MECH3750 - Tutorial 1

Question 1.

Show that:

$$\frac{u(x+h)-2u(x)+u(x-h)}{h^2} = u^{''}(x) + \frac{u^4(\zeta_1)h^2}{24} + \frac{u^4(\zeta_2)h^2}{24}$$

For $x \le \zeta_1 \le x+h, \quad x-h \le \zeta_2 \le x$

Take Taylor series of $u(x)$ about $x \pm h$

$$u(x+h) = u(x) + hu'(x) + \frac{h^2 u''(x)}{2!} + \frac{h^3 u'''(x)}{3!} + \frac{h^4 u^{(4)}(\xi_1)}{4!}$$

$$u(x-h) = u(x) + (-h)u'(x) + \frac{(-h)^2 u''(x)}{2!} + \frac{(-h)^3 u'''(x)}{3!} + \frac{(-h)^4 u^{(4)}(\xi_2)}{4!}$$

for $x \le \xi_1 \le x+h \quad , \quad x-h \le \xi_2 \le x$

now take $u(x+h) - 2u(x) + u(x-h)$

$$= \left(\cancel{u(x)} + \cancel{hu'(x)} + \frac{h^2}{2!}u''(x) + \cancel{\frac{h^3}{3!}u'''(x)} + \frac{h^4 u^{(4)}(\xi_1)}{4!} \right)$$

$$- \left(2\cancel{u(x)} \right)$$

$$+ \left(\cancel{u(x)} - \cancel{hu'(x)} + \frac{h^2}{2!}u''(x) - \cancel{\frac{h^3}{3!}u'''(x)} + \frac{h^4 u^{(4)}(\xi_2)}{4!} \right)$$

$$= \frac{2}{2!}h^2 u''(x) + \frac{h^4 u^{(4)}(\xi_2)}{4!} + \frac{h^4 u^{(4)}(\xi_2)}{4!}$$

$$\therefore \quad \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u''(x) + \frac{h^2 u^{(4)}(\xi_1)}{4!} + \frac{h^2 u^{(4)}(\xi_2)}{4!}$$

This may be written as

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u''(x) + \frac{h^2 u^{(4)}(\xi)}{4!} \quad , x-h \le \xi \le x+h$$

Question 2.

(a) Find the Terms in the Taylor Series, up to quadratic order of the function $f(x,y) = \ln(x^2+y^2)$ about the point $(x,y) = (1,1)$.

(b) Evaluate both $f(x,y)$ and your Taylor series approximation at the points $(x,y) = (1.1, 0.9)$ and $(x,y) = (0.2, 0.2)$. Check the order of the error at each point. Is it what you expect?

(a) Find Taylor series (quadratic order) of $f(x,y) = \ln(x^2+y^2)$

at $(1,1)$

$$f_x(x,y) = \frac{2x}{x^2+y^2} = 1$$

$$f_{xx}(x,y) = \frac{2}{x^2+y^2} - \frac{4x}{(x^2+y^2)^2} = 0$$

$$f_y(x,y) = \frac{2y}{x^2+y^2} = 1$$

$$f_{yy}(x,y) = \frac{2}{x^2+y^2} - \frac{4y}{(x^2+y^2)^2} = 0$$

$$f_{xy}(x,y) = -\frac{4xy}{(x^2+y^2)^2} = -1$$

$$\therefore f(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) + \frac{f_{xx}(1,1)(x-1)^2}{2!} + \frac{f_{yy}(1,1)(y-1)^2}{2!} + f_{xy}(1,1)(x-1)(y-1) + H.O.T$$

$$\Rightarrow f(x,y) \approx \ln(2) + (x-1) + (y-1) - (x-1)(y-1)$$

$$= \ln(2) - 3 + 2x + 2y - xy$$

exact $f(1.1, 0.9) = 0.703\,|\,097511413\ldots$

Taylor $f(1.1, 0.9) \approx 0.703\,|\,14\ldots$

exact $f(0.2, 0.2) = |\,-2.52572\,8644\ldots$

Taylor $f(0.2, 0.2) \approx |\,-1.5468\ldots$

Error is expected to be proportional to third order terms: $(x-1)^3, (y-1)^3, (x-1)^2(y-1)$ etc

for $(x,y) = (1.1, 0.9)$ $\quad (x-1)^3 = 10^{-3}$

$(y-1)^3 = 10^{-3}$ etc

error matches expectation $= O(10^{-3})$

for $(x,y) = (0.2, 0.2)$ $\quad (x-1)^3 = 0.8^3 \approx 0.512$

$= O(10^{-1}) \sim O(1)$

error high, but within expectations

Question 3.

Demonstrate the the following expression is a fourth order centered approximation of the second derivative $u''(x)$

$$u''(x) \approx \frac{-u_{i-2} + 16u_{i-1} - 30u_i + 16u_{u+1} - u_{i+2}}{12h^2}$$

Take Taylor Series

$$u(x+h) = u(x) + h u'(x) + \frac{h^2}{2} u''(x) + \frac{h^3}{3!} u'''(x) + \frac{h^4}{4!} u^{(4)}(x)$$

$$+ \frac{h^5}{5!} u^{(5)}(x) + O(h^6)$$

$$u(x-h) = u(x) - h u'(x) + \frac{h^2}{2} u''(x) - \frac{h^3}{3!} u'''(x) + \frac{h^4}{4!} u^{(4)}(x)$$

$$-\frac{h^5}{5!} u^{(5)}(x) + O(h^6)$$

$$u(x+2h) = u(x) + 2h u'(x) + \frac{4h^2 u''(x)}{2} + \frac{8}{6} h^3 u'''(x) + \frac{16}{12} h^4 u^{(4)}(x)$$

$$+ \frac{32h^5 u^{(5)}(x)}{5!} + O(h^6)$$

$$u(x-2h) = u(x) - 2h u'(x) + \frac{4h^2 u''(x)}{2} - \frac{8}{6} h^3 u'''(x) + \frac{16 h^4}{12} u^{(4)}(x)$$

$$-\frac{32h^5 u^{(5)}(x)}{5!} + O(h^6)$$

now take $-u(x-2h) + 16u(x-h) - 30u(x) + 16u(x+h)$
$- u(x+2h) =$

$$u(x)\underbrace{\cancel{\left(-1+16-30+16-1\right)}}_{=0}+$$

$$u'(x)h\underbrace{\cancel{\left(2-16+16-2\right)}}_{=0}+$$

$$u''(x)h^2\left(-\frac{4}{2}+\frac{16}{2}+\frac{16}{2}-\frac{4}{2}\right)+$$

$$u'''(x)h^3\underbrace{\cancel{\left(\frac{8}{6}-\frac{16}{6}+\frac{16}{6}-\frac{8}{6}\right)}}_{=0}+$$

$$u^{(4)}(x)h^4\underbrace{\cancel{\left(\frac{-16}{12}+\frac{16}{12}+\frac{16}{12}-\frac{16}{2}\right)}}_{=0}+$$

$$u^{(5)}(x)h^5\underbrace{\cancel{\left(\frac{+32}{5!}-\frac{16}{5!}+\frac{16}{5!}-\frac{32}{5!}\right)}}_{=0}+O(h^6)$$

$$=12h^2u''(x)$$

$$\therefore\ u''(x)=\frac{-u_{i-2}+16u_{i-1}-30u_i+16u_{i+1}-u_{i-2}}{12h^2}$$

$$+O(h^4)$$

where

$$u_{i-2}=u(x-2h)$$
$$u_{i-1}=u(x-h)$$
$$u_i=u(x)$$
$$u_{i+1}=u(x+h)$$
$$u_{i+2}=u(x+2h)$$

$$\frac{1}{2}(x-x_0, y-y_0)\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{(x_0,y_0)}$$

$$= \Big(\frac{1}{2} f_{xx}(x_0,y_0)(x-x_0) + \frac{1}{2} f_{xy}(x_0,y_0)(y-y_0),$$

$$\frac{1}{2} f_{xy}(x_0,y_0)(x-x_0) + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)\Big) = b$$

$$b\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} = \frac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2$$

$$+ 2 \cdot \frac{1}{2} f_{xy}(x_0,y_0)(x-x_0)(y-y_0)$$

$$+ \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2$$

Question 4.

4.1

$$x^4 + y^4 = 1$$
$$x^2 - y^2 = -1$$

(a) Using the initial guess $\mathbf{x_0} = (x_0, y_0) = (1, 1)$, what is the approxiation produced by one step of Newton's method?
(b) Using a graph in the xy plane, how many solutions do we expect to this problem?
(c) For what initial guesses $\mathbf{x_0}$ will Newton's method fail due to the Jacobian matrix $J(\mathbf{x_0})$ being singular?

Solution:
a) Newton's method is given by the update rule:

$$\mathbf{x_{n+1}} = \mathbf{x_n} - [J(\mathbf{x_n})]^{-1} F(\mathbf{x}) \tag{1}$$

Where, J is the Jacobian, and J and F are defined as:

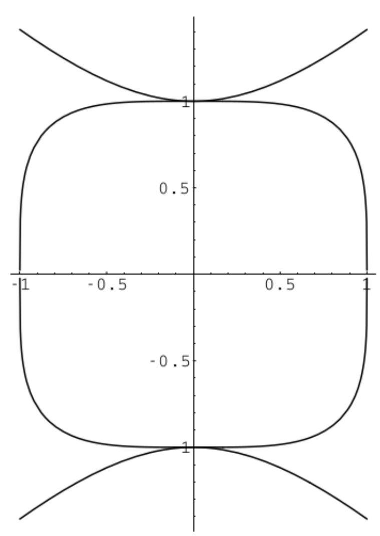
$$F(\mathbf{x}) = \begin{pmatrix} x^4 + y^4 - 1 \\ x^2 - y^2 + 1 \end{pmatrix} \tag{2}$$
$$J(\mathbf{x}) = \begin{pmatrix} 4x^3 & 4y^3 \\ 2x & -2y \end{pmatrix} \tag{3}$$

Therefore, if we start from our initial guess $\mathbf{x_0}$, we get:

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 4 & 4 \\ 2 & -2 \end{pmatrix}\right]^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4}$$
$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{-16} \left[\begin{pmatrix} -2 & -4 \\ -2 & 4 \end{pmatrix}\right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5}$$
$$= \begin{pmatrix} 5/8 \\ 9/8 \end{pmatrix} \tag{6}$$

Try to code this up and see how many iterations it takes to converge.

b) The plot below shows 2 expected solutions:



c) Newton's method will fail for $\mathbf{x_0} = (0, 0)$ as the Jacobian will become singular.

4.2

$$x^2 + cos(y) = 5$$
$$sinh(x) + sin(y) = 4$$

(a) What is the vector function $F(x, y)$ for which we require roots?
(b) Evaluate the Jacobian ($J(\mathbf{x})$) for this system.
(c) Let $\mathbf{x_0} = (1, 1)$ and $h = 0.1$, evaluate $J(\mathbf{x})$ using a forward difference.
(d) Using your answer from (c), compute $\mathbf{x_1}$ and $\mathbf{x_2}$.

Solution:
a) Newton's method is given by the update rule:

$$\mathbf{x_{n+1}} = \mathbf{x_n} - [J(\mathbf{x_n})]^{-1} F(\mathbf{x}) \tag{7}$$

For this problem, J is the Jacobian, and J and F are defined as:

$$F(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} x^2 + cos(y) - 5 \\ sinh(x) + sin(y) - 4 \end{pmatrix} \tag{8}$$

$$J(\mathbf{x}) = \begin{pmatrix} 2x & -sin(y) \\ cosh(x) & cos(y) \end{pmatrix} \tag{9}$$

b) We haven't really touched on finite differences yet, but they are a way of approximating derivatives. Here the forward difference is defined as:

$$\frac{df}{dx} \approx \frac{f(x+h) - f(x)}{\delta x} \tag{10}$$

Using this definition, our Jacobian at $\mathbf{x_0}$ is:

$$J(1,1) \approx \frac{1}{0.1} \begin{pmatrix} [f_1(1+0.1,1) - f_1(1,1)] & [f_1(1,1+0.1) - f_1(1,1)] \\ [f_2(1+0.1,1) - f_2(1,1)] & [f_2(1,1+0.1) - f_2(1,1)] \end{pmatrix} \tag{11}$$

$$= \begin{pmatrix} 2.1 & -0.86706 \\ 1.60446 & 0.49736 \end{pmatrix} \tag{12}$$

c) Using the answer above, we can find:

$$J(1,1)^{-1} = \begin{pmatrix} 0.2042 & 0.3560 \\ -0.6587 & 0.8622 \end{pmatrix}$$

Therefore we have:

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0.2042 & 0.3560 \\ -0.6587 & 0.8622 \end{pmatrix} \begin{pmatrix} -3.4597 \\ -1.9833 \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} 2.4125 \\ 0.4310 \end{pmatrix} \tag{14}$$

We then need to re-evaluate F, J and J^{-1} for our new point and we find:

$$\mathbf{x_2} = \begin{pmatrix} 2.4125 \\ 0.4310 \end{pmatrix} - \begin{pmatrix} 0.1349 & 0.0620 \\ -0.8354 & 0.7165 \end{pmatrix} \begin{pmatrix} 1.7289 \\ 1.9540 \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} 2.0581 \\ 0.4752 \end{pmatrix} \tag{16}$$