MECH3750 - Tutorial 3

Question 1.

Suppose $p_1,\ p_2,\ p_3$ are three vectors (or functions). Suppose we wish to find another vector (or function) $y = a_1p_1 + a_2p_2 + a_3p_3$ which minimizes the distance squared:

$$d^2\left(y,\ f\right) = \left(y - f, y - f\right)$$

Where d is the "distance" as introduced in lectures, and u, v is a well defined inner product.

Obtain $a_1,\ a_2,\ a_3$ (to find the best approximation) for the following cases:

(a) $p_1 = (1,\ 0,\ -1,\ 0)$, $p_2 = (1,\ 1,\ 1,\ 0)$, $p_3 = (1,\ -2,\ 1,\ 0)$. With inner product: $(u, v) = \sum_i u_i v_i$ and:

$$\text{(i) } f = (4,\ 0,\ 2,\ 0) \qquad \text{(ii) } f = (0,\ 0, 0,\ 1)$$

(b) $p_1 = \sin x$, $p_2 = \sin 2x$, $p_3 = \sin 3x$. With inner product: $(u, v) = \int_0^\pi u(x)v(x)\ \mathrm{d}x$ and:

$$\text{(i) } f(x) = \sin 2x + \sin 3x \qquad \text{(ii) } f(x) = \sin 4x$$

Hint: Use the result established in lectures that:

$$(\sin nx,\ \sin mx) = \begin{cases} \frac{\pi}{2} & m = n \\ 0 & m \neq n \end{cases}$$

(c) $p_1 = 1$, $p_2 = x$, $p_3 = \frac{1}{2}\left(3x^2 - 1\right)$. With inner product: $(u, v) = \int_{-1}^1 u(x)v(x)\ \mathrm{d}x$ and:

$$\text{(i) } f(x) = x^2 \qquad \text{(ii) } f(x) = 5x^3 - 3x$$

(d) Comment on why $d^2\left(y,\ f\right) = \left(y - f, y - f\right) = 0$ in all (i) cases. Comment on (ii) results and why all these cases are non zero.

(e) There is an alternative way of solving all (i) cases. Why?

Question 2.

(a) Find the best approximation to the function $f(x) = 1$ on the interval $[0,\ \pi]$ using the set $\sin nx,\ n = 1, 2, ..., N$. On the worksheet, you are asked to plot your answer for different values of N.

(b) Find the best approximation to the function $f(x) = 1$ on the interval $[0,\ L]$ using the set $\sin \frac{\pi n x}{L},\ n = 1, 2, ..., N$. You may use the results of part (a) by applying a suitable substitution in the integrals.

(c) Find the best approximation to the function $f(x) = x$ on the interval $[0,\ \pi]$ using the set $\sin nx,\ n = 1, 2, ..., N$. On the worksheet, you are asked to plot your answer for different values of N. Show how the results can be adapted to represent the function $f(x) = x$ on the interval $[0,\ L]$.

(d) Find the best approximation to the function $f(x) = x$ on the interval $[0,\ \pi]$ using the set $1,\ \cos nx,\ n = 1, 2, ..., N$. You will need to check that the function 1 is orthogonal to all other members of the set.

Note, if we take $N \to \infty$, we call our solutions the Fourier sine, or cosine series for $f(x)$. These are important functions in many partial differential equations useful in engineering applications such as the heat transfer (MECH3400), and vibration of strings and beams (MECH3200).