MECH3750 - Tutorial 3

Question 1.

Suppose $p_1,\ p_2,\ p_3$ are three vectors (or functions). Suppose we wish to find another vector (or function) $y = a_1p_1 + a_2p_2 + a_3p_3$ which minimizes the distance squared:

$$d^2\,(y,\ f) = \|y - f\|^2 = (y - f, y - f)$$

Where d is the "distance" as introduced in lectures, and (u, v) is a well defined inner product.

Obtain $a_1,\ a_2,\ a_3$ (to find the best approximation) for the following cases:

(a) $p_1 = (1,\ 0,\ -1,\ 0)$, $p_2 = (1,\ 1,\ 1,\ 0)$, $p_3 = (1,\ -2,\ 1,\ 0)$. With inner product: $(u, v) = \sum_i u_iv_i$ and:

$$\text{(i)}\ f = (4,\ 0,\ 2,\ 0) \qquad \text{(ii)}\ f = (0,\ 0, 0,\ 1)$$

SOLUTIONS

In all following questions, the coefficients may be solved from the normal equations as:

$$\begin{bmatrix} (p_1, p_1) & (p_1, p_2) & (p_1, p_3) \\ (p_2, p_1) & (p_2, p_2) & (p_2, p_3) \\ (p_3, p_1) & (p_3, p_2) & (p_3, p_3) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (p_1, f) \\ (p_2, f) \\ (p_3, f) \end{bmatrix} \tag{1}$$

(i) The inner product for vectors defines the *dot* product. Using Equation 1, and evaluating of each inner product, we obtain the diagonal matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3x \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}$$

So $a_1 = 1$, $a_2 = 2$ and $a_3 = 1$ and $y = p_1 + 2p_2 + p_3 = (4,\ 0,\ 2,\ 0) = f$ with $d^2 = 0$.

(ii) As the p vectors are unchanged, the matrix takes the same value as in (i), evaluation of each inner product on the RHS of 1, results in:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $a_1 = 0$, $a_2 = 0$ and $a_3 = 0$ and $y = (0,\ 0,\ 0,\ 0) = f$ and $d^2 = 1$.

(b) $p_1 = \sin x$, $p_2 = \sin 2x$, $p_3 = \sin 3x$. With inner product: $(u, v) = \int_0^\pi u(x)v(x)\ \mathrm{d}x$ and:

$$\text{(i)}\ f(x) = \sin 2x + \sin 3x \qquad \text{(ii)}\ f(x) = \sin 4x$$

Hint: Use the result established in lectures that:

$$(\sin nx,\ \sin mx) = \begin{cases} \frac{\pi}{2} & m = n \\ 0 & m \neq n \end{cases}$$

SOLUTIONS

(i) Note that $f(x) = p_2 + p_3$. Hence substitution into the normal equations (1), results in:

$$\begin{bmatrix} (p_1, p_1) & (p_1, p_2) & (p_1, p_3) \\ (p_2, p_1) & (p_2, p_2) & (p_2, p_3) \\ (p_3, p_1) & (p_3, p_2) & (p_3, p_3) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (p_1, p_2 + p_3) \\ (p_2, p_2 + p_3) \\ (p_3, p_2 + p_3) \end{bmatrix}$$

And observe that $(p_n,\ p_m + p_o) = (p_n,\ p_m) + (p_n,\ p_o)$ using properties (iii) and (i) of inner products as defined in lecture week 3a to give:

$$\begin{bmatrix} (p_1, p_1) & (p_1, p_2) & (p_1, p_3) \\ (p_2, p_1) & (p_2, p_2) & (p_2, p_3) \\ (p_3, p_1) & (p_3, p_2) & (p_3, p_3) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (p_1, p_2) + (p_1, p_3) \\ (p_2, p_2) + (p_2, p_3) \\ (p_3, p_2) + (p_3, p_3) \end{bmatrix}$$

Using the suggested hint, we have that:

$$(p_n,\ p_m) = \begin{cases} \frac{\pi}{2} & m = n \\ 0 & m \neq n \end{cases}$$

So that the normal equations simplify to:

$$\begin{bmatrix} \frac{\pi}{2} & 0 & 0 \\ 0 & \frac{\pi}{2} & 0 \\ 0 & 0 & \frac{\pi}{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$$

Therefore: $a_1 = 0,\ a_2 = 1,\ a_3 = 1$ and $y = \sin 2x + \sin 3x = f$ with $d^2 = 0$.

(ii) Reusing the LHS diagonal matrix from (i) since the p terms remain unchanged and substitution of p's f into the normal equations results in:

$$\begin{bmatrix} \frac{\pi}{2} & 0 & 0 \\ 0 & \frac{\pi}{2} & 0 \\ 0 & 0 & \frac{\pi}{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (\sin x,\ \sin 4x) \\ (\sin 2x,\ \sin 4x) \\ (\sin 3x,\ \sin 4x) \end{bmatrix}$$

However, using the same identity for the inner product of sin terms, every element of the RHS vector is equal to zero, hence we obtain $a_1 = a_2 = a_3 = 0$ and therefore $y = 0$ with

$$d^2 = \int_0^{\pi} \sin^2 4x \ \mathrm{d}x = \frac{\pi}{2}$$

.

(c) $p_1 = 1$, $p_2 = x$, $p_3 = \frac{1}{2}\left(3x^2 - 1\right)$. With inner product: $(u, v) = \int_{-1}^{1} u(x)v(x)\ \mathrm{d}x$ and:

$$\text{(i) } f(x) = x^2 \qquad \text{(ii) } f(x) = 5x^3 - 3x$$

SOLUTIONS

From (1) we have the normal equations:

$$\begin{bmatrix} (p_1,p_1) & (p_1,p_2) & (p_1,p_3) \\ (p_2,p_1) & (p_2,p_2) & (p_2,p_3) \\ (p_3,p_1) & (p_3,p_2) & (p_3,p_3) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (p_1,f) \\ (p_2,f) \\ (p_3,f) \end{bmatrix}$$

Now observe however, that $p_1,\ p_2,\ p_3$ are the *legendre polynomials* from lectures and Tutorial 2 (with subscript shifted up by 1). Recall from Tutorial 2, question 4c, we derived the following results:

$$\int_{-1}^{1} p_1 p_1 = 2$$
$$\int_{-1}^{1} p_1 p_2 = 0$$
$$\int_{-1}^{1} p_1 p_3 = 0$$
$$\int_{-1}^{1} p_2 p_2 = \frac{2}{3}$$
$$\int_{-1}^{1} p_3 p_3 = \frac{2}{5}$$

And also note $\int_{-1}^{1} p_2 p_3 \, \mathrm{d}x = 0$ since $p_2 p_3$ is odd. Therefore the normal equations simplify to:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (p_1,f) \\ (p_2,f) \\ (p_3,f) \end{bmatrix}$$

Where the property of inner products such that $(u,v) = (v,u)$ was used to express the remaining inner products in terms of known results.

(i). Substituting $p_1 = 1,\ p_2 = x,\ p_3 = \frac{1}{2}\left(3x^2 - 1\right), f = x^2$ into the above gives:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (1,x^2) \\ (x,x^2) \\ (\frac{1}{2}\left(3x^2-1\right), x^2) \end{bmatrix}$$

Evaluating the LHS inner products and solving the diagonal system results in:

$$a_1 = \frac{1}{3},\ a_2 = 0,\ a_3 = \frac{2}{3}$$

Therefore $y = x^2$ and $d^2 = 0$

(ii). With $f = 5x^3 - 3x$ we have:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (1, 5x^3-3x) \\ (x, 5x^3-3x) \\ (\frac{1}{2}\left(3x^2-1\right), 5x^3-3x) \end{bmatrix}$$

The first and third inner products can be observed to be equal to 0, since the resulting integrals are odd. Evaluating the second inner product as an integral using wolfram alpha reveals that $(x, 5x^3 - 3x) = 0$ also, and therefore:

$$a_1 = 0,\ a_2 = 0,\ a_3 = 0$$

Therefore $y = 0$ and $d^2 = \int_{-1}^{1} \left(5x^3 - 3x\right)^2 \, \mathrm{d}x = \frac{8}{7}$

(d) Comment on why $d^2\left(y,\ f\right) = \left(y - f, y - f\right) = 0$ in all (i) cases. Comment on (ii) results and why all these cases are non zero.

SOLUTIONS

In all (i) cases, f is in the set covered by $\{p_1,\ p_2,\ p_3\}$, thus it can be approximated exactly by y.

In all (ii) cases, we find $y = 0$. Problem a(ii) shows what is going on. The vector f is orthogonal to $\{p_1,\ p_2,\ p_3\}$. The best approximation is in fact zero. Think of the closest point in the x-y plane to $(0, 0, 1)$. It is the point $0, 0, 0$ directly "below" $(0, 0, 1)$. It is not a very good approximation. Since it has no components in the set $\{p_1,\ p_2,\ p_3\}$ it cannot be approximated by anything in the set.
(e) There is an alternative way of solving all (i) cases. Why?

SOLUTIONS

If f can be represented exactly by the set, then we can just let:

$$f = a_1 p_1 + a_2 p_2 + a_3 p_3$$

and solve the equation for three unknowns.

Question 2.

(a) Find the best approximation to the function $f(x) = 1$ on the interval $[0,\ \pi]$ using the set $\sin nx,\ n = 1, 2, ..., N$. On the worksheet, you are asked to plot your answer for different values of N.

SOLUTIONS

We seek an approximation of the form of Equation 2 of $f(x) = 1$ on $[0, \pi]$:

$$y = \sum_{n=1}^{N} a_n \sin nx \tag{2}$$

We wish to minimize:

$$\|y - f\|^2 = \int_0^{\pi} \left(f(x) - \sum_{n=1}^{N} a_n \sin(nx)\right)^2 \, \mathrm{d}x$$

This results in the normal equations:

$$\begin{bmatrix} (\sin x, \sin x) & (\sin x, \sin 2x) & \dots & (\sin x, \sin Nx) \\ (\sin 2x, \sin x) & (\sin 2x, \sin 2x) & \dots & (\sin 2x, \sin Nx) \\ \vdots & \vdots & \ddots & \vdots \\ (\sin Nx, \sin x) & (\sin Nx, \sin 2x) & \dots & (\sin Nx, \sin Nx) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} (\sin x, f) \\ (\sin 2x, f) \\ \vdots \\ (\sin Nx, f) \end{bmatrix} \tag{3}$$

We can now use the result established in lectures that:

$$\int_0^{\pi} \sin(nx) \sin(mx) \, \mathrm{d}x = \begin{cases} \frac{\pi}{2} & m = n \\ 0 & m \neq n \end{cases}$$

Hence every non-diagonal entry of the LHS matrix is zero, and the n'th coefficient may be solved for directly as equation 4:

$$a_n = \frac{2}{\pi}\left(\sin nx, f\right) \tag{4}$$

Hence, for $f(x) = 1$, equation (4) becomes:

$$a_n = \frac{2}{\pi}\int_0^\pi \sin nx \ \mathrm{d}x = \frac{2}{\pi n}\left(-\cos(\pi n) + 1\right) = \begin{cases} 0 & n \text{ even} \\ 4/\pi n & n \text{ odd} \end{cases}$$

Finally, substituting each a_n into equation 2, we obtain:

$$y = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots + \frac{\sin Nx}{N}\right) \qquad N \text{ odd}$$

$$y = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots + \frac{\sin((N-1)x)}{N-1}\right) \qquad N \text{ even}$$

on $[0, \pi]$

(b) Find the best approximation to the function $f(x) = 1$ on the interval [0, L] using the set $\sin \frac{\pi n x}{L}$, $n = 1, 2, ..., N$. You may use the results of part (a) by applying a suitable substitution in the integrals.

SOLUTIONS

We seek an approximation of the form of Equation 5 of $f(x) = 1$ on $[0, L]$:

$$y = \sum_{n=1}^{N} a_n \sin \frac{\pi n x}{L} \tag{5}$$

We wish to minimize:

$$d^2 = \|y - f\|^2 = \int_0^L \left(f(x) - \sum_{n=1}^{N} a_n \sin \frac{\pi n x}{L}\right)^2 \mathrm{d}x$$

If we now substitute $u = \pi x/L, \quad du = \pi/L \ dx, \quad u(0) = 0, \quad u(L) = \pi \quad$ into d^2, we obtain:

$$d^2 = \|y - f\|^2 = \frac{L}{\pi}\int_0^\pi \left(f(u) - \sum_{n=1}^{N} a_n \sin(nu)\right)^2 \mathrm{d}u$$

Since $f(x) = f(u) = 1$. This is the same d^2 as in (a) except scaled by the constant $\frac{L}{\pi}$. Hence d^2 will be minimized by the same coefficients as (a), resulting in the solution:

$$y = \frac{4}{\pi}\left(\frac{1}{1}\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \cdots + \frac{1}{N}\sin\frac{N\pi x}{L}\right) \qquad N \text{ odd}$$

$$y = \frac{4}{\pi}\left(\frac{1}{1}\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \cdots + \frac{1}{N-1}\sin\frac{(N-1)\pi x}{L}\right) \qquad N \text{ odd}$$

on $[0, L]$

(c) Find the best approximation to the function $f(x) = x$ on the interval $[0,\ \pi]$ using the set $\sin nx,\ n = 1, 2, ..., N$. On the worksheet, you are asked to plot your answer for different values of N. Show how the results can be adapted to represent the function $f(x) = x$ on the interval $[0,\ L]$.

SOLUTION

The normal equations are identical to (a), hence we may proceed from Equation 4 of (a), substitution of $f(x) = x$ into 4 results in:

$$a_n = \frac{2}{\pi}\left(\sin nx, x\right)$$

Evaluating the inner product results in:

$$a_n = \frac{2}{\pi}\left(\frac{\sin(\pi n) - \pi n\cos(\pi n)}{n^2}\right)$$

$$a_n = \frac{2}{n}\begin{cases} -1 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

Hence substituting a_n into Equation 2, we obtain:

$$y = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \cdots + (-1)^{N+1}\frac{\sin Nx}{N}\right) \qquad \text{on } [0, \pi]$$

We can generalize the results to $[0, L]$ by considering the set $\sin\left(\frac{\pi n x}{L}\right),\ n = 1, 2, ..., N$. We now wish to minimize:

$$d^2 = \|y - f\|^2 = \int_0^L \left(x - \sum_{n=1}^{N} a_n \sin\frac{\pi n x}{L}\right)^2 \mathrm{d}x$$

If we now substitute $u = \pi x/L, \quad du = \pi/L\ dx, \quad u(0) = 0, \quad u(L) = \pi \quad$ into d^2, we obtain:

$$d^2 = \|y - f\|^2 = \frac{L}{\pi}\int_0^{\pi} \left(\frac{L}{\pi}u - \sum_{n=1}^{N} a_n \sin(nu)\right)^2 \mathrm{d}u$$

The constant factor $\frac{L}{\pi}$ out the front of the integral has no effect upon the coefficients. The integral is now in a similar form to (a) except with $f(x) = x$ replaced with $f(u) = \frac{L}{\pi}u$, hence the coefficients may be given explicitly from Equation 4 as:

$$a_n = \frac{2}{\pi}\left(\sin nu, \frac{L}{\pi}u\right) = \frac{2L}{\pi^2}\left(\sin nu, u\right)$$

Where the factor $\frac{L}{\pi}$ may be taken out of the inner product using property (ii) of inner products as defined in lecture week 3a. The solution to the inner product is already known from earlier in the question, hence the coefficients are given by:

$$a_n = \frac{2L}{\pi n}\begin{cases} -1 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

Hence substituting a_n into y, we obtain:

$$y = \frac{2L}{\pi}\left(\sin\frac{\pi x}{L} - \frac{1}{2}\frac{2\pi x}{L} + \cdots + (-1)^{N+1}\frac{1}{N}\frac{N\pi x}{L}\right) \qquad \text{on } [0, L]$$

(d) Find the best approximation to the function $f(x) = x$ on the interval $[0, \pi]$ using the set $1, \cos nx, n = 1, 2, ..., N$. You will need to check that the function 1 is orthogonal to all other members of the set.

SOLUTION

First confirm that 1 and $\cos nx$ are orthogonal:

$$(1, \cos nx) = \int_0^\pi \cos nx \, \mathrm{d}x = 0$$

Hence, orthogonal. Now we seek an approximation of Equation 6 of $f(x) = x$ on $[0, \pi]$.

$$y = a_0 + \sum_{n=1}^{N} a_n \cos nx \tag{6}$$

Wish to minimize:

$$d^2 = \int_0^\pi \left(f(x) - a_0 - \sum_{n=1}^{N} a_n \cos nx \right)^2$$

Hence we have the normal equations:

$$\begin{bmatrix} (1,1) & (1, \cos x) & \dots & (1, \cos Nx) \\ (\cos x, 1) & (\cos x, \cos x) & \dots & (\cos x, \cos Nx) \\ \vdots & \vdots & \ddots & \vdots \\ (\cos Nx, 1) & (\cos Nx, \cos x) & \dots & (\cos Nx, \cos Nx) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} (1, f) \\ (\cos x, f) \\ \vdots \\ (\cos Nx, f) \end{bmatrix} \tag{7}$$

Recall, we established in tutorial 2 the identity:

$$\int_0^\pi \cos(nx)\cos(mx) \, \mathrm{d}x = \begin{cases} \frac{\pi}{2} & m = n \\ 0 & m \neq n \end{cases}$$

Hence, all non-diagonal entries of the matrix vanish, resulting in an explicit expression for the 0'th coefficient:

$$a_0 = \frac{1}{\pi} \int_0^\pi x \, \mathrm{d}x = \frac{\pi}{2}$$

and for the n'th coefficient $(n > 0)$

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, \mathrm{d}x = \frac{2}{\pi n^2} (\pi n \sin(\pi n) + \cos(\pi n) - 1)$$

$$a_n = \begin{cases} 0 & n \text{ even} \\ -4/\pi n^2 & n \text{ odd} \end{cases}$$

Finally substituting all a_n into Equation 6 gives the solution:

$$y = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos x}{1} + \frac{\cos 3x}{9} + \dots + \frac{\cos Nx}{N^2} \right) \qquad N \text{ odd}$$

$$y = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos x}{1} + \frac{\cos 3x}{9} + \dots + \frac{\cos (N-1)x}{(N-1)^2} \right) \qquad N \text{ even}$$

on $[0, \pi]$.