MECH3750 - Tutorial 4 (week 5)

Question 1.
Show that:

$$\overline{\exp(iy)} = \exp(-iy)$$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}$$

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

Question 2.

Using the complex inner product defined as:

$$(u, v) = \sum_{i=0}^{4} \overline{u_i} v_i$$

consider the vectors:

$$q_n^{(k)} = \exp\left(ik\frac{2\pi n}{M}\right) \qquad n = 0, 1, 2, 3$$

(a) Write $q^{(k)}$ explicitly for $k = 0, 1, 2, 3$.

(b) Use the inner product to find $\|q^{(k)}\|$ for $k = 0, 1, 2, 3$.

(c) Verify: $(q^{(0)}, q^{(1)}) = 0$; $(q^{(2)}, q^{(3)}) = 0$; $(q^{(0)}, q^{(2)}) = 0$

Question 3.

In our interpretation of the DFT, the values a_{p_k} represent the coefficients of the vector:

$$p_n^{(k)} = \frac{1}{N}\exp\left(i\frac{2\pi nk}{N}\right)$$

in the signal f_n for $k = 0, 1, \ldots, N-1$.

Verify that $p_n^{(N-1)} = p_n^{(-1)}$ and also $p_n^{(N-m)} = p_n^{(-m)}$. This is important for interpreting the values of the DFT for large k.

SOLUTIONS

$$p_n^{(N-1)} = \frac{1}{N}\exp\left(i\frac{2\pi n(N-1)}{N}\right) = \frac{1}{N}\exp\left(i\frac{2\pi nN}{N} - i\frac{2\pi n}{N}\right) = \frac{1}{N}\exp\left(i2\pi n\right)\exp\left(-i\frac{2\pi n}{N}\right)$$

But $\exp(i2\pi n) = 1$, since it is a rotation around the unit circle n times, hence:

$$p_n^{(N-1)} = \frac{1}{N}\exp\left(-i\frac{2\pi n}{N}\right) = p_n^{-1}$$

Under a similar argument

$$p_n^{(N-m)} = \frac{1}{N}\exp\left(i\frac{2\pi n(N-m)}{N}\right) = \frac{1}{N}\exp\left(i\frac{2\pi nN}{N} - i\frac{2\pi nm}{N}\right) = \frac{1}{N}\exp\left(i2\pi n\right)\exp\left(-i\frac{2\pi nm}{N}\right)$$

So

$$p_n^{(N-m)} = \frac{1}{N}\exp\left(-i\frac{2\pi nm}{N}\right) = p_n^{-m}$$

Question 4.

Find the DFT of:

(a) $\boldsymbol{f} = (1,\ 2,\ 0,\ 1)$

(b) $\boldsymbol{f} = (1,\ 1,\ \ldots,\ 1)$, for $N = 8$

SOLUTIONS

The normal equations are given by:

$$\begin{bmatrix} (\boldsymbol{p}^0, \boldsymbol{p}^0) & (\boldsymbol{p}^0, \boldsymbol{p}^1) & \dots & (\boldsymbol{p}^0, \boldsymbol{p}^{N-1}) \\ (\boldsymbol{p}^1, \boldsymbol{p}^0) & (\boldsymbol{p}^1, \boldsymbol{p}^1) & \dots & (\boldsymbol{p}^1, \boldsymbol{p}^{N-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{p}^{N-1}, \boldsymbol{p}^0) & (\boldsymbol{p}^{N-1}, \boldsymbol{p}^1) & \dots & (\boldsymbol{p}^{N-1}, \boldsymbol{p}^{N-1}) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{p}^0, f) \\ (\boldsymbol{p}^1, f) \\ \vdots \\ (\boldsymbol{p}^{N-1}, f) \end{bmatrix}$$

For:

$$p_n^{(k)} = \frac{1}{N}\exp\left(i\frac{2\pi nk}{N}\right)$$

However, it is known from lectures that:

$$(\boldsymbol{p}^j, \boldsymbol{p}^k) = \begin{cases} 1/N & j = k \\ 0 & j \neq k \end{cases}$$

Hence, all off diagonal entries vanish, and the coefficients are given explicitly by (after expanding the RHS complex inner products):

$$a_k = \sum_{n=0}^{N-1} \exp(-ikx_n) f_n \quad \text{for } x_n = \frac{2\pi n}{N}$$

(a) For $\boldsymbol{f} = (1,\ 2,\ 0,\ 1)$, coefficients are given by:

$$a_k = \sum_{n=0}^{3} \exp\left(-ik\frac{\pi n}{2}\right) f_n$$

$$\begin{aligned}
a_0 &= 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 0 + 1 \cdot 1 && = 4 \\
a_1 &= 1 \cdot 1 + \exp\left(-i\frac{\pi}{2}\right) \cdot 2 + \exp\left(-i\pi\right) \cdot 0 + \exp\left(-i\frac{3\pi}{2}\right) \cdot 1 && = 1 - 2i + i \\
a_2 &= 1 \cdot 1 + \exp\left(-i2\frac{\pi}{2}\right) \cdot 2 + \exp\left(-i2\pi\right) \cdot 0 + \exp\left(-i2\frac{3\pi}{2}\right) \cdot 1 && = 1 - 2 - 1 \\
a_3 &= 1 \cdot 1 + \exp\left(-i3\frac{\pi}{2}\right) \cdot 2 + \exp\left(-i3\pi\right) \cdot 0 + \exp\left(-i3\frac{3\pi}{2}\right) \cdot 1 && = 1 + 2i - i
\end{aligned}$$

So $\boldsymbol{a} = (4, 1 - i. - 2, 1 + i)$

(b) For $\boldsymbol{f} = (1,\ 1,\ \ldots,\ 1)$, $N = 8$

All $f_n = 1$, coefficients simplify to:

$$a_k = \sum_{n=0}^{7} \exp\left(-ik\frac{\pi n}{4}\right)$$

$$\begin{aligned}
a_0 &= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 && = 8 \\
a_1 &= 1 + \frac{1 - i}{\sqrt{2}} - i + \frac{-1 - i}{\sqrt{2}} - 1 + \frac{-1 + i}{\sqrt{2}} + i + \frac{1 + i}{\sqrt{2}} && = 0 \\
a_2 &= 1 - i - 1 + i + 1 - i - 1 + i && = 0 \\
a_3 &= 1\frac{-1 - i}{\sqrt{2}} + i + \frac{1 - i}{\sqrt{2}} - 1 + \frac{1 + i}{\sqrt{2}} - i + \frac{-1 + i}{\sqrt{2}} && = 0 \\
a_4 &= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 && = 0 \\
a_5 &= 1 + \frac{-1 + i}{\sqrt{2}} - i + \frac{1 + i}{\sqrt{2}} - 1 + \frac{1 - i}{\sqrt{2}} + i + \frac{-1 - i}{\sqrt{2}} && = 0 \\
a_6 &= 1 + i - 1 - i + 1 + i - 1 - i && = 0 \\
a_7 &= 1 + \frac{1 + i}{\sqrt{2}} + i + \frac{-1 + i}{\sqrt{2}} - 1 + \frac{-1 - i}{\sqrt{2}} - i + \frac{1 - i}{\sqrt{2}} && = 0
\end{aligned}$$

$\boldsymbol{a} = (8, 0, 0, 0, 0, 0, 0, 0)$

Question 5.

Show that the DFT of: $\boldsymbol{f} = (f_0,\ f_1,\ \ldots,\ f_7)$ for:

$$f_n = \sin\frac{2\pi n}{8}$$

Is given by $(0,\ A, 0,\ 0,\ 0,\ 0, B)$, and determine $A,\ B$. You may use the orthogonality properties of $\boldsymbol{p}_n^{(k)}$.

SOLUTIONS

First note that:

$$\sin\frac{2\pi n}{8} = \frac{1}{2i}\left(\exp\left(\frac{i2\pi n}{8}\right) - \exp\left(\frac{-i2\pi n}{8}\right)\right)$$

And that:

$$\exp\left(\frac{-i2\pi n}{8}\right) = \exp\left(\frac{i(8-1)2\pi n}{8}\right) = \frac{1}{2i}\left(p_n^1 - p_n^7\right)$$

Therefore:

$$(\boldsymbol{p}^k, \boldsymbol{f}) = \begin{cases} 1/(8\cdot 2i) & k=1 \\ -1/(8\cdot 2i) & k=7 \\ 0 & k\neq 1,7 \end{cases}$$

And the normal equations simplify to:

$$\begin{bmatrix} 1/8 & 0 & \dots & 0 \\ 0 & 1/8 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/8 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/(8\cdot 2i) \\ \vdots \\ -1/(8\cdot 2i) \end{bmatrix}$$

Therefore $\boldsymbol{a} = (0, 1/2i, 0, 0, 0, 0, 0, -1/2i)$

Question 6.

The DFT of a signal $\boldsymbol{f}$ is: $(8,\ 4-8i,\ 2,\ -i,\ 0,\ i,\ 2,\ 4+8i)$

Determine the original signal $\boldsymbol{f}$.

Hint: Use the property that $p_n^{N-m} = p_n^{-m}$
SOLUTIONS

The inverse fourier transform is given by:

$$f_n = \frac{1}{N}\sum_{k=0}^{N-1} a_k \exp\left(ikx_n\right)$$