3750 Tutorial 1 - Error Analysis & Derivatives

Theory

Taylor series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$ note; $a^0 = 1 \ \forall a$, $0! = 1$

"evaluating a function f at a point 'x', about a known point 'a'"

$$= \frac{f^{(0)}(a)(x-a)^0}{0!} + \frac{f^{(1)}(a)(x-a)^1}{1!} + \frac{f^{(2)}(a)(x-a)^2}{2!} + \dots$$

truncating the taylor series: (two methods)

(i) $f(x) = \frac{f^{(0)}(a)(x-a)^0}{0!} + \frac{f^{(1)}(a)(x-a)^1}{1!} + \frac{f^{(2)}(\xi)(x-a)^2}{2!}$, $x \le \xi \le x+h$

→ equals sign. last term in truncation is evaluated at ξ, accounting for all error.

(ii) $f(x) \simeq \frac{f^{(0)}(a)(x-a)^0}{0!} + \frac{f^{(1)}(a)(x-a)^1}{1!} + \frac{f^{(2)}(a)(x-a)^2}{2!}$

→ approx. sign. dropping all following terms due to diminishing contribution. Not exact!
→ truncation error will be proportional to largest term truncated ($\varepsilon \propto \frac{f^{(3)}(a)(x-a)^3}{3!}$ in this case)

The aim is to minimize error. We require $h < 1$ ($h = (x-a)$) to ensure Taylor series converges to $f(x)$.
→ higher powers of h means less error.
$\varepsilon = O(h^n) \Rightarrow \varepsilon \downarrow$ as $n \uparrow$
'error' ↑   ← 'order of'

Taylor series in two variables;
$f(x,y) \rightarrow df(x,y) = \frac{\partial f(x,y)}{\partial x} dx + \frac{\partial f(x,y)}{\partial y} dy$

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \tfrac{1}{2}f_{xx}(a,b)(x-a)^2 + \tfrac{1}{2}f_{xy}(a,b)(x-a)(y-b) + \tfrac{1}{2}f_{yx}(a,b)(x-a)(y-b) + \tfrac{1}{2}f_{yy}(a,b)(y-b)^2 + \dots$$

note: $f_{xy} = f_{yx}$ if function f is twice differentiable and continuous.

Backwords difference example

show: $u'(x) \simeq \dfrac{u(x) - u(x-h)}{h}$ and find the error associated

taylor series expansions:

$$u(x) = u(x)$$

$$u(x-h) = u(x) - u'(x)h + u''(\xi)\frac{h^2}{2} \quad \text{for } x-h \leq \xi \leq x$$

$$u(x) - u(x-h) = \cancel{u(x)} - \left[\cancel{u(x)} - u'(x)h + u''(\xi)\frac{h^2}{2}\right]$$

$$u(x) - u(x-h) = u'(x)h - u''(\xi)\frac{h^2}{2}$$

$$\boxed{u'(x) = \frac{u(x) - u(x-h)}{h}} + \underbrace{u''(\xi)\frac{h}{2}}_{\substack{\text{Error associated with} \\ \text{backwords difference.}}} \quad \Rightarrow \varepsilon = O(h)$$

Common 1st order approximations

forward diff; $f'(x) \simeq \dfrac{f(x+h) - f(x)}{h}$, $\varepsilon = \dfrac{h}{2} f''(x)$

central diff; $f'(x) \simeq \dfrac{f(x+h) - f(x-h)}{2h}$, $\varepsilon = \dfrac{h^2}{6} f'''(x)$

Generating approximations example:

show $f'(x) \approx \frac{1}{6h}\left[4f(x+h) - 3f(x) - f(x-2h)\right]$ & define error

General Solving Method

1) write out $f(x \pm k)$ terms with undetermined coeffts

(i) $f'(x) = Af(x+h) + Bf(x) + Cf(x-2h)$

2) take taylor approximations for each term, evaluated at $'x' = x \pm k$, $'a' = x$
→ always expand to 1 or 2 more terms than # unknowns in method of undetermined coeffts.

(ii) $f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$, $a = x$, $x = x \pm h$

$\approx f^{(0)}(x) + f^{(1)}(x)h + \frac{f^{(2)}(x)h^2}{2!} + \frac{f^{(3)}(x)h^3}{3!}$

3) write out all expanded terms so columns line up for each derivative

(iii) $f'(x) = Af(x) + Ahf'(x) + \frac{Ah^2}{2}f''(x) + \frac{Ah^3}{6}f'''(x)$

$+ Bf(x)$

$+ Cf(x) - 2Chf'(x) + 2Ch^2f''(x) - \frac{4}{3}Ch^3f'''(x)$

4) compare terms to form simultaneous equations

(iv) $\quad = 0 \qquad = f'(x) \qquad = \boxed{0}$ ← have 3 unknowns, need 3 equations can force this to be zero.

5) formulate system of equations as a matrix

(v) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1/2 & 0 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 1/h \\ 0 \end{bmatrix}$ ← dividing through by h's to reduce complexity.

6) solve matrix (or sim. eq's) and back-substitute to $f'(x)$ equation
→ first non-zero term is error due to truncation

(vi) $A = 2/3h$, $B = -1/2h$, $C = -1/6h$

$Af(x+h) + Bf(x) + Cf(x-2h)$
$= 0 + f'(x) + 0 + 3/9\, h^2 f'''(x)$

$f'(x) \approx \frac{1}{6h}\left[4f(x+h) - 3f(x) - f(x-2h)\right] - \frac{1}{3}h^2 f''(x)$

$\therefore \varepsilon \sim O(h^2)$

note: sign of error is unimportant, only magnitude is of interest