



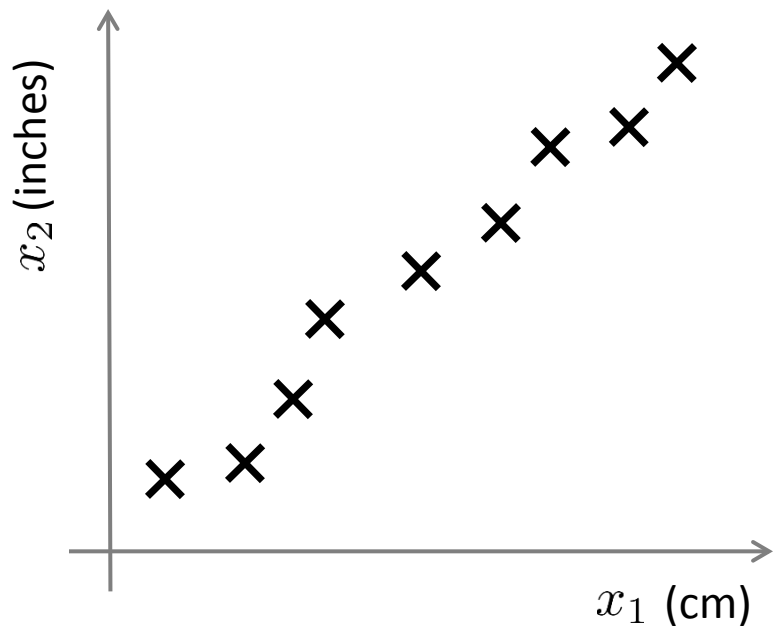
Machine Learning

# Dimensionality Reduction

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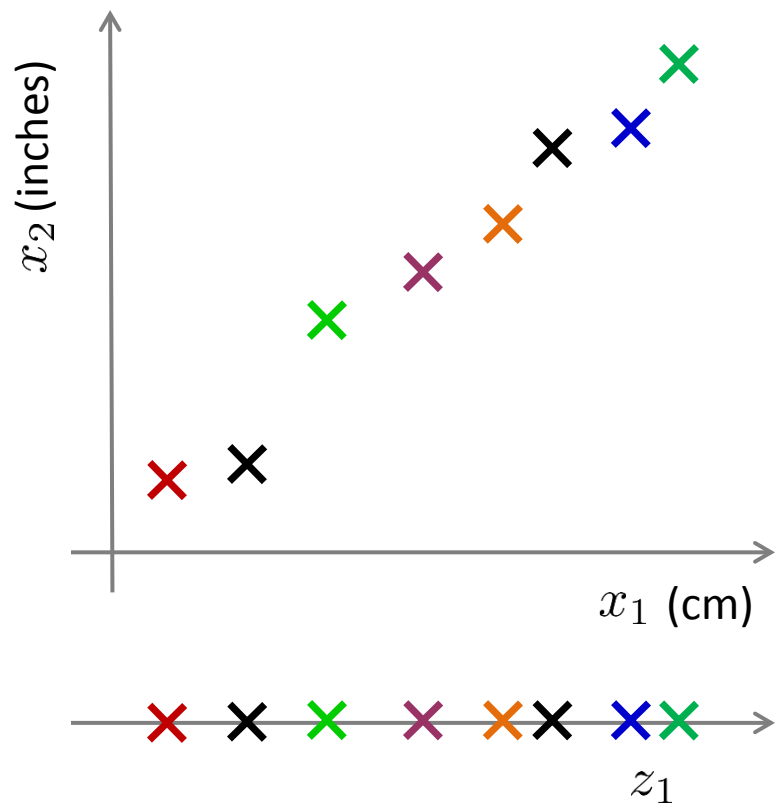
Motivation I:  
Data Compression

# Data Compression



Reduce data from  
2D to 1D

# Data Compression

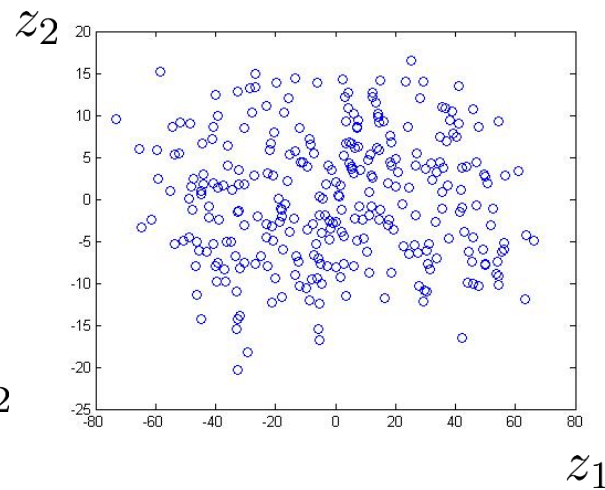
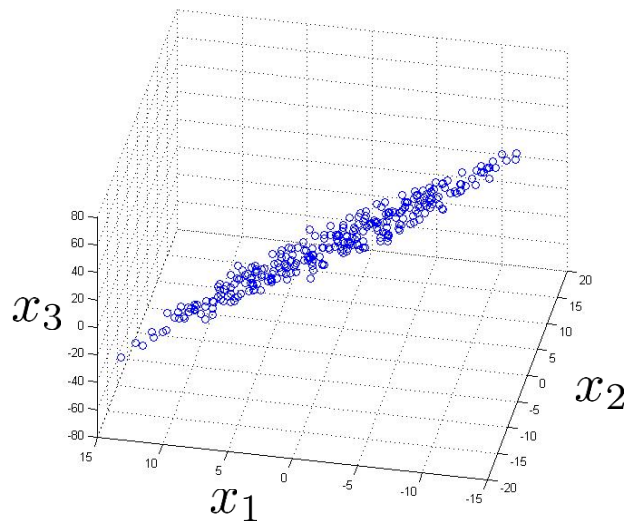
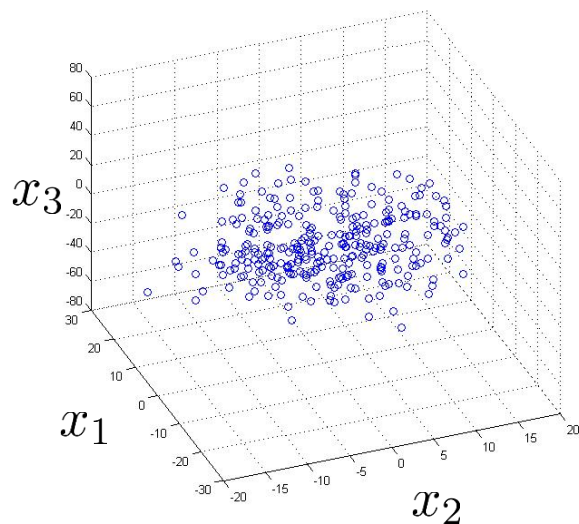


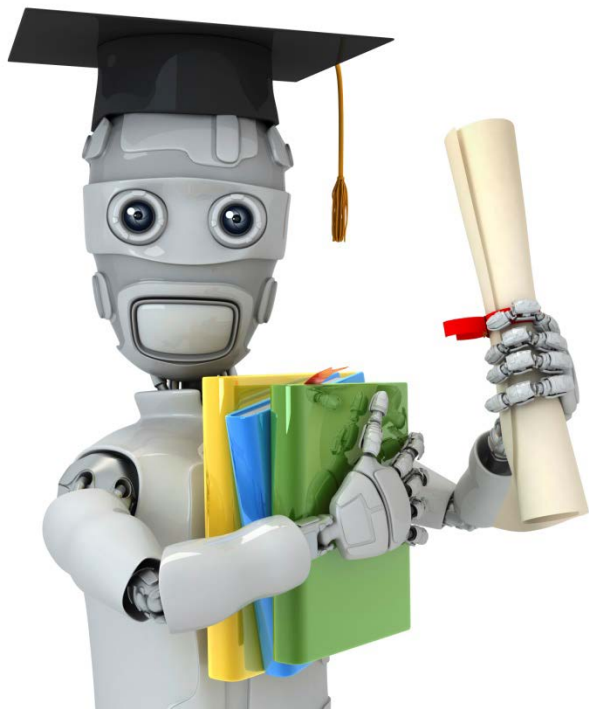
Reduce data from  
2D to 1D

$$\begin{array}{ccc} x^{(1)} & \rightarrow & z^{(1)} \\ x^{(2)} & \rightarrow & z^{(2)} \\ & \vdots & \\ x^{(m)} & \rightarrow & z^{(m)} \end{array}$$

# Data Compression

Reduce data from 3D to 2D





Machine Learning

# Dimensionality Reduction

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Motivation II:  
Data Visualization

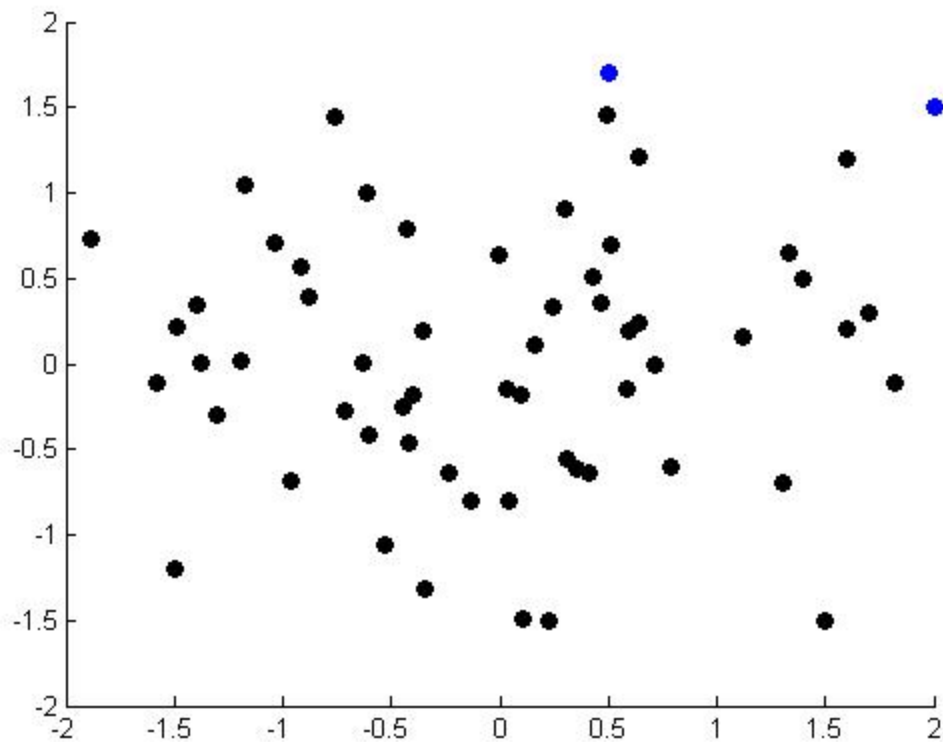
# Data Visualization

Country	GDP (trillions of US\$)	Per capita GDP (thousands of intl. \$)	Human Develop- ment Index	Life expectancy	Poverty Index (Gini as percentage)	Mean household income (thousands of US\$)	...
Canada	1.577	39.17	0.908	80.7	32.6	67.293	...
China	5.878	7.54	0.687	73	46.9	10.22	...
India	1.632	3.41	0.547	64.7	36.8	0.735	...
Russia	1.48	19.84	0.755	65.5	39.9	0.72	...
Singapore	0.223	56.69	0.866	80	42.5	67.1	...
USA	14.527	46.86	0.91	78.3	40.8	84.3	...
...	...	...	...	...	...	...	...

# Data Visualization

Country	$z_1$	$z_2$
Canada	1.6	1.2
China	1.7	0.3
India	1.6	0.2
Russia	1.4	0.5
Singapore	0.5	1.7
USA	2	1.5
...	...	...

# Data Visualization



1





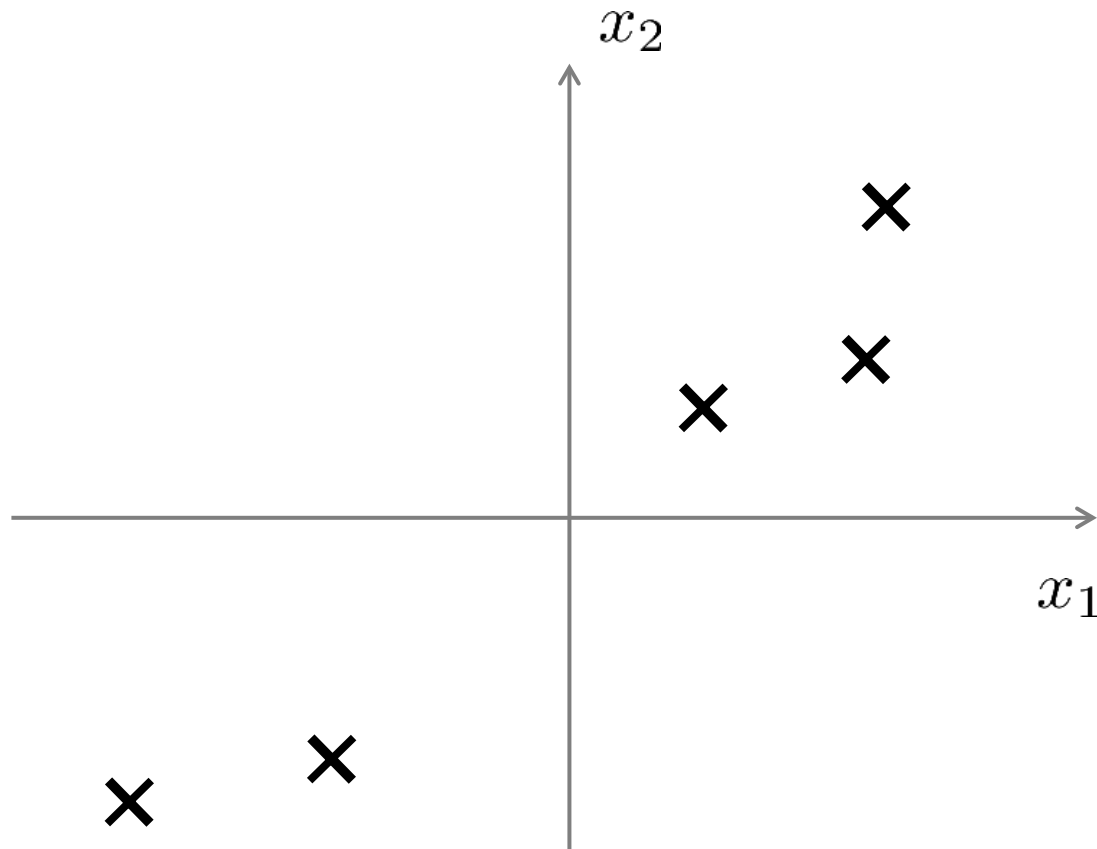
Machine Learning

# Dimensionality Reduction

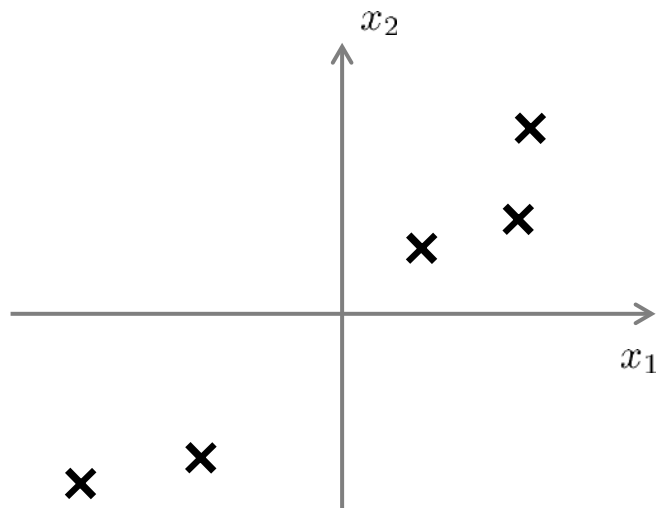
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Principal Component  
Analysis problem  
formulation

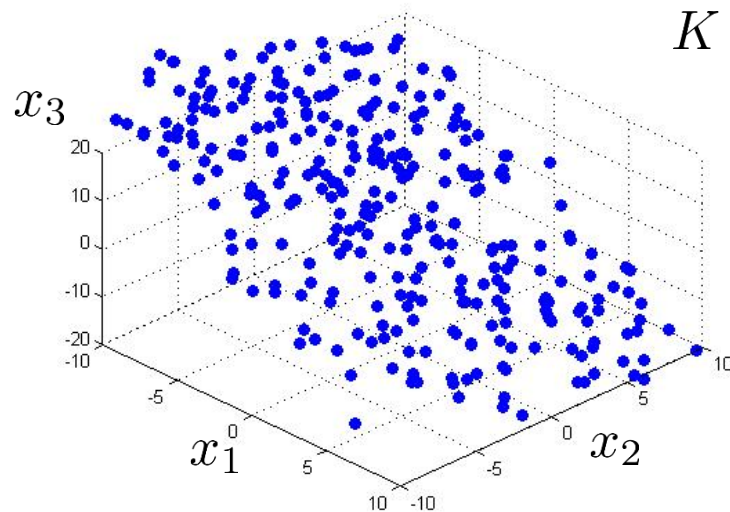
# Principal Component Analysis (PCA) problem formulation



## Principal Component Analysis (PCA) problem formulation



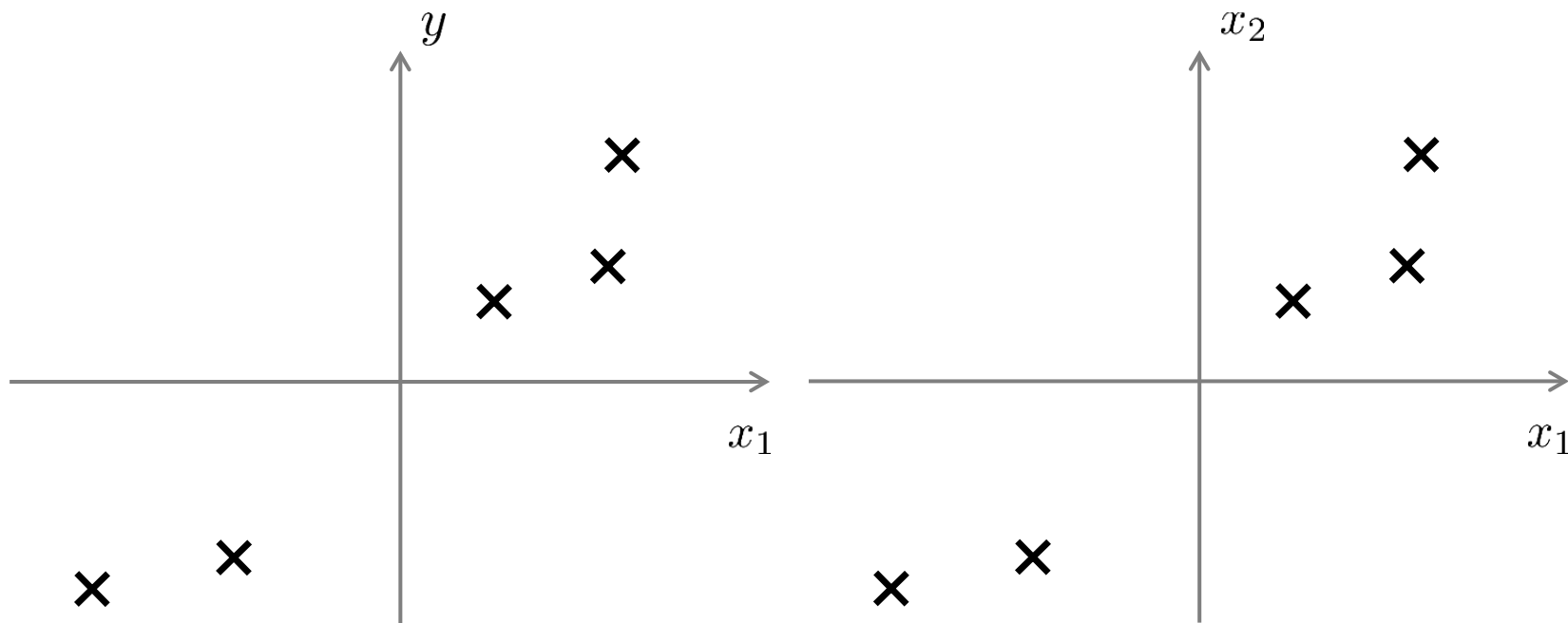
$$3D \rightarrow 2D$$
$$K = 2$$



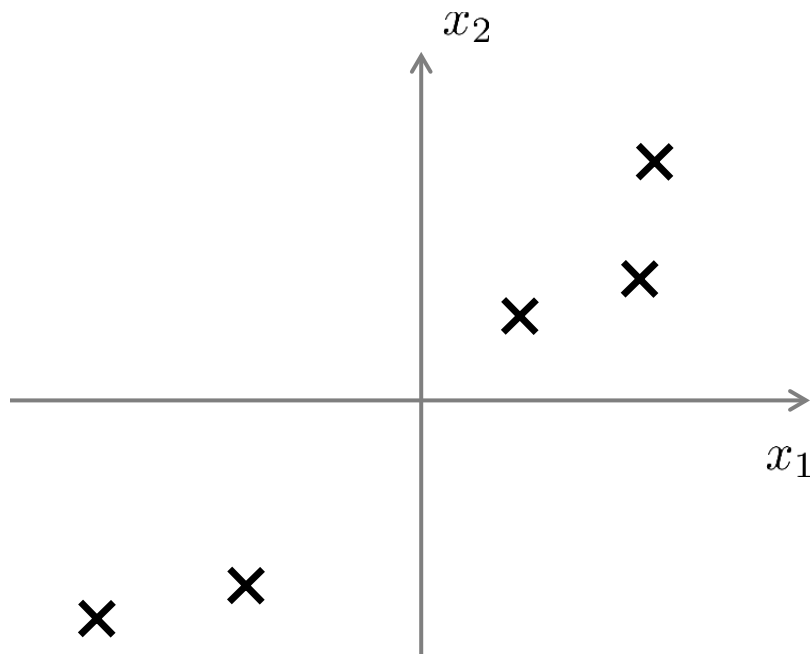
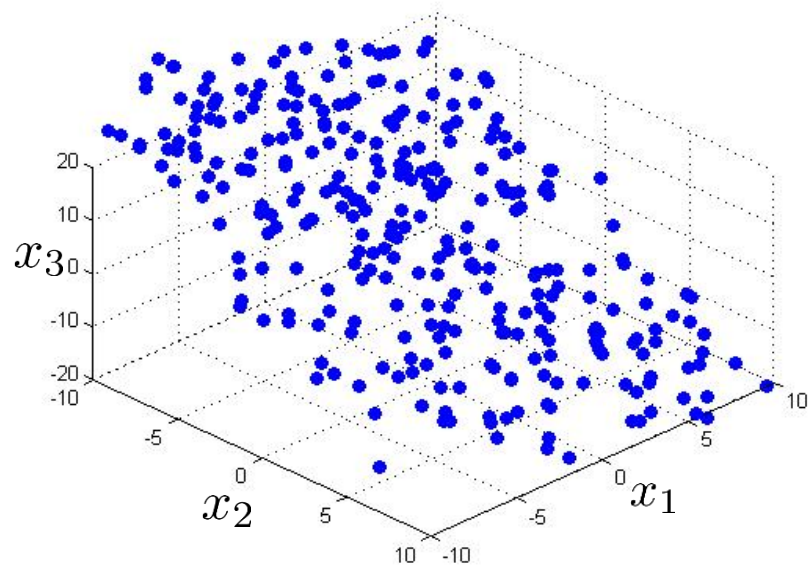
Reduce from 2-dimension to 1-dimension: Find a direction (a vector  $u^{(1)} \in \mathbb{R}^n$ ) onto which to project the data so as to minimize the projection error.

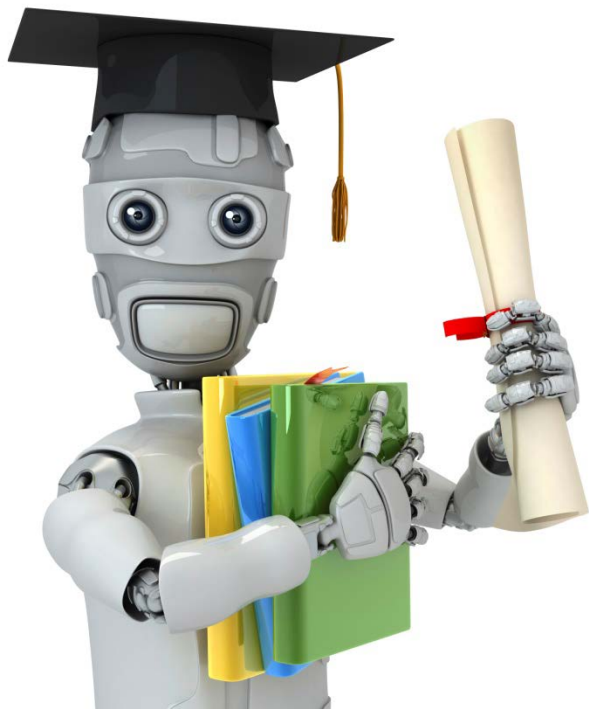
Reduce from  $n$ -dimension to  $k$ -dimension: Find  $k$  vectors  $u^{(1)}, u^{(2)}, \dots, u^{(k)}$  onto which to project the data, so as to minimize the projection error.

# PCA is not linear regression



# PCA is not linear regression





Machine Learning

# Dimensionality Reduction

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Principal Component  
Analysis algorithm

## Data preprocessing

Training set:  $x^{(1)}, x^{(2)}, \dots, x^{(m)}$

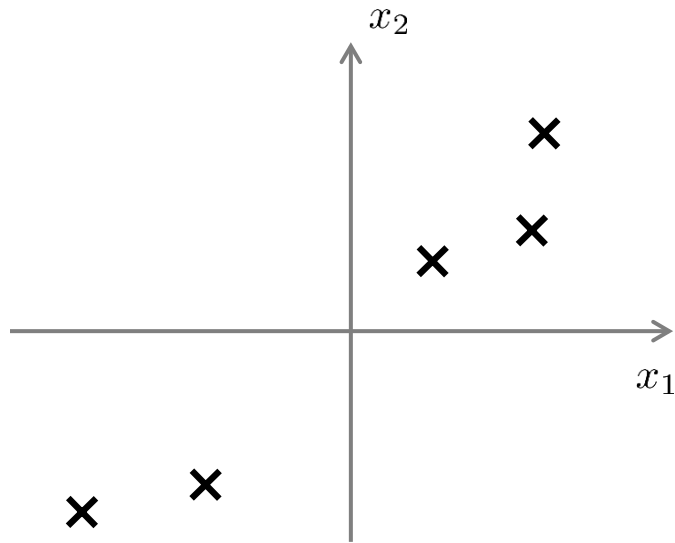
Preprocessing (feature scaling/mean normalization):

$$\mu_j = \frac{1}{m} \sum_{i=1}^m x_j^{(i)}$$

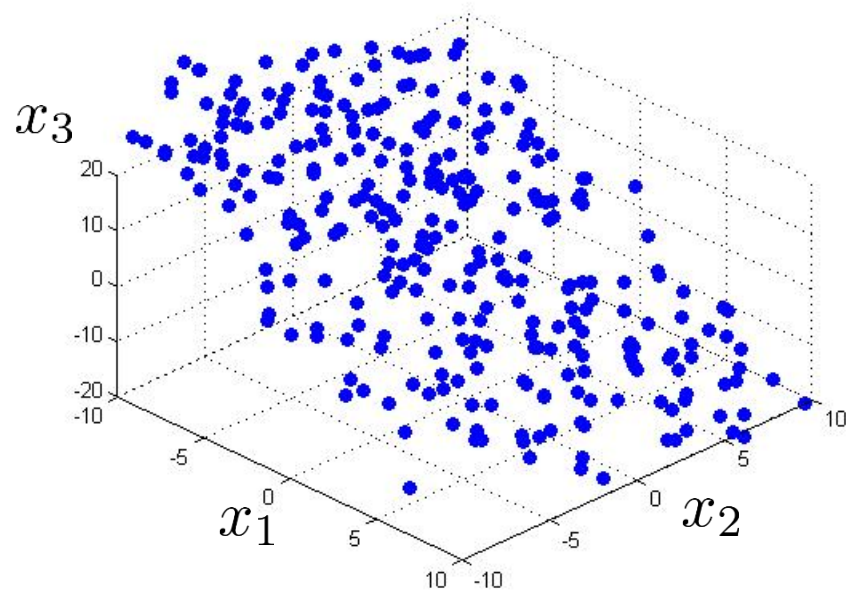
Replace each  $x_j^{(i)}$  with  $x_j - \mu_j$ .

If different features on different scales (e.g.,  $x_1$  = size of house,  $x_2$  = number of bedrooms), scale features to have comparable range of values.

# Principal Component Analysis (PCA) algorithm



Reduce data from 2D to 1D



Reduce data from 3D to 2D



## Principal Component Analysis (PCA) algorithm

Reduce data from  $n$ -dimensions to  $k$ -dimensions

Compute “covariance matrix”:

$$\Sigma = \frac{1}{m} \sum_{i=1}^n (x^{(i)})(x^{(i)})^T$$

Compute “eigenvectors” of matrix  $\Sigma$ :

**[U,S,V] = svd(Sigma);**

# Principal Component Analysis (PCA) algorithm

From `[U,S,V] = svd(Sigma)`, we get:

$$U = \begin{bmatrix} | & | & \dots & | \\ u^{(1)} & u^{(2)} & & u^{(n)} \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{n \times n}$$

# Principal Component Analysis (PCA) algorithm summary

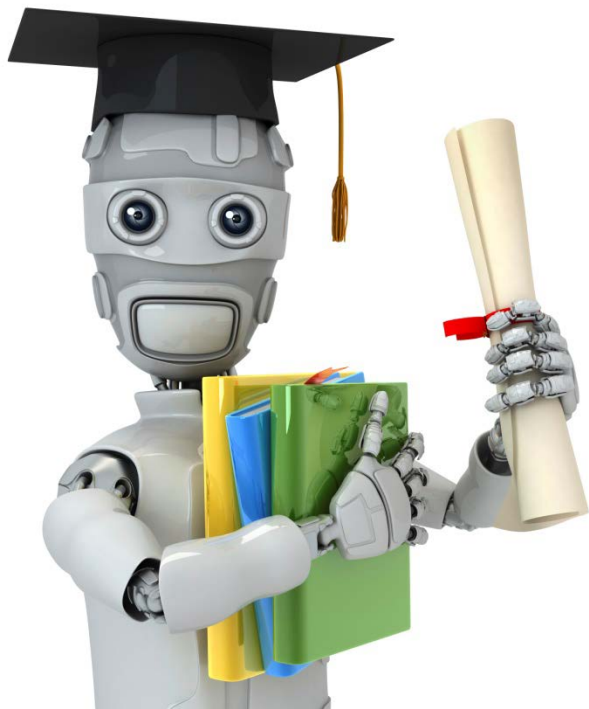
After mean normalization (ensure every feature has zero mean) and optionally feature scaling:

$$\mathbf{Sigma} = \frac{1}{m} \sum_{i=1}^m (x^{(i)})(x^{(i)})^T$$

```
[U,S,V] = svd(Sigma);
```

```
Ureduce = U(:,1:k);
```

```
z = Ureduce'*x;
```



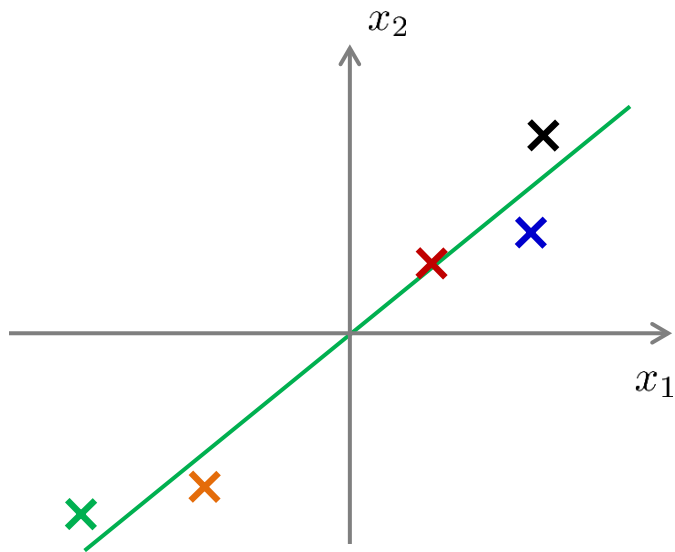
Machine Learning

# Dimensionality Reduction

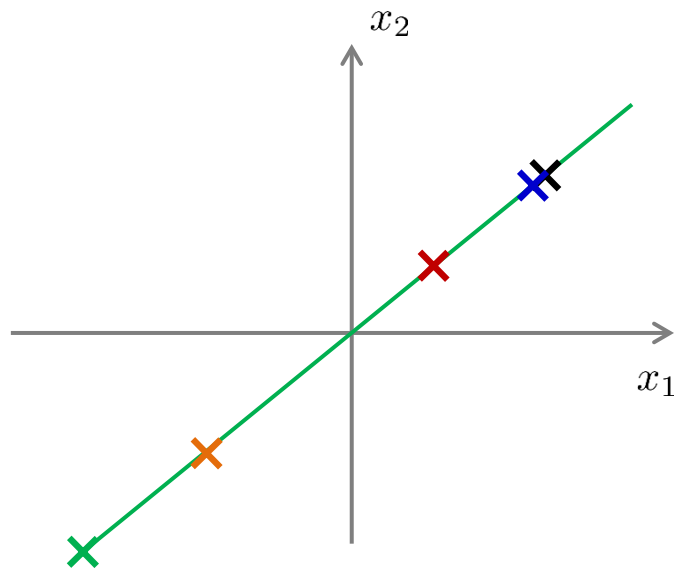
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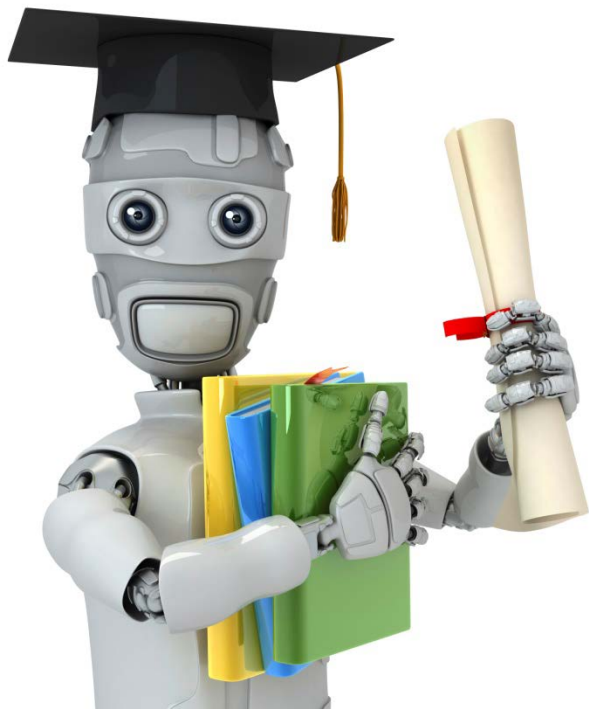
Reconstruction from  
compressed  
representation

# Reconstruction from compressed representation



$$z = U_{reduce}^T x$$





Machine Learning

# Dimensionality Reduction

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Choosing the number of principal components

## Choosing $k$ (number of principal components)

Average squared projection error:

Total variation in the data:

Typically, choose  $k$  to be smallest value so that

$$\frac{\frac{1}{m} \sum_{i=1}^m \|x^{(i)} - x_{approx}^{(i)}\|^2}{\frac{1}{m} \sum_{i=1}^m \|x^{(i)}\|^2} \leq 0.01 \quad (1\%)$$

“99% of variance is retained”

# Choosing $k$ (number of principal components)

Algorithm:

Try PCA with  $k = 1$

Compute  $U_{reduce}, z^{(1)}, z^{(2)}, \dots, z^{(m)}, x_{approx}^{(1)}, \dots, x_{approx}^{(m)}$

Check if

$$\frac{\frac{1}{m} \sum_{i=1}^m \|x^{(i)} - x_{approx}^{(i)}\|^2}{\frac{1}{m} \sum_{i=1}^m \|x^{(i)}\|^2} \leq 0.01?$$

$$[U, S, V] = \text{svd}(\text{Sigma})$$



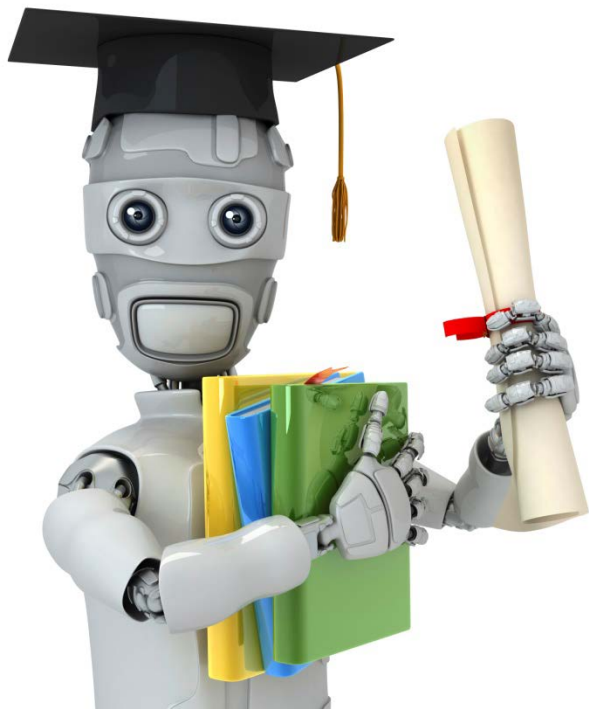
## Choosing $k$ (number of principal components)

`[U,S,V] = svd(Sigma)`

Pick smallest value of  $k$  for which

$$\frac{\sum_{i=1}^k S_{ii}}{\sum_{i=1}^m S_{ii}} \geq 0.99$$

(99% of variance retained)



Machine Learning

# Dimensionality Reduction

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## Advice for applying PCA

## Supervised learning speedup

$$(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \dots, (x^{(m)}, y^{(m)})$$

Extract inputs:

$$\text{Unlabeled dataset: } x^{(1)}, x^{(2)}, \dots, x^{(m)} \in \mathbb{R}^{10000}$$

$$\downarrow \text{PCA}$$

$$z^{(1)}, z^{(2)}, \dots, z^{(m)} \in \mathbb{R}^{1000}$$

New training set:

$$(z^{(1)}, y^{(1)}), (z^{(2)}, y^{(2)}), \dots, (z^{(m)}, y^{(m)})$$

Note: Mapping  $x^{(i)} \rightarrow z^{(i)}$  should be defined by running PCA only on the training set. This mapping can be applied as well to the examples  $x_{cv}^{(i)}$  and  $x_{test}^{(i)}$  in the cross validation and test sets.

# Application of PCA

- Compression
  - Reduce memory/disk needed to store data
  - Speed up learning algorithm
- Visualization

## Bad use of PCA: To prevent overfitting

Use  $z^{(i)}$  instead of  $x^{(i)}$  to reduce the number of features to  $k < n$ .

Thus, fewer features, less likely to overfit.

This might work OK, but isn't a good way to address overfitting. Use regularization instead.

$$\min_{\theta} \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2 + \frac{\lambda}{2m} \sum_{j=1}^n \theta_j^2$$

## PCA is sometimes used where it shouldn't be

Design of ML system:

- Get training set  $\{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \dots, (x^{(m)}, y^{(m)})\}$
- Run PCA to reduce  $x^{(i)}$  in dimension to get  $z^{(i)}$
- Train logistic regression on  $\{(z^{(1)}, y^{(1)}), \dots, (z^{(m)}, y^{(m)})\}$
- Test on test set: Map  $x_{test}^{(i)}$  to  $z_{test}^{(i)}$ . Run  $h_{\theta}(z)$  on  $\{(z_{test}^{(1)}, y_{test}^{(1)}), \dots, (z_{test}^{(m)}, y_{test}^{(m)})\}$

How about doing the whole thing without using PCA?

Before implementing PCA, first try running whatever you want to do with the original/raw data  $x^{(i)}$ . Only if that doesn't do what you want, then implement PCA and consider using  $z^{(i)}$ .