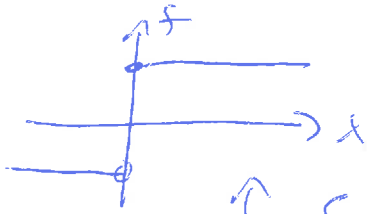
1). $f(x) = \begin{cases} -1 & x<0 \\ 1 & x \geq 0 \end{cases}$



f is odd.

Find Fourier Series coeffs.

as $f(x)$ is ODD $\Rightarrow a_0 = a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(nx)\,dx + \int_0^{\pi} \sin(nx)\,dx\right)$$

note integral domains.

$$= \frac{1}{\pi}\left(-\int_0^{\pi} -\sin(nx)\,dx + \int_0^{\pi}\sin(nx)\,dx\right) \quad \text{as sin odd.}$$

$$= \frac{2}{\pi}\left(\int_0^{\pi}\sin(nx)\,dx\right)$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} = \frac{-2}{n\pi}\left(\cos(n\pi) - 1\right)$$

While this b_n is correct, we can simplify further

$$\cos(n\pi) = (-1)^n \Rightarrow b_n = \frac{-2}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd.} \end{cases}$$

can shift so that sum only picks non-zero terms.

$$\rightarrow f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin(nx) = \frac{4\sin(x)}{\pi} + 0 + \frac{4\sin(3x)}{3\pi} + 0 + \dots$$

OR

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin((2n-1)x) = \frac{4\sin(x)}{\pi} + \frac{4\sin(3x)}{3\pi} + \frac{4\sin(5x)}{5\pi} + \frac{4\sin(7x)}{7\pi} + \dots$$

$n=1$ 2 3 4

Second form makes better use of the sum. Esp. if we truncate at some N ie $\sum_{n=1}^{N}$

2) $f(x)$ on $[0,1]$: Shifted Legendre Polynomials (Q_i) are the appropriate choice for this domain

$\Rightarrow$ approx. $f(x)$ w/ $y = \alpha_0 Q_0 + \alpha_1 Q_1 + \alpha_2 Q_2 + \alpha_3 Q_3 + \alpha_4 Q_4$

* Need up to Q_4 to get to order x^4.

Least squares approx. yields Normal Equations:

$$\begin{bmatrix} (Q_0,Q_0) & & & & 0 \\ & (Q_1,Q_1) & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & (Q_4,Q_4) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} (f,Q_0) \\ (f,Q_1) \\ \vdots \\ (f,Q_4) \end{bmatrix}$$

as $\{Q_i\}$ orthogonal

$$(Q_i,Q_i) = \int_0^1 Q_i Q_i \, dx = 1/2i+1$$

$$\rightarrow \begin{bmatrix} 1 & & & & 0 \\ & 1/3 & & & \\ & & 1/5 & & \\ & & & 1/7 & \\ 0 & & & & 1/9 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \int_0^1 f Q_0 \, dx \\ \int_0^1 f Q_1 \, dx \\ \vdots \\ \int_0^1 f Q_4 \, dx \end{bmatrix}$$

* I made an error here in tutorial on 22/8.

$\Rightarrow \alpha_0 = \int_0^1 f Q_0 \, dx$

$\frac{\alpha_1}{3} = \int_0^1 f Q_1 \, dx \rightarrow \alpha_1 = 3\int_0^1 f Q_1 \, dx$

$\alpha_2/5 = \int_0^1 f Q_2 \, dx \rightarrow \alpha_2 = 5\int_0^1 f Q_2 \, dx$

$\alpha_3/7 = \int_0^1 f Q_3 \, dx \rightarrow \alpha_3 = 7\int_0^1 f Q_3 \, dx$

$\alpha_4/9 = \int_0^1 f Q_4 \, dx \rightarrow \alpha_4 = 9\int_0^1 f Q_4 \, dx$

Coefficients for L.S. approximations.

Can evaluate when $f(x)$ known.

N.B. $Q_0 - Q_3$ are on formula sheet.

3) $\underset{\sim}{a} = (1+i,\ 1-i,\ 2+4i)$

$\underset{\sim}{b} = (2+i,\ 4-i,\ 4+i)$

$$(\underset{\sim}{a},\underset{\sim}{b}) = \bar{\underset{\sim}{a}}\cdot\underset{\sim}{b} = (1-i,\ 1+i,\ 2-4i)\cdot(2+i,\ 4-i,\ 4+i)$$

$$= 3-i+5+3i+12-14i = 20-12i$$

$$\|\underset{\sim}{a}\| = \sqrt{(\underset{\sim}{a},\underset{\sim}{a})} = \sqrt{\bar{\underset{\sim}{a}}\cdot\underset{\sim}{a}} = \sqrt{(1-i,\ 1+i,2-4i)\cdot(1+i,\ 1-i,\ 2+4i)}$$

$$= \sqrt{2+2+20} = \sqrt{24} = 2\sqrt{6}$$

4) Evaluate w/ $e^{i\theta} = \cos(\theta)+i\sin(\theta)$

$\rightarrow$ $q^{(0)} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ $q^{(1)} = \begin{pmatrix}1\\i\\-1\\-i\end{pmatrix}$ $q^{(2)} = \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$ $q^{(3)} = \begin{pmatrix}1\\-i\\-1\\i\end{pmatrix}$

5). Use $\alpha_k = \sum_{n=0}^{N-1} f_n e^{-ikx_n}$ $\quad x_n = \frac{2\pi n}{N}$, $n=0,\dots,N-1$

DFT $= \underset{\sim}{\alpha}$

* See Tutorial 4 Solutions - Q5 (same Question)

Result $A = \frac{1}{2i}$ $\quad B = -\frac{1}{2i}$

see also Q1: $\overline{e^{iy}} = e^{-iy}$ & Q3: $p_n^{(k)} = \frac{1}{N} e^{\left(\frac{i2\pi nk}{N}\right)}$

$\sin(y) = \frac{1}{2i}(e^{iy} - e^{-iy})$ $\quad p_n^{(N-m)} = p_n^{(-m)}$

$\cos(y) = \frac{1}{2}(e^{iy} + e^{-iy})$