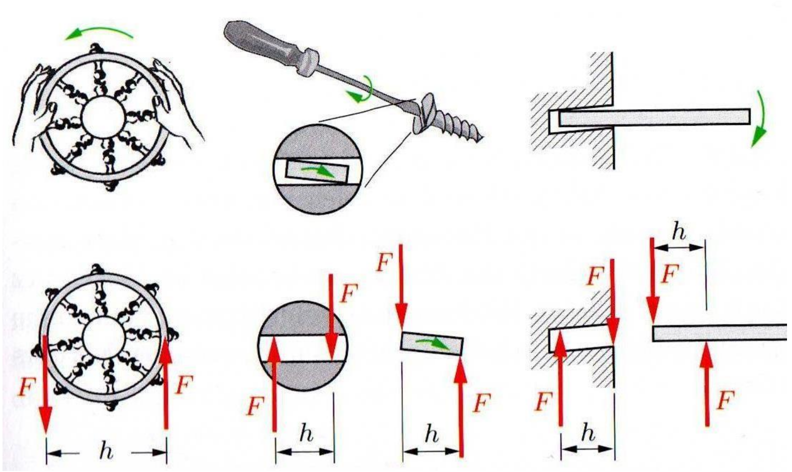
$$\boxed{M = F(a+d) - Fa = Fd}$$

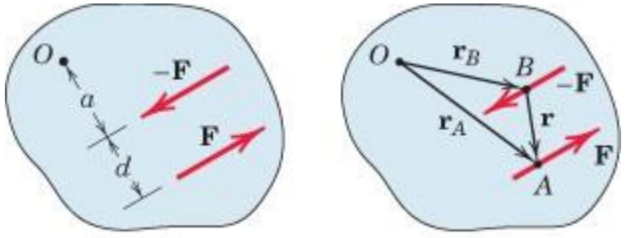
- Vector algebra method

$$\boxed{\begin{aligned} \mathbf{M} &= \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F} \\ \mathbf{r} &= \mathbf{r}_A - \mathbf{r}_B \\ \mathbf{M} &= \mathbf{r} \times \mathbf{F} \end{aligned}}$$

=> Thus moment of a force couple is a free vector.

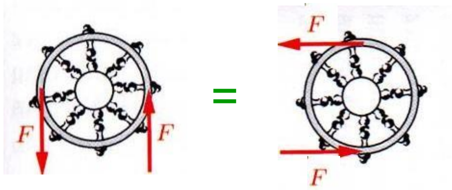






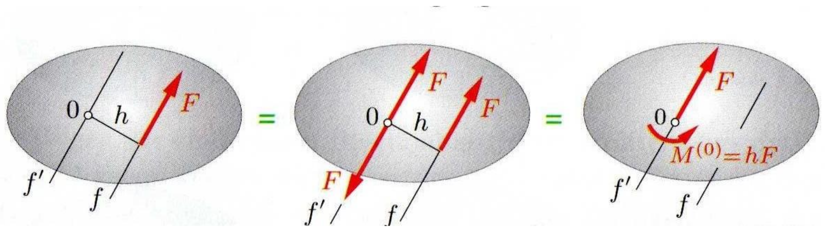
(a) (b)

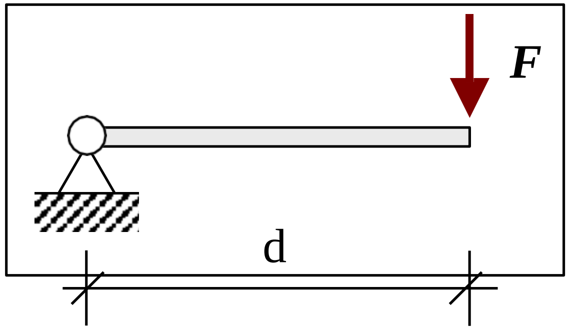
Equivalent Couples



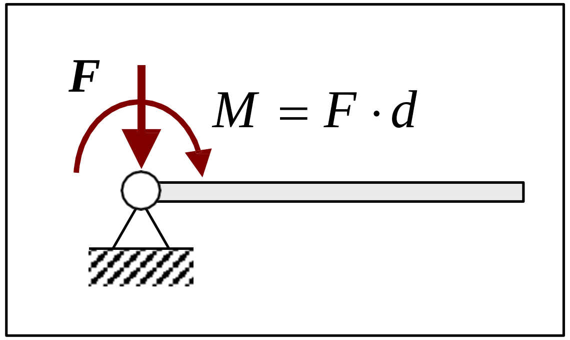
Moment M [Nm]

- **Force-Couple systems**: The translation of a force perpendicular to its direction creates a moment.





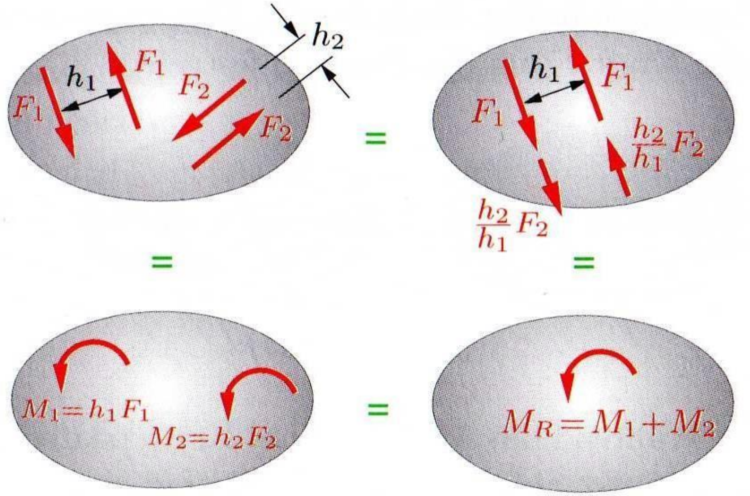
equivalent



- The resultant moment is defined as sum of the single moments

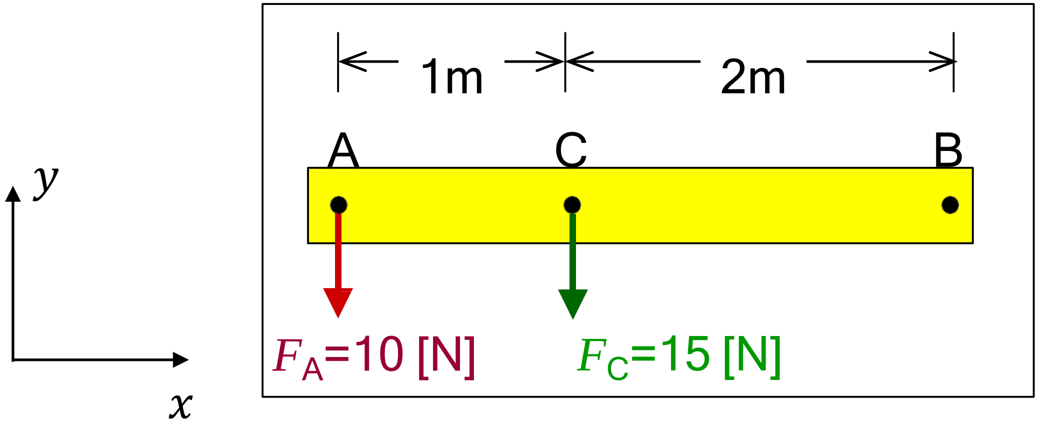
$$\boxed{\mathbf{M}_R = \sum_{i=1}^{n} \mathbf{M}_i = \mathbf{M}_1 + \mathbf{M}_2}$$

- The magnitude of the moment depends on the distance h



Example: Moment (1)

- Consider the rod AB:



- **Question:** What are the moments of the force F_A with respect to the points B and C?
- **Answer**
  For the force acting on Point A, the moments are:

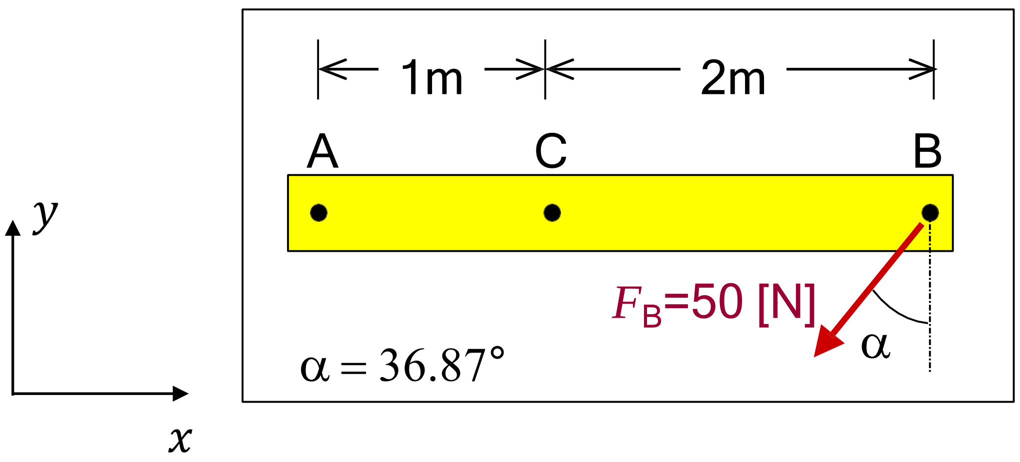
$$M_B = 10 \times 3 = 30 \text{ [Nm]}$$
$$M_C = 10 \times 1 = 10 \text{ [Nm]}$$

Try it yourself:

- **Question:** What are the moments of the force F_C with respect to the points A and B?
- **Answer**
  ...

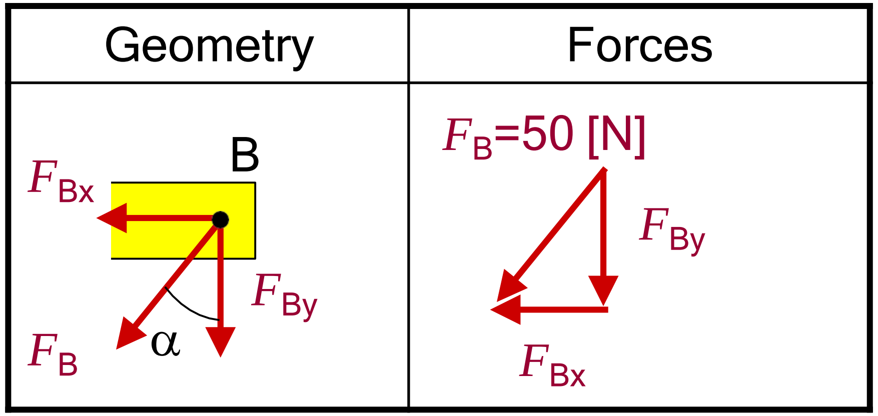
Example: Moment (2)

- Consider the rod AB:



- **Question 1:** What are the horizontal and vertical components of F_B?
- **Question 2:** What is the moment due to the force F_B with respect to Point A?

- **Answer 1:**



$$F_{Bx} = F_B \sin\alpha = \text{-}30\,[\text{N}]$$
$$F_{By} = F_B \cos\alpha = \text{-}40\,[\text{N}]$$

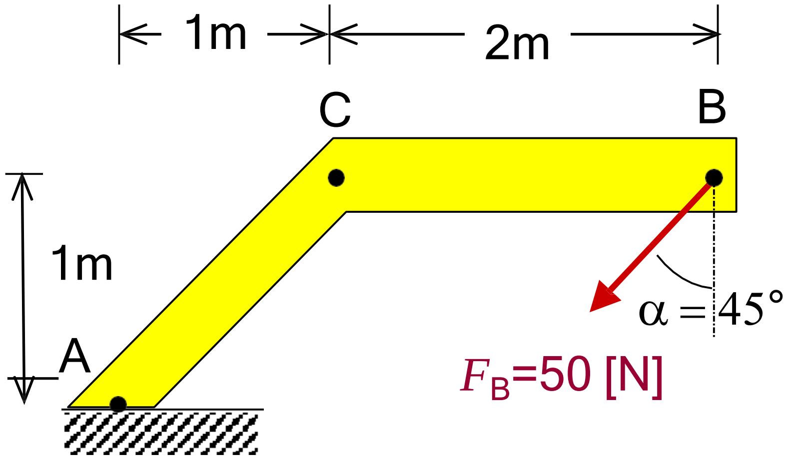
- **Answer 2:**

$$M_A = F_{Bx} \cdot 0\,\text{m} + F_{By} \cdot 3\,\text{m}$$
$$= \text{-}120\,[\text{Nm}]$$

The moment of F_{By} is clockwise, therefore negative value

Try it yourself…

- Consider the structure ACB:



- **Question 1:** Determine the moment with respect to Point A due to the force F_B?

- **Answer 1:**

…

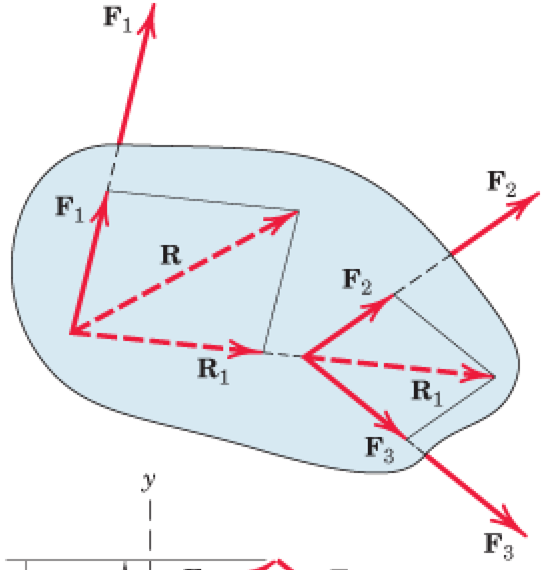
Resultants

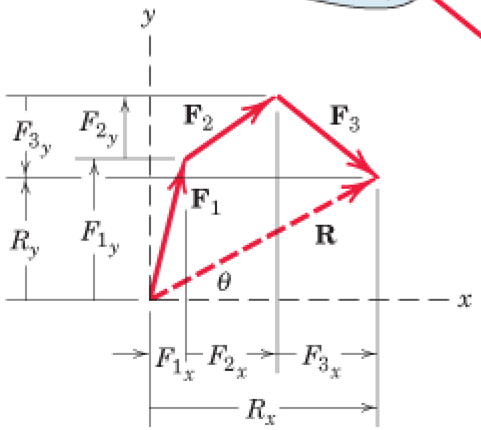
- **Resultant force** for a system of forces acting on a body (system does not have to be concurrent)

$$R = F_1 + F_2 + F_3 + \quad = \sum F$$

$$R_x = \sum F_x \quad R_y = \sum F_y \quad R = \sqrt{\left(\sum F_x\right)^2 + \left(\sum F_y\right)^2}$$

$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{\sum F_y}{\sum F_x}$$

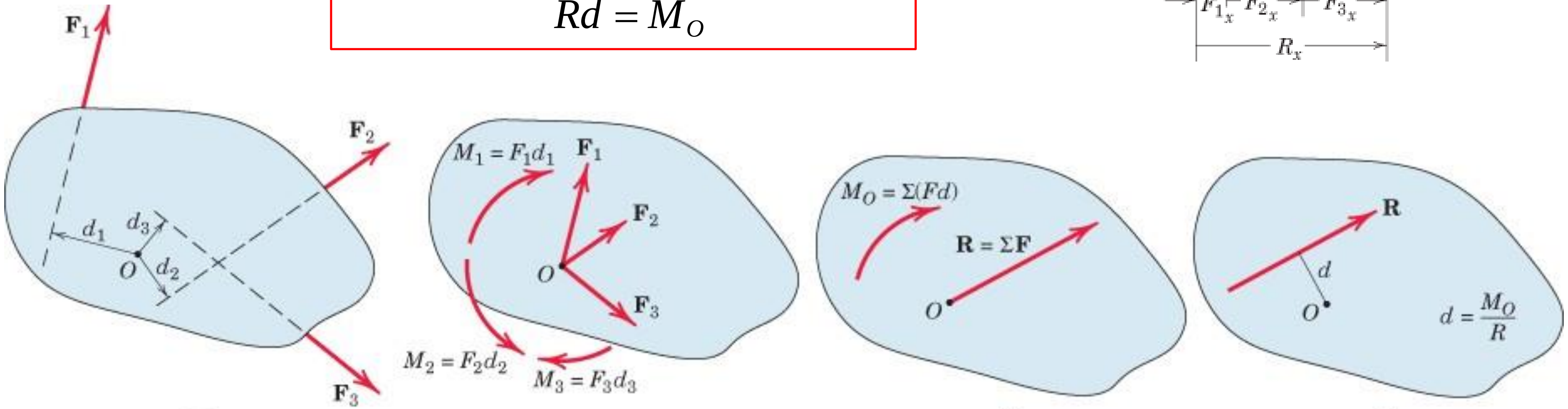




- **Principle of moments**

$$R = \sum F \quad M_O = \sum M = \sum (Fd)$$

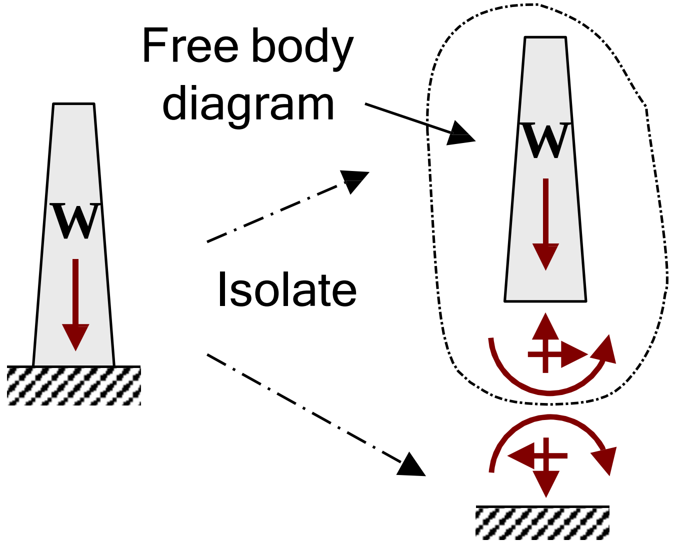
$$Rd = M_O$$

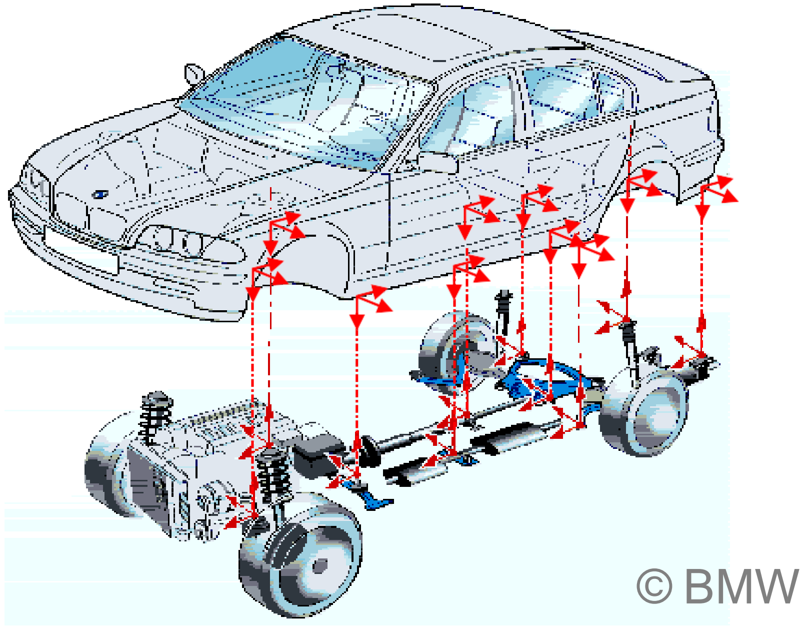


(a) (b) (c) (d)

Full Free Body Diagram with Forces and Moments

- Definition: A free body diagram is a sketch that shows a body “free” from its surroundings with all the forces and moments that act on the body. In other words, “Isolate” the body !
- When isolated, the forces and moments at the points where the body was linked to other objects should be considered.
- Referring to Newton’s 3rd principle “action = reaction”, the forces and moments at the two sides where the object was separated from the environment have the same values but opposite orientations.
- It is strongly recommended to draw for all problems these free body diagrams!





Lessons Learned

You should have learned and understood:

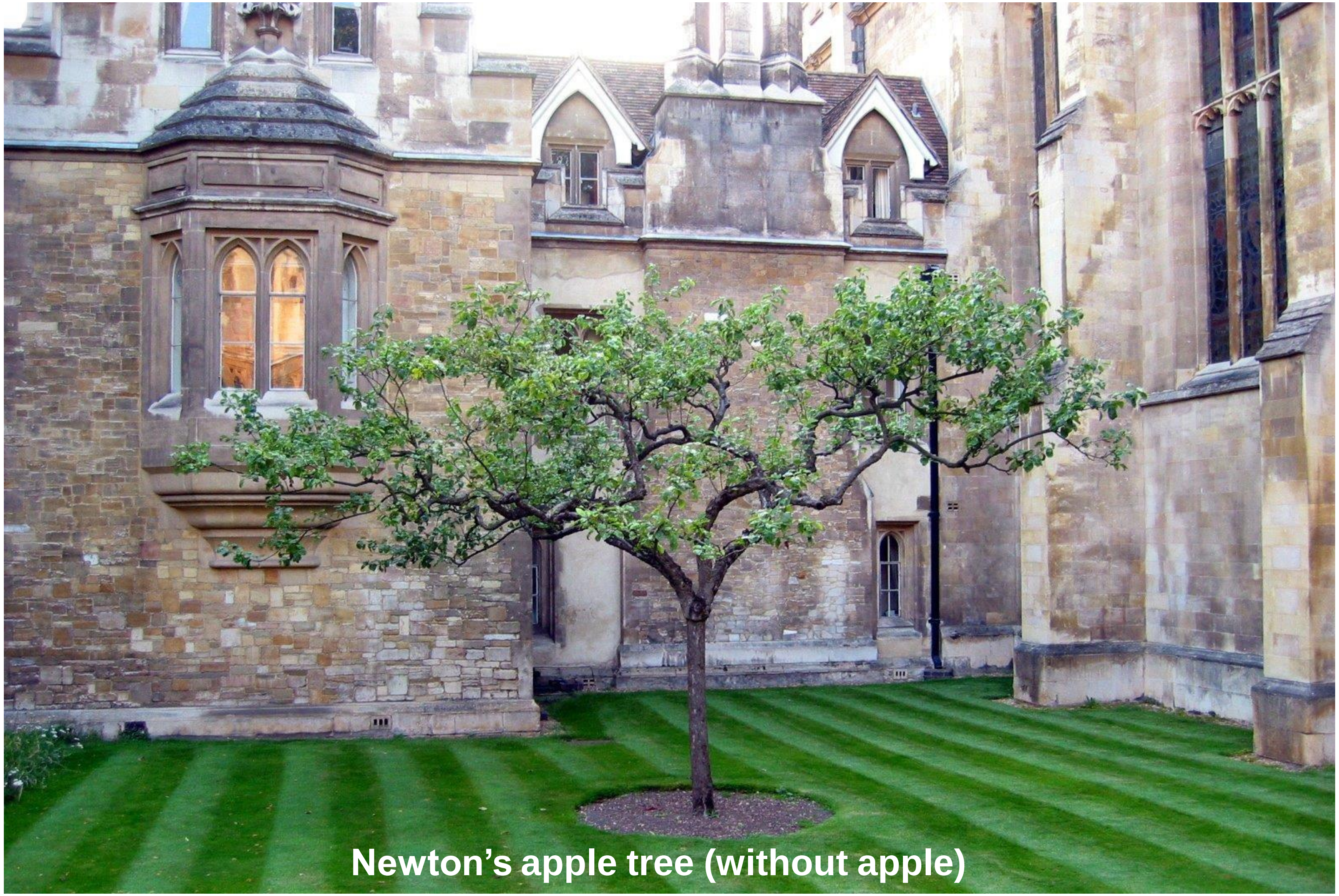
1. What is a scalar and a vector?
2. Which physical quantities are scalars and which are vectors?
3. Have you understood the three Laws of Motion of Newton?
4. What is a force and a moment?
5. How is a free body diagram derived? Which forces and moments have to be introduced?
6. How is a resultant of a group of forces or of a distributed force computed?

7. How is the moment computed from a couple of forces, which are parallel and pointing in opposite directions?

- How is a resultant of two moments computed?
- How is the moment with respect to a particular point computed, which is generated by several forces?

8. What is the “right-hand rule” for moments? Why do you need that?

End.



Fundamentals

Decimal multiples and sub-multiples

1,000,000,000	10^9	Giga	G
1,000,000	10^6	Mega	M
1,000	10^3	Kilo	k
0.001	10^{-3}	Milli	m
0.000 001	10^{-6}	Mikro	μ
0.000 000 001	10^{-9}	Nano	n

Unit conversion

	Unit (FPS)	Unit (SI)
Force	1 lb	4.482 N
Mass	1 slug	14.5938 kg
Length	1 ft	0.3048 m
Length	1 inch	0.0254 m
Length	1 mile	1,609.0 m
Liquid	1 pint	0.568 l
Liquid	1 gallon	4.546 l
Temperature	$x\,°F = \frac{5}{9}(x-32)°C$	

Greek alphabet

Alpha	α A	Ny	ν N
Beta	β B	Xi	ξ Ξ
Gamma	γ Γ	Omikron	o O
Delta	δ Δ	Pi	π Π
Epsilon	ε E	Rho	ρ P
Zeta	ζ Z	Sigma	σ Σ
Eta	η H	Tau	τ T
Theta	θ ϑ	Ypsilon	υ Y
Jota	ι I	Phi	ϕ Φ
Kappa	κ K	Chi	χ X
Lambda	λ Λ	Psi	ψ Ψ
My	μ M	Omega	ω Ω

SI = International Units System

EMS412U - MATHEMATICAL AND COMPUTATIONAL MODELLING 1

Degrees of freedom

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Lecture notes adapted from: Dr. Emiliano Bilotti

Queen Mary University of London

Degrees of Freedom

"Degrees of Freedom" (DOF):
Any of the minimum number of **coordinates** required to specify completely the **motion** of a mechanical system.

DOF= $d \cdot \frac{(d+1)}{2}$, in **d** dimensions
of which:

d translations
and
$d \cdot \frac{(d-1)}{2}$ **rotations**

3 degrees of freedom
2 translations + 1 rotation

1 degree of freedom
1 translation on the rails

1 degree of freedom
1 rotation

3 degrees of freedom
2 translation + 1 rotation

Degrees of Freedom Suppression – Reactions

If a number of degrees of freedom are restricted (e.g. by an external support), you need to insert the same number of reactions (forces or moments) to the free body diagram

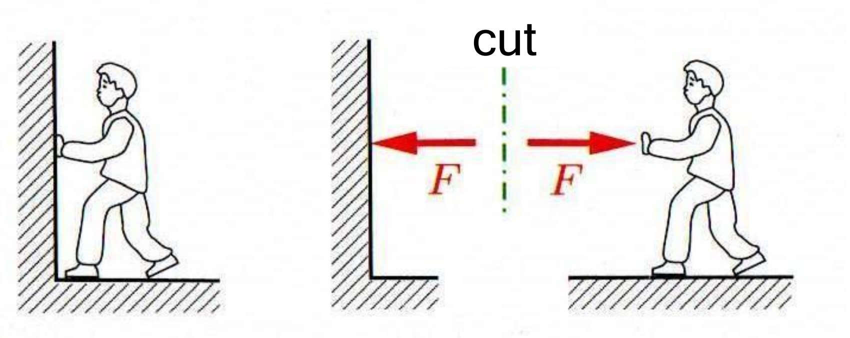
The reactions (forces or moments) are calculated by applying the equilibrium conditions

Previously...

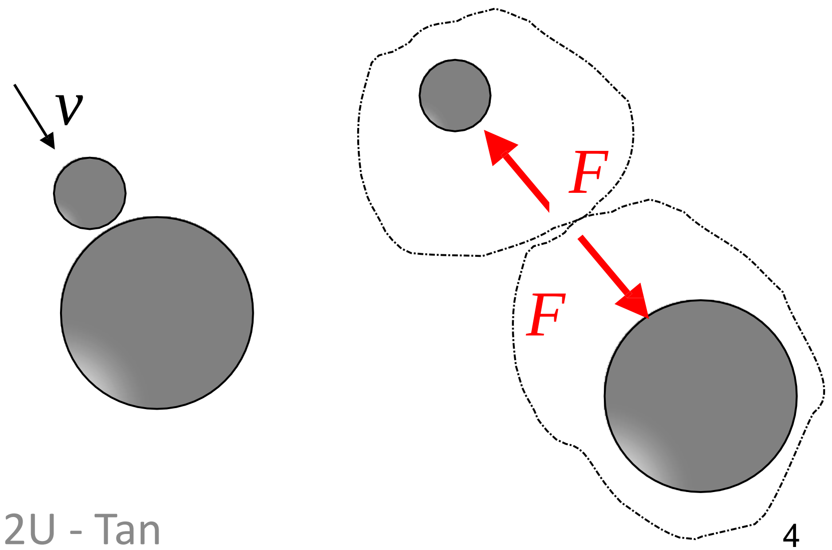
- Newton's 3rd law states that:
  "To every action (force exerted) there is an equal and opposite reaction", which is:

  action = reaction (3)

- This principle is realized by **cutting** the objects from one another; the so-called **free-body diagram** is obtained.
- A free body diagram means:
  1. A closed cut around the object
  2. At each point where the object was separated, reaction forces (and eventually moments) have to be inserted
  3. The object should be totally isolated.

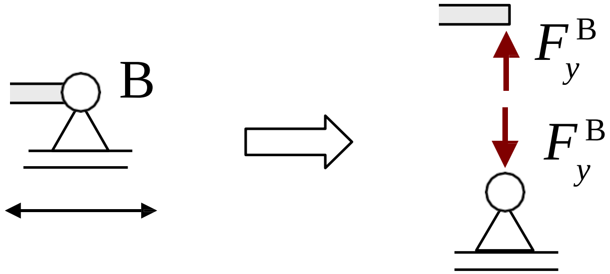


Free Body Diagram



Reaction to different types of support for elastic beams

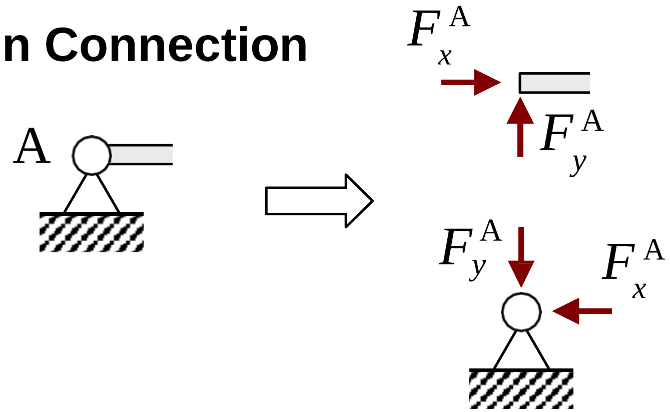
1. Non-fixed support



- It can not move vertically
- It can move horizontally
- It can rotate

→ 1 degree of freedom restricted = 1 reaction

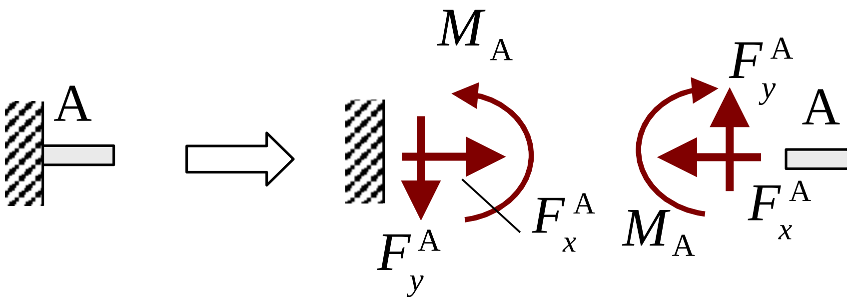
2. Pin Connection



- It cannot move horizontally
- It cannot move vertically
- It can rotate

→ 2 degrees of freedom restricted = 2 reactions

3. Clamped/Fixed support



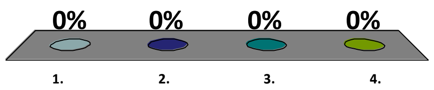
- It cannot move horizontally
- It cannot move vertically
- It cannot rotate

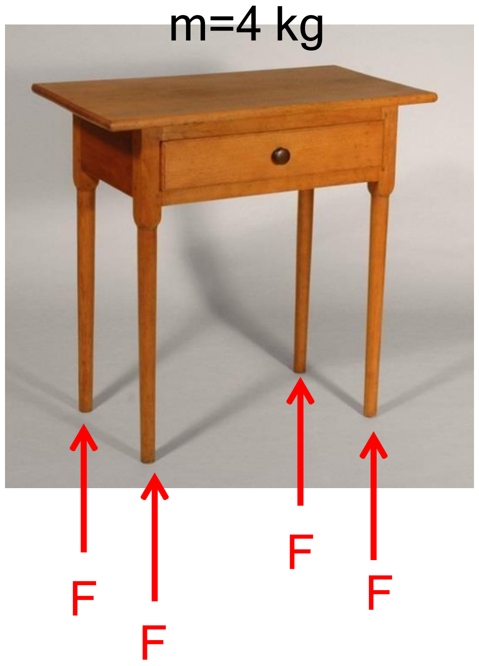
→ 3 degrees of freedom restricted = 3 reactions

Support Reactions

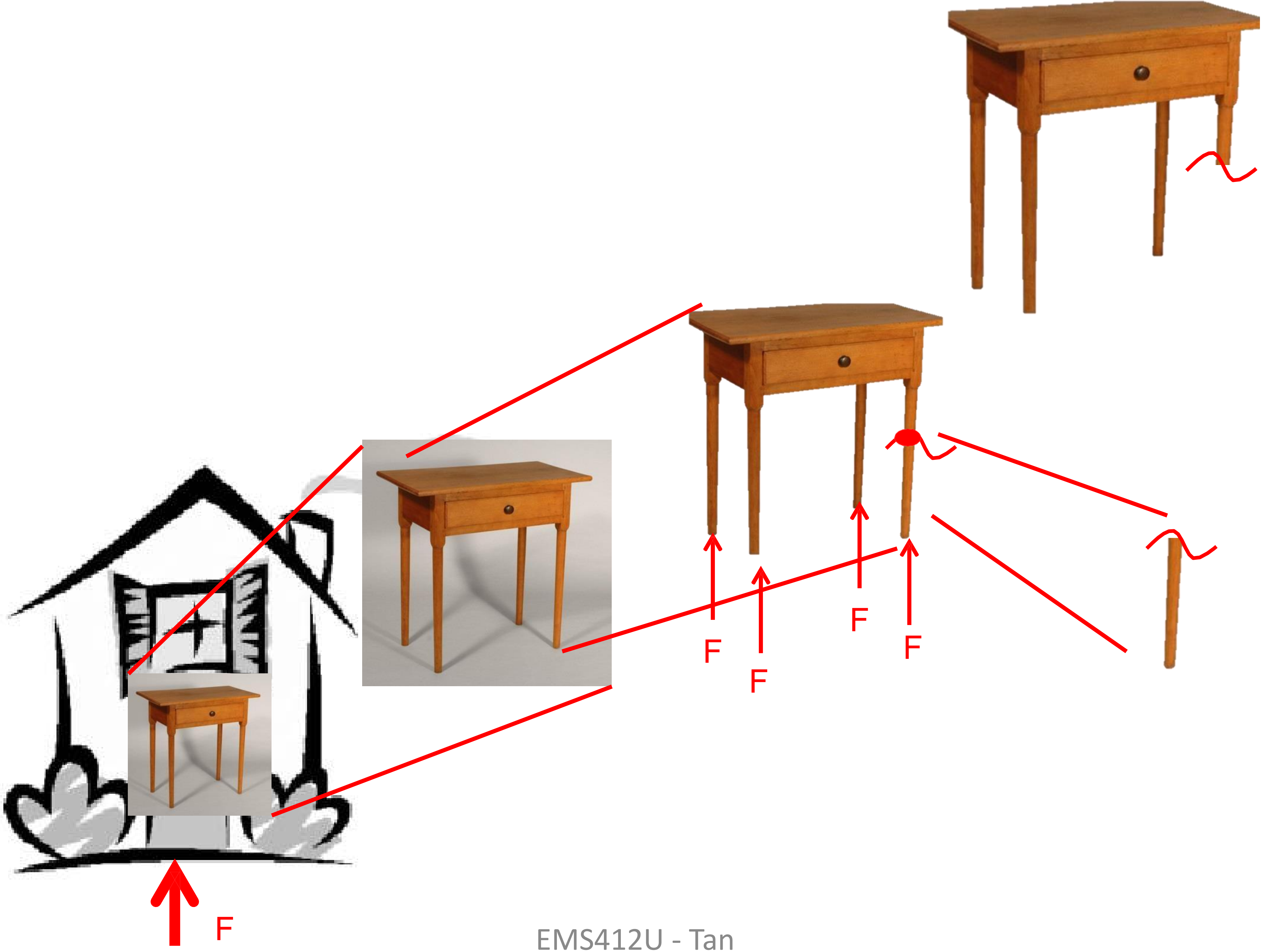
The table in figure below has a mass of 4kg. Which force is acting on the floor under each leg (the table is equally supported by all 4 legs)?

1. 98.1 N
2. 9.81 N
3. 981 N
4. 0.981 N

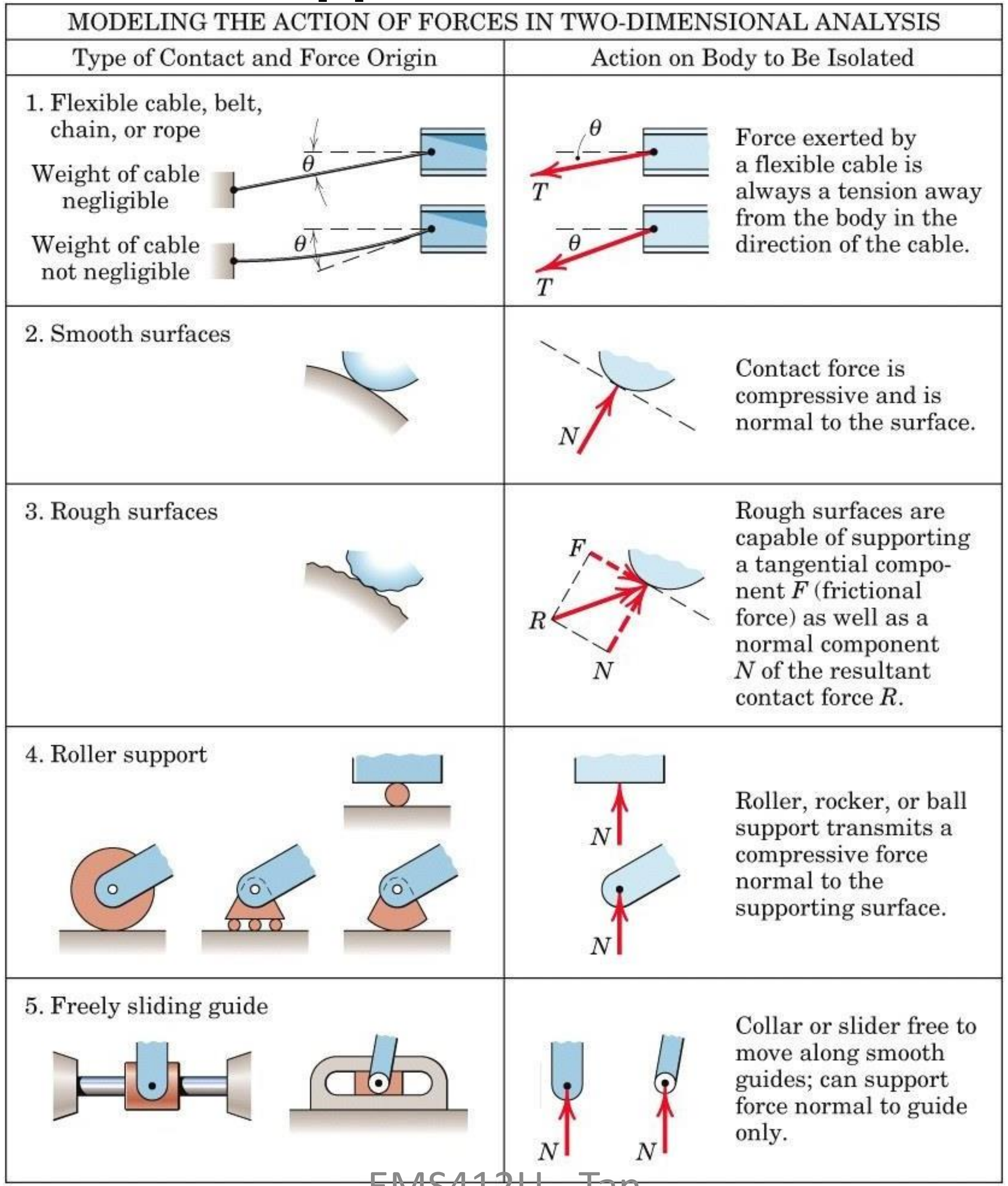




Another table similar to the one above has an unknown mass. Which force is acting on the floor under each leg (the table is equally supported by all 4 legs)?


F
F
F
F
F

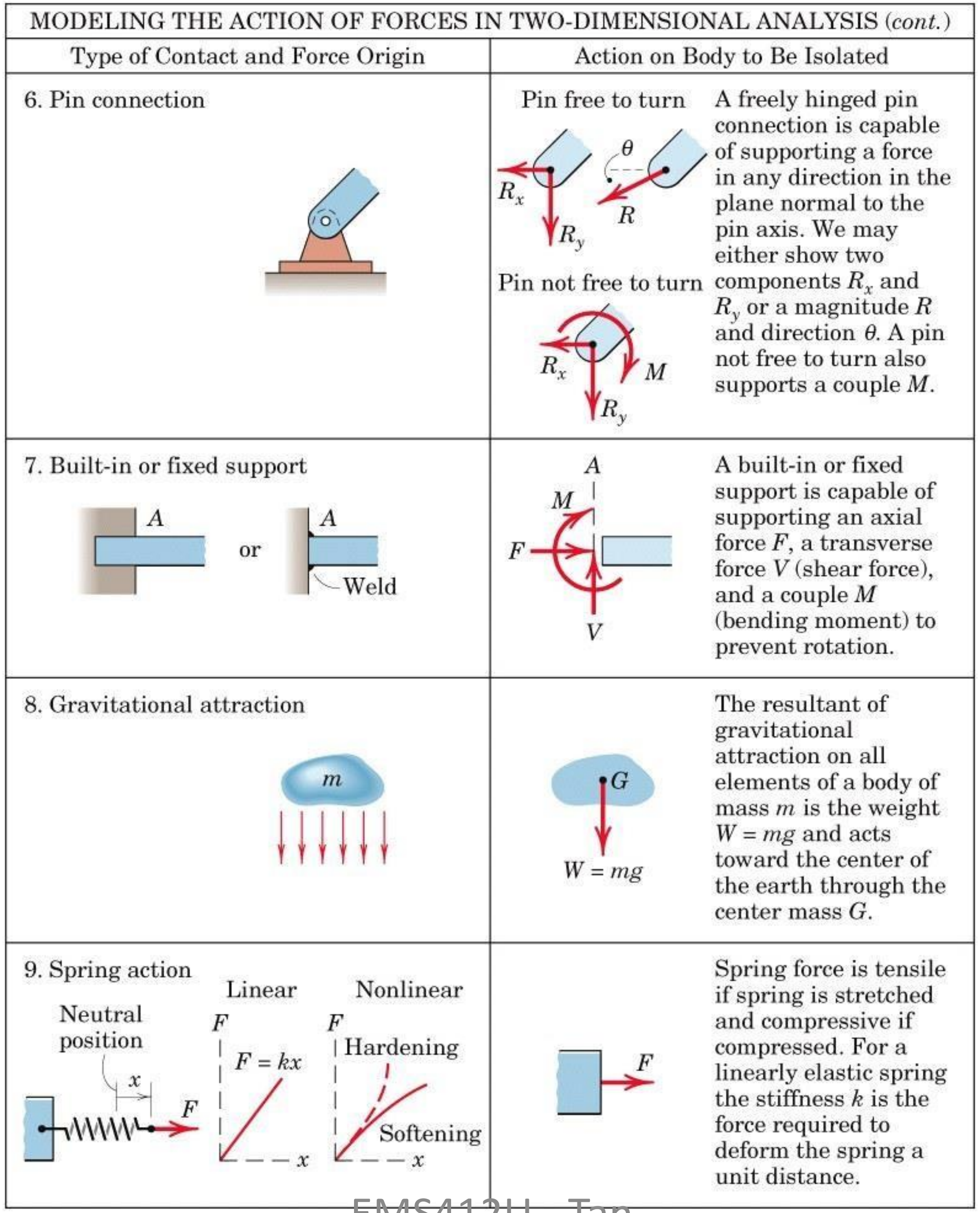
Support Reactions



MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
1. Flexible cable, belt, chain, or rope Weight of cable negligible (θ) Weight of cable not negligible (θ)	(θ, T) (θ, T) Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces	(N) Contact force is compressive and is normal to the surface.
3. Rough surfaces	(F, R, N) Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.
4. Roller support	(N) (N) Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide	(N) (N) Collar or slider free to move along smooth guides; can support force normal to guide only.

Figure 3-1a

Support Reactions



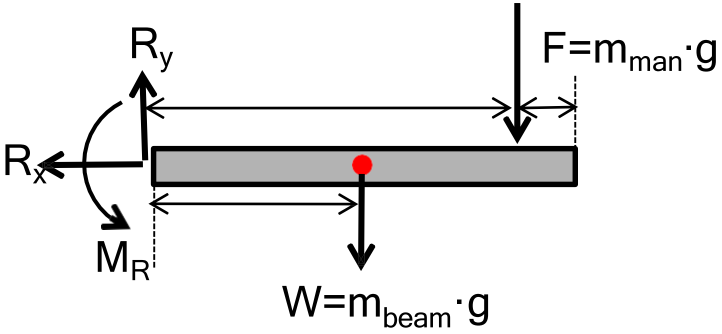
MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (*cont.*)		
Type of Contact and Force Origin	**Action on Body to Be Isolated**	
6. Pin connection	Pin free to turn R_x, R_y, R, θ Pin not free to turn R_x, R_y, M	A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.
7. Built-in or fixed support A or A Weld	A, M, F, V	A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction m	G $W = mg$	The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action Neutral position, x, F Linear: F, $F = kx$, x Nonlinear: F, Hardening, Softening, x	F	Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

Figure 3-1b

Free Body Diagram – Procedure

1. **Draw an outlined shape of the body**
Imagines the body to be isolated of "free cut" from its constrains and connections and sketch its outlined shape

2. **Show all force and couple moments acting on the body**
Identify all the external forces and couple moments acting on the body. Usually they are: a) applied loadings, b) reactions occurring at the supports and points of contact with other bodies, c) the weight of the body.



3. **Label all the loadings and specify their direction relative to the x and y axis.**

4. **Indicate the dimensions of the body necessary for computing the moments of forces**

Free Body Diagram – Procedure

1. **Draw an outlined shape of the body**
Imagines the body to be isolated of “free cut” from its constrains and connections and sketch its outlined shape

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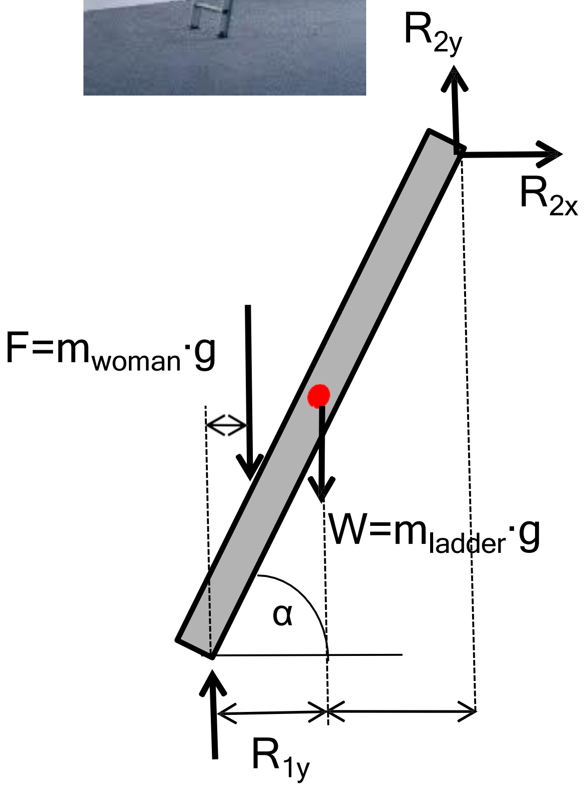
Identify all the external forces and couple moments acting on the body. Usually they are: a) applied loadings, b) reactions occurring at the supports and points of contact with other bodies, c) the weight of the body.



3. **Label all the loadings and specify their direction relative to the x and y axis.**

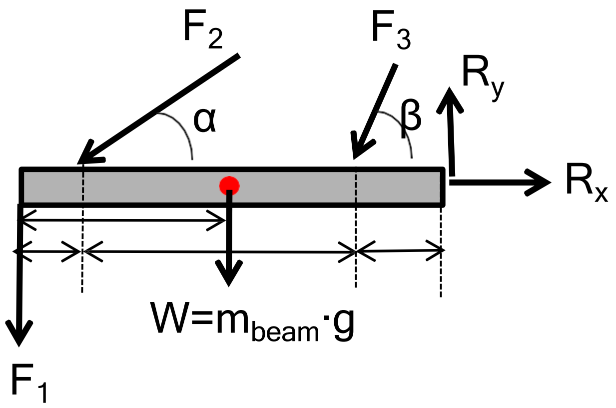
4. **Indicate the dimensions of the body necessary for computing the moments of forces**

Free Body Diagram – Procedure (Try yourself)

1. **Draw an outlined shape of the body**
Imagines the body to be isolated of "free cut" from its constrains and connections and sketch its outlined shape

2. **Show all force and couple moments acting on the body**
Identify all the external forces and couple moments acting on the body. Usually they are: a) applied loadings, b) reactions occurring at the supports and points of contact with other bodies, c) the weight of the body.

3. **Label all the loadings and specify their direction relative to the x and y axis.**



4. **Indicate the dimensions of the body necessary for computing the moments of forces**

Free Body Diagram

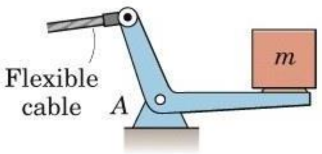
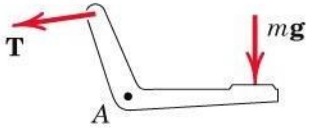
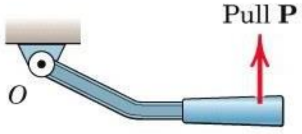
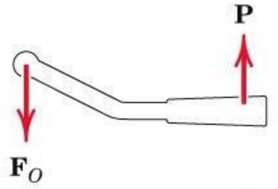
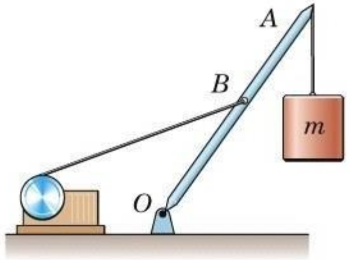
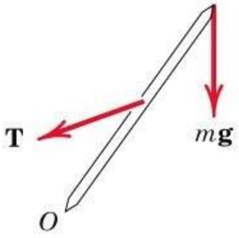
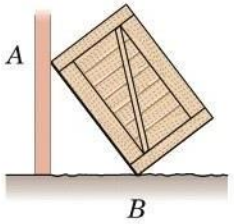
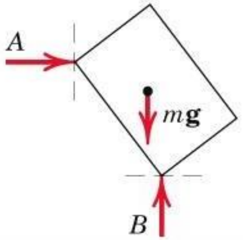
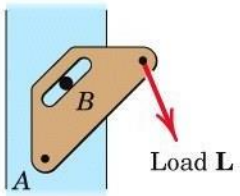
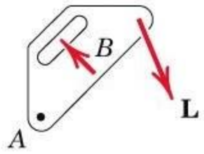
	Body	Incomplete FBD
1. Bell crank supporting mass m with pin support at A.		
2. Control lever applying torque to shaft at O.		
3. Boom OA, of negligible mass compared with mass m. Boom hinged at O and supported by hoisting cable at B.		
4. Uniform crate of mass m leaning against smooth vertical wall and supported on a rough horizontal surface.		
5. Loaded bracket supported by pin connection at A and fixed pin in smooth slot at B.		

Figure 3-A

Free Body Diagram

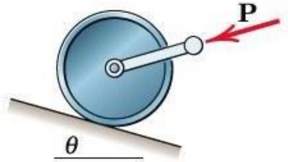
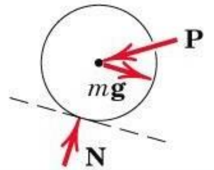
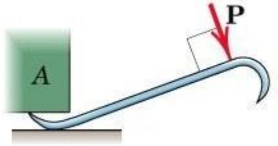
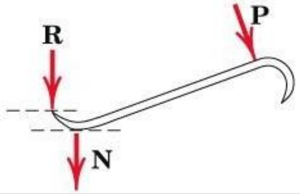
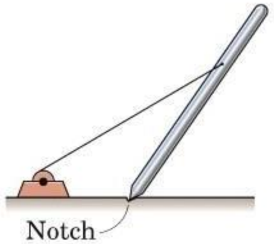
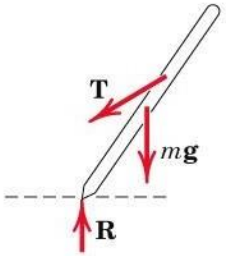
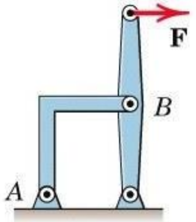
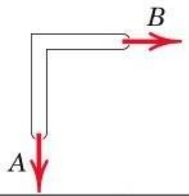
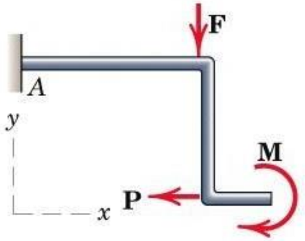
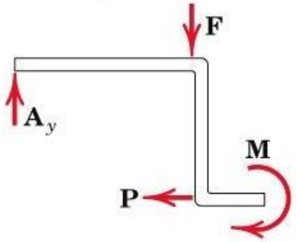
	Body	Incomplete FBD
1. Lawn roller of mass m being pushed up incline θ.		
2. Prybar lifting body A having smooth horizontal surface. Bar rests on horizontal rough surface.		
3. Uniform pole of mass m being hoisted into position by winch. Horizontal supporting surface notched to prevent slipping of pole.		
4. Supporting angle bracket for frame; pin joints.		
5. Bent rod welded to support at A and subjected to two forces and couple.		

Figure 3-B

EMS412U - MATHEMATICAL AND COMPUTATIONAL MODELLING 1

Static equilibrium

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Lecture notes adapted from: Dr. Emiliano Bilotti

Queen Mary
University of London

Static Equilibrium

- A system is in **static equilibrium** if the effects of all forces and moments create an overall balance;
- This means: the resultant force and the resultant moment are vanishing:

$$\sum_i^m F_{ix} = F_{1x} + F_{2x} + \ldots + F_{mx} = 0$$
$$\sum_i^n F_{iy} = F_{1y} + F_{2y} + \ldots + F_{ny} = 0$$
$$\sum_i^p M_{iz} = M_{1z} + M_{2z} + \ldots + M_{pz} = 0$$

in 2D

and in 3D:

$$\sum_i^m F_{ix} = 0 \quad \sum_i^n F_{iy} = 0 \quad \sum_i^p F_{iz} = 0$$

$$\sum_i^m M_{ix} = 0 \quad \sum_i^n M_{iy} = 0 \quad \sum_i^p M_{iz} = 0$$

Equilibrium means that nothing is moving (or is moving at constant velocity

Static Equilibrium

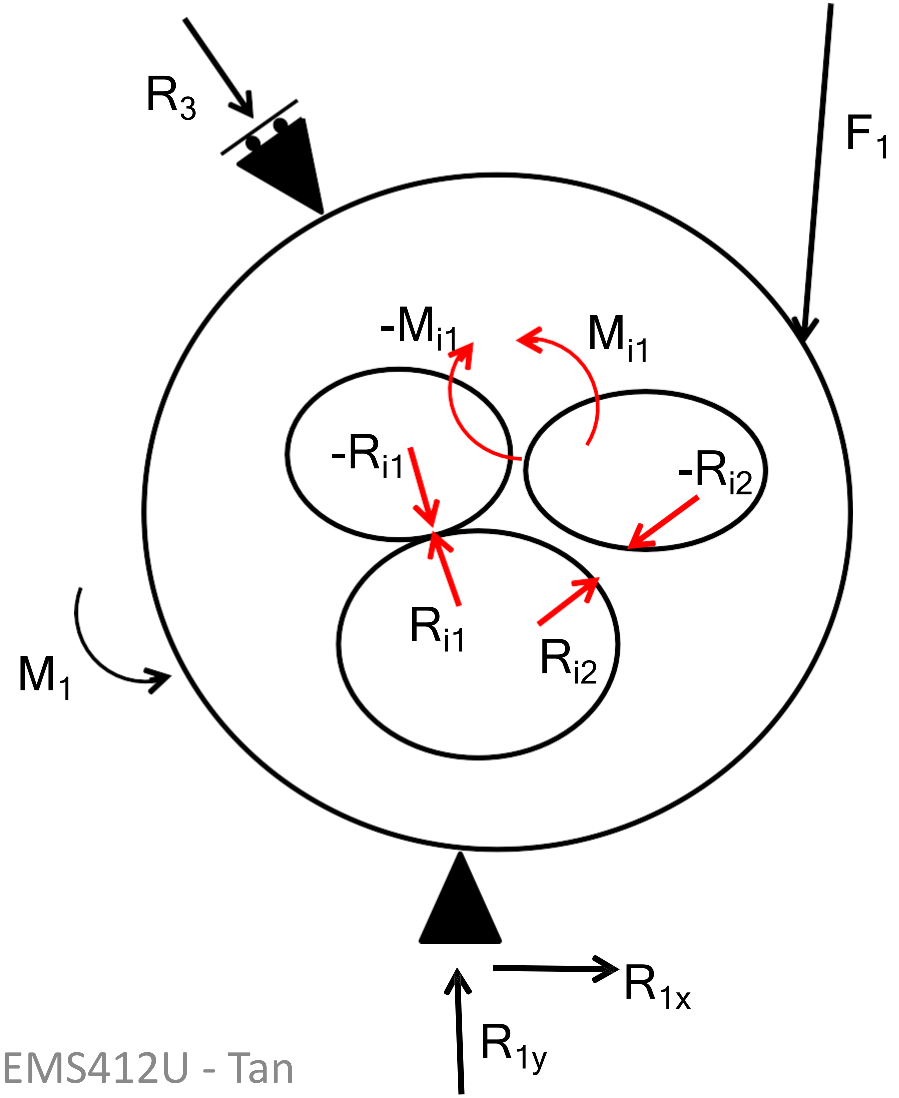
- Only external forces!

$$\sum_{i}^{m} F_{ix} = F_{1x} + F_{2x} + \ldots + F_{mx} = 0$$

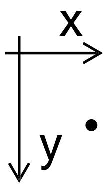
$$\sum_{i}^{n} F_{iy} = F_{1y} + F_{2y} + \ldots + F_{ny} = 0$$

$$\sum_{i}^{p} M_{iz} = M_{1z} + M_{2z} + \ldots + M_{pz} = 0$$

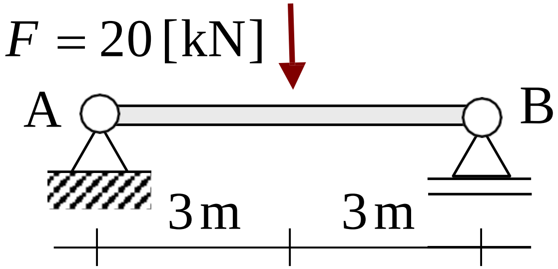
- Internal forces balance out!



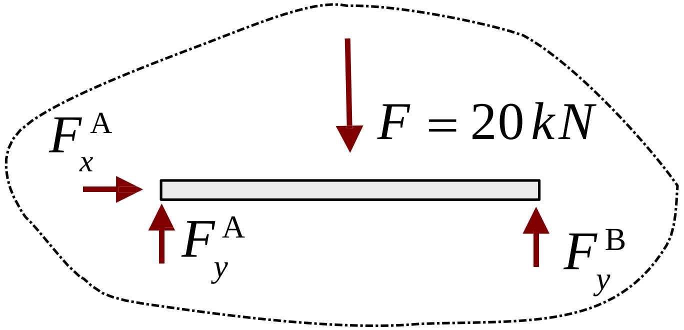
Equilibrium – Elastic Beam Example (1)



- Consider a beam under a single force acting on the middle:

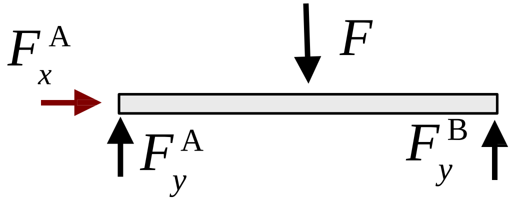


- Draw the free body diagram with all support reactions:



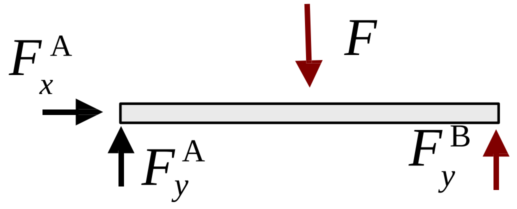
- The system shall be in equilibrium, i.e. it shall not move. Determine the unknown support reactions:

$\boxed{1.\ \sum F_x = 0}$



$$F_x^A = 0$$

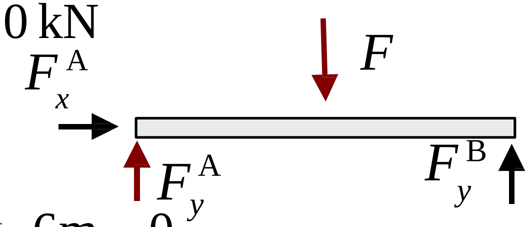
$\boxed{2.\ \sum M_z^A = 0}$



$$-F \cdot 3\mathrm{m} + F_y^B \cdot 6\mathrm{m} = 0$$

$$F_y^B = F/2 = 10\,\mathrm{kN}$$

$\boxed{3.\ \sum M_z^B = 0}$



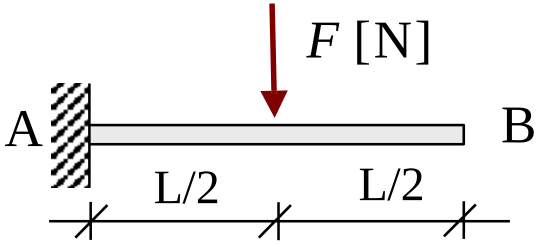
$$+F \cdot 3\mathrm{m} - F_y^A \cdot 6\mathrm{m} = 0$$

$$F_y^A = F/2 = 10\,\mathrm{kN}$$

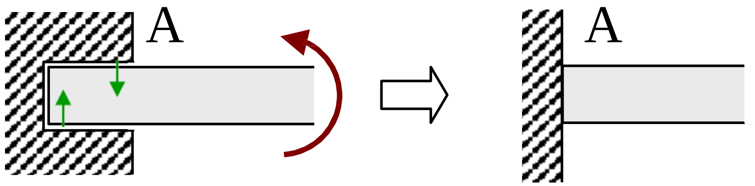
$\boxed{4.\ \text{Control via}\ \sum F_y = 0}$ $\quad -F_y^A + F - F_y^B = 0$

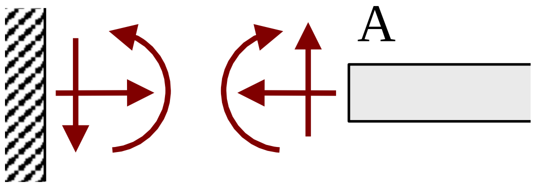
Equilibrium – Elastic Beam Example (2)

- A special type of support is the ***clamped support***, which can transfer vertical and horizontal forces **as well as moments**:



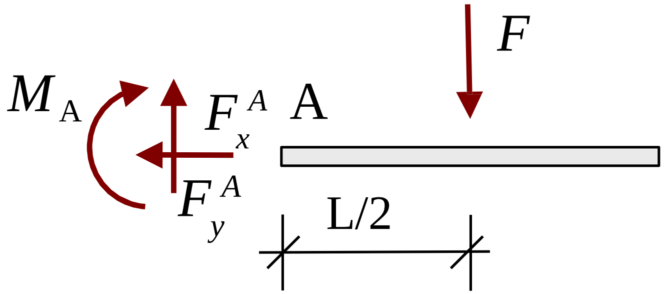
- It can be regarded as a simplification of:





- The moment equilibrium with respect to point A is:

$$\sum_{\mathrm{A}} M = 0: \quad \rightarrow \quad M_{\mathrm{A}} + F \cdot \frac{L}{2} = 0$$



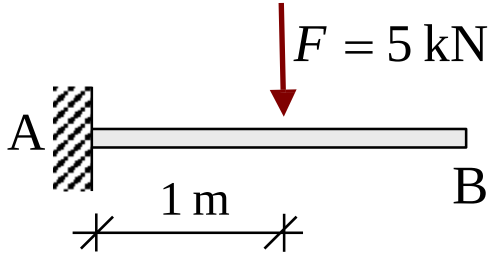
- For the moment equilibrium, every point can be chosen, i.e. the system is for all points in equilibrium.
  For example, moment equilibrium with respect to point B is

$$\sum_{\mathrm{B}} M = 0:$$

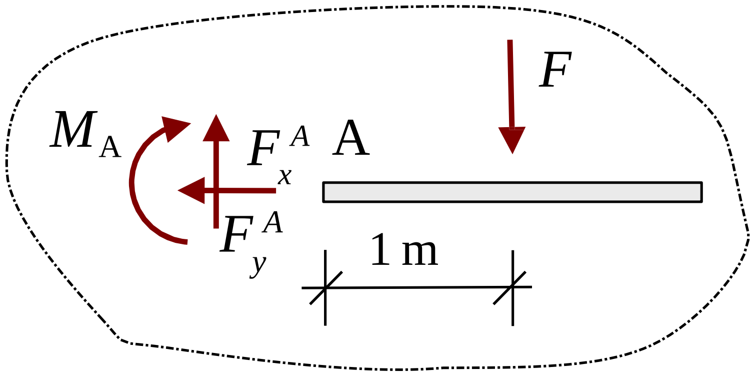
$$\rightarrow \quad M_{\mathrm{A}} - F \cdot \frac{L}{2} + F_y^A \cdot L = 0$$

Equilibrium – Elastic Beam Example (2)

- Compute the support reactions at point A of a simple clamped beam, 2m long, subjected to a load of F = 5 kN.



1. Draw a free body diagram:



Make sure that you have considered all forces, exterior loads and the support reactions.

2. Compute equilibrium of horizontal forces: $$\sum_i F_x = 0: \quad \rightarrow \quad F_x^A = 0$$

   It follows that the horizontal support at point A is zero.

3. Compute equilibrium of vertical forces: $$\sum_i F_y = 0: \quad \rightarrow \quad F_y^A - F = 0$$

   It follows that the vertical support at point A is: $F_y^A = F = 5\,\text{kN}$

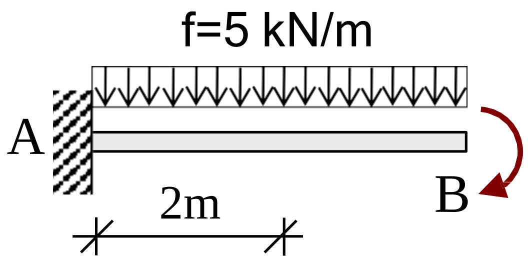
4. Compute the moment equilibrium with respect to point A $$\sum_A M = 0: \quad \rightarrow \quad M_A + F \cdot \frac{L}{2} = 0$$

   It follows that the moment at the left end of the beam is:

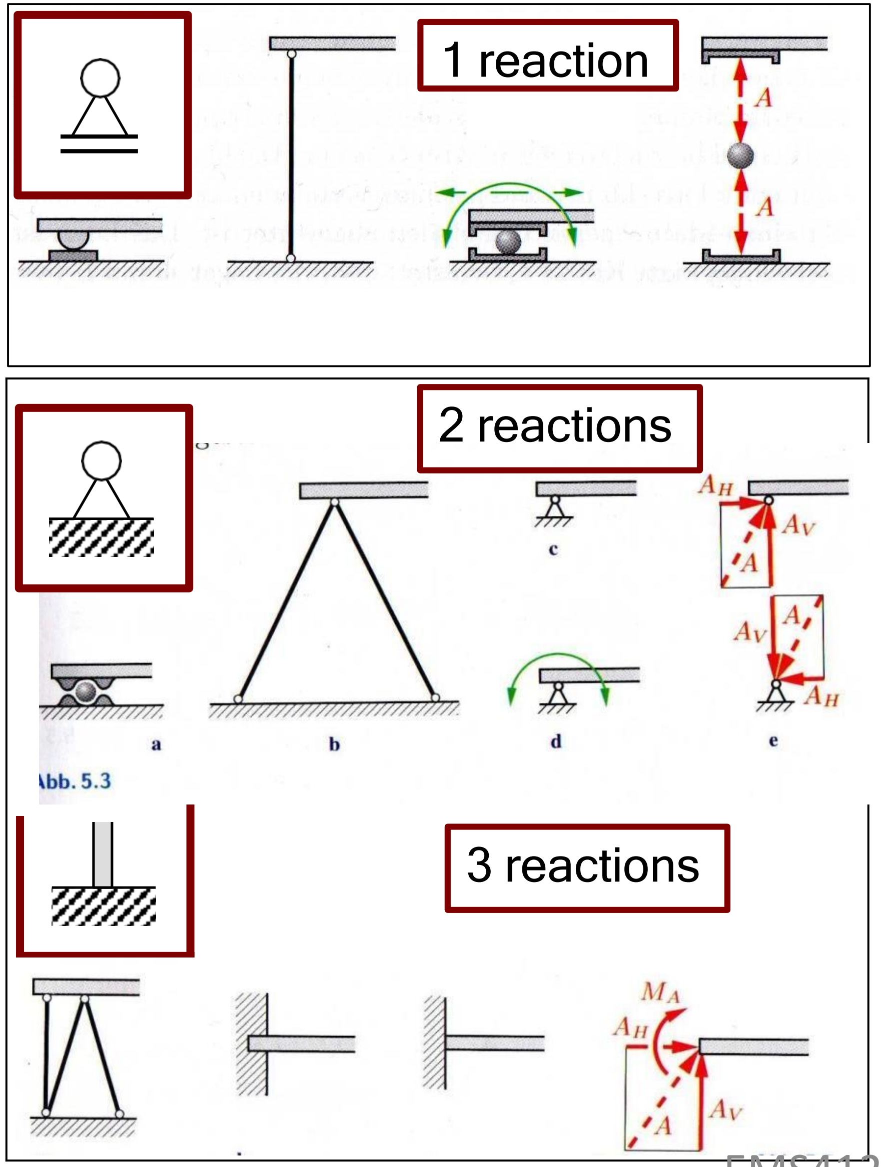
$$M_A = -FL/2 = -5kNm$$

Try it yourself…

- Compute the support reactions at point A of a simple clamped beam 4m long, subjected to a distributed load f= 5 kN/m and a couple moment in B of 5 Nm.



Support Reactions and Statically Determinate

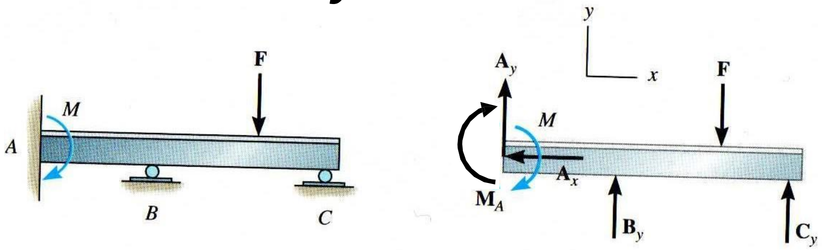


- In 2D-problems we have **3 independent equilibrium equations:**

$$\boxed{\sum_{i}^{m} F_{ix} = 0; \sum_{i}^{n} F_{iy} = 0; \sum_{i}^{p} M_{iz} = 0}$$

- Hence we can determine three unknown support reactions for a structure consisting only of a single part. If more supports are used, the structure is called

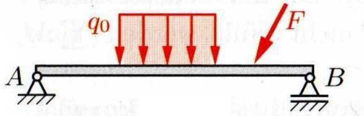
  statically indeterminate

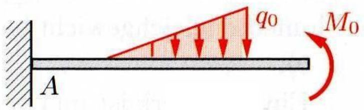


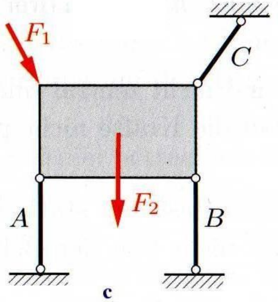
- If the number of support reactions equals the number of equilibrium equations, the structure is ***statically determinate***.

Statically Determinate / Indeterminate

Statically determinate

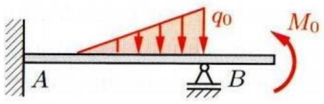


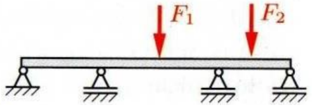




The structure can support the loads without moving and the number of support reactions equals the number of equilibrium conditions

Statically indeterminate





The structure can support the loads without moving. But because the number of support reactions is not equal to the number of equilibrium conditions, you need additional conditions to solve the problem (see later).

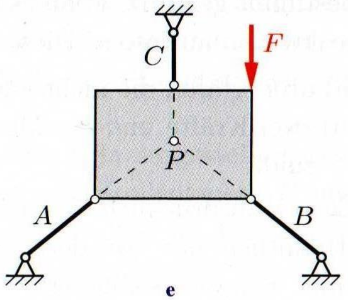
Improper constraints – 2D

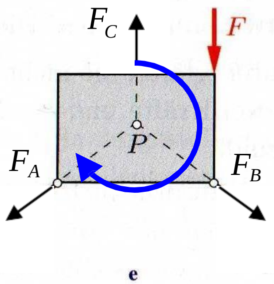
In some special cases, the support reactions
a) all meet in one point
b) are all parallel

Then the structure can move (rotate or translate)

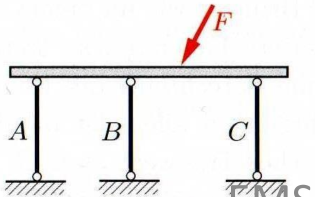
AVOID THESE DESIGNS!

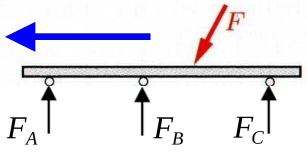
Compare a bad design of a bookshelf without diagonal stiffening.





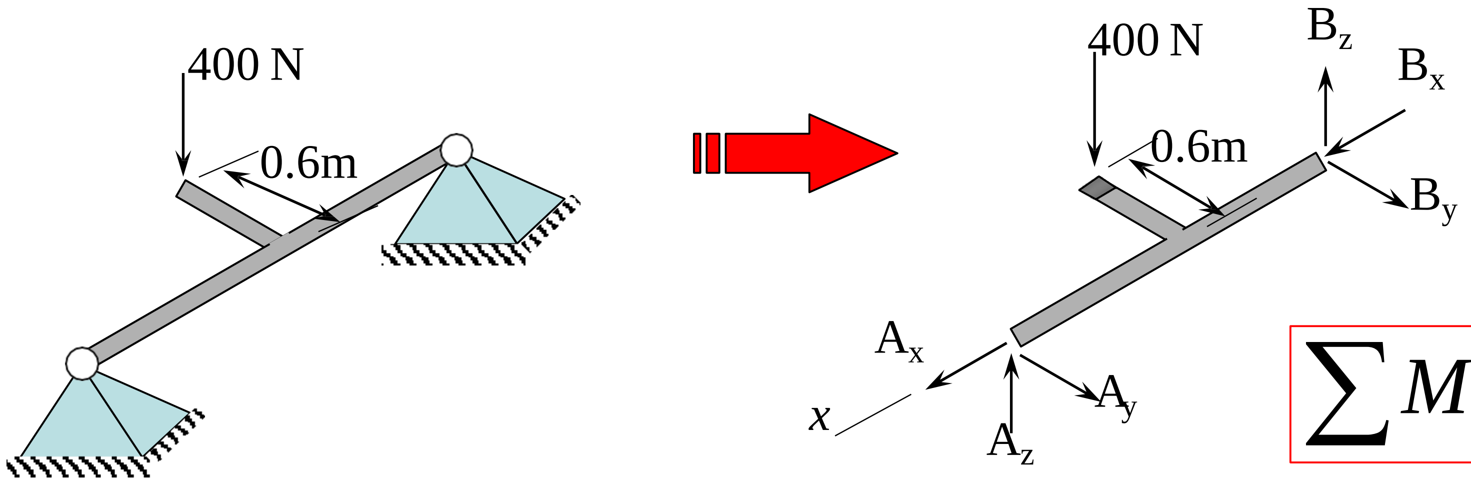
Because all support reactions go through the point P, a moment around this point coming from the force F can not be equilibrated.





Because all support reactions are vertical, no horizontal force can be equilibrated.

Improper constraints – 3D



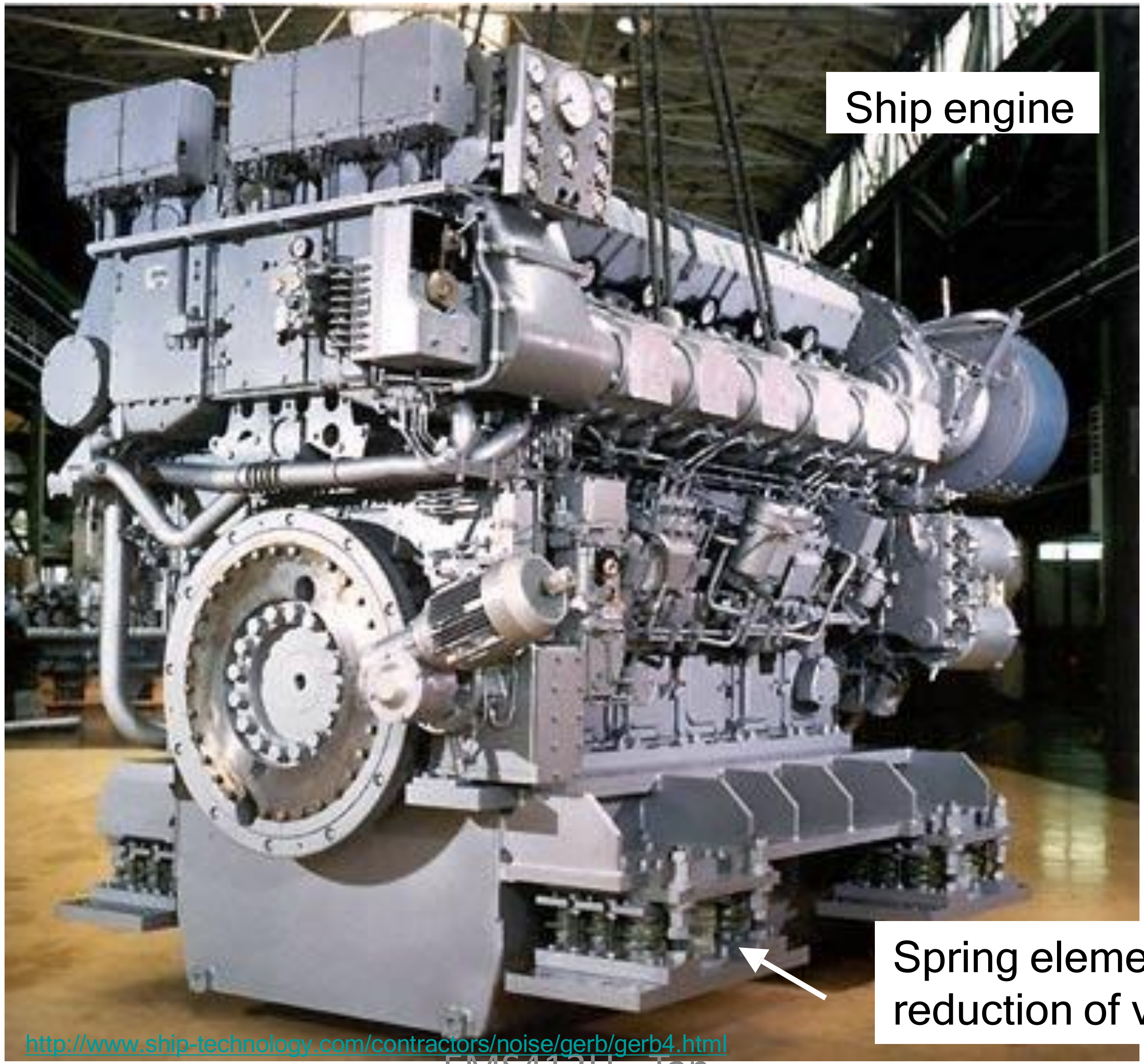
$$\boxed{\sum M_x \neq 0}$$

- In this case, all the support reactions intersect the *x*-axis and therefore produce no moment about *x*.
- The 400 [N] applied force, however, does produce a moment about *x*.
- The beam will rotate and so is improperly constrained.

Lessons Learned

- What is the meaning of equilibrium or balance?
- How many independent equilibrium conditions do exist in two and in three dimensions?
- What types of supports are used in two dimensions?
- Which support reactions correspond to which type of support?
- Have you understood how pulleys and cables are modelled and how they are used in free body diagrams?
- What is Hooke's law for linear elastic springs (translational and rotational)?
- Have you understood how springs are modelled and how they are used in free body diagrams?
- What is a statically determinate and a statically indeterminate structure in two and three dimensions?
- How can I identify if a structure is statically determinate or statically indeterminate?
- What are structures with improper constraints? Why should you avoid to design these?
- Are you able to compute the support reactions using equilibrium equations for two- and three-dimensional problems?

End.


Ship engine
Spring elements for the reduction of vibrations
http://www.ship-technology.com/contractors/noise/gerb/gerb4.html