Question 1.
Let $f = (4, 0, 2, 1)$ and $p_1 = (1,\ 0,\ -1,\ 0)$, $p_2 = (1,\ 1,\ 1,\ 1)$ and $p_3 = (1,\ -2,\ 1,\ 0)$.

a. (7 marks) Find the least-squares approximation to f of the form $y = a_1p_1 + a_2p_2 + a_3p_3$. You may use the *normal* equations without proof.

b. (3 marks) State the mathematical problem that your approximation solves.

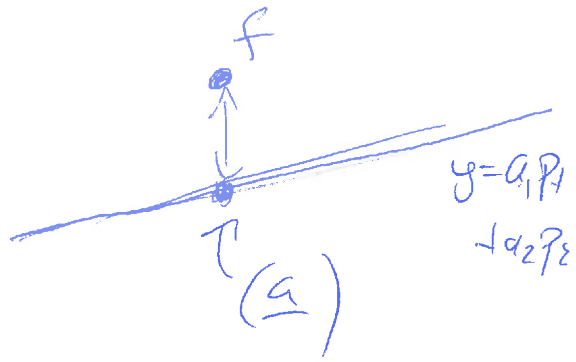
$$A\,\underline{a} = \underset{\sim}{b}$$

$$A = \begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & p_1 \cdot p_3 \\ & p_2 \cdot p_2 & p_2 \cdot p_3 \\ & & p_3 \cdot p_3 \end{pmatrix} \qquad \underline{a}\ , \quad \underset{\sim}{b} = \begin{pmatrix} f \cdot p_1 \\ f \cdot p_2 \\ f \cdot p_3 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \qquad \underset{\sim}{b} = \begin{pmatrix} 2 \\ 7 \\ 6 \end{pmatrix}$$

$$2a_1 = 2 \rightarrow a_1 = 1$$
$$4a_2 = 7 \rightarrow a_2 = 7/4$$
$$6a_3 = 6 \rightarrow a_3 = 1$$

b. closest point in space y to f.



Question 2.

Find the Taylor series to second order (ignore terms of $O(x-1)^3$ etc.) of $f(x,y) = x^2y^2$ near the point $(x,y) = (1,1)$.

$\hookrightarrow (a,b)$

$$f(x,y) \simeq f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+\tfrac{1}{2}\Big(f_{xx}(x-a)^2 + 2f_{xy}(x-a)(y-b) + f_{yy}(y-b)^2\Big)$$
$$+ HOT$$

$@(1,1)$

$f = x^2y^2 = 1$

$f_x = 2xy^2 = 2$

$f_y = 2x^2y = 2$

$f_{xx} = 2y^2 = 2$

$f_{xy} = 4xy = 4$

$f_{yy} = 2x^2 = 2$

$$f(x,y) \simeq 1 + 2(x-1) + 2(y-1)$$
$$+\tfrac{1}{2}\Big(2(x-1)^2 + 8(x-1)(y-1) + 2(y-1)^2\Big)$$
$$\simeq 1 + 2(x-1) + 2(y-1)$$
$$+ (x-1)^2 + 4(x-1)(y-1)$$
$$+ (y-1)^2$$
$$+ HOT.$$

Question 3.

Consider the error integral: $E(a_1, a_2) = \int_0^1 (a_1 + a_2 x - x^7)^2 \; dx$

Your task is to show how the error $E(a_1, a_2)$ can be minimised. Determine $\frac{\partial E}{\partial a_1}$. Explain using formulas how you would find a_1, a_2. **Do not carry out any numerical calculations to do so.**

$$\frac{\partial E}{\partial a_1} = \frac{\partial}{\partial a_1} \int_0^1 (a_1 + a_2 x - x^7)^2 \, dx$$

$$= \int_0^1 \frac{\partial}{\partial a_1} (a_1 + a_2 x - x^7)^2 \, dx = \int_0^1 2(a_1 + a_2 x - x^7) \, dx$$

~~8k~~ Explain strategy for solving:

set $\frac{\partial E}{\partial a_1} = 0$ & $\frac{\partial E}{\partial a_2} = 0$ as we are minimising E.

Rearrange s.t. all known terms on RHS

ie $\frac{\partial E}{\partial a_1} = 0 \rightarrow \int_0^1 1(a_1 + a_2 x) \, dx = \int_0^1 1 \cdot x^7 \, dx$

$\frac{\partial E}{\partial a_2} = 0 \rightarrow \int_0^1 x(a_1 + a_2 x) \, dx = \int_0^1 x \cdot x^7 \, dx$

Rearrange ~~Recognise~~ as normal form w/ $P_0 = 1$ & $P_1 = x$

$$\underbrace{\begin{pmatrix} \int_0^1 1 \cdot 1 \, dx & \int_0^1 1 \cdot x \, dx \\ \int_0^1 x \cdot 1 \, dx & \int_0^1 x^2 \, dx \end{pmatrix}}_{A \,=\, \text{known}} \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\underline{a}} = \underbrace{\begin{pmatrix} \int_0^1 1 \cdot x^7 \, dx \\ \int_0^1 x \cdot x^7 \, dx \end{pmatrix}}_{\underline{b} \,=\, \text{known}}$$

$$\Rightarrow \underline{a} = A^{-1} \underline{b}$$

Question 4.
Carry out one step of Newton's method to solve the equations $x^2 + y^2 - 1 = 0$ and $y - x^3 = 0$ using $(x, y) = (1, 2)$ as the starting point.

See also Tony's lecture notes for similar problem

$$\underline{x}^{n+1} = \underline{x}^n - J^{-1}(\underline{x}_n)\,\underline{F}(\underline{x}_n)$$

$$\text{w/ } \underline{F} = \begin{pmatrix} x^2 + y^2 - 1 \\ y - x^3 \end{pmatrix} \quad \& \quad \underline{x}^0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$J = \begin{pmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ -3x^2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}\Bigg|_{\underline{x}^0}$$

$$J^{-1} = \frac{1}{14}\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}$$

$$\underline{F}(\underline{x}^0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\underline{x}^1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{14}\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{14}\begin{pmatrix} 0 \\ 14 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In lectures Tony had F_1 & F_2 swapped