MECH3750 - Tutorial 2 Solutions

Question 1.

Suppose you have data (x_i, f_i) which can be fitted to a quadratic of the form: $y(x) = a_0 + a_1 x + a_2 x^2$
The least square error to be minimized is: $\sum_{i=1}^{N} \left(a_0 + a_1 x_i + a_2 x_i^2 - f_i\right)^2$
Show that the solution to this problem satisfies:

$$P^T P \boldsymbol{a} = P^T \boldsymbol{f}$$

So that

$$\boldsymbol{a} = \left(P^T P\right)^{-1} P^T \boldsymbol{f}$$

For a suitable matrix P, and where $\boldsymbol{a}$ is the vector of unknown coefficients (a_0, a_1, a_2). In the worksheet, you will be asked to use this formulation.

Question 1a

$$P = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 \end{bmatrix}, \quad P^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \\ x_1^2 & x_2^2 & \cdots & x_N^2 \end{bmatrix}$$

$$\therefore P^T P = \begin{bmatrix} \sum_{i=1}^{N} 1 & \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i^3 \\ \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i^3 & \sum_{i=1}^{N} x_i^4 \end{bmatrix}$$

This may be written more compactly as

$$P = \left(p^{(1)} \quad p^{(2)} \quad p^{(3)}\right), \quad P^T = \begin{pmatrix} (p^{(1)})^T \\ (p^{(2)})^T \\ (p^{(3)})^T \end{pmatrix}$$

$$\therefore P^T P = \begin{bmatrix} (p^{(1)})^T p^{(1)} & (p^{(1)})^T p^{(2)} & (p^{(1)})^T p^{(3)} \\ (p^{(2)})^T p^{(1)} & (p^{(2)})^T p^{(2)} & (p^{(2)})^T p^{(3)} \\ (p^{(3)})^T p^{(1)} & (p^{(3)})^T p^{(2)} & (p^{(3)})^T p^{(3)} \end{bmatrix}$$

Similarly for $P^T \underset{\sim}{f}$

$$\begin{bmatrix} (p^{(1)})^T \\ (p^{(2)})^T \\ (p^{(3)})^T \end{bmatrix} \underset{\sim}{f} = \begin{bmatrix} (p^{(1)})^T \underset{\sim}{f} \\ (p^{(2)})^T \underset{\sim}{f} \\ (p^{(3)})^T \underset{\sim}{f} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} f_i \\ \sum_{i=1}^{N} x_i f_i \\ \sum_{i=1}^{N} x_i^2 f_i \end{bmatrix}$$

Question 1 b

Want to choose a_0, a_1, a_2 such that

$$R = \sum_{i=1}^{N} (a_0 + a_1 x_i + a_2 x_i^2 - f_i) \quad \text{is minimized}$$

Solution satisfies:

$$\frac{\partial R}{\partial a_0} = 0 = 2 \sum_{i=1}^{N} (a_0 + a_1 x_i + a_2 x_i^2 - f_i)$$

$$\frac{\partial R}{\partial a_1} = 0 = 2 \sum_{i=1}^{N} x_i (a_0 + a_1 x_i + a_2 x_i^2 - f_i)$$

$$\frac{\partial R}{\partial a_2} = 0 = 2 \sum_{i=1}^{N} x_i^2 (a_0 + a_1 x_i + a_2 x_i^2 - f_i)$$

expanding and rearranging

$$\Rightarrow \quad a_0 \sum 1 + a_1 \sum x_i + a_2 \sum x_i^2 = \sum f_i$$

$$a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 = \sum x_i f_i$$

$$a_0 \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4 = \sum x_i^2 f_i$$

$$\overbrace{\begin{bmatrix} \sum 1 & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix}} \overset{\underset{\sim}{a}}{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}} = \overset{\underset{\sim}{y}}{\begin{bmatrix} \sum f_i \\ \sum x_i f_i \\ \sum x_i^2 f_i \end{bmatrix}}$$

from observation of the above system, the left side matrix takes the form of $P^T P$ from (a) and the right side vector $P^T \underset{\sim}{t}$

Hence: $$P^T P \, \underset{\sim}{a} = P^T t$$

$$\Rightarrow \quad \underset{\sim}{a} = (P^T P)^{-1} P^T \underset{\sim}{t}$$

Question 2.

(a) Evaluate (by hand) the following integral:

$$\int_0^1 (6x^2 - 6x + 1)\, e^x \; \mathrm{d}x$$

(b) Demonstrate that the functions $P_1(x) = 2x - 1$ and $P_2(x) = 6x^2 - 6x + 1$ are orthogonal, by showing that:

$$\int_0^1 P_1(x)P_2(x) \; \mathrm{d}x = 0$$

(c) You are given that $P_0(x) = 1$ and $P_1(x) = 2x - 1$. Show how to find a function of the form $P_2(x) = ax^2 + bx + c$, which satisfies:

$$\int_0^1 P_0(x)P_2(x) \; \mathrm{d}x = 0 \quad \text{and} \quad \int_0^1 P_1(x)P_2(x) \; \mathrm{d}x = 0$$

Question 2.a

$$I = \int_0^1 (6x^2 - 6x + 1) e^x \, dx$$

integrate by parts

$u = 6x^2 - 6x + 1 \qquad dv = e^x dx$

$du = (12x - 6) dx \qquad v = e^x$

$$I = \Big| e^x (6x^2 - 6x + 1) \Big|_0^1 - \int_0^1 e^x (12x - 6) \, dx$$

$u_2 = 12x - 6 \qquad dv = e^x dx$

$du_2 = 12 \, dx \qquad v = e^x$

$$I = (e(6 - 6 + 1) - 1) - \Big| e^x (12x - 6) \Big|_0^1 + 12 \int_0^1 e^x dx$$

$$= (e - 1) - (e(12 - 6) - (-6)) - 12 \Big| e^x dx \Big|_0^1$$

$$= (e - 1) - 6(e + 1) + 12(e - 1)$$

$$= 7e - 19$$

b) $\int_0^1 (2x-1)(6x^2-6x+1)\,dx$

$= \int_0^1 (12x^3 - 12x^2 + 2x - 6x^2 + 6x - 1)\,dx$

$= \int_0^1 (12x^3 - 18x^2 + 8x - 1)\,dx$

$= \Big| 3x^4 - 6x^3 + 4x^2 - x \Big|_0^1$

$= 3 - 6 + 4 - 1 = 0$ $\therefore$ orthogonal

c) want

[1] $\int_0^1 1 \times (ax^2 + bx + c)\,dx = 0$

[2] $\int_0^1 (2x-1)(ax^2+bx+c)\,dx = 0$

[1] $\Rightarrow$ $\Big| \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx \Big|_0^1 = 0$

$\Rightarrow \frac{a}{3} + \frac{b}{2} + c = 0$ [4]

[2] $\Rightarrow$ $\int_0^1 (2ax^3 + 2bx^2 + 2cx - ax^2 - bx - c)\,dx$

$$\left. \frac{a}{2}x^4 + \frac{2}{3}bx^3 + cx^2 - \frac{a}{3}x^3 - \frac{b}{2}x^2 - cx \right|_0^1$$

$$= \frac{a}{6} + \frac{b}{6} = 0 \Rightarrow a = -b \text{ , sub into [4]}$$

$$\Rightarrow -\frac{1}{3}b + \frac{b}{2} + c = 0 \Rightarrow \frac{b}{6} + c = 0$$

$$\Rightarrow b = -6c \Rightarrow a = 6c$$

for any $c \neq 0$, choose $c = 1$

$$\Rightarrow P_2(x) = 6x^2 - 6x + 1$$

Question 3. (***formulate in tutorial, code in practical***)
A hot-wire anemometer calibration gave the following voltage velocity points:

U (m/s)	91.08	85.04	78.54	68.51	58.35	49.63	41.45	34.32
E (V)	0.8560	0.8414	0.8375	0.8160	0.7977	0.7762	0.7535	0.7347

The model function $E = \alpha_0 + \alpha_1 U + \alpha_2 \sqrt{U}$ is to be fitted to the calibration data using the "least-squares" error criterion.

(a) Write the design matrix for the given model equation and data points
(b) Formulate the normal equations in matrix form and obtain values for the coefficients $\alpha_0, \alpha_1, \alpha_2$
(c) Plot E vs U over the range $35m/s \leq U \leq 90m/s$ for both the original data and the model function

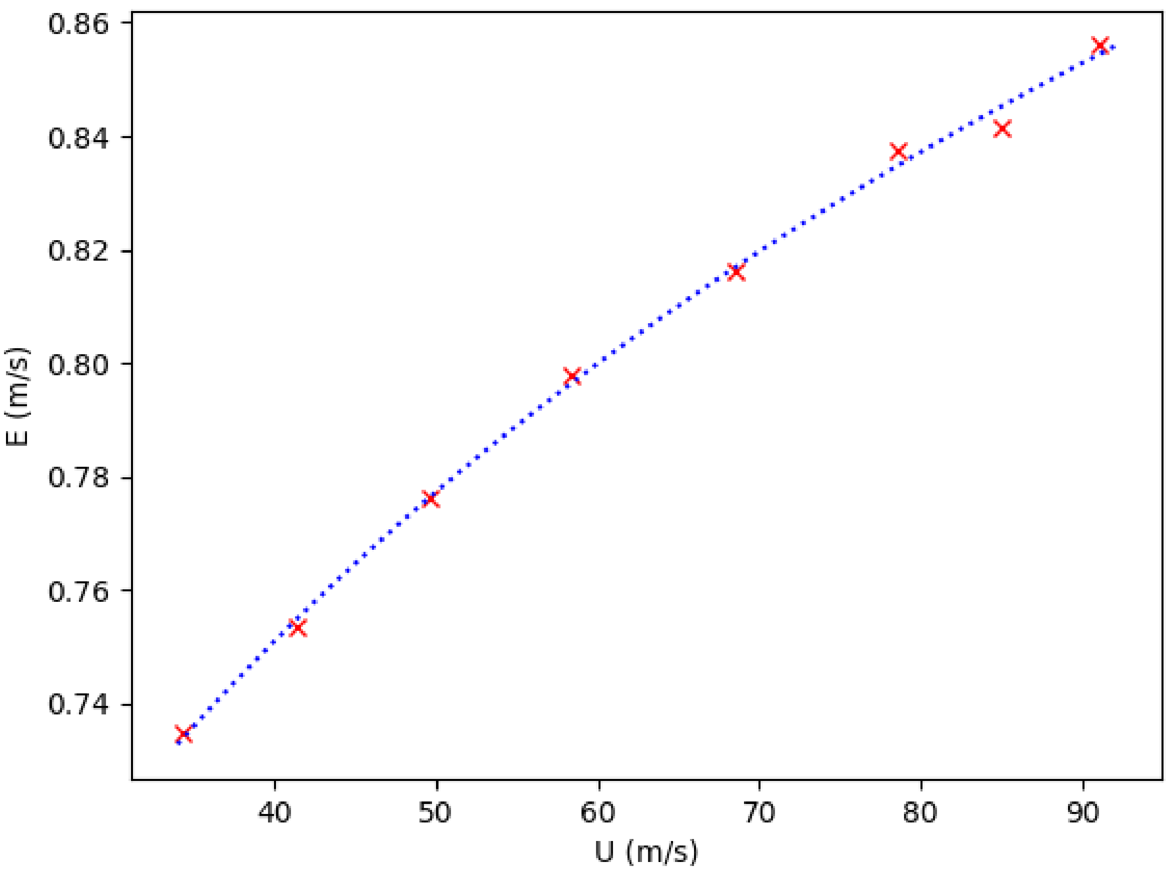
Solution
(a) Here we can write our error $dE = \sum_i \left(\alpha_0 + \alpha_1 U_i + \alpha_2 \sqrt{U_i} - E_i\right)^2$, then by taking partial derivatives with respect to α_j we obtain our design matrix:

$$\begin{pmatrix} 1 & U_0 & \sqrt{U_0} \\ \vdots & \vdots & \vdots \\ 1 & U_7 & \sqrt{U_7} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} E_0 \\ \vdots \\ E_7 \end{pmatrix} \tag{1}$$

(b)

```
#MECH3750:  Tutorial  Sheet  2
#  Week  2  problem
#  Updated:  06.08.18
#  Maintainer:  @TravisMitchell

from math import sqrt
from numpy import zeros,linalg,dot,transpose,linspace
from matplotlib.pyplot import plot,xlabel,ylabel,show

#  data
U = [91.08, 85.04, 78.54, 68.51, 58.35, 49.63, 41.45, 34.32]
E = [0.8560, 0.8414, 0.8375, 0.8160, 0.7977, 0.7762, 0.7535, 0.7347]

m = len(U)
A = zeros((m,3), float) #Design  Matrix
b = zeros((m), float)
for k in range(m):
    A[k,0] = 1.0
    A[k,1] = U[k]
    A[k,2] = sqrt(U[k])
    b[k]   = E[k]

alpha = linalg.solve(dot(transpose(A),A), dot(transpose(A),b))
print("A = ",A)
print("b^T = ",b)
print("alpha = ",alpha)

plot(U,E, 'xr')
x_plot = linspace(34,92, 100)
y_plot = [alpha[0] +alpha[1]*x+ alpha[2]*sqrt(x) for x in x_plot]
plot(x_plot, y_plot, 'b:')
xlabel("U (m/s)")
ylabel("E (m/s)")
show()
```

(c)



Additional Practice

Question 4.

(a) A function $f(x)$, is *odd* if $f(x) = -f(-x)$ and *even* if $f(x) = f(-x)$. Show that $\sin(x)$ is odd and $\cos(x)$ is even.

(b) Demonstrate with the aid of sketches (or analytically) that for any odd function $f(x)$ and any even function $g(x)$. The following identities hold:

$$(i)\quad \int_{-a}^{a} f(x)\ \mathrm{d}x = 0$$

$$(ii)\quad \int_{-a}^{a} g(x)\ \mathrm{d}x = 2\int_{0}^{a} g(x)\ \mathrm{d}x$$

$$(iii)\quad \int_{-a}^{a} f(x)g(x)\ \mathrm{d}x = 0$$

$$(iv)\quad \int_{-a}^{a} f_1(x)f_2(x)\ \mathrm{d}x = 2\int_{0}^{a} f_1(x)f_2(x)\ \mathrm{d}x \quad f_1,\ f_2 \text{ odd}$$

Question 5.

Consider the Legendre polynomials:

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right)\\
P_3(x) &= \frac{4}{5}\left(5x^3 - 3x\right)
\end{aligned}$$

(a) Confirm that P_1, P_3 are odd

(b) Confirm that P_0, P_2 are even

(c) Use the properties of even and odd functions to evaluate:

$$\int_{-1}^{1} P_0P_0\ \mathrm{d}x \qquad \int_{-1}^{1} P_0P_1\ \mathrm{d}x \qquad \int_{-1}^{1} P_0P_2\ \mathrm{d}x \qquad \int_{-1}^{1} P_0P_3\ \mathrm{d}x$$

$$\int_{-1}^{1} P_1P_1\ \mathrm{d}x \qquad \int_{-1}^{1} P_2P_3\ \mathrm{d}x \qquad \int_{-1}^{1} P_2P_2\ \mathrm{d}x$$

Question 6.

For the following questions, assume $m, n \neq 0$.

(a) Determine the indefinite integral of:

$$\int \cos(mx)\cos(nx)\ \mathrm{d}x$$

(b) Use the results of part (a) to evaluate:

$$\int_0^{\pi} \cos(mx)\cos(nx)\ \mathrm{d}x$$

(c) Use the properties of even and odd functions, and the results established in part (b) and lectures to evaluate:

$$(i)\ \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ \mathrm{d}x$$

$$(ii)\ \int_{-\pi}^{\pi} \sin(mx)\cos(nx)\ \mathrm{d}x$$

$$(iii)\ \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ \mathrm{d}x$$