# MECH3750 - Tutorial 2

**Question 1.**

Suppose you have data $(x_i, f_i)$ which can be fitted to a quadratic of the form:

$$y(x) = a_0 + a_1 x + a_2 x^2$$

The least square error to be minimized is:

$$\sum_{i=1}^{N} \left(a_0 + a_1 x_i + a_2 x_i^2 - f_i\right)^2$$

Show that the solution to this problem satisfies:

$$P^T P \boldsymbol{a} = P^T \boldsymbol{f}$$

So that

$$\boldsymbol{a} = \left(P^T P\right)^{-1} P^T \boldsymbol{f}$$

For a suitable matrix $P$, and where $\boldsymbol{a}$ is the vector of unknown coefficients $(a_0, a_1, a_2)$. In the worksheet, you will be asked to use this formulation.

**Question 2.**

(a) Evaluate (by hand) the following integral:

$$\int_0^1 \left(6x^2 - 6x + 1\right) e^x \, \mathrm{d}x$$

(b) Demonstrate that the functions $P_1(x) = 2x - 1$ and $P_2(x) = 6x^2 - 6x + 1$ are orthogonal, by showing that:

$$\int_0^1 P_1(x) P_2(x) \, \mathrm{d}x = 0$$

(c) You are given that $P_0(x) = 1$ and $P_1(x) = 2x - 1$. Show how to find a function of the form $P_2(x) = ax^2 + bx + c$, which satisfies:

$$\int_0^1 P_0(x) P_2(x) \, \mathrm{d}x = 0 \quad \text{and} \quad \int_0^1 P_1(x) P_2(x) \, \mathrm{d}x = 0$$

**Question 3. (*formulate in tutorial, code in practical*)**
A hot-wire anemometer calibration gave the following voltage velocity points:

| $U$ $(m/s)$ | 91.08 | 85.04 | 78.54 | 68.51 | 58.35 | 49.63 | 41.45 | 34.32 |
|---|---|---|---|---|---|---|---|---|
| $E$ $(V)$ | 0.8560 | 0.8414 | 0.8375 | 0.8160 | 0.7977 | 0.7762 | 0.7535 | 0.7347 |

The model function $E = \alpha_0 + \alpha_1 U + \alpha_2 \sqrt{U}$ is to be fitted to the calibration data using the "least-squares" error criterion.

(a) Write the design matrix for the given model equation and data points
(b) Formulate the normal equations in matrix form and obtain values for the coefficients $\alpha_0, \alpha_1, \alpha_2$
(c) Plot $E$ vs $U$ over the range $35m/s \leq U \leq 90m/s$ for both the original data and the model function

## Additional Practice

**Question 4.**

(a) A function $f(x)$, is *odd* if $f(x) = -f(-x)$ and *even* if $f(x) = f(-x)$. Show that $\sin(x)$ is odd and $\cos(x)$ is even.

(b) Demonstrate with the aid of sketches (or analytically) that for any odd function $f(x)$ and any even function $g(x)$. The following identities hold:

$$(i)\ \int_{-a}^{a} f(x)\ \mathrm{d}x = 0$$

$$(ii)\ \int_{-a}^{a} g(x)\ \mathrm{d}x = 2\int_{0}^{a} g(x)\ \mathrm{d}x$$

$$(iii)\ \int_{-a}^{a} f(x)g(x)\ \mathrm{d}x = 0$$

$$(iv)\ \int_{-a}^{a} f_1(x)f_2(x)\ \mathrm{d}x = 2\int_{0}^{a} f_1(x)f_2(x)\ \mathrm{d}x \quad f_1,\ f_2 \text{ odd}$$

**Question 5.**

Consider the Legendre polynomials:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \\ P_3(x) &= \frac{4}{5}\left(5x^3 - 3x\right) \end{aligned}$$

(a) Confirm that $P_1, P_3$ are odd

(b) Confirm that $P_0, P_2$ are even

(c) Use the properties of even and odd functions to evaluate:

$$\int_{-1}^{1} P_0P_0\ \mathrm{d}x \qquad \int_{-1}^{1} P_0P_1\ \mathrm{d}x \qquad \int_{-1}^{1} P_0P_2\ \mathrm{d}x \qquad \int_{-1}^{1} P_0P_3\ \mathrm{d}x$$

$$\int_{-1}^{1} P_1P_1\ \mathrm{d}x \qquad \int_{-1}^{1} P_2P_3\ \mathrm{d}x \qquad \int_{-1}^{1} P_2P_2\ \mathrm{d}x$$

**Question 6.**

For the following questions, assume $m, n \neq 0$.

(a) Determine the indefinite integral of:

$$\int \cos(mx)\cos(nx)\ \mathrm{d}x$$

(b) Use the results of part (a) to evaluate:

$$\int_0^{\pi} \cos(mx)\cos(nx)\ \mathrm{d}x$$

(c) Use the properties of even and odd functions, and the results established in part (b) and lectures to evaluate:

$$(i)\ \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ \mathrm{d}x$$
$$(ii)\ \int_{-\pi}^{\pi} \sin(mx)\cos(nx)\ \mathrm{d}x$$
$$(iii)\ \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ \mathrm{d}x$$