

# Normaliz 3.2.0

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## Contents

1. Introduction	7
1.1. The objectives of Normaliz	7
1.2. Platforms, implementation and access from other systems	8
1.3. Major changes relative to version 3.1.1	9
1.4. Future extensions	9
1.5. Download and installation	9
2. Normaliz by examples	10
2.1. Terminology	10
2.2. Practical preparations	11
2.3. A cone in dimension 2	12
2.3.1. The Hilbert basis	13
2.3.2. The cone by inequalities	14
2.3.3. The interior	15
2.4. A lattice polytope	17
2.4.1. Only the lattice points	20
2.5. A rational polytope	20
2.5.1. The rational polytope by inequalities	23
2.6. Magic squares	24
2.6.1. With even corners	27
2.6.2. The lattice as input	30

2.7.	Decomposition in a numerical semigroup . . . . .	30
2.8.	A job for the dual algorithm . . . . .	31
2.9.	A dull polyhedron . . . . .	32
2.9.1.	Defining it by generators . . . . .	34
2.10.	The Condorcet paradoxon . . . . .	35
2.10.1.	Excluding ties . . . . .	36
2.10.2.	At least one vote for every preference order . . . . .	37
2.11.	Testing normality . . . . .	38
2.11.1.	Computing just a witness . . . . .	39
2.12.	Inhomogeneous congruences . . . . .	40
2.12.1.	Lattice and offset . . . . .	41
2.12.2.	Variation of the signs . . . . .	41
2.13.	Integral closure and Rees algebra of a monomial ideal . . . . .	42
2.13.1.	Only the integral closure . . . . .	43
2.14.	Convex hull computation/vertex enumeration . . . . .	43
2.15.	The integer hull . . . . .	45
2.16.	Starting from a binomial ideal . . . . .	47
3.	The input file . . . . .	49
3.1.	Input items . . . . .	50
3.1.1.	The ambient space and lattice . . . . .	50
3.1.2.	Plain vectors . . . . .	50
3.1.3.	Formatted vectors . . . . .	51
3.1.4.	Plain matrices . . . . .	51
3.1.5.	Formatted matrices . . . . .	52
3.1.6.	Constraints in tabular format . . . . .	53
3.1.7.	Constraints in symbolic format . . . . .	53
3.1.8.	Computation goals and algorithmic variants . . . . .	54
3.1.9.	Comments . . . . .	54
3.1.10.	Restrictions . . . . .	54
3.1.11.	Homogeneous and inhomogeneous input . . . . .	55
3.1.12.	Default values . . . . .	55
3.1.13.	Normaliz takes intersections (almost always) . . . . .	55
3.2.	Homogeneous generators . . . . .	56
3.2.1.	Cones . . . . .	56
3.2.2.	Lattices . . . . .	56
3.3.	Homogeneous Constraints . . . . .	56
3.3.1.	Cones . . . . .	56
3.3.2.	Lattices . . . . .	57
3.4.	Inhomogeneous generators . . . . .	57
3.4.1.	Polyhedra . . . . .	57
3.4.2.	Lattices . . . . .	58

3.5.	Inhomogeneous constraints . . . . .	58
3.5.1.	Cones . . . . .	58
3.5.2.	Lattices . . . . .	58
3.6.	Tabular constraints . . . . .	59
3.6.1.	Forced homogeneity . . . . .	59
3.7.	Symbolic constraints . . . . .	60
3.8.	Relations . . . . .	60
3.9.	Unit vectors . . . . .	60
3.10.	Grading . . . . .	60
3.10.1.	lattice_ideal . . . . .	61
3.11.	Dehomogenization . . . . .	61
3.12.	Pointedness . . . . .	62
3.13.	The zero cone . . . . .	62
3.14.	Additional input file for NmzIntegrate . . . . .	62
4.	Computation goals and algorithmic variants . . . . .	62
4.1.	Default choices and basic rules . . . . .	63
4.2.	The choice of algorithmic variants . . . . .	63
4.2.1.	Primal vs. dual . . . . .	63
4.2.2.	Lattice points by approximation . . . . .	64
4.2.3.	Bottom decomposition . . . . .	64
4.2.4.	Symmetrization . . . . .	64
4.2.5.	Subdivision of simplicial cones . . . . .	65
4.3.	Computation goals . . . . .	65
4.3.1.	Lattice data . . . . .	65
4.3.2.	Support hyperplanes and extreme rays . . . . .	65
4.3.3.	Hilbert basis and related data . . . . .	65
4.3.4.	Enumerative data . . . . .	65
4.3.5.	Combined computation goals . . . . .	66
4.3.6.	The class group . . . . .	66
4.3.7.	Integer hull . . . . .	66
4.3.8.	Triangulation and Stanley decomposition . . . . .	66
4.3.9.	NmzIntegrate . . . . .	66
4.3.10.	Boolean valued computation goals . . . . .	67
4.4.	Integer type . . . . .	67
4.5.	Control of computations and communication with interfaces . . . . .	68
4.6.	Rational and integer solutions in the inhomogeneous case . . . . .	69
5.	Running Normaliz . . . . .	69
5.1.	Basic rules . . . . .	70
5.2.	Info about Normaliz . . . . .	70
5.3.	Control of execution . . . . .	70
5.4.	Control of output files . . . . .	71

5.5. Overriding the options in the input file . . . . .	71
6. More examples . . . . .	71
6.1. Lattice points by approximation . . . . .	72
6.2. The bottom decomposition . . . . .	73
6.3. Subdivision of large simplicial cones . . . . .	74
6.4. Primal vs. dual – division of labor . . . . .	76
6.5. Symmetrization . . . . .	76
6.5.1. Restrictions in MS Windows . . . . .	78
6.6. Explicit dehomogenization . . . . .	79
6.7. Nonpointed cones . . . . .	80
6.7.1. A nonpointed cone . . . . .	80
6.7.2. A polyhedron without vertices . . . . .	82
6.7.3. Checking pointedness first . . . . .	83
6.7.4. Input of a subspace . . . . .	84
6.7.5. Data relative to the original monoid . . . . .	85
6.8. Exporting the triangulation . . . . .	86
6.8.1. Nested triangulations . . . . .	87
6.8.2. Disjoint decomposition . . . . .	87
6.9. Exporting the Stanley decomposition . . . . .	88
6.10. Module generators over the original monoid . . . . .	89
6.10.1. An inhomogeneous example . . . . .	90
6.11. Precomputed support hyperplanes . . . . .	92
6.12. Shift, denominator, quasipolynomial and multiplicity . . . . .	92
7. Optional output files . . . . .	95
7.1. The homogeneous case . . . . .	95
7.2. Modifications in the inhomogeneous case . . . . .	96
8. Performance . . . . .	96
8.1. Parallelization . . . . .	96
8.2. Running large computations . . . . .	97
9. Distribution and installation . . . . .	98
10. Building Normaliz yourself . . . . .	99
10.1. Compiler prerequisites . . . . .	99
10.2. Required libraries . . . . .	100
10.3. Optional packages . . . . .	100
10.4. autotools . . . . .	101
10.5. cmake . . . . .	101
10.6. Mac OS X . . . . .	102
10.7. Windows . . . . .	102

11. Copyright and how to cite	103
A. Mathematical background and terminology	104
A.1. Polyhedra, polytopes and cones . . . . .	104
A.2. Cones . . . . .	105
A.3. Polyhedra . . . . .	105
A.4. Affine monoids . . . . .	107
A.5. Affine monoids from binomial ideals . . . . .	108
A.6. Lattice points in polyhedra . . . . .	108
A.7. Hilbert series . . . . .	109
A.8. The class group . . . . .	111
B. Annotated console output	111
B.1. Primal mode . . . . .	111
B.2. Dual mode . . . . .	113
C. Normaliz 2 input syntax	115
D. libnormaliz	116
D.1. Integer type as a template parameter . . . . .	116
D.1.1. Alternative integer types . . . . .	116
D.2. Construction of a cone . . . . .	117
D.3. Computations in a cone . . . . .	119
D.4. Retrieving results . . . . .	122
D.4.1. Checking computations . . . . .	122
D.4.2. Rank, index and dimension . . . . .	122
D.4.3. Support hyperplanes and constraints . . . . .	123
D.4.4. Extreme rays and vertices . . . . .	123
D.4.5. Generators . . . . .	123
D.4.6. Lattice points in polytopes and elements of degree 1 . . . . .	124
D.4.7. Hilbert basis . . . . .	124
D.4.8. Module generators over original monoid . . . . .	124
D.4.9. Grading and dehomogenization . . . . .	124
D.4.10. Enumerative data . . . . .	124
D.4.11. Triangulation and disjoint decomposition . . . . .	125
D.4.12. Stanley decomposition . . . . .	126
D.4.13. Coordinate transformation . . . . .	126
D.4.14. Class group . . . . .	127
D.4.15. Integer hull . . . . .	127
D.4.16. Excluded faces . . . . .	128
D.4.17. Boolean valued results . . . . .	128
D.5. Exception handling . . . . .	129
D.6. Control of output . . . . .	129

D.7. A simple program . . . . .	130
E. PyNormaliz	134
F. QNormaliz	134
F.1. Prerequisites . . . . .	135
F.2. Restrictions . . . . .	135
F.3. Results . . . . .	136
F.4. Installation . . . . .	136

# 1. Introduction

## 1.1. The objectives of Normaliz

The program Normaliz is a tool for computing the Hilbert bases and enumerative data of rational cones and, more generally, sets of lattice points in rational polyhedra. The mathematical background and the terminology of this manual are explained in Appendix A. For a thorough treatment of the mathematics involved we refer the reader to [4]. The terminology follows [4]. For algorithms of Normaliz see [5], [7], [8] and [9]. Some new developments are briefly explained in this manual.

Both polyhedra and lattices can be given by

- (1) systems of generators and/or
- (2) constraints.

Since version 3.1, cones need not be pointed and polyhedra need not have vertices, but are allowed to contain a positive-dimensional affine subspace.

In addition to generators and constraints, affine monoids can be defined by lattice ideals, in other words, by binomial equations.

In order to describe a rational polyhedron by generators, one specifies a finite set of vertices  $x_1, \dots, x_n \in \mathbb{Q}^d$  and a set  $y_1, \dots, y_m \in \mathbb{Z}^d$  generating a rational cone  $C$ . The polyhedron defined by these generators is

$$P = \text{conv}(x_1, \dots, x_n) + C, \quad C = \mathbb{R}_+ y_1 + \dots + \mathbb{R}_+ y_m.$$

An affine lattice defined by generators is a subset of  $\mathbb{Z}^d$  given as

$$L = w + L_0, \quad L_0 = \mathbb{Z}z_1 + \dots + \mathbb{Z}z_r, \quad w, z_1, \dots, z_r \in \mathbb{Z}^d.$$

Constraints defining a polyhedron are affine-linear inequalities with integral coefficients, and the constraints for an affine lattice are affine-linear diophantine equations and congruences. The conversion between generators and constraints is an important task of Normaliz.

The first main goal of Normaliz is to compute a system of generators for

$$P \cap L.$$

The minimal system of generators of the monoid  $M = C \cap L_0$  is the Hilbert basis  $\text{Hilb}(M)$  of  $M$ . The homogeneous case, in which  $P = C$  and  $L = L_0$ , is undoubtedly the most important one, and in this case  $\text{Hilb}(M)$  is the system of generators to be computed. In the general case the system of generators consists of  $\text{Hilb}(M)$  and finitely many points  $u_1, \dots, u_s \in P \cap L$  such that

$$P \cap L = \bigcup_{j=1}^s u_j + M.$$

The second main goal are enumerative data that depend on a grading of the ambient lattice. Normaliz computes the Hilbert series and the Hilbert quasipolynomial of the monoid or set

of lattice points in a polyhedron. In combinatorial terminology: Normaliz computes Ehrhart series and quasipolynomials of rational polyhedra. Via its offspring NmzIntegrate [11], Normaliz computes generalized Ehrhart series and Lebesgue integrals of polynomials over rational polytopes.

Normaliz now has a variant called QNormaliz. Its basic number class are elements of a subfield of  $\mathbb{R}$  instead of integers. This extends the scope of Normaliz to certain nonrational polyhedra, but limits its applicability to convex hull computations and triangulations since finite generation of the lattice points in such polyhedra is no longer given.

The computation goals of Normaliz can be set by the user. In particular, they can be restricted to subtasks, such as the lattice points in a polytope or the leading coefficient of the Hilbert (quasi)polynomial.

Performance data of Normaliz can be found in [8].

*Acknowledgement.* In 2013–2016 the development of Normaliz has been supported by the DFG SPP 1489 “Algorithmische und experimentelle Methoden in Algebra, Geometrie und Zahlentheorie”.

## 1.2. Platforms, implementation and access from other systems

Executables for Normaliz are provided for Mac OS, Linux and MS Windows. If the executables prepared cannot be run on your system, then you can compile Normaliz yourself (see Section 10). The statically linked Linux binaries provided on our homepage can be run in the Linux subsystem that is available in certain versions of MS Windows 10.

Normaliz is written in C++, and should be compilable on every system that has a GCC compatible compiler. It uses the standard packages Boost and GMP (see Section 10). The parallelization is based on OpenMP.

The executables provided by us use the integer optimization program SCIP [1] for certain subtasks, but the inclusion of SCIP must be activated at compile time.

Normaliz consists of two parts: the front end normaliz for input and output and the C++ library libnormaliz that does the computations.

Normaliz can be accessed from the interactive general purpose system PYTHON via the interface PYNORMALIZ written by Sebastian Gutsche.

Normaliz can be accessed from the following systems:

- SINGULAR via the library `normaliz.lib`,
- MACAULAY 2 via the package `Normaliz.m2`,
- COCOA via an external library and `libnormaliz`,
- GAP via the GAP package `NORMALIZINTERFACE` [12] which uses `libnormaliz`,
- POLYMAKE (thanks to the POLYMAKE team),
- SAGEMATH via `PyNormaliz` (on preparation, thanks to Matthias Köppe).

The Singular and Macaulay 2 interfaces are contained in the Normaliz distribution. At present, their functionality is limited to Normaliz 2.10. Nevertheless they can profit from the newer



versions.

Furthermore, Normaliz is used by the B. Burton's system REGINA and in SECDEC by S. Borowka et al.

Normaliz does not have its own interactive shell. We recommend the access via PyNormaliz, GAP for SageMath (in preparation) for interactive use.

### 1.3. Major changes relative to version 3.1.1

- (1) installation via autotools (written by Matthias Köppe),
- (2) automatic choice of algorithmic variants (can be switched off),
- (3) sparse format for vectors and matrices,
- (4) constraints in symbolic form,
- (5) Hilbert series with denominator corresponding to a homogeneous system of parameters,
- (6) choice of an output directory,
- (7) improvements of libnormaliz: extension of enumeration ConeProperty, constructors based on `Matrix<Integer>`, additional functions for retrieval of results,
- (8) a better implementation of Approximate and its use in the inhomogeneous case,
- (9) option Symmetrize that produces symmetrized input for nmzIntegrate and runs nmzIntegrate on this input,
- (10) QNormaliz, a version of Normaliz using coordinates in an extension of  $\mathbb{Q}$  (restricted to convex hull computations and triangulation).

See the file CHANGELOG in the basic package for more information on the history of Normaliz.

### 1.4. Future extensions

- (1) Exploitation of symmetries,
- (2) integration of NmzIntegrate into libnormaliz,
- (3) multigraded Hilbert series,
- (4) access from further systems,
- (5) Gröbner and Graver bases.

### 1.5. Download and installation

Download

- the zip file with the Normaliz source, documentation, examples and further platform independent components, and
- the zip file containing the executable(s) for your system

from the Normaliz website

<http://normaliz.uos.de>

and unzip both in the same directory of your choice. In it, a directory `normaliz-3.2.0` (called Normaliz directory in the following) is created with several subdirectories.

See Section 9 for more details on the distribution and consult Section 10 if you want to compile Normaliz yourself.

## 2. Normaliz by examples

### 2.1. Terminology

For the precise interpretation of parts of the Normaliz output some terminology is necessary, but this section can be skipped at first reading, and the user can come back to it when it becomes necessary. We will give less formal descriptions along the way.

As pointed out in the introduction, Normaliz “computes” intersections  $P \cap L$  where  $P$  is a rational polyhedron in  $\mathbb{R}^d$  and  $L$  is an affine sublattice of  $\mathbb{Z}^d$ . It proceeds as follows:

- (1) If the input is inhomogeneous, then it is homogenized by introducing a homogenizing coordinate: the polyhedron  $P$  is replaced by the cone  $C(P)$ : it is the closure of  $\mathbb{R}_+(P \times \{1\})$  in  $\mathbb{R}^{d+1}$ . Similarly  $L$  is replaced by  $\tilde{L} = \mathbb{Z}(L \times \{1\})$ . In the homogeneous case in which  $P$  is a cone and  $L$  is a subgroup of  $\mathbb{Z}^d$ , we set  $C(P) = P$  and  $\tilde{L} = L$ .
- (2) The computations take place in the *efficient lattice*

$$\mathbb{E} = \tilde{L} \cap \mathbb{R}C(P).$$

where  $\mathbb{R}C(P)$  is the linear subspace generated by  $C(P)$ . The internal coordinates are chosen with respect to a basis of  $\mathbb{E}$ . The *efficient cone* is

$$\mathbb{C} = \mathbb{R}_+(C(P) \cap \mathbb{E}).$$

- (3) Inhomogeneous computations are truncated using the dehomogenization (defined implicitly or explicitly).
- (4) The final step is the conversion to the original coordinates. Note that we must use the coordinates of  $\mathbb{R}^{d+1}$  if homogenization has been necessary, simply because some output vectors may be non-integral.

Normaliz computes inequalities, equations and congruences defining  $\mathbb{E}$  and  $\mathbb{C}$ . The output contains only those constraints that are really needed. They must always be used jointly: the equations and congruences define  $\mathbb{E}$ , and the equations and inequalities define  $\mathbb{C}$ . Altogether they define the monoid  $M = \mathbb{C} \cap \mathbb{E}$ . In the homogeneous case this is the monoid to be computed. In the inhomogeneous case we must intersect  $M$  with the dehomogenizing hyperplane to obtain  $P \cap L$ .

In this section, only pointed cones (and polyhedra with vertices) will be discussed. Nonpointed cones will be addressed in Section 6.7.

## 2.2. Practical preparations

You may find it comfortable to run Normaliz via the GUI jNormaliz [2]. In the Normaliz directory open jNormaliz by clicking `jNormaliz.jar` in the appropriate way. (We assume that Java is installed on your machine.) In the jNormaliz file dialogue choose one of the input files

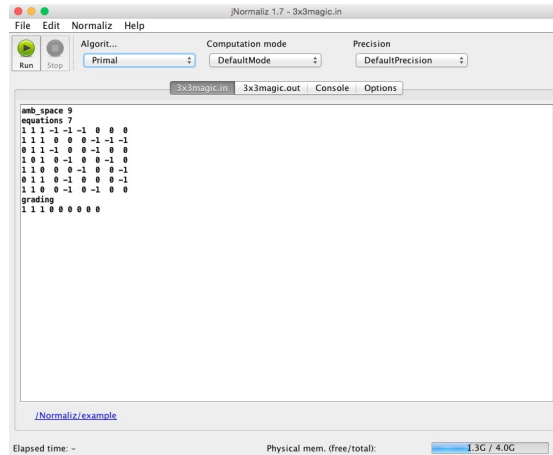


Figure 1: jNormaliz

in the subdirectory `example`, say `small.in`, and press Run. In the console window you can watch Normaliz at work. Finally inspect the output window for the results.

The menus and dialogues of jNormaliz are self explanatory, but you can also consult the documentation [2] via the help menu.

*Remark* The jNormaliz drop down menus do presently not cover all options of Normaliz. But since all computation goals and algorithmic variants can be set in the input file, there is no real restriction in using jNormaliz. The only option not reachable by jNormaliz is the output directory (see Section 5.4).

Moreover, one can, and often will, run Normaliz from the command line. This is fully explained in Section 5. At this point it is enough to call Normaliz by typing

```
normaliz -c <project>
```

where `<project>` denotes for the project to be computed. Normaliz will load the file `project.in`. The option `-c` makes Normaliz to write a progress report on the terminal. Normaliz writes its results to `<project>.out`.

Note that you may have to prefix `normaliz` by a path name, and `<project>` must contain a path to the input file if it is not in the current directory. Suppose the Normaliz directory is the current directory and we are using a Linux or Mac system. Then

```
./normaliz -c example/small
```

will run `small.in` from the directory `example`. On Windows we must change this to

```
.\normaliz -c example\small
```

The commands given above will run Normaliz with the full parallelization that your system can deliver. For the very small examples in this tutorial you may want to add `-x=1` to suppress parallelization.

As long as you don't specify a computation goal on the command line or in the input file, Normaliz will use the *default computation goals*:

```
HilbertBasis
HilbertSeries
ClassGroup
```

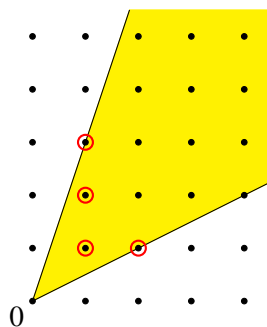
The computation of the Hilbert series requires the explicit or implicit definition of a grading. Normaliz does only complain that a computation goal cannot be reached if the goal has been set explicitly. For example, if you say `HilbertSeries` and there is no grading, an exception will be thrown and Normaliz terminates, but an output file with the already computed data will be written.

Normaliz will always print the results that are obtained on the way to the computation goals and do not require extra effort.

Appendix B helps you to read the console output that you have demanded by the option `-c`.

## 2.3. A cone in dimension 2

We want to investigate the cone  $C = \mathbb{R}_+(2, 1) + \mathbb{R}_+(1, 3) \subset \mathbb{R}^2$ :



This cone is defined in the input file `2cone.in`:

```
amb_space 2
cone 2
1 3
2 1
```

The input tells Normaliz that the ambient space is  $\mathbb{R}^2$ , and then a cone with 2 generators is defined, namely the cone  $C$  from above.

The figure indicates the Hilbert basis, and this is our first computation goal.

If you prefer to consider the columns of a matrix as input vectors (or have got a matrix in this format from another system) you can use the input

```
amb_space 2
cone transpose 2
1 2
3 1
```

Note that the number 2 following transpose is now the number of *columns*. Later on we will also show the use of formatted matrices.

### 2.3.1. The Hilbert basis

In order to compute the Hilbert basis, we run Normaliz from jNormaliz or by

```
./normaliz -c example/2cone
```

and inspect the output file:

```
4 Hilbert basis elements
2 extreme rays
2 support hyperplanes
```

Self explanatory so far.

```
embedding dimension = 2
rank = 2 (maximal)
external index = 1
internal index = 5
original monoid is not integrally closed
```

The embedding dimension is the dimension of the space in which the computation is done. The rank is the rank of the lattice  $\mathbb{E}$  (notation as in Section 2.1). In fact, in our example  $\mathbb{E} = \mathbb{Z}^2$ , and therefore has rank 2.

For subgroups  $G \subset U \subset \mathbb{Z}^d$  we denote the order of the torsion subgroup of  $U/G$  by the *index* of  $G$  in  $U$ . The *external index* is the index of the lattice  $\mathbb{E}$  in  $\mathbb{Z}^d$ . In our case  $\mathbb{E} = \mathbb{Z}^d$ , and therefore the external index is 1. Note: the external index is 1 exactly when  $\mathbb{E}$  is a direct summand of  $\mathbb{Z}^d$ .

For this example and many others the *original monoid* is well defined: the generators of the cone used as input are contained in  $\mathbb{E}$ . (This need not be the case if  $\mathbb{E}$  is a proper sublattice of  $\mathbb{Z}^d$ , and we let the original monoid undefined in inhomogeneous computations.) Let  $G$  be the subgroup generated by the original monoid. The *internal index* is the index of  $G$  in  $\mathbb{E}$ .

The original monoid is *integrally closed* if and only if it contains the Hilbert basis, and this is evidently false for our example. We go on.

```
size of triangulation = 1
```

```
resulting sum of |det|s = 5
```

The primal algorithm of Normaliz relies on a (partial) triangulation. In our case the triangulation consists of a single simplicial cone, and (the absolute value of) its determinant is 5.

```
No implicit grading found
```

If you do not define a grading explicitly, Normaliz tries to find one itself: It is defined if and only if there is a linear form  $\gamma$  on  $\mathbb{E}$  under which all extreme rays of the efficient cone  $\mathbb{C}$  have value 1, and if so,  $\gamma$  is the implicit grading. Such does not exist in our case.

The last information before we come to the vector lists:

```
rank of class group = 0
finite cyclic summands:
5: 1
```

The class group of the monoid  $M$  has rank 0, in other words, it is finite. It has one finite cyclic summand of order 5.

This is the first instance of a multiset of integers displayed as a sequence of pairs

$\langle n \rangle$ :  $\langle m \rangle$

Such an entry says: the multiset contains the number  $\langle n \rangle$  with multiplicity  $\langle m \rangle$ .

Now we look at the vector lists (typeset in two columns to save space):

```
4 Hilbert basis elements:      2 extreme rays:
1 1                             1 3
1 2                             2 1
1 3
2 1                             2 support hyperplanes:
                                -1 2
                                3 -1
```

The support hyperplanes are given by the linear forms (or inner normal vectors):

$$\begin{aligned} -x_1 + 2x_2 &\geq 0, \\ 3x_1 - x_2 &\geq 0. \end{aligned}$$

If the order is not fixed for some reason, Normaliz sorts vector lists as follows : (1) by degree if a grading exists and the application makes sense, (2) lexicographically.

### 2.3.2. The cone by inequalities

Instead by generators, we can define the cone by the inequalities just computed (2cone\_ineq.in). We use this example to show the input of a formamtted matrix:

```
amb_space auto
inequalities
```

```
[[ -1  2] [ 3 -1]]
```

A matrix of input type `inequalities` contains *homogeneous* inequalities. `Normaliz` can determine the dimension of the ambient space from the formatted matrix. Therefore we can declare the ambient space as being “auto determined” (but `amb_space 2` is not forbidden).

We get the same result as with `2cone.in` except that the data depending on the original monoid cannot be computed: the internal index and the information on the original monoid are missing since there is no original monoid.

### 2.3.3. The interior

Now we want to compute the lattice points in the interior of our cone. If the cone  $C$  is given by the inequalities  $\lambda_i(x) \geq 0$  (within  $\text{aff}(C)$ ), then the interior is given by the inequalities  $\lambda_i(x) > 0$ . Since we are interested in lattice points, we work with the inequalities  $\lambda_i(x) \geq 1$ .

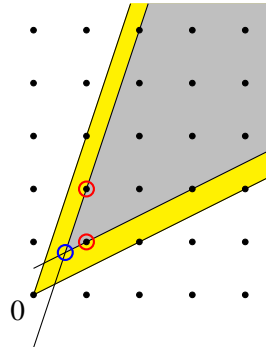
The input file `2cone_int.in` says

```
amb_space 2
strict_inequalities 2
-1  2
3 -1
```

The strict inequalities encode the conditions

$$\begin{aligned} -x_1 + 2x_2 &\geq 1, \\ 3x_1 - x_2 &\geq 1. \end{aligned}$$

This is our first example of inhomogeneous input.



Alternatively we could use the following two equivalent input files, in a more intuitive notation:

```
amb_space 2
constraints 2
-1  2 > 0
3 -1 > 0
```

```

amb_space 2
constraints 2
-1 2 >= 1
3 -1 >= 1

```

There is an even more intuitive way to type the input file using symbolic constraints that we will introduce in Section 2.6.1.

Normaliz homogenizes inhomogeneous computations by introducing an auxiliary homogenizing coordinate  $x_{d+1}$ . The polyhedron is obtained by intersecting the homogenized cone with the hyperplane  $x_{d+1} = 1$ . The recession cone is the intersection with the hyperplane  $x_{d+1} = 0$ . The recession monoid is the monoid of lattice points in the recession cone, and the set of lattice points in the polyhedron is represented by its system of module generators over the recession monoid.

Note that the homogenizing coordinate serves as the denominator for rational vectors. In our example the recession cone is our old friend that we have already computed, and therefore we need not comment on it.

```

2 module generators
4 Hilbert basis elements of recession monoid
1 vertices of polyhedron
2 extreme rays of recession cone
3 support hyperplanes of polyhedron (homogenized)

embedding dimension = 3
affine dimension of the polyhedron = 2 (maximal)
rank of recession monoid = 2

```

The only surprise may be the embedding dimension: Normaliz always takes the dimension of the space in which the computation is done. It is the number of components of the output vectors. Because of the homogenization it has increased by 1.

```

size of triangulation    = 1
resulting sum of |det|s = 25

```

In this case the homogenized cone has stayed simplicial, but the determinant has changed.

```

dehomogenization:
0 0 1

```

The dehomogenization is the linear form  $\delta$  on the homogenized space that defines the hyperplanes from which we get the polyhedron and the recession cone by the equations  $\delta(x) = 1$  and  $\delta(x) = 0$ , respectively. It is listed since one can also work with a user defined dehomogenization.

```

module rank = 1

```

This is the rank of the module of lattice points in the polyhedron over the recession monoid.



In our case the module is an ideal, and so the rank is 1.

The output of inhomogeneous computations is always given in homogenized form. The last coordinate is the value of the dehomogenization on the listed vectors, 1 on the module generators, 0 on the vectors in the recession monoid:

2 module generators:	4 Hilbert basis elements of recession monoid:
1 1 1	1 1 0
1 2 1	1 2 0
	1 3 0
	2 1 0

The module generators are  $(1,1)$  and  $(1,2)$ .

1 vertices of polyhedron:
3 4 5

Indeed, the polyhedron has a single vertex, namely  $(3/5, 4/5)$ .

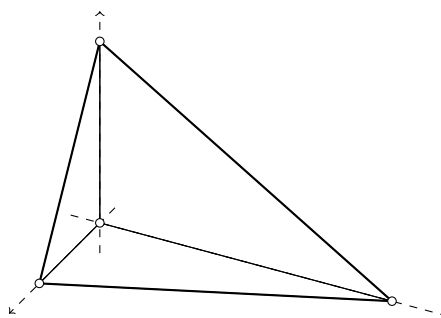
2 extreme rays of recession cone:	3 support hyperplanes of polyhedron (homogenized):
1 3 0	-1 2 -1
2 1 0	0 0 1
	3 -1 -1

Two support hyperplanes are exactly those that we have used to define the polyhedron – and it has only 2. But Normaliz always outputs the support hyperplanes that are needed for the cone that one obtains by homogenizing the polyhedron, as indicated by ‘homogenized’. The homogenizing variable is always  $\geq 0$ . In this case the support hyperplane  $(0,0,1)$  is essential for the description of the cone. Note that it need not always appear.

## 2.4. A lattice polytope

The file `polytope.in` contains

amb_space 4
polytope 4
0 0 0
2 0 0
0 3 0
0 0 5



This is a good place to mention that Normaliz also accepts matrices (and vectors) in sparse format:

amb_space 4
polytope 4
;
1:2;

```
2:3;  
3:5;
```

Each input row, concluded by ;, lists the indices and the corresponding nonzero values in that row of the matrix.

The Ehrhart monoid of the integral polytope with the 4 vertices

$$(0,0,0), \quad (2,0,0), \quad (0,3,0) \quad \text{and} \quad (0,0,5)$$

in  $\mathbb{R}^3$  is to be computed. The generators of the Ehrhart monoid are obtained by attaching a further coordinate 1 to the vertices, and this explains `amb_space 4`. In fact, the input type polytope is nothing but a convenient (perhaps superfluous) version of

```
amb_space 4  
cone 4  
0 0 0 1  
2 0 0 1  
0 3 0 1  
0 0 5 1
```

Running `normaliz` produces the file `polytope.out`:

```
19 Hilbert basis elements  
18 Hilbert basis elements of degree 1  
4 extreme rays  
4 support hyperplanes  
  
embedding dimension = 4  
rank = 4 (maximal)  
external index = 1  
internal index = 30  
original monoid is not integrally closed
```

Perhaps a surprise: the lattice points of the polytope do not yield all Hilbert basis elements.

```
size of triangulation   = 1  
resulting sum of |det|s = 30
```

Nothing really new so far. But now `Normaliz` finds a grading given by the last coordinate. See 3.10 below for general information on gradings.

```
grading:  
0 0 0 1  
  
degrees of extreme rays:  
1: 4
```

Again we encounter the notation  $\langle n \rangle$ :  $\langle m \rangle$ : we have 4 extreme rays, all of degree 1.

```
Hilbert basis elements are not of degree 1
```

Perhaps a surprise: the polytope is not integrally closed as defined in [4]. Now we see the enumerative data defined by the grading:

```

multiplicity = 30

Hilbert series:
1 14 15
denominator with 4 factors:
1: 4

degree of Hilbert Series as rational function = -2

Hilbert polynomial:
1 4 8 5
with common denominator = 1

```

The polytope has  $\mathbb{Z}^3$ -normalized volume 30 as indicated by the multiplicity. The Hilbert (or Ehrhart) function counts the lattice points in  $kP$ ,  $k \in \mathbb{Z}_+$ . The corresponding generating function is a rational function  $H(t)$ . For our polytope it is

$$\frac{1 + 14t + 15t^2}{(1 - t)^4}.$$

The denominator is given in multiset notation: 1: 4 say that the factor  $(1 - t^1)$  occurs with multiplicity 4.

The Ehrhart polynomial (again we use a more general term in the output file) of the polytope is

$$p(k) = 1 + 4k + 8k^2 + 5k^3.$$

In our case it has integral coefficients, a rare exception. Therefore one usually needs a denominator.

Everything that follows has already been explained.

```

rank of class group = 0
finite cyclic summands:
30: 1

*****

18 Hilbert basis elements of degree 1:
0 0 0 1
...
2 0 0 1

1 further Hilbert basis elements of higher degree:
1 2 4 2

```

4 extreme rays:	4 support hyperplanes:
0 0 0 1	-15 -10 -6 30
0 0 5 1	0 0 1 0
0 3 0 1	0 1 0 0
2 0 0 1	1 0 0 0

The support hyperplanes give us a description of the polytope by inequalities: it is the solution of the system of the 4 inequalities

$$x_3 \geq 0, \quad x_2 \geq 0, \quad x_1 \geq 0 \quad \text{and} \quad 15x_1 + 10x_2 + 6x_3 \leq 30.$$

### 2.4.1. Only the lattice points

Suppose we want to compute only the lattice points in our polytope. In the language of graded monoids these are the degree 1 elements, and so we add `Deg1Elements` to our input file (`polytope_deg1.in`):

```
amb_space 4
polytope 4
0 0 0
2 0 0
0 3 0
0 0 5
Deg1Elements
/* This is our first explicit computation goal*/
```

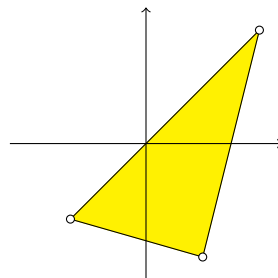
We have used this opportunity to include a comment in the input file.

We lose all information on the Hilbert series, and from the Hilbert basis we only retain the degree 1 elements.

## 2.5. A rational polytope

Normaliz has no special input type for rational polytopes. In order to process them one uses the type cone together with a grading. Suppose the polytope is given by vertices

$$v_i = (r_{i1}, \dots, r_{id}), \quad i = 1, \dots, m, \quad r_{ij} \in \mathbb{Q}.$$



Then we write  $v_i$  with a common denominator:

$$v_i = \left( \frac{p_{i1}}{q_i}, \dots, \frac{p_{id}}{q_i} \right), \quad p_{ij}, q_i \in \mathbb{Z}, q_i > 0.$$

The generator matrix is given by the rows

$$\tilde{v}_i = (p_{i1}, \dots, p_{id}, q_i), \quad i = 1, \dots, m.$$

We must add a grading since Normaliz cannot recognize it without help (unless all the  $q_i$  are equal to 1). The grading linear form has coordinates  $(0, \dots, 0, 1)$ .

We want to investigate the Ehrhart series of the triangle  $P$  with vertices

$$(1/2, 1/2), (-1/3, -1/3), (1/4, -1/2).$$

For this example the procedure above yields the input file `rational.in`:

```
amb_space 3
cone 3
1 1 2
-1 -1 3
1 -2 4
grading
unit_vector 3
HilbertSeries
```

This is the first time that we used the shortcut `unit_vector <n>` which represents the  $n$ th unit vector  $e_n \in \mathbb{R}^d$  and is only allowed for input types which require a single vector.

From the output file we only list the data of the Ehrhart series.

```
multiplicity = 5/8

Hilbert series:
1 0 0 3 2 -1 2 2 1 1 1 1 2
denominator with 3 factors:
1: 1 2: 1 12: 1

degree of Hilbert Series as rational function = -3

Hilbert series with cyclotomic denominator:
-1 -1 -1 -3 -4 -3 -2
cyclotomic denominator:
1: 3 2: 2 3: 1 4: 1

Hilbert quasi-polynomial of period 12:
0: 48 28 15 7: 23 22 15
1: 11 22 15 8: 16 28 15
```

2:	-20	28	15	9:	27	22	15
3:	39	22	15	10:	-4	28	15
4:	32	28	15	11:	7	22	15
5:	-5	22	15	with common denominator = 48			
6:	12	28	15				

The multiplicity is a rational number. Since in dimension 2 the normalized area (of full-dimensional polytopes) is twice the Euclidean area, we see that  $P$  has Euclidean area  $5/16$ .

Unlike in the case of a lattice polytope, there is no canonical choice of the denominator of the Ehrhart series. Normaliz gives it in 2 forms. In the first form the numerator polynomial is

$$1 + 3t^3 + 2t^4 - t^5 + 2t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11} + 2t^{12}$$

and the denominator is

$$(1-t)(1-t^2)(1-t^{12}).$$

As a rational function,  $H(t)$  has degree  $-3$ . This implies that  $3P$  is the smallest integral multiple of  $P$  that contains a lattice point in its interior.

Normaliz gives also a representation as a quotient of coprime polynomials with the denominator factored into cyclotomic polynomials. In this case we have

$$H(t) = -\frac{1 + t + t^2 + t^3 + 4t^4 + 3t^5 + 2t^6}{\zeta_1^3 \zeta_2^2 \zeta_3 \zeta_4}$$

where  $\zeta_i$  is the  $i$ -th cyclotomic polynomial ( $\zeta_1 = t - 1$ ,  $\zeta_2 = t + 1$ ,  $\zeta_3 = t^2 + t + 1$ ,  $\zeta_4 = t^2 + 1$ ).

Normaliz transforms the representation with cyclotomic denominator into one with denominator of type  $(1 - t^{e_1}) \cdots (1 - t^{e_r})$ ,  $r = \text{rank}$ , by choosing  $e_r$  as the least common multiple of all the orders of the cyclotomic polynomials appearing,  $e_{r-1}$  as the lcm of those orders that have multiplicity  $\geq 2$  etc.

There are other ways to form a suitable denominator with 3 factors  $1 - t^e$ , for example  $g(t) = (1 - t^2)(1 - t^3)(1 - t^4) = -\zeta_1^3 \zeta_2^2 \zeta_3 \zeta_4$ . Of course,  $g(t)$  is the optimal choice in this case. However,  $P$  is a simplex, and in general such optimal choice may not exist. We will explain the reason for our standardization below.

Let  $p(k)$  be the number of lattice points in  $kP$ . Then  $p(k)$  is a quasipolynomial:

$$p(k) = p_0(k) + p_1(k)k + \cdots + p_{r-1}(k)k^{r-1},$$

where the coefficients depend on  $k$ , but only to the extent that they are periodic of a certain period  $\pi \in \mathbb{N}$ . In our case  $\pi = 12$  (the lcm of the orders of the cyclotomic polynomials).

The table giving the quasipolynomial is to be read as follows: The first column denotes the residue class  $j$  modulo the period and the corresponding line lists the coefficients  $p_i(j)$  in ascending order of  $i$ , multiplied by the common denominator. So

$$p(k) = 1 + \frac{7}{12}k + \frac{5}{16}k^2, \quad k \equiv 0 \pmod{12}, \quad (12),$$

etc. The leading coefficient is the same for all residue classes and equals the Euclidean volume. Our choice of denominator for the Hilbert series is motivated by the following fact:  $e_i$  is the common period of the coefficients  $p_{r-i}, \dots, p_{r-1}$ . The user should prove this fact or at least verify it by several examples.

Especially in the case of a simplex the representation of the Hilbert series shown so far may not be the expected one. In fact, there is a representation in which the exponents of  $t$  in the denominator are the degrees of the integral extreme generators. So one would expect the denominator to be  $(1-t^2)(1-t^3)(1-t^4)$  in our case. The generalization to the nonsimplicial case uses the degrees of a homogeneous system of parameters (see [4, p. 200]). Normaliz can compute such a denominator if the computation goal HSOP is set (`rationalHSOP.in`):

```
Hilbert series (HSOP):
1 1 1 3 4 3 2
denominator with 3 factors:
2: 1 3: 1 4: 1
```

Note that the degrees of the elements in a homogeneous system of parameters are by no means unique and that there is no optimal choice in general. To find a suitable sequence of degrees Normaliz must compute the face lattice of the cone to some extent. Therefore be careful not to ask for HSOP if the cone has many support hyperplanes.

*Warning:* It is tempting, but not a good idea to define the polytope by the input type vertices. It would make Normaliz compute the lattice points in the polytope, but not in the cone over the polytope, and we need these to determine the Ehrhart series.

### 2.5.1. The rational polytope by inequalities

We extract the support hyperplanes of our polytope from the output file and use them as input (`poly_ineq.in`):

```
amb_space 3
inequalities 3
-8 2 3
1 -1 0
2 7 3
grading
unit_vector 3
HilbertSeries
```

These data tell us that the polytope, as a subset of  $\mathbb{R}^2$ , is defined by the inequalities

$$\begin{aligned} -8x_1 + 2x_2 + 3 &\geq 0, \\ x_1 - x_2 + 0 &\geq 0, \\ 2x_1 + 7x_2 + 3 &\geq 0. \end{aligned}$$

These inequalities are inhomogeneous, but we are using the homogeneous input type `inequalities` which amounts to introducing the grading variable  $x_3$ , as we have done it for the generators.

Why don't we define it by the “natural” inhomogeneous inequalities using `inhom_inequalities`? We could do it, but then only the polytope itself would be the object of computation, and we would have no access to the Ehrhart series. We could just compute the lattice points in the polytope. (Try it.)

The inequalities as written above look somewhat artificial. It is certainly more natural to write them in the form

$$\begin{aligned} 8x_1 - 2x_2 &\leq 3, \\ x_1 - x_2 &\geq 0, \\ 2x_1 + 7x_2 &\geq -3. \end{aligned}$$

and for the direct transformation into Normaliz input we have introduced the type constraints. But Normaliz would then interpret the input as inhomogeneous and we run into the same problem as with `inhom_inequalities`. The way out: we tell Normaliz that we want a homogeneous computation (`poly_hom_const.in`):

```
amb_space 3
hom_constraints 3
8 -2 <= 3
1 -1 >= 0
2 7 >= -3
grading
unit_vector 3
HilbertSeries
```

## 2.6. Magic squares

Suppose that you are interested in the following type of “square”

$x_1$	$x_2$	$x_3$
$x_4$	$x_5$	$x_6$
$x_7$	$x_8$	$x_9$

and the problem is to find nonnegative values for  $x_1, \dots, x_9$  such that the 3 numbers in all rows, all columns, and both diagonals sum to the same constant  $\mathcal{M}$ . Sometimes such squares are



called *magic* and  $\mathcal{M}$  is the *magic constant*. This leads to a linear system of equations

$$\begin{aligned}x_1 + x_2 + x_3 &= x_4 + x_5 + x_6; \\x_1 + x_2 + x_3 &= x_7 + x_8 + x_9; \\x_1 + x_2 + x_3 &= x_1 + x_4 + x_7; \\x_1 + x_2 + x_3 &= x_2 + x_5 + x_8; \\x_1 + x_2 + x_3 &= x_3 + x_6 + x_9; \\x_1 + x_2 + x_3 &= x_1 + x_5 + x_9; \\x_1 + x_2 + x_3 &= x_3 + x_5 + x_7.\end{aligned}$$

This system is encoded in the file `3x3magic.in`:

```
amb_space 9
equations 7
1 1 1 -1 -1 -1 0 0 0
1 1 1 0 0 0 -1 -1 -1
0 1 1 -1 0 0 -1 0 0
1 0 1 0 -1 0 0 -1 0
1 1 0 0 0 -1 0 0 -1
0 1 1 0 -1 0 0 0 -1
1 1 0 0 -1 0 -1 0 0
grading
sparse 1:1 2:1 3:1;
```

The input type `equations` represents *homogeneous* equations. The first equation reads

$$x_1 + x_2 + x_3 - x_4 - x_5 - x_6 = 0,$$

and the other equations are to be interpreted analogously. The magic constant is a natural choice for the grading. It is given in sparse form, equivalent to the dense form

```
grading
1 1 1 0 0 0 0 0 0
```

It seems that we have forgotten to define the cone. This may indeed be the case, but doesn't matter: if there is no input type that defines a cone, `Normaliz` chooses the positive orthant, and this is exactly what we want in this case.

The output file contains the following:

```
5 Hilbert basis elements
5 Hilbert basis elements of degree 1
4 extreme rays
4 support hyperplanes

embedding dimension = 9
```

```

rank = 3
external index = 1

size of triangulation = 2
resulting sum of |det|s = 4

grading:
1 1 1 0 0 0 0 0 0
with denominator = 3

```

The input degree is the magic constant. However, as the denominator 3 shows, the magic constant is always divisible by 3, and therefore the effective degree is  $\mathcal{M}/3$ . This degree is used for the multiplicity and the Hilbert series.

```

degrees of extreme rays:
1: 4

Hilbert basis elements are of degree 1

```

This was not to be expected (and is no longer true for  $4 \times 4$  squares).

```

multiplicity = 4

Hilbert series:
1 2 1
denominator with 3 factors:
1: 3

degree of Hilbert Series as rational function = -1

Hilbert polynomial:
1 2 2
with common denominator = 1

```

The Hilbert series is

$$\frac{1 + 2t + t^2}{(1 - t)^3}.$$

The Hilbert polynomial is

$$P(k) = 1 + 2k + 2k^2,$$

and after substituting  $\mathcal{M}/3$  for  $k$  we obtain the number of magic squares of magic constant  $\mathcal{M}$ , provided 3 divides  $\mathcal{M}$ . (If  $3 \nmid \mathcal{M}$ , there is no magic square of magic constant  $\mathcal{M}$ .)

```

rank of class group = 1
finite cyclic summands:
2: 2

```

So the class group is  $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ .

```

5 Hilbert basis elements of degree 1:
0 2 1 2 1 0 1 0 2
1 0 2 2 1 0 0 2 1
1 1 1 1 1 1 1 1 1
1 2 0 0 1 2 2 0 1
2 0 1 0 1 2 1 2 0

0 further Hilbert basis elements of higher degree:

```

The 5 elements of the Hilbert basis represent the magic squares

2	0	1
0	1	2
1	2	0

1	0	2
2	1	0
0	2	1

1	1	1
1	1	1
1	1	1

1	2	0
0	1	2
2	0	1

0	2	1
2	1	0
1	0	2

All other solutions are linear combinations of these squares with nonnegative integer coefficients. One of these 5 squares is clearly in the interior:

```

4 extreme rays:
0 2 1 2 1 0 1 0 2
1 0 2 2 1 0 0 2 1
1 2 0 0 1 2 2 0 1
2 0 1 0 1 2 1 2 0

4 support hyperplanes:
-2 -1 0 0 4 0 0 0 0
0 -1 0 0 2 0 0 0 0
0 1 0 0 0 0 0 0 0
2 1 0 0 -2 0 0 0 0

```

These 4 support hyperplanes cut out the cone generated by the magic squares from the linear subspace they generate. Only one is reproduced as a sign inequality. This is due to the fact that the linear subspace has submaximal dimension and there is no unique lifting of linear forms to the full space.

```

6 equations:
1 0 0 0 0 1 -2 -1 1
0 1 0 0 0 1 -2 0 0
0 0 1 0 0 1 -1 -1 0
0 0 0 1 0 -1 2 0 -2
0 0 0 0 1 -1 1 0 -1
0 0 0 0 0 3 -4 -1 2

3 basis elements of lattice:
1 0 -1 -2 0 2 1 0 -1
0 1 -1 -1 0 1 1 -1 0
0 0 3 4 1 -2 -1 2 2

```

So one of our equations has turned out to be superfluous (why?). Note that also the equations are not reproduced exactly. Finally, Normaliz lists a basis of the efficient lattice  $\mathbb{E}$  generated by the magic squares.

### 2.6.1. With even corners

We change our definition of magic square by requiring that the entries in the 4 corners are all even. Then we have to augment the input file by the following (3x3magiceven.in):

```

congruences 4 sparse
1:1 10:2;
3:1 10:2;
7:1 10:2;
9:1 10:2;

```

This sparse form is equivalent to the dense form

```

congruences 4
1 0 0 0 0 0 0 0 0 2
0 0 1 0 0 0 0 0 0 2
0 0 0 0 0 0 1 0 0 2
0 0 0 0 0 0 0 0 1 2

```

The first 9 entries in each row represent the coefficients of the coordinates in the homogeneous congruences, and the last is the modulus:

$$x_1 \equiv 0 \pmod{2}$$

is the first congruence etc.

We could also define these congruences as symbolic constraints:

```

constraints 4 symbolic
x[1] ~ 0(2);
x[3] ~ 0(2);
x[7] ~ 0(2);
x[9] ~ 0(2);

```

The output changes accordingly:

```

9 Hilbert basis elements
0 Hilbert basis elements of degree 1
4 extreme rays
4 support hyperplanes

embedding dimension = 9
rank = 3
external index = 4

size of triangulation = 2
resulting sum of |det|s = 8

grading:
1 1 1 0 0 0 0 0 0
with denominator = 3

degrees of extreme rays:
2: 4

```

```

multiplicity = 1

Hilbert series:
1 -1 3 1
denominator with 3 factors:
1: 1  2: 2

degree of Hilbert Series as rational function = -2

Hilbert series with cyclotomic denominator:
-1 1 -3 -1
cyclotomic denominator:
1: 3  2: 2

Hilbert quasi-polynomial of period 2:
0:  2 2 1
1:  -1 0 1
with common denominator = 2

```

After the extensive discussion in Section 2.5 it should be easy for you to write down the Hilbert series and the Hilbert quasipolynomial. (But keep in mind that the grading has a denominator.)

```

rank of class group = 1
finite cyclic summands:
4: 2

*****

0 Hilbert basis elements of degree 1:

9 further Hilbert basis elements of higher degree:
...

4 extreme rays:
0 4 2 4 2 0 2 0 4
2 0 4 4 2 0 0 4 2
2 4 0 0 2 4 4 0 2
4 0 2 0 2 4 2 4 0

```

We have listed the extreme rays since they have changed after the introduction of the congruences, although the cone has not changed. The reason is that Normaliz always chooses the extreme rays from the efficient lattice  $\mathbb{E}$ .

```

4 support hyperplanes:
...

```

6 equations:	
...	3 basis elements of lattice:
	2 0 -2 -4 0 4 2 0 -2
2 congruences:	0 1 2 3 1 -1 0 1 2
1 0 0 0 0 0 0 0 2	0 0 6 8 2 -4 -2 4 4
0 1 0 0 1 0 0 0 2	

The rank of the lattice has of course not changed, but after the introduction of the congruences the basis has changed.

### 2.6.2. The lattice as input

It is possible to define the lattice by generators. We demonstrate this for the magic squares with even corners. The lattice has just been computed (3x3magiceven\_lat.in):

```
amb_space 9
lattice 3
2 0 -2 -4 0 4 2 0 -2
0 1 2 3 1 -1 0 1 2
0 0 6 8 2 -4 -2 4 4
grading
1 1 1 0 0 0 0 0 0
```

It produces the same output as the version starting from equations and congruences.

lattice has a variant that takes the saturation of the sublattice generated by the input vectors (3x3magic\_sat.in):

```
amb_space 9
saturation 3
2 0 -2 -4 0 4 2 0 -2
0 1 2 3 1 -1 0 1 2
0 0 6 8 2 -4 -2 4 4
grading
1 1 1 0 0 0 0 0 0
```

Clearly, we remove the congruences by this choice and arrive at the output of 3x3magic.in.

## 2.7. Decomposition in a numerical semigroup

Let  $S = \langle 6, 10, 15 \rangle$ , the numerical semigroup generated by 6, 10, 15. How can 97 be written as a sum in the generators?

In other words: we want to find all nonnegative integral solutions to the equation

$$6x_1 + 10x_2 + 15x_3 = 97$$

Input (NumSemi.in):

```
amb_space 3
constraints 1 symbolic
6x[1] + 10x[2] + 15x[3] = 97;
```



The equation cuts out a triangle from the positive orthant.

The set of solutions is a module over the monoid  $M$  of solutions of the homogeneous equation  $6x_1 + 10x_2 + 15x_3 = 0$ . So  $M = 0$ .

```
6 module generators:
2 1 5 1
2 4 3 1
2 7 1 1
7 1 3 1
7 4 1 1
12 1 1 1

0 Hilbert basis elements of recession monoid:
```

The last line is as expected, and the 6 module generators are the goal of the computation.

Normaliz is smart enough to recognize that it must compute the lattice points in a polygon, and does exactly this. You can recognize it in the console output: it contains the line

```
Computing approximating polytopee
```

Moreover, the line indicates that Normaliz uses its approximation algorithm to find the lattice points in this polytope. See Section 6.1. Further inspection shows that the approximating lattice polytope has 6 vertices, namely for each original vertex three neighboring lattice points on its coordinate axis.

For those who like to play: add the option `--NoSymmetrization` to the command line. Then the terminal output will change, and the polytope is computed exactly as it comes.

## 2.8. A job for the dual algorithm

We increase the size of the magic squares to  $5 \times 5$ . Normaliz can do the same computation as for  $3 \times 3$  squares, but this will take some minutes. Suppose we are only interested in the Hilbert basis, we should use the dual algorithm for this example. The input file is `5x5dual.in`:

```
amb_space 25
equations 11
1 1 1 1 1 -1 -1 -1 -1 -1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
...
1 1 1 1 0 0 0 0 -1 0 0 0 -1 0 0 0 -1 0 0 0 -1 0 0 0 0
grading
1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

The input file does not say anything about the dual algorithm mentioned in the section title. With this input it is chosen automatically. See Section 6.4 for a discussion of when this happens. But you can insist on the dual algorithm by adding `DualMode` to the input (or `-d` to the command line). Or, if you want to compare it to the primal algorithm add `PrimalMode` (or `-P` to the command line).

The Hilbert basis contains 4828 elements, too many to be listed here.

If you want to run this example with default computation goals, use the file `5x5.in`. It will compute the Hilbert basis and the Hilbert series, and the latter with HSOP:

```
Hilbert series (HSOP):
1 15 356 4692 36324 198467 ... 198467 36324 4692 356 15 1
denominator with 15 factors:
1: 5  2: 3  6: 2  12: 1  60: 2  420: 1  1260: 1

degree of Hilbert Series as rational function = -5

The numerator of the Hilbert Series is symmetric.
```

In view of the length of the numerator of the Hilbert series it may be difficult to observe the symmetry. So `Normaliz` does it for you.

The size  $6 \times 6$  is out of reach for the Hilbert series, but the Hilbert basis can be computed (in the automatically chosen dual mode). It takes some hours.

## 2.9. A dull polyhedron

We want to compute the polyhedron defined by the inequalities

$$\begin{aligned}\xi_2 &\geq -1/2, \\ \xi_2 &\leq 3/2, \\ \xi_2 &\leq \xi_1 + 3/2.\end{aligned}$$

They are contained in the input file `InhomIneq.in`:

```
amb_space 2
constraints 3
  0 2 >= -1
  0 2 <=  3
 -2 2 <=  3
grading
unit_vector 1
```

The grading says that we want to count points by the first coordinate.





It yields the output

```

2 module generators
1 Hilbert basis elements of recession monoid
2 vertices of polyhedron
1 extreme rays of recession cone
3 support hyperplanes of polyhedron (homogenized)

embedding dimension = 3
affine dimension of the polyhedron = 2 (maximal)
rank of recession monoid = 1

size of triangulation   = 1
resulting sum of |det|s = 8

dehomogenization:
0 0 1

grading:
1 0 0

```

The interpretation of the grading requires some care in the inhomogeneous case. We have extended the input grading vector by an entry 0 to match the embedding dimension. For the computation of the degrees of *lattice points* in the ambient space you can either use only the first 2 coordinates or take the full scalar product of the point in homogenized coordinates and the extended grading vector.

```

module rank = 2
multiplicity = 2

```

The module rank is 2 in this case since we have two “layers” in the solution module that are parallel to the recession monoid. This is of course also reflected in the Hilbert series.

```

Hilbert series:
1 1
denominator with 1 factors:
1: 1

shift = -1

```

We haven’t seen a shift yet. It is always printed (necessarily) if the Hilbert series does not start

in degree 0. In our case it starts in degree  $-1$  as indicated by the shift  $-1$ . We thus get the Hilbert series

$$t^{-1} \frac{t+t}{1-t} = \frac{t^{-1}+1}{1-t}.$$

Note: We used the opposite convention for the shift in Normaliz 2.

Note that the Hilbert (quasi)polynomial is always computed for the unshifted monoid defined by the input data. (This was different in previous versions of Normaliz.)

```
degree of Hilbert Series as rational function = -1

Hilbert polynomial:
2
with common denominator = 1

*****

2 module generators:
-1 0 1
0 1 1

1 Hilbert basis elements of recession monoid:
1 0 0

2 vertices of polyhedron:
-4 -1 2
0 3 2

1 extreme rays of recession cone:
1 0 0

3 support hyperplanes of polyhedron (homogenized):
0 -2 3
0 2 1
2 -2 3
```

The dual algorithm that was used in Section 2.8 can also be applied to inhomogeneous computations. We would of course lose the Hilbert series. In certain cases it may be preferable to suppress the computation of the vertices of the polyhedron if you are only interested in the integer points; see Section 4.6.

### 2.9.1. Defining it by generators

If the polyhedron is given by its vertices and the recession cone, we can define it by these data (InhomIneq\_gen.in):

```

amb_space 2
vertices 2
-4 -1 2
0 3 2
cone 1
1 0
grading
unit_vector 1

```

The output is identical to the version starting from the inequalities.

## 2.10. The Condorcet paradoxon

In social choice elections each of the  $k$  voters picks a linear preference order of the  $n$  candidates. There are  $n!$  such orders.

We say that candidate  $A$  *beats* candidate  $B$  if the majority of the voters prefers  $A$  to  $B$ . As the Marquis de *Condorcet* (and others) observed, “beats” is not transitive, and an election may exhibit the *Condorcet paradoxon*: there is no Condorcet winner. (See [10] and the references given there for more information.)

We want to find the probability for  $k \rightarrow \infty$  that there is a Condorcet winner for  $n = 4$  candidates. The event that  $A$  is the Condorcet winner can be expressed by linear inequalities on the election outcome (a point in 24-space). The wanted probability is the lattice normalized volume of the polytope cut out by the inequalities at  $k = 1$ . The file `Condorcet.in`:

```

amb_space 24
inequalities 3
1 1 1 1 1 1 -1 -1 -1 -1 -1 -1 1 1 -1 -1 1 -1 1 1 -1 -1 1 -1
1 1 1 1 1 1 1 1 -1 -1 1 -1 -1 -1 -1 -1 -1 1 1 1 -1 -1 -1
1 1 1 1 1 1 1 1 1 -1 -1 -1 1 1 1 -1 -1 -1 -1 -1 -1 -1 -1
nonnegative
total_degree
Multiplicity

```

The first inequality expresses that  $A$  beats  $B$ , the second and the third say that  $A$  beats  $C$  and  $D$ . (So far we do not exclude ties, and they need not be excluded for probabilities as  $k \rightarrow \infty$ .)

In addition to these inequalities we must restrict all variables to nonnegative values, and this is achieved by adding the attribute `nonnegative`. The grading is set by `total_degree`. It replaces the grading vector with 24 entries 1. Finally `Multiplicity` sets the computation goal.

From the output file we only mention the quantity we are out for:

```

multiplicity = 1717/8192

```

Since there are 4 candidates, the probability for the existence of a Condorcet winner is  $1717/2048$ .

We can refine the information on the Condorcet paradoxon by computing the Hilbert series. Either we delete Multiplicity from the input file or, better, we add `--HilbertSeries` (or simply `-q`) on the command line. The result:

```
Hilbert series:
1 5 133 363 4581 8655 69821 100915 ... 12346 890 481 15 6
denominator with 24 factors:
1: 1 2: 14 4: 9

degree of Hilbert Series as rational function = -25
```

For this example the option `Symmetrize`, `-Y` makes sense (see Section 6.5), but it is too small to demonstrate the power of symmetrization. For others, symmetrization can convert days into seconds.

### 2.10.1. Excluding ties

Now we are more ambitious and want to compute the Hilbert series for the Condorcet paradoxon, or more precisely, the number of election outcomes having  $A$  as the Condorcet winner depending on the number  $k$  of voters. Moreover, as it is customary in social choice theory, we want to exclude ties. The input file changes to `CondorcetSemi.in`:

```
amb_space 24
excluded_faces 3
1 1 1 1 1 1 -1 -1 -1 -1 -1 -1 1 1 -1 -1 1 -1 1 1 -1 -1 1 -1
1 1 1 1 1 1 1 1 -1 -1 1 -1 -1 -1 -1 -1 -1 1 1 1 -1 -1 -1
1 1 1 1 1 1 1 1 1 -1 -1 -1 1 1 1 -1 -1 -1 -1 -1 -1 -1 -1
nonnegative
total_degree
HilbertSeries
```

We could omit `HilbertSeries`, and the computation would include the Hilbert basis. The type `excluded_faces` only affects the Hilbert series. In every other respect it is equivalent to inequalities.

From the file `CondorcetSemi.out` we only display the Hilbert series:

```
Hilbert series:
6 15 481 890 12346 ... 100915 69821 8655 4581 363 133 5 1
denominator with 24 factors:
1: 1 2: 14 4: 9

shift = 1

degree of Hilbert Series as rational function = -24
```

Surprisingly, this looks like the Hilbert series in the previous section read backwards, roughly speaking. This is true, and one can explain it as we will see below.

It is justified to ask why we don't use `strict_inequalities` instead of `excluded_faces`. It does of course give the same Hilbert series. If there are many excluded faces, then it is better to use `strict_inequalities`. However, at present `NmzIntegrate` can only work with `excluded_faces`.

### 2.10.2. At least one vote for every preference order

Suppose we are only interested in elections in which every preference order is chosen by at least one voter. This can be modeled as follows (`Condorcet_one.in`):

```
amb_space 24
inequalities 3
1 1 1 1 1 1 -1 -1 -1 -1 -1 -1 1 1 -1 -1 1 -1 1 1 -1 -1 1 -1
1 1 1 1 1 1 1 1 -1 -1 1 -1 -1 -1 -1 -1 -1 1 1 1 -1 -1 -1
1 1 1 1 1 1 1 1 1 -1 -1 -1 1 1 1 -1 -1 -1 -1 -1 -1 -1 -1
strict_signs
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
total_degree
HilbertSeries
```

The entry 1 at position  $i$  of the vector `strict_signs` imposes the inequality  $x_i \geq 1$ . A  $-1$  would impose the inequality  $x_i \leq -1$ , and the entry 0 imposes no condition on the  $i$ -th coordinate.

```
Hilbert series:
1 5 133 363 4581 8655 69821 100915 ... 12346 890 481 15 6
denominator with 24 factors:
1: 1 2: 14 4: 9

shift = 24

degree of Hilbert Series as rational function = -1
```

Again we encounter (almost) the Hilbert series of the Condorcet paradoxon (without side conditions). It is time to explain this coincidence. Let  $C$  be the Condorcet cone defined by the nonstrict inequalities,  $M$  the monoid of lattice points in it,  $I_1 \subset M$  the ideal of lattice points avoiding the 3 facets defined by ties,  $I_2$  the ideal of lattice points with strictly positive coordinates, and finally  $I_3$  the ideal of lattice points in the interior of  $C$ . Moreover, let  $\mathbb{1} \in \mathbb{Z}^{24}$  be the vector with all entries 1.

Since  $\mathbb{1}$  lies in the three facets defining the ties, it follows that  $I_2 = M + \mathbb{1}$ . This explains why we obtain the Hilbert series of  $I_2$  by multiplying the Hilbert series of  $M$  by  $t^{24}$ , as just observed. Generalized Ehrhart reciprocity (see [4, Theorem 6.70]) then explains the Hilbert series of  $I_1$  that we observed in the previous section. Finally, the Hilbert series of  $I_3$  that we don't have displayed is obtained from that of  $M$  by “ordinary” Ehrhart reciprocity. But we can also obtain

$I_1$  from  $I_3$ :  $I_1 = I_3 - \mathbb{1}$ , and generalized reciprocity follows from ordinary reciprocity in this very special case.

The essential point in these arguments (apart from reciprocity) is that  $\mathbb{1}$  lies in all support hyperplanes of  $C$  except the coordinate hyperplanes.

You can easily compute the Hilbert series of  $I_3$  by making all inequalities strict.

## 2.11. Testing normality

We want to test the monoid  $A_{4 \times 4 \times 3}$  defined by  $4 \times 4 \times 3$  contingency tables for normality (see [5] for the background). The input file is A443.in:

```
amb_space 40
cone_and_lattice 48
1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0
...
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1
HilbertBasis
```

Why cone\_and\_lattice? Well, we want to find out whether the monoid is normal, i.e., whether  $M = C(M) \cap \text{gp}(M)$ . If  $M$  is even integrally closed in  $\mathbb{Z}^{24}$ , then it is certainly integrally closed in the evidently smaller lattice  $\text{gp}(M)$ , but the converse does not hold in general, and therefore we work with the lattice generated by the monoid generators.

It turns out that the monoid is indeed normal:

```
original monoid is integrally closed
```

Actually the output file reveals that  $M$  is even integrally closed in  $\mathbb{Z}^{24}$ : the external index is 1, and therefore  $\text{gp}(M)$  is integrally closed in  $\mathbb{Z}^{24}$ .

The output files also shows that there is a grading on  $\mathbb{Z}^{24}$  under which all our generators have degree 1. We could have seen this ourselves: Every generator has exactly one entry 1 in the first 16 coordinates. (This is clear from the construction of  $M$ .)

A noteworthy detail from the output file:

```
size of partial triangulation    = 48
```

It shows that Normaliz uses only a partial triangulation in Hilbert basis computations; see [5].

It is no problem to compute the Hilbert series as well if you are interested in it. Simply add -q to the command line or remove HilberBasis from the input file. Then a full triangulation is needed (size 2,654,272).

Similar examples are A543, A553 and A643. The latter is not normal, as we will see below. Even on a standard PC or laptop, the Hilbert basis computation does not take very long because Normaliz uses only a partial triangulation. The Hilbert series can still be determined, but the computation time will grow considerably since it requires a full triangulation. See [8] for timings.

### 2.11.1. Computing just a witness

If the Hilbert basis is large and there are many support hyperplanes, memory can become an issue for Normaliz, as well as computation time. Often one is only interested in deciding whether the given monoid is integrally closed (or normal). In the negative case it is enough to find a single element that is not in the original monoid – a witness disproving integral closedness. As soon as such a witness is found, Normaliz stops the Hilbert basis computation (but will continue to compute other data if they are asked for). We look at the example `A643.in` (for which the full Hilbert basis is not really a problem):

```
amb_space 54
cone_and_lattice 72
1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 ...
...
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 ...
IsIntegrallyClosed
```

Don't add `HilbertBasis` because it will overrule `IsIntegrallyClosed`!

The output:

```
72 extreme rays
153858 support hyperplanes

embedding dimension = 54
rank = 42
external index = 1
internal index = 1
original monoid is not integrally closed
witness for not being integrally closed:
0 0 1 0 1 1 1 1 0 0 1 0 0 1 0 1 0 1 1 0 1 1 0 0 1 1 1 0 0 1 1 0 0 1 1 ...

grading:
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 ...

degrees of extreme rays:
1: 72

*****

72 extreme rays:
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ...
...
```

If you repeat such a computation, you may very well get a different witness if several parallel threads find witnesses. Only one of them is delivered.

## 2.12. Inhomogeneous congruences

We want to compute the nonnegative solutions of the simultaneous inhomogeneous congruences

$$\begin{aligned}x_1 + 2x_2 &\equiv 3 \pmod{7}, \\ 2x_1 + 2x_2 &\equiv 4 \pmod{13}\end{aligned}$$

in two variables. The input file `InhomCong.in` is

```
amb_space 2
constraints 2 symbolic
x[1] + 2x[2] ~ 3 (7);
2x[1] + 2x[2] ~ 4 (13);
```

This is an example of input of symbolic constraints. We use `~` as the best ASCII character for representing the congruence sign  $\equiv$ .

Alternatively one can use a matrix in the input `As` for which we must move the right hand side over to the left.

```
amb_space 2
inhom_congruences 2
1 2 -3 7
2 2 -4 13
```

It is certainly harder to read.

The first vector list in the output:

```
3 module generators:
0 54 1
1 1 1
80 0 1
```

Easy to check: if  $(1, 1)$  is a solution, then it must generate the module of solutions together with the generators of the intersections with the coordinate axes. Perhaps more difficult to find:

```
6 Hilbert basis elements of recession monoid:
0 91 0
1 38 0
3 23 0
5 8 0
12 1 0
91 0 0

1 vertices of polyhedron:
0 0 91
```

Strange, why is  $(0, 0, 1)$ , representing the origin in  $\mathbb{R}^2$ , not listed as a vertex as well? Well the vertex shown represents an extreme ray in the lattice  $\mathbb{E}$ , and  $(0, 0, 1)$  does not belong to  $\mathbb{E}$ .

```
2 extreme rays of recession cone:
```



```

0 91 0
91 0 0

3 support hyperplanes of polyhedron (homogenized)
0 0 1
0 1 0
1 0 0

1 congruences:
58 32 1 91

```

Normaliz has simplified the system of congruences to a single one.

```

3 basis elements of lattice:
1 0 33
0 1 -32
0 0 91

```

Again, don't forget that Normaliz prints a basis of the efficient lattice  $\mathbb{E}$ .

### 2.12.1. Lattice and offset

The set of solutions to the inhomogeneous system is an affine lattice in  $\mathbb{R}^2$ . The lattice basis of  $\mathbb{E}$  above does not immediately let us write down the set of solutions in the form  $w + L_0$  with a subgroup  $L_0$ , but we can easily transform the basis of  $\mathbb{E}$ : just add the first and the second vector to obtain  $(1, 1, 1)$  – we already know that it belongs to  $\mathbb{E}$  and any element in  $\mathbb{E}$  with last coordinate 1 would do. Try the file `InhomCongLat.in`:

```

amb_space 2
offset
1 1
lattice 2
32 33
91 91

```

### 2.12.2. Variation of the signs

Suppose we want to solve the system of congruences under the condition that both variables are negative (`InhomCongSigns.in`):

```

amb_space 2
inhom_congruences 2
1 2 -3 7
2 2 -4 13
signs
-1 -1

```

The two entries of the sign vector impose the sign conditions  $x_1 \leq 0$  and  $x_2 \leq 0$ .

From the output we see that the module generators are more complicated now:

```
4 module generators:
-11  0 1
-4   -7 1
-2 -22 1
 0 -37 1
```

The Hilbert basis of the recession monoid is simply that of the nonnegative case multiplied by  $-1$ .

## 2.13. Integral closure and Rees algebra of a monomial ideal

Next, let us discuss the example `MonIdeal.in` (typeset in two columns):

```
amb_space 5
rees_algebra 9
1 2 1 2          1 0 3 4
3 1 1 3          5 1 0 1
2 5 1 0          2 4 1 5
0 2 4 3          2 2 2 4
0 2 3 4
```

The input vectors are the exponent vectors of a monomial ideal  $I$  in the ring  $K[X_1, X_2, X_3, X_4]$ . We want to compute the normalization of the Rees algebra of the ideal. In particular we can extract from it the integral closure of the ideal. Since we must introduce an extra variable  $T$ , we have `amb_space 5`.

In the Hilbert basis we see the exponent vectors of the  $X_i$ , namely the unit vectors with last component 0. The vectors with last component 1 represent the integral closure  $\bar{I}$  of the ideal. There is a vector with last component 2, showing that the integral closure of  $I^2$  is larger than  $\bar{I}^2$ .

```
16 Hilbert basis elements:
0 0 0 1 0
...
5 1 0 1 1
6 5 2 2 2

11 generators of integral closure of the ideal:
0 2 3 4
...
5 1 0 1
```

The output of the generators of  $\bar{I}$  is the only place where we suppress the homogenizing variable for “historic” reasons. If we extract the vectors with last component 1 from the extreme rays, then we obtain the smallest monomial ideal that has the same integral closure as  $I$ .

```

10 extreme rays:
0 0 0 1 0
...
5 1 0 1 1

```

The support hyperplanes which are not just sign conditions describe primary decompositions of all the ideals  $\overline{I}^k$  by valuation ideals. It is not hard to see that none of them can be omitted for large  $k$  (for example, see: W. Bruns and G. Restuccia, Canonical modules of Rees algebras. J. Pure Appl. Algebra 201, 189–203 (2005)).

```

23 support hyperplanes:
0 0 0 0 1
0 ...
6 0 1 3 -13

```

### 2.13.1. Only the integral closure

If only the integral closure of the ideal is to be computed, one can choose the input as follows (IntClMonId.in):

```

amb_space 4
vertices 9
1 2 1 2 1
...
2 2 2 4 1
cone 4
1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1

```

The generators of the integral closure appear as module generators in the output and the generators of the smallest monomial ideal with this integral closure are the vertices of the polyhedron.

## 2.14. Convex hull computation/vertex enumeration

Normaliz computes convex hulls as should be very clear by now, and the only purpose of this section is to emphasize that Normaliz can be restricted to this task by setting an explicit computation goal. By convex hull computation we mean the determination of the support hyperplanes of a polyhedron is given by generators (or vertices). The converse operation is vertex enumeration. Both amount to the dualization of a cone, and can therefore be done by the same algorithm.

As an example we take the input file `cyclicpolytope30-15.in`, the cyclic polytope of dimension 15 with 30 vertices (suggested by D. Avis and Ch. Jordan):

```
/* cyclic polytope of dimension 15 with 30 vertices */
amb_space 16
polytope 30
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
2 4 8 16 32 64 128 256 512 1024 2048 4096 8192 16384 32768
...
30 900 27000 810000 ... 478296900000000000000000 143489070000000000000000
SupportHyperplanes
```

Already the entries of the vertices show that the computation cannot be done in 64 bit arithmetic. But you need not be worried. Just start Normaliz as usual. It will simply switch to infinite precision by itself, as shown by the terminal output (use the option `-c` or `--Verbose`).

```

                                     \.....|
Normaliz 3.2.0                      \....|
                                     \...|
(C) The Normaliz Team, University of Osnabrueck \..|
January 2017                                \.|
                                             \|
*****
Compute: SupportHyperplanes
Could not convert 15181127029874798299.
Arithmetic Overflow detected, try a bigger integer type!
Restarting with a bigger type.
*****
starting primal algorithm (only support hyperplanes) ...
Generators sorted lexicographically
Start simplex 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
gen=17, 72 hyp
gen=18, 240 hyp
gen=19, 660 hyp
gen=20, 1584 hyp
gen=21, 3432 hyp
gen=22, 6864 hyp
gen=23, 12870 hyp
gen=24, 22880 hyp
gen=25, 38896 hyp
gen=26, 63648 hyp
gen=27, 100776 hyp
gen=28, 155040 hyp
gen=29, 232560 hyp
gen=30, 341088 hyp
Pointed since graded
Select extreme rays via comparison ... done.
```

```
-----  
transforming data... done.
```

Have a look at the output file if you are not afraid of 341088 linear forms.

If you have looked closely at the terminal output above, you should have stumbled on the lines

```
Could not convert 15181127029874798299.  
Arithmetic Overflow detected, try a bigger integer type!
```

They show that Normaliz has tried the computation in 64 bit integers, but encountered a number that is too large for this precision. It has automatically switched to infinite precision. (See Section 4.4 for more information on integer types.)

## 2.15. The integer hull

The integer hull of a polyhedron  $P$  is the convex hull of the set of lattice points in  $P$  (despite of its name, it usually does not contain  $P$ ). Normaliz computes by first finding the lattice points and then computing the convex hull. The computation of the integer hull is requested by the computation goal `IntegerHull`.

The computation is somewhat special since it creates a second cone (and lattice)  $C_{\text{int}}$ . In homogeneous computations the degree 1 vectors generate  $C_{\text{int}}$  by an input matrix of type `cone_and_lattice`. In inhomogeneous computations the module generators and the Hilbert basis of the recession cone are combined and generate  $C_{\text{int}}$ . Therefore the recession cone is reproduced, even if the polyhedron should not contain a lattice point.

The integer hull computation itself is always inhomogeneous. The output file for  $C_{\text{int}}$  is `<project>.IntHull.out`.

As a very simple example we take `rationalIH.in` (`rational.in` augmented by `IntegerHull`):

```
amb_space 3  
cone 3  
1 1 2  
-1 -1 3  
1 -2 4  
grading  
unit_vector 3  
HilbertSeries  
IntegerHull
```

It is our rational polytope from Section 2.5. We know already that the origin is the only lattice point it contains. Nevertheless let us have a look at `rationalIH.IntHull.out`:

```
1 vertices of polyhedron  
0 extreme rays of recession cone  
1 support hyperplanes of polyhedron (homogenized)
```

```

embedding dimension = 3
affine dimension of the polyhedron = 0
rank of recession monoid = 0
internal index = 1

*****

1 vertices of polyhedron:
0 0 1

0 extreme rays of recession cone:

1 support hyperplanes of polyhedron (homogenized):
0 0 1

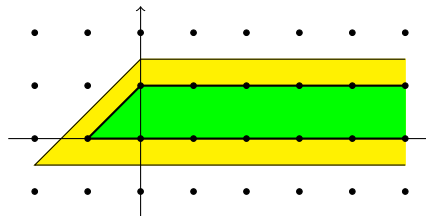
2 equations:
1 0 0
0 1 0

1 basis elements of lattice:
0 0 1

```

Since the lattice points in  $P$  are already known, the goal was to compute the constraints defining the integer hull. Note that all the constraints defining the integer hull can be different from those defining  $P$ . In this case the integer hull is cut out by the 2 equations.

As a second example we take the polyhedron of Section 2.9. The integer hull is the "green" polyhedron:



The input is `InhomIneqIH.in` (`InhomIneq.in` augmented by `IntegerHull`). The data of the integer hull are found in `InhomIneqIH.IntHull.out`:

```

...
2 vertices of polyhedron:
-1 0 1
0 1 1

1 extreme rays of recession cone:
1 0 0

```

```

3 support hyperplanes of polyhedron (homogenized):
0 -1 1
0 1 0
1 -1 1

```

## 2.16. Starting from a binomial ideal

As an example, we consider the binomial ideal generated by

$$X_1^2 X_2 - X_4 X_5 X_6, \quad X_1 X_4^2 - X_3 X_5 X_6, \quad X_1 X_2 X_3 - X_5^2 X_6.$$

We want to find an embedding of the toric ring it defines and the normalization of the toric ring. The input vectors are obtained as the differences of the two exponent vectors in the binomials. So the input ideal `lattice_ideal.in` is

```

amb_space 6
lattice_ideal 3
2 1 0 -1 -1 -1
1 0 -1 2 -1 -1
1 1 1 0 -2 -1

```

In order to avoid special input rules for this case in which our object is not defined as a subset of an ambient space, but as a quotient of type *generators/relations*, we abuse the name `amb_space`: it determines the space in which the input vectors live.

We get the output

```

6 original generators of the toric ring

```

namely the residue classes of the indeterminates.

```

9 Hilbert basis elements
9 Hilbert basis elements of degree 1

```

So the toric ring defined by the binomials is not normal. Normaliz found the standard grading on the toric ring. The normalization is generated in degree 1, too (in this case).

```

5 extreme rays
5 support hyperplanes

embedding dimension = 3
rank = 3 (maximal)
external index = 1
internal index = 1
original monoid is not integrally closed

```

We saw that already.

```

size of triangulation    = 5
resulting sum of |det|s = 10

grading:
-2 1 1

```

This is the grading on the ambient space (or polynomial ring) defining the standard grading on our subalgebra. The enumerative data that follow are those of the normalization!

```

degrees of extreme rays:
1: 5

Hilbert basis elements are of degree 1

multiplicity = 10

Hilbert series:
1 6 3
denominator with 3 factors:
1: 3

degree of Hilbert Series as rational function = -1

Hilbert polynomial:
1 3 5
with common denominator = 1

rank of class group = 2
class group is free

*****

6 original generators:
0 0 1
3 5 2
0 1 0
1 2 1
1 3 0
1 0 3

```

This is an embedding of the toric ring defined by the binomials. There are many choices, and Normaliz has taken one of them. You should check that the generators in this order satisfy the binomial equations. Turning to the ring theoretic interpretation, we can say that the toric ring defined by the binomial equations can be embedded into  $K[Y_1, Y_2, Y_3]$  as a monomial subalgebra that is generated by  $Y_1^0 Y_2^0 Y_3^1, \dots, Y_1^1 Y_2^0 Y_3^3$ .



Now the generators of the normalization:

9 Hilbert basis elements of degree 1:	5 extreme rays:
0 0 1	0 0 1
0 1 0	0 1 0
1 0 3	1 0 3
1 1 2	1 3 0
1 2 1	3 5 2
1 3 0	
2 3 2	5 support hyperplanes:
2 4 1	-15 7 5
3 5 2	-3 1 2
	0 0 1
	0 1 0
	1 0 0
0 further Hilbert basis elements of higher degree:	

### 3. The input file

The input file <project>.in consists of one or several items. There are several types of items:

- (1) definition of the ambient space,
- (2) matrices with integer entries,
- (3) vectors with integer entries,
- (4) constraints in or symbolic format,
- (5) computation goals and algorithmic variants,
- (6) comments.

An item cannot include another item. In particular, comments can only be included between other items, but not within another item. Matrices and vectors can have two different formats, plain and formatted.

Matrices and vectors are classified by the following attributes:

- (1) generators, constraints, accessory,
- (2) cone/polyhedron, (affine) lattice,
- (3) homogeneous, inhomogeneous.

In this classification, equations are considered as constraints on the lattice because Normaliz treats them as such – for good reason: it is very easy to intersect a lattice with a hyperplane.

The line structure is irrelevant for the interpretation of the input, but it is advisable to use it for the readability of the input file.

The input syntax of Normaliz 2 can still be used. It is explained in Appendix C.

## 3.1. Input items

### 3.1.1. The ambient space and lattice

The ambient space is specified as follows:

```
amb_space <d>
```

where  $\langle d \rangle$  stands for the dimension  $d$  of the ambient vector space  $\mathbb{R}^d$  in which the geometric objects live. The *ambient lattice*  $\mathbb{A}$  is set to  $\mathbb{Z}^d$ .

Alternatively one can define the ambient space implicitly by

```
amb_space auto
```

In this case the dimension of the ambient space is determined by Normaliz from the first formatted vector or matrix in the input file. It is clear that any input item that requires the knowledge of the dimension can only follow after the first formatted vector or matrix.

*In the following the letter  $d$  will always denote the dimension set with `amb_space`.*

An example:

```
amb_space 5
```

indicates that polyhedra and lattices are subobjects of  $\mathbb{R}^5$ . The ambient lattice is  $\mathbb{Z}^5$ .

*The first non-comment input item must specify the ambient space.*

### 3.1.2. Plain vectors

A plain vector is built as follows:

```
<T>  
<x>
```

Again  $\langle T \rangle$  denotes the type and  $\langle x \rangle$  is the vector itself. The number of components is determined by the type of the vector and the dimension of the ambient space. At present, all vectors have length  $d$ .

Example:

```
grading  
1 0 0
```

Normaliz allows also the input of sparse vectors. Sparse input is signaled by the key word `sparse` as the first entry. It is followed by entries of type  $\langle \text{col} \rangle : \langle \text{val} \rangle$  where  $\langle \text{pos} \rangle$  denotes the column and  $\langle \text{val} \rangle$  the value in that column. (The unspecified columns have entry 0.) A sparse vector is terminated by the character `;`.

Example:

```
grading
```

```
sparse 1:1;
```

For certain vectors there also exist shortcuts. Examples:

```
total_degree  
unit_vector 25
```

### 3.1.3. Formatted vectors

A formatted vector is built as follows:

```
<T>  
[ <x> ]
```

where  $\langle T \rangle$  denotes the type and  $\langle x \rangle$  is the vector itself. The components can be separated by white space, commas or semicolons. An example showing all possibilities (not recommended):

```
grading  
[1,0; 0 5]
```

### 3.1.4. Plain matrices

A plain matrix is built as follows:

```
<T> <m>  
<x_1>  
...  
<x_m>
```

Here  $\langle T \rangle$  denotes the type of the matrix,  $\langle m \rangle$  the number of rows, and  $\langle x_1 \rangle, \dots, \langle x_m \rangle$  are the rows. The number of columns is implicitly defined by the dimension of the ambient space and the type of the matrix. Example (with `amb_space 3`):

```
cone 3  
1 2 3  
4 5 6  
11 12 13
```

Normaliz allows the input of matrices in transposed form:

```
<T> transpose <n>  
<x_1>  
...  
<x_m>
```

Note that  $\langle n \rangle$  is now the number of *columns* of the matrix that follows it (assumed to be the number of input vectors). The number of rows is determined by the dimension of the ambient

space and the type of the matrix. Example:

```
cone transpose 3
1 0 3
0 1 4
```

is equivalent to

```
cone 3
1 0
0 1
3 4
```

Like evectors, matrices have a sparse input variant, again signaled by the key word `sparse`. The rows are sparse vectors with entries `<col>:<val>`, and each row is concluded by the character `;`.

Example:

```
inequalities 3 sparse
1:1;
2:1;
3:1;
```

chooses the  $3 \times 3$  unit matrix as a matrix of type `inequalities`. Note that also in case of transposed matrices, sparse entry is row by row.

*Matrices may have zero rows.* Such empty matrices like

```
inhom_inequalities 0
```

can be used to make the input inhomogeneous (Section 3.1.11) or to avoid the automatic choice of the positive orthant in certain cases (Section 3.1.12). (The empty `inhom_inequalities` hve both effects simultaneously.) Apart from these effects, empty matrices have no influence on the computation.

### 3.1.5. Formatted matrices

A formatted matrix is built as follows:

```
<T>
[ [<x_1>]
...
[<x_m>] ]
```

Here `<T>` denotes the type of the matrix and `<x_1>`, ..., `<x_m>` are vectors. Legal separators are white space, commas and semicolons. An example showing all possibilities (not really recommended):

```
cone [
[ 2 1][3 4];
```

```
[0 1],
[9 10] [11 12]
]
```

Similarly as plain matrices, formatted matrices can be given in transposed form, and they can be empty.

### 3.1.6. Constraints in tabular format

This input type is somewhat closer to standard notation than the encoding of constraints in matrices. The general type of equations and inequalities is

```
<x> <rel> <int>;
```

where <x> denotes a vector of length  $d$ , <rel> is one of the relations =, <=, >=, <, > and <int> is an integer.

Congruences have the form

```
<x> ~ <int> (<mod>);
```

where <mod> is a nonzero integer.

Examples:

```
1 -2 >= 5
1 -1 = 0
-1 1 ~ 7 (9)
```

Note: all numbers and relation signs must be separated by white space.

### 3.1.7. Constraints in symbolic format

This input type is even closer to standard notation than the encoding of constraints in matrices or in tabular format. It is especially useful if the constraints are sparse. Instead of assigning a value to a coordinate via its position in a vector, it uses coordinates named  $x[\langle n \rangle]$  where  $\langle n \rangle$  is the index of the coordinate. The index is counted from 1.

The general type of equations and inequalities is

```
<lhs> <rel> <rhs>;
```

where <lhs> and <rhs> denote linear function of the  $x[\langle n \rangle]$  with integer coefficients. As above, <rel> is one of the relations =, <=, >=, <, >. (An empty <lhs> or <rhs> has the value 0.) Note the terminating semicolon.

Congruences have the form

```
<lhs> ~ <rhs> (<mod>);
```

where <mod> is a nonzero integer.

Examples:

```
x[1] >= 2x[2] + 5;  
x[1]+1=x[2] ;  
-x[1] + x[2] ~ 7 (9);
```

There is no need to insert white space for separation, but it may be inserted anywhere where it does not disrupt numbers or relation signs.

### 3.1.8. Computation goals and algorithmic variants

These are single or compound words, such as

```
HilbertBasis  
Multiplicity
```

The file can contain several computation goals, as in this example.

### 3.1.9. Comments

A comment has the form

```
/* <text> */
```

where <text> stands for the text of the comment. It can have arbitrary length and stretch over several lines. Example:

```
/* This is a comment  
*/
```

Comments are only allowed at places where also a new keyword would be allowed, especially not between the entries of a matrix or a vector. Comments can not be nested.

### 3.1.10. Restrictions

Input items can almost freely be combined, but there are some restrictions:

- (1) Every input type can appear only once.
- (2) The types  
                    cone, cone\_and\_lattice, polytope, rees\_algebra  
exclude each other mutually.
- (3) The input type subspace excludes polytope and rees\_algebra.
- (4) The types  
                    lattice, saturation, cone\_and\_lattice  
exclude each other mutually.
- (5) polytope can not be combined with grading.
- (6) The only type that can be combined with lattice\_ideal is grading.

- (7) The following types cannot be combined with inhomogeneous types or dehomogenization:  
polytope, rees\_algebra, excluded\_faces
- (8) The following types cannot be combined with inhomogeneous types:  
dehomogenization, support\_hyperplanes

### 3.1.11. Homogeneous and inhomogeneous input

Apart from the restrictions listed in the previous section, homogeneous and inhomogeneous types can be combined as well as generators and constraints. A single inhomogeneous type or dehomogenization in the input triggers an inhomogeneous computation. The input item of inhomogeneous type may be an empty matrix.

### 3.1.12. Default values

If there is no lattice defining item, Normaliz (virtually) inserts the unit matrix as an input item of type `lattice`. If there is no cone defining item, the unit matrix is (additionally) inserted as an input item of type `cone`.

If the input is inhomogeneous, then Normaliz provides default values for vertices and the offset as follows:

- (1) If there is an input matrix of lattice type `lattice`, but no offset, then the offset 0 is inserted.
- (2) If there is an input matrix of type `cone`, but no vertices, then the vertex 0 is inserted.

**An important point.** If the input does not contain any cone generators or inequalities, Normaliz automatically assumes that you want to compute in the positive orthant. In order to avoid this choice you can add an empty matrix of inequalities. This will not affect the results, but avoid the sign restriction.

### 3.1.13. Normaliz takes intersections (almost always)

The input may contain several cone defining items and several lattice defining items.

The sublattice  $L$  defined by the lattice input items is the *intersection* of the sublattices defined by the single items. The polyhedron  $P$  is defined as the intersection of all polyhedra defined by the single polyhedron defining items. The object then computed by Normaliz is

$$P \cap L.$$

There are three notable exceptions to the rule that Normaliz takes intersections:

- (1) vertices and cone form a unit. Together they define a polyhedron.
- (2) The same applies to offset and lattice that together define an affine lattice.
- (3) The subspace is added to cone or cone\_and\_lattice.

## 3.2. Homogeneous generators

### 3.2.1. Cones

The main type is `cone`. The other two types are added for special computations.

**cone** is a matrix with  $d$  columns. Every row represents a vector, and they define the cone generated by them. Section 2.3, `2cone.in`

**subspace** is a matrix with  $d$  columns. The linear subspace generated by the rows is added to the cone. Section 6.7.4.

**polytope** is a matrix with  $d - 1$  columns. It is internally converted to cone extending each row by an entry 1. Section 2.4, `polytope.in`

**rees\_algebra** is a matrix with  $d - 1$  columns. It is internally converted to type `cone` in two steps: (i) each row is extended by an entry 1 to length  $d$ . (ii) The first  $d - 1$  unit vectors of length  $d$  are appended. Section 2.13, `MonIdeal.in`.

Moreover, it is possible to define a cone and a lattice by the same matrix:

**cone\_and\_lattice** The vectors of the matrix with  $d$  columns define both a cone and a lattice. Section 2.11, `A443.in`.

If `subspace` is used in combination with `cone_and_lattice`, then the sublattice generated by its rows is added to the lattice generated by `cone_and_lattice`.

The `Normaliz 2` types `integral_closure` and `normalization` can still be used. They are synonyms for `cone` and `cone_and_lattice`, respectively.

### 3.2.2. Lattices

There are 3 types:

**lattice** is a matrix with  $d$  columns. Every row represents a vector, and they define the lattice generated by them. Section 2.6.2, `3x3magiceven_lat.in`

**saturation** is a matrix with  $d$  columns. Every row represents a vector, and they define the *saturation* of the lattice generated by them. Section 2.6.2, `3x3magic_sat.in`.

**cone\_and\_lattice** See Section 3.2.1.

## 3.3. Homogeneous Constraints

### 3.3.1. Cones

**inequalities** is a matrix with  $d$  columns. Every row  $(\xi_1, \dots, \xi_d)$  represents a homogeneous inequality

$$\xi_1 x_1 + \dots + \xi_d x_d \geq 0, \quad \xi_i \in \mathbb{Z},$$

for the vectors  $(x_1, \dots, x_d) \in \mathbb{R}^d$ . Sections 2.3.2, 2.5.1, `2cone_ineq.in`, `poly_ineq.in`

**signs** is a vector with  $d$  entries in  $\{-1, 0, 1\}$ . It stands for a matrix of type `inequalities` composed of the sign inequalities  $x_i \geq 0$  for the entry 1 at the  $i$ -th component and the



inequality  $x_i \leq 0$  for the entry  $-1$ . The entry  $0$  does not impose an inequality. See 2.12.2, `InhomCongSigns.in`.

**nonnegative** It stands for a vector of type `sign` with all entries equal to  $1$ . See Section 2.10, `Condorcet.in`.

**excluded\_faces** is a matrix with  $d$  columns. Every row  $(\xi_1, \dots, \xi_d)$  represents an inequality

$$\xi_1 x_1 + \dots + \xi_d x_d > 0, \quad \xi_i \in \mathbb{Z},$$

for the vectors  $(x_1, \dots, x_d) \in \mathbb{R}^d$ . It is considered as a homogeneous input type though it defines inhomogeneous inequalities. The faces of the cone excluded by the inequalities are excluded from the Hilbert series computation, but `excluded_faces` behaves like inequalities in every other respect. Section 2.10.1, `CondorcetSemi.in`.

**support\_hyperplanes** is a matrix with  $d$  columns. It requires homogeneous input. It is the input type for precomputed support hyperplanes. Therefore `Normaliz` checks if all input generators satisfy the inequalities defined by them. Apart from this extra check, it behaves like inequalities. Note that it overrides all other inequalities in the input, but `excluded_faces` still exclude the faces defined by them. Section 6.11, `2cone_supp.in`.

### 3.3.2. Lattices

**equations** is a matrix with  $d$  columns. Every row  $(\xi_1, \dots, \xi_d)$  represents an equation

$$\xi_1 x_1 + \dots + \xi_d x_d = 0, \quad \xi_i \in \mathbb{Z},$$

for the vectors  $(x_1, \dots, x_d) \in \mathbb{R}^d$ . Section 2.6, `3x3magic.in`

**congruences** is a matrix with  $d + 1$  columns. Each row  $(\xi_1, \dots, \xi_d, c)$  represents a congruence

$$\xi_1 z_1 + \dots + \xi_d z_d \equiv 0 \pmod{c}, \quad \xi_i, c \in \mathbb{Z},$$

for the elements  $(z_1, \dots, z_d) \in \mathbb{Z}^d$ . Section 2.6.1, `3x3magiceven.in`.

## 3.4. Inhomogeneous generators

### 3.4.1. Polyhedra

**vertices** is a matrix with  $d + 1$  columns. Each row  $(p_1, \dots, p_d, q)$ ,  $q > 0$ , specifies a generator of a polyhedron (not necessarily a vertex), namely

$$v_i = \left( \frac{p_1}{q}, \dots, \frac{p_n}{q} \right), \quad p_i \in \mathbb{Z}, q \in \mathbb{Z}_{>0},$$

Section 2.9.1, `InhomIneq_gen.in`

**Note:** `vertices` and `cone` together define a polyhedron. If `vertices` is present in the input, then the default choice for `cone` is the empty matrix.

The `Normaliz 2` input type `polyhedron` can still be used.

### 3.4.2. Lattices

**offset** is a vector with  $d$  entries. It defines the origin of the affine lattice. Section 2.12.1, `InhomCongLat.in`.

**Note:** `offset` and `lattice` (or `saturation`) together define an affine lattice. If `offset` is present in the input, then the default choice for `lattice` is the empty matrix.

## 3.5. Inhomogeneous constraints

### 3.5.1. Cones

**inhom\_inequalities** is a matrix with  $d + 1$  columns. We consider inequalities

$$\xi_1 x_1 + \cdots + \xi_d x_d \geq \eta, \quad \xi_i, \eta \in \mathbb{Z},$$

rewritten as

$$\xi_1 x_1 + \cdots + \xi_d x_d + (-\eta) \geq 0$$

and then represented by the input vectors

$$(\xi_1, \dots, \xi_d, -\eta).$$

Section 2.9, `InhomIneq.in`.

**strict\_inequalities** is a matrix with  $d$  columns. We consider inequalities

$$\xi_1 x_1 + \cdots + \xi_d x_d \geq 1, \quad \xi_i \in \mathbb{Z},$$

represented by the input vectors

$$(\xi_1, \dots, \xi_d).$$

Section 2.3.3, `2cone_int.in`.

**strict\_signs** is a vector with  $d$  components in  $\{-1, 0, 1\}$ . It is the "strict" counterpart to `signs`. An entry 1 in component  $i$  represents the inequality  $x_i > 0$ , an entry  $-1$  the opposite inequality, whereas 0 imposes no condition on  $x_i$ . 2.10.2, `Condorcet_one.in`

### 3.5.2. Lattices

**inhom\_equations** is a matrix with  $d + 1$  columns. We consider equations

$$\xi_1 x_1 + \cdots + \xi_d x_d = \eta, \quad \xi_i, \eta \in \mathbb{Z},$$

rewritten as

$$\xi_1 x_1 + \cdots + \xi_d x_d + (-\eta) = 0$$

and then represented by the input vectors

$$(\xi_1, \dots, \xi_d, -\eta).$$

See 2.7 `NumSemi.in`.

**inhom\_congruences** We consider a matrix with  $d+2$  columns. Each the row  $(\xi_1, \dots, \xi_d, -\eta, c)$  represents a congruence

$$\xi_1 z_1 + \dots + \xi_d z_d \equiv \eta \pmod{c}, \quad \xi_i, \eta, c \in \mathbb{Z},$$

for the elements  $(z_1, \dots, z_d) \in \mathbb{Z}^d$ . Section 2.12, `InhomCongSigns.in`.

### 3.6. Tabular constraints

**constraints** allows the input of equations, inequalities and congruences in a format that is close to standard notation. As for matrix types the keyword `constraints` is followed by the number of constraints. The syntax of tabular constraints has been described in Section 3.2.1. If  $(\xi_1, \dots, \xi_d)$  is the vector on the left hand side and  $\eta$  the integer on the right hand side, then the constraint defines the set of vectors  $(x_1, \dots, x_d)$  such that the relation

$$\xi_1 x_1 + \dots + \xi_d x_d \text{ rel } \eta$$

is satisfied, where `rel` can take the values  $=, \leq, \geq, <, >$  with the represented by input strings `=, <=, >=, <, >`, respectively.

The input string `~` represents a congruence  $\equiv$  and requires the additional input of a modulus. It represents the congruence

$$\xi_1 x_1 + \dots + \xi_d x_d \equiv \eta \pmod{c}.$$

Sections 2.3.3, `2cone_int.in`, 2.6.1, `3x3magiceven.in`, 2.9, `InhomIneq.in`.

A right hand side  $\neq 0$  makes the input inhomogeneous, as well as the relations  $<$  and  $>$ . Strict inequalities are always understood as conditions for integers. So

$$\xi_1 x_1 + \dots + \xi_d x_d < \eta$$

is interpreted as

$$\xi_1 x_1 + \dots + \xi_d x_d \leq \eta - 1,$$

#### 3.6.1. Forced homogeneity

It is often more natural to write constraints in inhomogeneous form, even when one wants the computation to be homogeneous. The type `constraints` does not allow this. Therefore we have introduced

**hom\_constraints** for the input of equations, non-strict inequalities and congruences in the same format as `constraints`, except that these constraints are meant to be for a homogeneous computation. It is clear that the left hand side has only  $d-1$  entries now. See Section 2.5.1, `poly_hom_const.in`.

### 3.7. Symbolic constraints

The input syntax is

**constraints** <n> **symbolic** where <n> is the number of constraints in symbolic form that follow.

The constraints have the form described in Section 3.1.7. Note that every symbolic constraint (including the last) must be terminated by a semicolon.

Swew 2.7, NumSemi.in, 2.12, InhomCong.in.

The interpretation of homogeneity follows the same rules as for tabular constraints. The variant `hom_constraints` is allowed and works as for tabular constraints.

### 3.8. Relations

Relations do not select a sublattice of  $\mathbb{Z}^d$  or a subcone of  $\mathbb{R}^d$ , but define a monoid as a quotient of  $\mathbb{Z}_+^d$  modulo a system of congruences (in the semigroup sense!).

The rows of the input matrix of this type are interpreted as generators of a subgroup  $U \subset \mathbb{Z}^d$ , and `Normaliz` computes an affine monoid and its normalization as explained in Section A.5.

Set  $G = \mathbb{Z}^d / U$  and  $L = G / \text{torsion}(G)$ . Then the ambient lattice is  $\mathbb{A} = \mathbb{Z}^r$ ,  $r = \text{rank } L$ , and the efficient lattice is  $L$ , realized as a sublattice of  $\mathbb{A}$ . `Normaliz` computes the image of  $\mathbb{Z}_+^d$  in  $L$  and its normalization.

**lattice\_ideal** is a matrix with  $d$  columns containing the generators of the subgroup  $U$ .  
Section 2.16, `lattice_ideal.in`.

The type `lattice_ideal` cannot be combined with any other input type (except `grading`)—such a combination would not make sense. (See Section 3.10.1 for the use of a `grading` in this case.)

### 3.9. Unit vectors

A `grading` or a `dehomogenization` is often given by a unit vector:

**unit\_vector** <n> represents the  $n$ th unit vector in  $\mathbb{R}^d$  where  $n$  is the number given by <n>.

This shortcut cannot be used as a row of a matrix. It can be used whenever a single vector is asked for, namely after `grading`, `dehomogenization`, `signs` and `strict_signs`. See Section 2.5, `rational.in`

### 3.10. Grading

This type is accessory. A  $\mathbb{Z}$ -valued `grading` can be specified in two ways:

- (1) *explicitly* by including a `grading` in the input, or

- (2) *implicitly*. In this case Normaliz checks whether the extreme integral generators of the monoid lie in an (affine) hyperplane  $A$  given by an equation  $\lambda(x) = 1$  with a  $\mathbb{Z}$ -linear form  $\lambda$ . If so, then  $\lambda$  is used as the grading.

*Implicit gradings are only possible for homogeneous computations.*

Explicit definition of a grading:

**grading** is a vector of length  $d$  representing the linear form that gives the grading. Section 2.5, `rational.in`.

**total\_degree** represents a vector of length  $d$  with all entries equal to 1. Section 2.10, `Condorcet.in`.

Before Normaliz can apply the degree, it must be restricted to the effective lattice  $\mathbb{E}$ . Even if the entries of the grading vector are coprime, it often happens that all degrees of vectors in  $\mathbb{E}$  are divisible by a greatest common divisor  $g > 1$ . Then  $g$  is extracted from the degrees, and it will appear as denominator in the output file.

Normaliz checks whether all generators of the (recession) monoid have positive degree (after passage to the quotient modulo the unit group in the nonpointed case). Vertices of polyhedra may have degrees  $\leq 0$ .

### 3.10.1. `lattice_ideal`

In this case the unit vectors correspond to generators of the monoid. Therefore the degrees assigned to them must be positive. Moreover, the vectors in the input represent binomial relations, and these must be homogeneous. In other words, both monomials in a binomial must have the same degree. This amounts to the condition that the input vectors have degree 0. Normaliz checks this condition.

## 3.11. Dehomogenization

Like `grading` this is an accessory type.

Inhomogeneous input for objects in  $\mathbb{R}^d$  is homogenized by an additional coordinate and then computed in  $\mathbb{R}^{d+1}$ , but with the additional condition  $x_{d+1} \geq 0$ , and then dehomogenizing all results: the substitution  $x_{d+1} = 1$  acts as the *dehomogenization*, and the inhomogeneous input types implicitly choose this dehomogenization.

Like the grading, one can define the dehomogenization explicitly:

**dehomogenization** is a vector of length  $d$  representing the linear form  $\delta$ .

The dehomogenization can be any linear form  $\delta$  satisfying the condition  $\delta(x) \geq 0$  on the cone that is truncated. (In combination with constraints, the condition  $\delta(x) \geq 0$  is automatically satisfied since  $\delta$  is added to the constraints.)

The input type `dehomogenization` can only be combined with homogeneous input types, but makes the computation inhomogeneous, resulting in inhomogeneous output. The polyhedron computed is the intersection of the cone  $\mathbb{C}$  (and the lattice  $\mathbb{E}$ ) with the hyperplane given by

$\delta(x) = 1$ , and the recession cone is  $\mathbb{C} \cap \{x : \delta(x) = 0\}$ .

A potential application is the adaptation of other input formats to Normaliz. The output must then be interpreted accordingly.

Section 6.6, `dehomogenization.in`.

### 3.12. Pointedness

Since version 3.1 Normaliz can also compute nonpointed cones and polyhedra without vertices.

### 3.13. The zero cone

The zero cone with an empty Hilbert basis is a legitimate object for Normaliz. Nevertheless a warning message is issued if the zero cone is encountered.

### 3.14. Additional input file for NmzIntegrate

NmzIntegrate, whether called by Normaliz or from the command line, needs an input file `<project>.pnm` that contains the polynomial for which the generalized Ehrhart series or the integral is to be computed. See [11].

## 4. Computation goals and algorithmic variants

The library `libnormaliz` contains a class `ConeProperties` that collects computation goals, algorithmic variants and additional data that are used to control the work flow in `libnormaliz` as well as the communication with other programs. The latter are not important for the Normaliz user, but are listed as a reference for `libnormaliz`. See Appendix D for a description of `libnormaliz`.

All computation goals and algorithmic variants can be communicated to Normaliz in two ways:

- (1) in the input file, for example `HilbertBasis`,
- (2) via a verbatim command line option, for example `--HilbertBasis`.

For the most important choices there are single letter command line options, for example `-N` for `HilbertBasis`. The single letter options ensure backward compatibility to Normaliz 2. In `jNormaliz` they are also accessible via their full names.

Some computation goals apply only to homogeneous computations, and some others make sense only for inhomogeneous computations.

Some single letter command line options combine two or more computation goals, and some algorithmic variants imply computation goals.

Normaliz can call `NmzIntegrate`. The three computation goals that require `NmzIntegrate` do not belong to the class `ConeProperties`.

## 4.1. Default choices and basic rules

If several computation goals are set, all of them are pursued. In particular, computation goals in the input file and on the command line are accumulated. But

**--ignore, -i** on the command line switches off the computation goals and algorithmic variants set in the input file.

The default computation goal is set if neither the input file nor command line contains a computation goal or an algorithmic variant that implies a computation goal. It is

`HilbertBasis + HilbertSeries + ClassGroup`.

If set explicitly in the input file or on the command line the following adds these computation goals:

### **DefaultMode**

It is possible to set `DefaultMode` explicitly in addition to other computation goals. If it is set, implicitly or explicitly, Normaliz will not complain about unreachable computation goals.

Normaliz collects all computation goals and algorithmic variants set in the input file and on the command line.

## 4.2. The choice of algorithmic variants

For its main computation goals Normaliz has algorithmic variants. It tries to choose the variant that seems best for the given input data (except symmetrization). This automatic choice may however be a bad one. Therefore the user can completely control which algorithmic variant is used.

### 4.2.1. Primal vs. dual

For the computation of Hilbert bases Normaliz has two algorithms, the primal algorithm that is based on triangulations, and the dual algorithm that is of type “pair completion”. We have seen both in Section 2. Roughly speaking, the primal algorithm is the first choice for generator input, and the dual algorithm is usually better for constraint input. The choice also applies to the computation of degree 1 elements. However, for them there exists a third variant. See Section 4.2.2. The conditions under which the dual algorithm is chosen are specified there.

The choice of the algorithm can be fixed or blocked:

**DualMode, -d** activates the dual algorithm for the computation of the Hilbert basis and degree 1 elements. Includes `HilbertBasis`, unless `Deg1Elements` is set. It overrules `IsIntegrallyClosed`.

**PrimalMode**, **-P** blocks the use of the dual algorithm.

The automatic choice can of course fail. See Section 6.4 for an example for which it is bad.

#### 4.2.2. Lattice points by approximation

This is often the best choice for the computation of lattice points in a polytope. The polytope can be defined as the cross-section of a pointed cone with the hyperplane of degree 1 points (homogeneous input) or as a polyhedron (inhomogeneous input). The basic idea to escape the arithmetical complexity of rational vertices with large denominators by approximating a rational polytope by one with integral vertices. See Section 6.1 for further discussion and examples, and also for the conditions under which **Approximate** is chosen automatically by **Normaliz**.

You can force approximation or block it:

**Approximate**, **-r** activates the approximation algorithm for the computation of degree 1 elements. It implies **Deg1Elements**,

**NoApproximation** blocks approximation.

Approximation is also blocked by **DualMode**, but not by **PrimalMode**.

#### 4.2.3. Bottom decomposition

Bottom decomposition is a way to produce an optimal triangulation for a given set of generators. It is discussed in Section 6.2. The criterion for its automatic choice is explained there. It can be forced or blocked:

**BottomDecomposition**, **-b** tells **Normaliz** to use bottom decomposition in the primal algorithm.

**NoBottomDec**, **-o** forbids **Normaliz** to use bottom decomposition in the primal algorithm, even if it would otherwise be chosen because of large roughness (see Section 6.2).

An option to be mentioned in this context is

**KeepOrder**, **-k** forbids **Normaliz** to reorder the generators of the efficient cone  $\mathbb{C}$ . Only useful if original monoid generators are defined. Also blocks **BottomDecomposition**.

**KeepOrder** is only allowed if **OriginalMonoidGenerators** are defined. It is rarely a good idea to set **KeepOrder** (try it). It is primarily used internally when data must be computed in an auxiliary cone.

#### 4.2.4. Symmetrization

In rare cases **Normaliz** can delegate the computation of multiplicities or Hilbert series to **Nmz-Integrate**. If applicable, this is a very strong tool. We discuss it in Section 6.5. At present it will not be chosen automatically, but can be forced:



**Symmetrize**, **-Y** lets Normaliz compute the multiplicity and/or the Hilbert series via NmzIntegrate. (It does not imply a computation goal!)

**NoSymmetrization** blocks symmetrization. Allowed, but superfluous at present.

#### 4.2.5. Subdivision of simplicial cones

Under certain conditions Normaliz tries to speed up computations by subdividing large simplicial cones. This requires enlarging the set of generators and can lead to a nested triangulation (see Sections 6.3 and 6.8.1). The subdivision can be blocked by

**NoSubdivision**

### 4.3. Computation goals

The computation goal **Sublattice** does not imply any other computation goal. All other computation goals include **Sublattice** and **SupportHyperplanes**, apart from certain computation goals based on the dual algorithm; see Section 4.6.

#### 4.3.1. Lattice data

**Sublattice**, **-S** (upper case S) asks Normaliz to compute the coordinate transformation to and from the efficient sublattice.

#### 4.3.2. Support hyperplanes and extreme rays

**SupportHyperplanes**, **-s** triggers the computation of support hyperplanes and extreme rays. Normaliz tries to find a grading.

#### 4.3.3. Hilbert basis and related data

**HilbertBasis**, **-N** triggers the computation of the Hilbert basis. In inhomogeneous computations it asks for the Hilbert basis of the recession monoid *and* the module generators.

**Deg1Elements**, **-1** restricts the computation to the degree 1 elements of the Hilbert basis. Requires the presence of a grading. Forbidden in inhomogeneous computations.

**ModuleGeneratorsOverOriginalMonoid**, **-M** computes a minimal system of generators of the integral closure over the original monoid (see Section 6.10). Requires the existence of original monoid generators.

The boolean valued computation goal **IsIntegrallyClosed** is also related to the Hilbert basis; see Section 4.3.10.

#### 4.3.4. Enumerative data

The computation goals in this section require a grading. They include **SupportHyperplanes**.

**HilbertSeries, -q** triggers the computation of the Hilbert series.

**Multiplicity, -v** restricts the computation to the multiplicity. ]

**HSOP** lets Normaliz compute the degrees in a homogeneous system of parameters and the induced representation of the Hilbert series.

#### 4.3.5. Combined computation goals

Can only be set by single letter command line options:

- n HilbertBasis + Multiplicity
- h HilbertBasis + HilbertSeries
- p Deg1Elements + HilbertSeries

#### 4.3.6. The class group

**ClassGroup, -C** is self explanatory, includes SupportHyperplanes. Not allowed in inhomogeneous computations.

#### 4.3.7. Integer hull

**IntegerHull, -H** computes the integer hull of a polyhedron. Implies the computation of the lattice points in it.

More precisely: in homogeneous computations it implies Deg1Elements, in inhomogeneous computations it implies HilbertBasis. See Section 2.15.

#### 4.3.8. Triangulation and Stanley decomposition

**Triangulation, -T** makes Normaliz compute, store and export the full triangulation.

**ConeDecomposition, -D** Normaliz computes a disjoint decomposition of the cone by semiopen simplicial cones. Implies Triangulation.

**TriangulationSize, -t** makes Normaliz count the simplicial cones in the full triangulation.

**TriangulationDetSum** makes Normaliz additionally sum the absolute values of their determinants.

**StanleyDec, -y** makes Normaliz compute, store and export the Stanley decomposition. Only allowed in homogeneous computations.

The triangulation and the Stanley decomposition are treated separately since they can become very large and may exhaust memory if they must be stored for output.

#### 4.3.9. NmzIntegrate

The computation goals that require calling NmzIntegrate can only be set on the command line.

- E makes Normaliz compute a generalized Ehrhart series.

- L** makes Normaliz compute the leading coefficient of a generalized Ehrhart series.
- I** makes Normaliz compute an integral.

These computation goals require a homogeneous computation.

Don't confuse these options with symmetrization. The latter symmetrizes (if possible) the given data and calls `NmzIntegrate` on the symmetrized object. The options above ask for a call of `NmzIntegrate` on the given data which must include a polynomial. See the `NmzIntegrate` manual.

#### 4.3.10. Boolean valued computation goals

They tell Normaliz to find out the answers to the questions they ask. Two of them are more important than the others since they may influence the course of the computations:

**IsIntegrallyClosed, -w** : is the original monoid integrally closed? Normaliz stops the Hilbert basis computation as soon as it can decide whether the original monoid contains the Hilbert basis (see Section 2.11.1).

**IsPointed** : is the efficient cone  $\mathbb{C}$  pointed? This computation goal is sometimes useful to give Normaliz a hint that a nonpointed cone is to be expected. See Section 6.7.3.

The remaining ones:

**IsDeg1ExtremeRays** : do the extreme rays have degree 1?

**IsDeg1HilbertBasis** : do the Hilbert basis elements have degree 1?

**IsReesPrimary** : for the input type `rees_algebra`, is the monomial ideal primary to the irrelevant maximal ideal?

The last three computation goals are not really useful for Normaliz since they will be answered automatically. Note that they may trigger extensive computations.

### 4.4. Integer type

There is no need to worry about the integer type chosen by Normaliz. All preparatory computations use infinite precision. The main computation is then tried with 64 bit integers. If it fails, it will be restarted with infinite precision.

The amount of computations done with infinite precision is usually very small, but the transformation of the computation results from 64 bit integers to infinite precision may take some time. If you need the highest possible speed, you can suppress infinite precision completely by

**LongLong**

However, in this case Normaliz cannot restart a failed computation.

On the other hand, the 64 bit attempt can be bypassed by

**BigInt, -B**

Note that Normaliz tries to avoid overflows by intermediate results (even if `LongLong` is set).

If such overflow should happen, the computation is repeated locally with infinite precision. (The number of such GMP transitions is shown in the terminal output.) If a final result is too large, Normaliz must restart the computation globally.

LongLong is not a cone property.

## 4.5. Control of computations and communication with interfaces

In addition to the computation goals in Section 4.3, the following elements of ConeProperties control the work flow in libnormaliz and can be used by programs calling Normaliz to ensure the availability of the data that are controlled by them.

**Generators** controls the generators of the efficient cone.

**OriginalMonoidGenerators** controls the generators of the original monoid.

**ModuleGenerators** controls the module generators in inhomogeneous computation.

**ExtremeRays** controls the extreme rays.

**VerticesOfPolyhedron** controls the vertices of the polyhedron in the inhomogeneous case.

**MaximalSubspace** controls the maximal linear subspace of the (homogenized) cone.

**EmbeddingDim** controls the embedding dimension.

**Rank** controls the rank.

**RecessionRank** controls the rank of the recession monoid in inhomogeneous computations.

**AffineDim** controls the affine dimension of the polyhedron in inhomogeneous computations.

**ModuleRank** in inhomogeneous computations it controls the rank of the module of lattice points in the polyhedron as a module over the recession monoid.

**ExcludedFaces** controls the excluded faces.

**InclusionExclusionData** controls data derived from the excluded faces.

**Grading** controls the grading.

**GradingDenom** controls its denominator.

**Dehomogenization** controls the dehomogenization.

**ReesPrimaryMultiplicity** controls the multiplicity of a monomial ideal, provided it is primary to the maximal ideal generated by the indeterminates. Used only with the input type rees\_algebra.

**WitnessNotIntegrallyClosed** controls witness against integral closedness.

**Equations** controls the equations.

**Congruences** controls the congruences.

**ExternalIndex** controls the external index.

**InternalIndex** controls the internal index.

**UnitGroupIndex** controls the unit group index.

**IsInhomogeneous** controls the inhomogeneous case..

**HilbertQuasiPolynomial** controls the Hilbert quasipolynomial.

**IsTriangulationNested** controls the indicator of this property.

**IsTriangulationPartial** similar.

## 4.6. Rational and integer solutions in the inhomogeneous case

The integer solutions of a homogeneous diophantine system generate the rational solutions as well: every rational solution has a multiple that is an integer solution. Therefore the rational solutions do not need an extra computation. If you prefer geometric language: a rational cone is generated by its lattice points.

This is no longer true in the inhomogeneous case where the computation of the rational solutions is an extra task for Normaliz. This extra step is inevitable for the primal algorithm, but not for the dual algorithm. In general, the computation of the rational solutions is much faster than the computation of the integral solutions, but this by no means always the case.

Therefore we have decoupled the two computations if the dual algorithm is applied to inhomogeneous systems or to the computation of degree 1 points in the homogeneous case. The combinations

**DualMode HilbertBasis, -dN**

**DualMode Deg1Elements, -d1**

**DualMode ModuleGenerators**

do not imply the computation goal **SupportHyperplanes** (and not even **Sublattice**) which would trigger the computation of the rational solutions (geometrically: the vertices of the polyhedron). If you want to compute them, you must add one of

**SupportHyperplanes, -s**

**ExtremeRays**

**VerticesOfPolyhedron**

The last choice is only possible in the inhomogeneous case. Another possibility in the inhomogeneous case is to use simply **DualMode**.

## 5. Running Normaliz

The standard form for calling Normaliz is

```
normaliz [options] <project>
```

where <project> is the name of the project, and the corresponding input file is <project>.in. Note that normaliz may require to be prefixed by a path name, and the same applies to <project>. A typical example on a Linux or Mac system:

```
./normaliz --verbose -x=5 ../example/big
```

that for MS Windows must be converted to

```
.\normaliz --verbose -x=5 example\big
```

Normaliz uses the standard conventions for calls from the command line:

- (1) the order of the arguments on the command line is arbitrary.
- (2) Single letter options are prefixed by the character - and can be grouped into one string.
- (3) Verbatim options are prefixed by the characters --.

The options for computation goals and algorithmic variants have been described in Section 4. In this section the remaining options for the control of execution and output are discussed, together with some basic rules on the use of the options.

## 5.1. Basic rules

The options for computation goals and algorithms variants have been explained in Section 4. The options that control the execution and the amount of output will be explained in the following. Basic rules for the use of options:

1. If no `<project>` is given, the program will terminate.
2. The option `-x` differs from the other ones: `<T>` in `-x=<T>` represents a positive number assigned to `-x`; see Section 5.3.
3. Similarly the option `--OutputDir=<outdir>` sets the output directory; see 5.4.
4. Normaliz will look for `<project>.in` as input file.

If you inadvertently typed `rafa2416.in` as the project name, then Normaliz will first look for `rafa2416.in.in` as the input file. If this file doesn't exist, `rafa2416.in` will be loaded.

5. The options can be given in arbitrary order. All options, including those in the input file, are accumulated, and syntactically there is no mutual exclusion. However, some options may block others during the computation. For example, `KeepOrder` blocks `BottomDecomposition`.
6. If Normaliz cannot perform a computation explicitly asked for by the user, it will terminate. Typically this happens if no grading is given although it is necessary.
7. In the options include `DefaultMode`, Normaliz does not complain about missing data (anymore). It will simply omit those computations that are impossible.
8. If a certain type of computation is not asked for explicitly, but can painlessly be produced as a side effect, Normaliz will compute it. For example, as soon as a grading is present and the Hilbert basis is computed, the degree 1 elements of the Hilbert basis are selected from it.

## 5.2. Info about Normaliz

`--help`, `-?` displays a help screen listing the Normaliz options.  
`--version` displays information about the Normaliz executable.

## 5.3. Control of execution

The options that control the execution are:

- verbose, -c** activates the verbose (“console”) behavior of Normaliz in which Normaliz writes additional information about its current activities to the standard output.
- x=<T>** Here <T> stands for a positive integer limiting the number of threads that Normaliz is allowed access on your system. The default value is set by the operating system. If you want to run Normaliz in a strictly serial mode, choose **-x=1**.

The number of threads can also be controlled by the environment variable `OMP_NUM_THREADS`. See Section 8.1 for further discussion.

## 5.4. Control of output files

In the default setting Normaliz writes only the output file `<project>.out` (and the files produced by Triangulation and StanleyDec). The amount of output files can be increased as follows:

- files, -f** Normaliz writes the additional output files with suffixes `gen`, `cst`, and `inv`, provided the data of these files have been computed.
- all-files, -a** includes Files, Normaliz writes all available output files (except `typ`, the triangulation or the Stanley decomposition, unless these have been requested).
- <suffix>** chooses the output file with suffix `<suffix>`.

For the list of potential output files, their suffixes and their interpretation see Section 7. There may be several options **--<suffix>**.

If the computation goal `IntegerHull` is set, Normaliz computes a second cone and lattice. The output is contained in `<project>.IntHull.out`. The options for the output of `<project>` are applied to `<project>.IntHull` as well. There is no way to control the output of the two computations individually.

Sometimes one wants the output to be written to another directory. The output directory can be set by

- OutputDir=<outdir>** . The path `<outdir>` is an absolute path or a path relative to the current directory (which is not necessarily the directory of `<project>.in`.)

Note that all output files will be written to the chosen directory. It must be created before Normaliz is started.

## 5.5. Overriding the options in the input file

Since Normaliz accumulates options, one cannot get rid of settings in the input file by command line options unless one uses

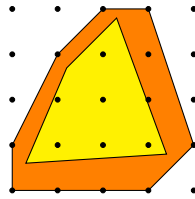
- ignore, -i** This option disables all settings in the input file.

## 6. More examples

## 6.1. Lattice points by approximation

In order to find the lattice points in a polytope  $P$ , old versions of Normaliz created the same vectors that are needed for the Hilbert basis computation, but used that no reduction is necessary to find the degree 1 points. They are automatically part of the Hilbert basis. While this observation speeded up the computations considerably, the number of vectors to be created could simply make the computation impossible. This was especially true for rational polytopes whose vertices have large denominators.

Normaliz 3.2.0 takes the following approach: simplicial subcones  $S$  with rational vertices are approximated by integral polytopes, and only the lattice points in  $S$  are kept. This is done automatically. Since version 2.11 the user can also choose a global approximation of the input polytope  $P$  by the option `Approximate`, and this is often a very good choice.



Now pproximation is chosen automatically by Normaliz if the polytope is not integral and has at most

100.

The reason for this limitation is that each rational vertex must be replaced dby several integral ones, and this increase can make the computation of the approximating polytope quite difficult. As mentioned in Section 6.1, `Approximate`, `-r` forces approximation (regardless of the bound `5e`) and `NoApproximation` blocks it.

Our demonstration example is `max_polytope_cand` with the input file

```
amb_space 5
inequalities 11
-3 -3 -10 14 88
26 -3 19 14 361
-4 25 6 -49 133
2 -3 1 5 17
-2 -3 -7 11 61
-6 23 9 -30 233
-2 7 3 -9 81
-8 17 12 -65 183
8 -15 4 23 65
-8 -13 -28 45 241
-8 27 12 -35 321
grading
unit_vector 5
Deg1Elements
```



This is not a random input file; it has come up in connection with the paper “Quantum jumps of normal polytopes” by W. Bruns, J. Gubeladze and M. Michałek, Discrete Comput. Geom. 56 (2016), no. 1, 181–215.

Despite of the large number of lattice points, the computation is extremely fast. If you insist in `NoApproximation`, it will take considerably longer, but will not overstretch your patience. Note that `Normaliz` passes to GMP integers for some computations, which becomes necessary because of the huge determinants of the simplicial cones (without approximation). Actually, approximation is used despite `NoApproximation`, namely ‘locally’ for each simplicial cone in the triangulation. (This cannot be switched off.)

The third choice using the dual algorithm is still o.k. though it takes the most time in this case. In other cases it may very well be the fastest.

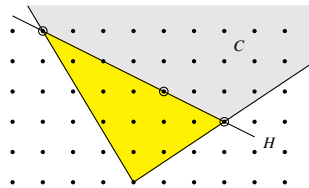
In the inhomogeneous case, approximation can be used if the polyhedron is actually a polytope and `HilbertBasis` (or just the default mode) is set. An example (`pedro2.in`, suggested by Paedro Garcia-Sanchez):

```
amb_space 5
constraints 1
1001 1211 1421 1631 2841 = 39249
```

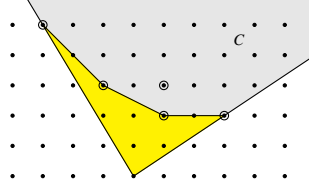
## 6.2. The bottom decomposition

The triangulation size and the determinant sum of the triangulation are critical size parameters in `Normaliz` computations. `Normaliz` always tries to order the generators in such a way that the determinant sum is close to the minimum, and on the whole this works out well. The use of the bottom decomposition by `BottomDecomposition`, `-b` enables `Normaliz` to compute a triangulation with the optimal determinant sum for the given set of generators, as we will explain in the following.

The determinant sum is independent of the order of the generators of the cone  $C$  if they lie in a hyperplane  $H$ . Then the determinant sum is exactly the normalized volume of the polytope spanned by  $0$  and  $C \cap H$ . The triangulation itself depends on the order, but the determinant sum is constant.



This observation helps to find a triangulation with minimal determinant sum in the general case. We look at the *bottom* (the union of the compact faces) of the polyhedron generated by  $x_1, \dots, x_n$  as vertices and  $C$  as recession cone, and take the volume underneath the bottom:



With the option `BottomDecomposition, -b`, `Normaliz` computes a triangulation that respects the bottom facets. This yields the optimal determinant sum for the given generators. If one can compute the Hilbert basis by the dual algorithm, it can be used as input, and then one obtains the absolute bottom of the cone, namely the compact facets of the convex hull of all nonzero lattice points.

`Normaliz` does not always use the bottom decomposition by default since its computation requires some time and administrative overhead. However, as soon as the input “profile” is considered to be “rough” it is invoked. The measure of roughness is the ratio between the maximum degree (or  $L_1$  norm without a grading) and the minimum. A ratio  $\geq 10$  activates the bottom decomposition.

If you have the impression that the bottom decomposition slows down your computation, you can suppress it by `NoBottomDec, -o`. (`KeepOrder, -k` not only suppresses the bottom decomposition, but also any reordering of the generators.)

The bottom decomposition is part of the subdivision of large simplicial cones discussed in the next section.

The example `StrictBorda.in` belongs to social choice theory (see `Condorcet.in` (see Section 2.10), `PluralityVsCutoff.in` and `CondeffPlur.in`). The last two profit enormously from symmetrization (see Section 6.5), but `StrictBorda.in` does not. Therefore we must compute the Hilbert series (or at least the multiOlicity) for a monoid in dimension 24 whose cone has 6363 extreme rays. It demonstrates the substantial gain that can be treached by bottom decomposition. Since the roughness is large enough, `Normaliz` chooses bottom decomposition automatically, unless we block it.

algorithm	triangulation size	determinant sum
bottom decomposition	30,399,162,846	75,933,588,203
standard order of extreme rays, -o	119,787,935,829	401,249,361,966

### 6.3. Subdivision of large simplicial cones

Especially in computations with rational polytopes one encounters very large determinants that can keep the `Normaliz` primal algorithm from terminating in reasonable time. As an example we take `hickerson-18.in` from the `LattE` distribution [3]. It is simplicial and the complexity is totally determined by the large determinant  $\approx 4.17 \times 10^{14}$  (computed with `-v`).

If we are just interested in the degree 1 points, `Normaliz` uses the approximation method of Section 6.1 and finds 44 degree 1 points very quickly. If we use these points together with the extreme rays of the simplex, then the determinant sum decreases to  $\approx 1.3 \times 10^{12}$ , and the

computation of the Hilbert basis and the Hilbert series is in reach. But it is better to pursue the idea of subdividing large simplicial cones systematically. Normaliz employs two methods:

- (1) computation of subdivision points by the IP solver SCIP,
- (2) computation of candidate subdivision points by approximation of the given simplicial cone by an overcone that is generated by vectors of “low denominator”.

Normaliz tries to subdivide a simplicial cone if it has determinant  $\geq 10^8$  or  $10^7$  if the Hilbert basis is computed. Both methods are used recursively via stellar subdivision until simplicial cones with determinant  $< 10^6$  have been reached or no further improvement is possible. Furthermore, if some big simplices are still remaining, method (2) is applied again in both cases with a higher approximation level. All subdivision points are then collected, and the start simplicial cone is subdivided with bottom decomposition, which in general leads to substantial further improvement.

The use of SCIP requires a Normaliz executable that is built with SCIP see Section 2.12 (compile). Without SCIP only the approximation method is used. However, it can happen that SCIP fails because the required precision cannot be reached by floating point calculations. In this case the approximation method will be tried as well.

*Recommendation: If you are applying Normaliz to arithmetically complicated examples, use Normaliz with SCIP.*

The following table contains some performance data for subdivisions based on SCIP (parallelization with 20 threads).

	hickerson-16	hickerson-18	knapsack_11_60
simplex volume	$9.83 \times 10^7$	$4.17 \times 10^{14}$	$2.8 \times 10^{14}$
volume under bottom	$8.10 \times 10^5$	$3.86 \times 10^7$	$2.02 \times 10^7$
volume used	$3.93 \times 10^6$	$5.47 \times 10^7$	$2.39 \times 10^7$
runtime without subdivision	2 s	>12 d	>8 d
runtime with subdivision	0.5 s	46 s	5.1 s

Performance data for approximation:

	hickerson-16	hickerson-18	knapsack_11_60
volume used	$3.9 \times 10^6$	$9.1 \times 10^7$	$2.3 \times 10^9$
runtime with subdivision	0.7 s	58 s	2 m 36 s

A good nonsimplicial example showing the subdivision at work is hickerson\_18plus1.in.

Note: After subdivision the decomposition of the cone may no longer be a triangulation in the strict sense, but a decomposition that we call a *nested triangulation*; see 6.8.1. If the creation of a nested triangulation must be blocked, one uses the option NoSubdivision. Inevitably it blocks the subdivision of large simplicial cones.

*Remark* The bounds mentioned above work well up to dimension  $\approx 10$ . In larger dimension they are too low because the probability of finding a subdivision point becomes too small.

## 6.4. Primal vs. dual – division of labor

As already mentioned several times, Normaliz has two main algorithms for the computation of Hilbert bases and degree 1 points, the primal algorithm and the dual algorithm. It is in general very hard to decide beforehand which of the two is better for a specific example. Nevertheless Normaliz tries to guess, unless `PrimalMode, -P` or `DualMode, -d` is explicitly chosen by the user. In first approximation one can say that the dual algorithm is chosen if the computation is based on constraints and the number of inequalities is neither too small nor too large. Normaliz chooses the dual algorithm if at the start of the Hilbert basis computation the cone is defined by  $s$  inequalities such that

$$r + \frac{50}{r} \leq s \leq 2e$$

where  $r$  is the rank of the monoid to be computed and  $e$  is the dimension of the space in which the data are embedded. These conditions are typically fulfilled for diophantine systems of equations whose nonnegative solutions are asked for. In the case of very few or many hyperplanes Normaliz prefers the primal algorithm. While this combinatorial condition is the only criterion for Normaliz, it depends also on the arithmetic of the example what algorithm is better. At present Normaliz makes no attempt to measure it in some way.

When both Hilbert basis and Hilbert series are to be computed, the best solution can be the combination of both algorithms. We recommend `2equations.in` as a demonstration example which combines the algorithmic variant `DualMode` and the computation goal `HilbertSeries`:

```
amb_space 9
equations 2
1 6 -7 -18 25 -36 6 8 -9
7 -13 15 6 -9 -8 11 12 -2
total_degree
DualMode
HilbertSeries
```

As you will see, the subdivision of large simplicial cones is very useful for such computations. Compare `2equations.in` and `2equations_default.in` for an impression on the relation between the algorithms.

## 6.5. Symmetrization

Under certain conditions one can count lattice points in a cone  $C$  by mapping  $C$  to a cone  $C'$  of lower dimension and then counting each lattice point  $y$  in  $C'$  with the number of its lattice preimages. This approach works well if the number of preimages is given by a polynomial in the coordinates of  $y$ . Since  $C'$  has lower dimension, one can hope that its combinatorial structure is much simpler than that of  $P$ . One must of course pay a price: instead of counting each lattice point with the weight 1, one must count it with a polynomial weight. The offspring `NmzIntegrate` of Normaliz has been created for this task. It can as well be applied to the computation of multiplicities; see [10] for the mathematical details and the algorithmic

approach of NmzIntegrate.

The availability of this approach depends on symmetries in the coordinates of  $C$ , and therefore we call it *symmetrization*. Normaliz tries symmetrization under the following condition:  $C$  is given by constraints (inequalities, equations, congruences, excluded faces) and the inequalities contain the sign conditions  $x_i \geq 0$  for all coordinates  $x_i$  of  $C$ . Then Normaliz groups coordinates that appear in all constraints and the grading (!) with the same coefficients and, roughly speaking, replaces them by their sum. The number of preimages that one must count for the vector  $y$  of sums is then a product of binomial coefficients – a polynomial as desired.

As an example we look again at the input for the Condorcet paradoxon:

```
amb_space 24
inequalities 3
1 1 1 1 1 1 -1 -1 -1 -1 -1 -1 1 1 -1 -1 1 -1 1 1 -1 -1 1 -1
1 1 1 1 1 1 1 1 -1 -1 1 -1 -1 -1 -1 -1 -1 -1 1 1 1 -1 -1 -1
1 1 1 1 1 1 1 1 1 -1 -1 -1 1 1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1
nonnegative
total_degree
Multiplicity
```

The grading is completely symmetric, and it is immediately clear that the input is symmetric in the first 6 coordinates. But also the column of three entries  $-1$  appears 6 times, and there are 6 more groups of 2 coordinates each (one group for each  $\pm 1$  pattern). With the suitable labeling, the number of preimages of  $(y_1, \dots, y_8)$  is given by

$$f(x) = \binom{x_1 + 5}{5} (y_2 + 1)(y_3 + 1)(y_4 + 1)(y_5 + 1)(y_6 + 1)(y_7 + 1) \binom{y_8 + 5}{5}.$$

Normaliz finds the groups of variables that appear with the same sign pattern, creates the input files for NmzIntegrate, runs it, and imports the results.

However, there is a restriction. Since NmzIntegrate must do the polynomial arithmetic in GMP integers, it makes no sense to apply it if the dimension does not drop by a reasonable amount. Therefore we require that

$$\dim C' \leq \min((2(\dim C)/3, 16).$$

If called with the options HilbertSeries, -q or Multiplicity, -v and Symmetrize, -Y, Normaliz will try symmetrization. If the inequality for  $\dim C'$  is not satisfied, it will stop the computation. But it writes the input files for NmzIntegrate into the same directory where it writes the output files of Normaliz. The file names are

`<project>.symm.in`    and    `<project>.symm.pnm`

You can then run Normaliz or NmzIntegrate on these input files with the options -E (for the Hilbert series or -L (for the multiplicity).

The Condorcet example is too small in order to demonstrate the power of symmetrization. A suitable example is `PluralityVsCutoff.in`:

```

winfried.../git_normaliz/source$ time ./normaliz -cY ../example/PluralityVsCutoff

                                     \.....|
Normaliz 3.2.0                       \....|
                                     \...|
(C) The Normaliz Team, University of Osnabrueck \..|
January 2017                                \.|
                                             \|

*****
Command line: -c ../example/PluralityVsCutoff
Compute: DefaultMode Symmetrize
executing: "./normaliz" -c -E -x=8 ../example/PluralityVsCutoff.symm
...
...
...
-----
transforming data... done.

real      0m2.697s
user      0m5.936s
sys       0m0.108s

```

The Hilbert series is computable without symmetrization, but you better make sure that there is no power failure for the next week if you try that. (The time above includes the Hilbert basis computed automatically in dual mode).

Another good example included in the distribution is `CondEffPlur.in`, but it takes some hours with symmetrization (instead of days without). For it, the dimension drops only from 24 to 13.

*Remarks* (1) Large `NmzIntegrate` computations with many parallel threads require much memory due to the fact that very long polynomials must be stored.

(2) At present `Normaliz` and `NmzIntegrate` communicate via files written to the directory where `Normaliz` writes its output files. These files are removed after the return to `Normaliz` from `NmzIntegrate`.

The communication via files does not allow the use of symmetrization via `libnormaliz`.

(3) You should think about the option `BottomDecomposition`. It will be applied to the symmetrized input. (Under the suitable conditions it is applied automatically.)

### 6.5.1. Restrictions in MS Windows

Unfortunately the last version of `NmzIntegrate` that we could compile under MS Windows is 1.3. This causes some restrictions in the use of `NmzIntegrate` and therefore of symmetrization in this operating system:

- (1) Due to a bug it is possible that a segmentation fault occurs if excluded faces are used.
- (2) The option `OutputDir` is not available.

An excellent way out is to run `Normaliz` and `NmzIntegrate` in the Linux subsystem of Windows 10.

## 6.6. Explicit dehomogenization

Inhomogeneous input for data in  $\mathbb{R}^d$  is homogenized by an extra  $(d + 1)$ th coordinate. The dehomogenization sets the last coordinate equal to 1. Other systems may prefer the first coordinate. By choosing an explicit dehomogenization `Normaliz` can be adapted to such input. The file `dehomogenization.in`

```
amb_space 3
inequalities 2
-1 1 0
-1 0 1
dehomogenization
unit_vector 1
```

indicates that in this case the first variable is the homogenizing one. The output file

```
1 module generators
2 Hilbert basis elements of recession monoid
1 vertices of polyhedron
2 extreme rays of recession cone
3 support hyperplanes of polyhedron (homogenized)

embedding dimension = 3
affine dimension of the polyhedron = 2 (maximal)
rank of recession monoid = 2

size of triangulation    = 0
resulting sum of |det|s = 0

dehomogenization:
1 0 0

module rank = 1

*****

1 module generators:
1 1 1

2 Hilbert basis elements of recession monoid:
```

0 0 1	
0 1 0	
1 vertices of polyhedron:	3 support hyperplanes of polyhedron (homogenized)
1 1 1	-1 0 1
	-1 1 0
2 extreme rays of recession cone:	1 0 0
0 0 1	
0 1 0	

shows that Normaliz does the computation in the same way as with implicit dehomogenization, except that now the first coordinate decides what is in the polyhedron and what belongs to the recession cone, roughly speaking.

Note that the dehomogenization need not be a coordinate. It can be any linear form that is nonnegative on the cone generators.

## 6.7. Nonpointed cones

Nonpointed cones and nonpositive monoids contain nontrivial invertible elements. The main effect is that certain data are no longer unique, or may even require a new definition. An important point to note is that cones always split off their unit groups as direct summands and the same holds for normal affine monoids. Since Normaliz computes only normal affine monoids, we can always pass to the quotient by the unit groups. Roughly speaking, all data are computed for the pointed quotient and then lifted back to the original cone and monoid. It is inevitable that some data are no longer uniquely determined, but are unique only modulo the unit group, for example the Hilbert basis and the extreme rays. Also the multiplicity and the Hilbert series are computed for the pointed quotient. From the algebraic viewpoint this means to replace the field  $K$  of coefficients by the group ring  $L$  of the unit group, which is a Laurent polynomial ring over  $K$ : instead of  $K$ -vector space dimensions one considers ranks over  $L$ .

### 6.7.1. A nonpointed cone

As a very simple example we consider the right halfplane (`halfspace2.in`):

```
amb_space 2
inequalities 1
1 0
```

When run in default mode, it yields the following output:

```
1 Hilbert basis elements
1 Hilbert basis elements of degree 1
1 extreme rays
1 support hyperplanes
```



```

embedding dimension = 2
rank = 2 (maximal)
external index = 1
dimension of maximal subspace = 1

size of triangulation   = 1
resulting sum of |det|s = 1

grading:
1 0

degrees of extreme rays:
1: 1

Hilbert basis elements are of degree 1

multiplicity = 1

Hilbert series:
1
denominator with 1 factors:
1: 1

degree of Hilbert Series as rational function = -1

Hilbert polynomial:
1
with common denominator = 1

rank of class group = 0
class group is free

*****

1 Hilbert basis elements of degree 1:
1 0

0 further Hilbert basis elements of higher degree:

1 extreme rays:
1 0

1 basis elements of maximal subspace:
0 1

```

```

1 support hyperplanes:
1 0

```

In the preamble we learn that the cone contains a nontrivial subspace. In this case it is the vertical axis, and close to the end we see a basis of this subspace, namely  $(0, 1)$ . This basis is always simultaneously a  $\mathbb{Z}$ -basis of the unit group of the monoid. The rest of the output is what we have gotten for the positive horizontal axis which in this case is a natural representative of the quotient modulo the maximal subspace. The quotient can always be embedded in the cone or monoid respectively, but there is no canonical choice. We could have gotten  $(1, 5)$  as the Hilbert basis as well.

Normaliz has found a grading. Of course it vanishes on the unit group, but is positive on the quotient monoid modulo the unit group.

Note that the data of type “dimension” (embedding dimension, rank, rank of recession monoid in the inhomogeneous case, affine dimension of the polyhedron)) are measured before the passage to the quotient modulo the maximal subspace. The same is true for equations and congruences (which are trivial for the example above).

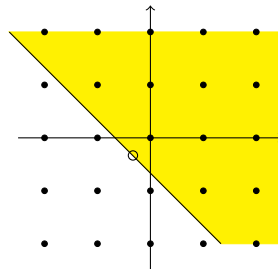
### 6.7.2. A polyhedron without vertices

We define the affine halfspace of the figure by `gen_inhom_nonpointed.in`:

```

amb_space 2
cone 3
1 -1
-1 1
0 1
vertices 1
-1 -1 3

```



It is clear that the “vertex” is not a vertex in the strict sense, but only gives a displacement of the cone. The output when run in default mode:

```

1 module generators
1 Hilbert basis elements of recession monoid
1 vertices of polyhedron
1 extreme rays of recession cone

```

```

2 support hyperplanes of polyhedron (homogenized)

embedding dimension = 3
affine dimension of the polyhedron = 2 (maximal)
rank of recession monoid = 2
internal index = 3
dimension of maximal subspace = 1

size of triangulation    = 1
resulting sum of |det|s = 3

dehomogenization:
0 0 1

module rank = 1

*****

1 module generators:
0 0 1

1 Hilbert basis elements of recession monoid:
0 1 0

1 vertices of polyhedron:
0 -2 3

1 extreme rays of recession cone:
0 1 0

1 basis elements of maximal subspace:
1 -1 0

2 support hyperplanes of polyhedron (homogenized):
0 0 1
3 3 2

```

The “vertex” of the polyhedron shown is of course the lifted version of the vertex modulo the maximal subspace. It is not the input “vertex”, but agrees with it up to a unit.

### 6.7.3. Checking pointedness first

Nonpointed cones will be an exception in Normaliz computations, and therefore Normaliz assumes that the (recession) cone it must compute is pointed. Only in rare circumstances it could be advisable to have this property checked first. There is no need to do so when the

dual algorithm is used since it does not require the cone to be pointed. Moreover, if an explicit grading is given or a grading dependent computation is asked for, one cannot save time by checking the pointedness first.

The exceptional case is a computation, say of a Hilbert basis, by the primal algorithm in which the computation of the support hyperplanes needs very long time to be completed. If you are afraid this may happen, you can force Normaliz to compute the support hyperplanes right away by adding `IsPointed` to the computation goals. This is a disadvantage only if the cone is unexpectedly pointed.

#### 6.7.4. Input of a subspace

If a linear subspace contained in the cone is known a priori, it can be given to Normaliz via the input type `subspace`. If Normaliz detects a subspace, it appends the rows of the matrix to the generators of the cone, and additionally the negative of the sum of the rows (since we must add the subspace as a cone). If `subspace` is combined with `cone_and_lattice`, then the rows of `subspace` are also appended to the generators of the lattice. It is not assumed that the vectors in `subspace` are linearly independent or generate the maximal linear subspace of the cone. A simple example (`subspace4.in`):

```
amb_space 4
cone 4
1 0 2 0
0 1 -2 1
0 0 0 1
0 0 0 -1
subspace 1
0 0 1 0
```

From the output:

```
2 Hilbert basis elements of degree 1:
0 1 0 0
1 0 0 0

0 further Hilbert basis elements of higher degree:

2 extreme rays:
0 1 0 0
1 0 0 0

2 basis elements of maximal subspace:
0 0 1 0
0 0 0 1

2 support hyperplanes:
0 1 0 0
```

```
1 0 0 0
```

One should note that the maximal subspace is generated by the smallest face that contains all invertible elements. Therefore, in order to make all vectors in a face invertible, it is enough to put a single vector from the interior of the face into subspace.

### 6.7.5. Data relative to the original monoid

If original monoid generators are defined, there are two data related to them that must be read with care.

First of all, we consider the original monoid generators as being built from the vectors in cone or cone\_and\_lattice plus the vectors in subspace and additionally the negative of the sum of the latter (as pointed out above).

The test for “Original monoid is integrally closed” is correct – it returns `true` if and only if the original monoid as just defined indeed equals the computed integral closure. (There was a mistake in version 3.0.)

The “module generators over the original monoid” only refer to the *image* of the original monoid and the image of the integral closure *modulo the maximal subspace*. They do not take into account that the unit group of the integral closure may not be generated by the original generators. An example in which the lack of integral closedness is located in the unit group (normface.in):

```
amb_space 5
cone 4
0 0 0 1 1
1 0 0 1 1
0 1 0 1 1
0 0 1 1 1
subspace 4
0 0 0 0 1
1 0 0 0 1
0 1 0 0 1
1 1 2 0 1
```

From the output file:

```
...
dimension of maximal subspace = 4
original monoid is not integrally closed
unit group index = 2
...

1 Hilbert basis elements of degree 1:
0 0 0 1 0
...
```

```
1 module generators over original monoid:
0 0 0 0 0
```

The original monoid is not integrally closed since the unit group of the integral closure is strictly larger than that of the original monoid: the extension has index 2, as indicated. The quotients modulo the unit groups are equal, as can be seen from the generator over the original monoid or the Hilbert basis (of the integral closure) that is contained in the original monoid.

## 6.8. Exporting the triangulation

The option `-T` asks Normaliz to export the triangulation by writing the files `<project>.tgn` and `<project>.tri`:

**tgn** The file `tgn` contains a matrix of vectors (in the coordinates of  $\mathbb{A}$ ) spanning the simplicial cones in the triangulation.

**tri** The file `tri` lists the simplicial subcones. There are two variants, depending on whether `ConeDecomposition` had been set. Here we assume that `ConeDecomposition` is not computed. See Section 6.8.2 for the variant with `ConeDecomposition`.

The first line contains the number of simplicial cones in the triangulation, and the next line contains the number  $m + 1$  where  $m = \text{rank } \mathbb{E}$ . Each of the following lines specifies a simplicial cone  $\Delta$ : the first  $m$  numbers are the indices (with respect to the order in the file `tgn`) of those generators that span  $\Delta$ , and the last entry is the multiplicity of  $\Delta$  in  $\mathbb{E}$ , i. e. the absolute value of the determinant of the matrix of the spanning vectors (as elements of  $\mathbb{E}$ ).

The following example is the 2-dimensional cross polytope with one excluded face (`cross2.in`). The excluded face is irrelevant for the triangulation.

```
amb_space 3
polytope 4
1 0
0 1
-1 0
0 -1
excluded_faces 1
1 1 -1
```

Its `tgn` and `tri` files are

tgn	tri
4	2
3	4
1 0 1	1 2 3 2
0 1 1	1 3 4 2
-1 0 1	plain
0 -1 1	

We see the 4 vertices  $v_1, \dots, v_4$  in homogenized coordinates in `tgn` and the 2 simplices (or the simplicial cones over them) in `tri`: both have multiplicity 2. The last word `plain` indicates that Normaliz has computed a triangulation in the strict sense, namely a simplicial subdivision in which neighboring simplicial cones match along common faces. The alternative is `nested` that we discuss below.

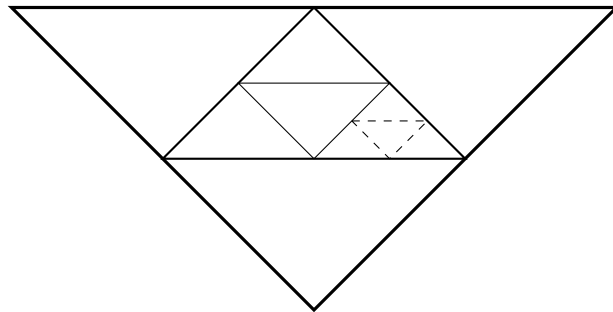
In addition to the files `<project>.tgn` and `<project>.tri`, also the file `<object>.inv` is written. It contains the data of the file `<project>.out` above the line of stars in a human and machine readable format.

### 6.8.1. Nested triangulations

If Normaliz has subdivided a simplicial cone of a triangulation of the cone  $C$ , the resulting decomposition of  $C$  may no longer be a triangulation in the strict sense. It is rather a *nested triangulation*, namely a map from a rooted tree to the set of full-dimensional subcones of  $C$  with the following properties:

- (1) the root is mapped to  $C$ ,
- (2) every other node is mapped to a full dimensional simplicial subcone,
- (3) the simplicial subcones corresponding to the branches at a node  $x$  form a triangulation of the simplicial cone corresponding to  $x$ .

The following figure shows a nested triangulation:



For the Normaliz computations, nested triangulations are as good as ordinary triangulations, but in other applications the difference may matter. With the option `-T`, Normaliz prints the leaves of the nested triangulation to the `tri` file. They constitute the simplicial cones that are finally evaluated by Normaliz.

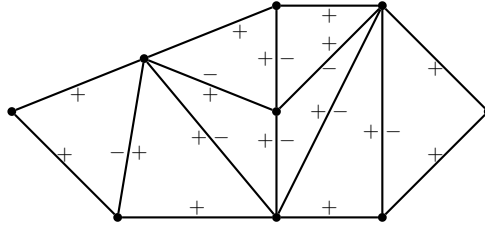
The triangulation is always plain if `-T` is the only computation goal or if it is just combined with `-v`. Otherwise it can only fail to be plain if it contains determinants  $\geq 10^8$ .

The subdivision can be blocked by `NoSubdivision`, independently of the computation goals.

### 6.8.2. Disjoint decomposition

Normaliz can export the disjoint decomposition of the cone that it has computed. This decomposition is always computed together with a full triangulation, unless only the multiplicity is

asked for. It represents the cone as the disjoint union of semiopen simplicial subcones. The corresponding closed cones constitute the triangulation, and from each of them some facets are removed so that one obtains a disjoint decomposition. See [8] for more information. In the following figure, the facets separating the triangles are omitted in the triangle on the – side.



If you want to access the disjoint decomposition, you must activate the computation goal `ConeDecomposition` or use the command line option `-D`. As an example we compute `cross2.in` with the computation goal `ConeDecomposition`. The file `cross2.tri` now looks as follows:

```
2
7
1 2 3    2    0 0 0
2 3 4    2    0 0 1
plain
```

As before the first line contains the size of the triangulation and the second is the number of entries of each row. The first 3 entries in each line are the indices of the extreme rays with respect to the `tn` file and the fourth entry is the determinant. They are followed by a 0/1 vector indicating the open facets in the order in which they are opposite to the extreme rays. If the corresponding entry is 1, the facet must be removed.

In our example all facets of the first simplicial cone are kept, and from the second simplicial cone the facet opposite to the third extreme ray (with index 4 relative to `tn`) must be removed.

## 6.9. Exporting the Stanley decomposition

The option `-y` makes `Normaliz` write the files `<project>.tn`, `<project>.dec` and `<project>.inv`. Stanley decomposition is contained in the file with the suffix `dec`. But this file also contains the inclusion/exclusion data if there are excluded faces:

(a) If there are any excluded faces, the file starts with the word `in_ex_data`. The next line contains the number of such data that follow. Each of these lines contains the data of a face and the coefficient with which the face is to be counted: the first number lists the number of generators that are contained in the face, followed by the indices of the generators relative to the `tn` file and the last number is the coefficient.

(b) The second block (the first if there are no excluded faces) starts with the word `Stanley_dec`, followed by the number of simplicial cones in the triangulation.

For each simplicial cone  $\Delta$  in the triangulation this file contains a block of data:



- (i) a line listing the indices  $i_1, \dots, i_m$  of the generators  $v_{i_1}, \dots, v_{i_m}$  relative to the order in `tn` (as in `tri`,  $m = \text{rank } \mathbb{E}$ );
- (ii) a  $\mu \times m$  matrix where  $\mu$  is the multiplicity of  $\Delta$  (see above).

In the notation of [8], each line lists an “offset”  $x + \varepsilon(x)$  by its coordinates with respect to  $v_{i_1}, \dots, v_{i_m}$  as follows: if  $(a_1, \dots, a_m)$  is the line of the matrix, then

$$x + \varepsilon(x) = \frac{1}{\mu}(a_1 v_{i_1} + \dots + a_m v_{i_m}).$$

The dec file of the example above is

```

in_ex_data
1
2 1 2 -1
Stanley_dec
2
1 3 4          1 2 3
2              2
3              3
0 0 2          0 0 0
1 1 2          1 0 1

```

There is 1 face in `in_ex_data` (namely the excluded one), it contains the 2 generators  $v_1$  and  $v_2$  and appears with multiplicity  $-1$ . The Stanley decomposition consists of 4 components of which each of the simplicial cone contains 2. The second offset in the second simplicial cone is

$$\frac{1}{2}(1v_1 + 0v_2 + 1v_3) = (0, 0, 1).$$

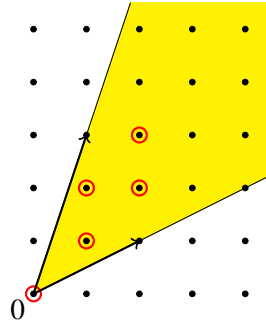
The file `3x3magiceven.in` has been processed with the option `-ahTy` activated. We recommend you to inspect all the output files in the subdirectory `example` of the distribution.

## 6.10. Module generators over the original monoid

Suppose that the original generators are well defined in the input. This is always the case when these consists just of a cone or a cone\_and\_lattice. Let  $M$  be the monoid generated by them. Then `Normaliz` computes the integral closure  $N$  of  $M$  in the effective lattice  $\mathbb{E}$ . It is often interesting to understand the difference set  $N \setminus M$ . After the introduction of a field  $K$  of coefficients, this amounts to understanding  $K[N]$  as a  $K[M]$ -module. With the option `ModuleGeneratorsOverOriginalMonoid`, `-M Normaliz` computes a minimal generating set  $T$  of this module. Combinatorially this means that we find an irreducible cover

$$N = \bigcup_{x \in T} x + M.$$

Note that  $0 \in T$  since  $M \subset N$ .



As an example, we can run `2cone.in` with the option `-M` on the command line. This yields the output

```
...
4 Hilbert basis elements:
 1 1
 1 2
 1 3
 2 1
2 extreme rays:
 1 3
 2 1
5 module generators over original monoid:
 0 0
 1 1
 1 2
 2 2
 2 3
```

In the nonpointed case `Normaliz` can only compute the module generators of  $N/N_0$  over  $M/(M \cap N_0)$  where  $N_0$  is the unit group of  $N$ . If  $M_0 \neq M_0$ , this is not a system of generators of  $M$  over  $N$ .

### 6.10.1. An inhomogeneous example

Let us have a look at a very simple input file (`genmod_inhom2.in`):

```
amb_space 2
cone 2
0 3
2 0
vertices 1
0 0 1
ModuleGeneratorsOverOriginalMonoid
```

The cone is the positive orthant that we have turned into a polyhedron by adding the vertex  $(0,0)$ . The original monoid is generated by  $(2,0)$  and  $(0,3)$ .

In addition to the original monoid  $M$  and its integral closure  $N$  we have a third object, namely the module  $P$  of lattice points in the polyhedron. We compute

1. the system of generators of  $P$  over  $N$  (the module generators) and
2. the system of generators of  $P$  over  $N$  (the module generators over original monoid).

We do not compute the system of generators of  $N$  over  $M$  (that we get in the homogeneous case).

The output:

```

1 module generators
2 Hilbert basis elements of recession monoid
1 vertices of polyhedron
2 extreme rays of recession cone
6 module generators over original monoid
3 support hyperplanes of polyhedron (homogenized)

embedding dimension = 3
affine dimension of the polyhedron = 2 (maximal)
rank of recession monoid = 2
internal index = 6

size of triangulation   = 1
resulting sum of |det|s = 6

dehomogenization:
0 0 1

module rank = 1

*****

1 module generators:
0 0 1

2 Hilbert basis elements of recession monoid:
0 1 0
1 0 0

1 vertices of polyhedron:
0 0 1

2 extreme rays of recession cone:
0 1 0
1 0 0

6 module generators over original monoid:
0 0 1
0 1 1
0 2 1
1 0 1

```

```

1 1 1
1 2 1

3 support hyperplanes of polyhedron (homogenized):
0 0 1
0 1 0
1 0 0

```

## 6.11. Precomputed support hyperplanes

Computing the support hyperplanes can be a very time consuming task, and if it has been the first step in the exploration of a difficult example, it may be desirable, to use the support hyperplanes as additional input in order to save computation time. This is especially true if Normaliz must do an intermediate computation of the support hyperplanes because of a large number of simplicial cones to be evaluated. The file `2cone_supp.in` is just a toy example:

```

amb_space 2
cone 2
2 1
1 3
support_hyperplanes 2
-1 2
3 -1

```

As pointed out in Section 3.3.1, Normaliz must trust you—here is no way of checking the correctness of this input without recomputing it.

## 6.12. Shift, denominator, quasipolynomial and multiplicity

In this section we discuss the interplay of shift, denominator of the grading and the quasipolynomial. As long as the denominator is 1, the situation is very simple and no ambiguity arises. See Section 2.9. We modify the example from that section as follows (`InhomIneq_7.in`):

```

amb_space 2
inhom_inequalities 3
0 2 1
0 -2 3
2 -2 3
grading
7 0

```

The output related to the grading is

```

grading:
7 0 0
with denominator = 7

```

```

module rank = 2
multiplicity = 2

Hilbert series:
1 1
denominator with 1 factors:
1: 1

shift = -1

degree of Hilbert Series as rational function = -1

Hilbert polynomial:
2
with common denominator = 1

```

The Hilbert series computed by hand is

$$\frac{t^{-7} + 1}{1 - t^7}.$$

We obtain it from the output as follows. The printed series is

$$\frac{1 + t}{1 - t}.$$

Now the shift is applied and yields

$$\frac{t^{-1} + 1}{1 - t}.$$

Finally we make the substitution  $t \mapsto t^7$ , and obtain the desired result.

Now we add the complication  $x_1 + x_2 \equiv -1 \pmod{8}$  ((InhomIneq\_7\_8.in):

```

amb_space 2
inhom_inequalities 3
0 2 1
0 -2 3
2 -2 3
grading
7 0
inhom_congruences 1
1 1 1 8

```

The result:

```

grading:
7 0 0

```

```
with denominator = 7
```

```
module rank = 2
```

```
multiplicity = 1/4
```

```
Hilbert series:
```

```
1 0 0 0 0 0 0 1
```

```
denominator with 1 factors:
```

```
8: 1
```

```
shift = -1
```

```
degree of Hilbert Series as rational function = -2
```

```
Hilbert series with cyclotomic denominator:
```

```
-1 1 -1 1 -1 1 -1
```

```
cyclotomic denominator:
```

```
1: 1 4: 1 8: 1
```

```
Hilbert quasi-polynomial of period 8:
```

```
0: 0 4: 0
```

```
1: 0 5: 0
```

```
2: 0 6: 1
```

```
3: 0 7: 1
```

```
with common denominator = 1
```

The printed Hilbert series is

$$\frac{1+t^7}{1-t^8}.$$

The application of the shift yields

$$\frac{t^{-1}+t^6}{1-t^8}.$$

the correct result for the divided grading. *The Hilbert quasipolynomial is computed for the divided grading*, as already explained in Section 2.6.1. As a last step, we can apply the substitution  $t \mapsto t^7$  in order obtain the Hilbert series

$$\frac{t^{-7}+t^{42}}{1-t^{56}}$$

for the original grading.

Like the quasipolynomial, *the multiplicity is computed for the divided grading*.

## 7. Optional output files

When one of the options `Files`, `-f` or `AllFiles`, `-a` is activated, `Normaliz` writes additional output files whose names are of type `<project>.<type>`. (Note that the options `-T`, `Triangulation` and `-y`, `StanleyDec` as well as the options `-E`, `-L`, `-I` calling `NmzIntegrate` also write files in addition to `<project>.out`. `Symmetrize` does not produce extra output files.) Moreover one can select the optional output files individually via command line options. Most of these files contain matrices in a simple format:

```
<m>
<n>
<x_1>
...
<x_m>
```

where each row has `<n>` entries. Exceptions are the files with suffixes `cst`, `inv`, `esp`.

Note that the files are only written if they would contain at least one row.

As pointed out in Section 5.4, the optional output files for the integer hull are the same as for the original computation, as far as their content has been computed.

### 7.1. The homogeneous case

The option `-f` makes `Normaliz` write the following files:

- gen** contains the Hilbert basis. If you want to use this file as an input file and reproduce the computation results, then you must make it a matrix of type `cone_and_lattice` (and add the dehomogenization in the inhomogeneous case).
- cst** contains the constraints defining the cone and the lattice in the same format as they would appear in the input: matrices of types *constraints* following each other. Each matrix is concluded by the type of the constraints. Empty matrices are indicated by 0 as the number of rows. Therefore there will always be at least 3 matrices.  
If a grading is defined, it will be appended. Therefore this file (with suffix `in`) as input for `Normaliz` will reproduce the Hilbert basis and all the other data computed, at least in principle.
- inv** contains all the information from the file `out` that is not contained in any of the other files.

If `-a` is activated, then the following files are written *additionally*:

- ext** contains the extreme rays of the cone.
- ht1** contains the degree 1 elements of the Hilbert basis if a grading is defined.
- egn, esp** These contain the Hilbert basis and support hyperplanes in the coordinates with respect to a basis of  $\mathbb{E}$ . `esp` contains the grading and the dehomogenization in the coordinates of  $\mathbb{E}$ . Note that no equations for  $\mathbb{C} \cap \mathbb{E}$  or congruences for  $\mathbb{E}$  are necessary.
- lat** contains the basis of the lattice  $\mathbb{E}$ .

**mod** contains the module generators of the integral closure modulo the original monoid.

**msp** contains the basis of the maximal subspace.

In order to select one or more of these files individually, add an option of type `--<suffix>` to the command line where `<suffix>` can take the values

gen, cst, inv, ext, ht1, egn, esp, lat, mod, msp, typ
---

The type `typ` is not contained in `Files` or `AllFiles` since it can be extremely large. It is of the matrix format described above. It is the product of the matrices corresponding to `egn` and the transpose of `esp`. In other words, the linear forms representing the support hyperplanes of the cone  $C$  are evaluated on the Hilbert basis. The resulting matrix, with the generators corresponding to the rows and the support hyperplanes corresponding to the columns, is written to this file.

The suffix `typ` is motivated by the fact that the matrix in this file depends only on the isomorphism type of monoid generated by the Hilbert basis (up to row and column permutations). In the language of [4] it contains the *standard embedding*.

Note: the explicit choice of an optional output file does *not* imply a computation goal. Output files that would contain unknown data are simply not written without a warning or error message.

## 7.2. Modifications in the inhomogeneous case

The optional output files are a subset of those that can be produced in the homogeneous case. The main difference is that the generators of the solution module and the Hilbert basis of the recession monoid appear together in the file `gen`. They can be distinguished by evaluating the dehomogenization on them (simply the last component with inhomogeneous input), and the same applies to the vertices of the polyhedron and extreme rays of the recession cone. The file `cst` contains the constraints defining the polyhedron and the recession module in conjunction with the dehomogenization, which is also contained in the `cst` file, following the constraints.

With `-a` the files `egn` and `esp` are produced. These files contain `gen` and the support hyperplanes of the homogenized cone in the coordinates of  $\mathbb{E}$ , as well as the dehomogenization.

## 8. Performance

### 8.1. Parallelization

The executables of Normaliz have been compiled for parallelization on shared memory systems with OpenMP. Parallelization reduces the “real” time of the computations considerably, even on relatively small systems. However, one should not underestimate the administrative overhead involved.

- It is not a good idea to use parallelization for very small problems.



- On multi-user systems with many processors it may be wise to limit the number of threads for Normaliz somewhat below the maximum number of cores.

The number of parallel threads can be limited by the Normaliz option `-x` (see Section 5.3) or by the commands

```
export OMP_NUM_THREADS=<T>      (Linux/Mac)
```

or

```
set OMP_NUM_THREADS=<T>      (Windows)
```

where `<T>` stands for the maximum number of threads accessible to Normaliz. For example, we often use

```
export OMP_NUM_THREADS=20
```

on a multi-user system with 24 cores.

Limiting the number of threads to 1 forces a strictly serial execution of Normaliz.

The paper [8] contains extensive data on the effect of parallelization. On the whole Normaliz scales very well. However, the dual algorithm often performs best with mild parallelization, say with 4 or 6 threads.

## 8.2. Running large computations

Normaliz can cope with very large examples, but it is usually difficult to decide a priori whether an example is very large, but nevertheless doable, or simply impossible. Therefore some exploration makes sense.

See [8] for some very large computations. The following hints reflect the authors' experience with them.

(1) Run Normaliz with the option `-cs` and pay attention to the terminal output. The number of extreme rays, but also the numbers of support hyperplanes of the intermediate cones are useful data.

(2) In many cases the most critical size parameter is the number of simplicial cones in the triangulation. It makes sense to determine it as the next step. Even with the fastest potential evaluation (option `-v`), finding the triangulation takes less time, say by a factor between 3 and 10. Thus it makes sense to run the example with `-t` in order to explore the size.

As you can see from [8], Normaliz has successfully evaluated triangulations of size  $\approx 5 \cdot 10^{11}$  in dimension 24.

(3) Another critical parameter are the determinants of the generator matrices of the simplicial cones. To get some feeling for their sizes, one can restrict the input to a subset (of the extreme rays computed in (1)) and use the option `-v` or the computation goal `TriangulationDetSum` if there is no grading.

The output file contains the number of simplicial cones as well as the sum of the absolute values of the determinants. The latter is the number of vectors to be processed by Normaliz in triangulation based calculations.

The number includes the zero vector for every simplicial cone in the triangulation. The zero vector does not enter the Hilbert basis calculation, but cannot be neglected for the Hilbert series.

Normaliz has mastered calculations with  $> 10^{15}$  vectors.

(4) If the triangulation is small, we can add the option `-T` in order to actually see the triangulation in a file. Then the individual determinants become visible.

(5) If a cone is defined by inequalities and/or equations consider the dual mode for Hilbert basis calculation, even if you also want the Hilbert series.

(6) The size of the triangulation and the size of the determinants are *not* dangerous for memory by themselves (unless `-T` or `-y` are set). Critical magnitudes can be the number of support hyperplanes, Hilbert basis candidates, or degree 1 elements.

## 9. Distribution and installation

In order to install Normaliz you should first download the basic package from

<http://normaliz.uos.de>.

It contains the documentation, examples, source code, jNormaliz, NmzIntegrate and the packages for PyNormaliz, Singular and Macaulay2. Then unzip the downloaded file `normaliz-3.2.0.zip` in a directory of your choice. (Any other downloaded zip file for Normaliz should be unzipped in this directory, too.)

This process will create a directory `normaliz-3.2.0` (called Normaliz directory) and several subdirectories in it. The names of the subdirectories created are self-explanatory. Nevertheless we give an overview:

- In the Normaliz directory you should find `jNormaliz.jar`, several files, for example `Copying`, and subdirectories.
- The subdirectory `source` contains the source files. The subdirectory `genEhrhart` contains the `NmzIntegrate` source.
- The subdirectory `doc` contains the file you are reading and further documentation.
- In the subdirectory `example` are the input files for some examples. It contains all named input files of examples of this manual.
- Automated tests which run Normaliz on different inputs and options are contained in the subdirectory `test`.
- The subdirectory `Singular` contains the SINGULAR library `normaliz.lib` and a PDF file with documentation.
- The subdirectory `Macaulay2` contains the MACAULAY2 package `Normaliz.m2`.
- The subdirectory `PyNormaliz` contains the source PYTHON interface.
- The subdirectory `lib` contains libraries for jNormaliz.

- Moreover, there are subdirectories whose name starts with Q. They contain the source code, examples and tests for QNormaliz (see Appendix F)

We provide executables for Windows, Linux and Mac. Download the archive file corresponding to your system `normaliz-3.2.0<systemname>.zip` and unzip it. This process will store the executables of Normaliz and NmzIntegrate in the Normaliz directory. In case you want to run Normaliz from the command line or use it from other systems, you may have to copy the executables to a directory in the search path for executables.

Please read the release notes on the download page. Especially on Mac you may want to build Normaliz yourself since it is impossible to provide a statically linked executable with parallelization.

## 10. Building Normaliz yourself

Normaliz offers the luxury of three build systems:

1. `autotools`,
2. `cmake`
3. “classical” make using the “handwritten” `Makefile.classic`.

The basic steps are the same for all three systems, namely

- configuration,
- compilation,
- installation.

The main difference is the way how the build system is configured whereas compilation and installation are essentially identical. In all cases the compilation of NmzIntegrate is included (as well as the compilation of the example program `maxsimplex` in Appendix D.7).

In the following we describe the basic steps of `autotools` and `cmake` for Linux 64 bit under certain standard assumptions. Comments on Mac OS follow in Section 10.6 The file `INSTALL` in the directory `source` contains more information. `Makefile.classic` is only meant for Linux and the development of Normaliz.

The `autotools` scripts have been written by Matthias Köppe. The Normaliz team thanks him cordially for his generous help.

### 10.1. Compiler prerequisites

We require some C++11 features (e.g. `std::exception_ptr`), supported by:

- GNU g++ 4.4,
- clang++ 2.9,
- Intel icpc 12.0

See <https://github.com/Normaliz/Normaliz/issues/26> for a more detailed discussion.

The mentioned compilers are also able to handle OpenMP 3.0, with the exception of clang++, there the first OpenMP support was introduced in 3.7.

## 10.2. Required libraries

For compiling Normaliz the following libraries are needed:

- GMP including the C++ wrapper (libgmpxx and libgmp)
- Boost (headers only)

We will only discuss the basic use of cmake for compilation, see the file `source/INSTALL` for additional information, especially on how to use customized paths. Also the use of the autotools system is explained in this file. The “classical” Makefile is meant for development. Therefore you should use autotools or cmake.

The installation will store the files in standard locations, and we assume in the following that they do not need individual include paths.

## 10.3. Optional packages

As discussed in the manual, Normaliz can profit from the use of SCIP. If you want to use it, SCIP must be installed before the compilation of Normaliz, independently of the method used for building Normaliz.

To build SCIP download the scipoptsuite at <http://scip.zib.de/>. Notice that SCIP is not distributed under GPL, but the ZIB Academic License (<http://scip.zib.de/academic.txt>). Unpack it and then compile it with

```
make ZLIB=false GMP=false READLINE=false scipoptlib
```

Another optional package is CoCoALib, but only to the extent to that NmzIntegrate is optional. If you want to compile NmzIntegrate, CoCoALib must be installed first. The following sequence of commands will install it in the subdirectory CoCoA of your home directory.

```
mkdir ~/CoCoA/  
cd ~/CoCoA/  
wget http://cocoa.dima.unige.it/cocoalib/tgz/CoCoALib-0.99543.tgz  
tar xvf CoCoALib-0.99543.tgz  
cd CoCoALib-0.99543  
./configure --threadsafe-hack --no-boost  
make library -j2
```

If CoCoALib-0.99543 should be no longer available, replace it by a newer version.

## 10.4. autotools

To build Normaliz with the autotools system, navigate to the Normaliz directory and issue the following sequence of commands:

```
./configure  
make
```

This will compile Normaliz, but most likely without SCIP and NmzIntegrate since they need the optional libraries mentioned above and these must be found. If they are not located at standard places, you must specify their paths. Examples (on the machine of a Normaliz team member):

```
./configure --with-scipoptsuite-src=$HOME/SCIP/scipoptsuite-3.2.0/
```

or

```
./configure --with-cocoalib=$HOME/CoCoA/CoCoALib-0.99543
```

or with both paths. If the libraries are found, Normaliz will be compiled with SCIP and NmzIntegrate will be made, respectively, by the make command. Check the terminal output of ./configure for success.

The next step is

```
make
```

After this step you will find normaliz (and nmzIntegrate) in the directory source (and maxsimplex in its directory).

The last, optional step is

```
sudo make install
```

It copies the header files, the library libnormaliz and the executables (except maxsimplex) into subdirectories of /usr/local. It is of course possible to specify another installation path in the call of ./configure.

Note: In case you have checked out Normaliz from GitHub, the very first step is ./bootstrap.sh. Unfortunately, the paths for SCIP are version dependent. We have tested versions 3.2.0 and 3.2.1.

## 10.5. cmake

You may need to install cmake first:

```
sudo apt-get cmake cmake-curses-gui
```

To build Normaliz with cmake, start by creating a build directory within the Normaliz directory, say BUILD. Then change the working directory to BUILD.

The basic configuration (equivalent to configure of autotools) is

```
cmake ../source
```

Then `make` and `make install` will complete the basic installation.

For the inclusion of SCIP, use (for example)

```
SCIP_DIR=$HOME/SCIP/scipoptsuite-3.2.0/ cmake ../source
```

replacing `$HOME/SCIP/scipoptsuite-3.2.0/` with your own path to SCIP if necessary. Similarly,

```
COCO_A_DIR=$HOME/CoCoA/CoCoALib-0.99543 cmake ../source/
```

Then `make` and `make install` will complete the work. After `make` the executables can be found in `BUILD` and its subdirectories `genEhrhart` and `maxsimplex`.

The main advantage of `cmake` is the existence of a GUI in which you can change most settings originally chosen by `cmake`. Call `ccmake ../source` (2 times `c`) or, for a more sophisticated version, `cmake-gui ../source`.

Note: Unfortunately, the paths for SCIP are version dependent. The configuration files for SCIP presently can find the versions 3.2.0 and 3.2.1. For another version you must edit the file `FindSCIP.cmake` in `source/cmake/Modules`.

## 10.6. Mac OS X

Currently Apple does not supply a compiler which supports OpenMP. We recommend the use of LLVM 3.9 or newer from Homebrew. See <http://brew.sh/> from where you can also download GMP and Boost. For details see `INSTALL` in the directory `source` of the distribution.

For building Normaliz under Mac OS we recommend autotools if you want to use SCIP.

You can then follow the instructions for Linux.

Note: Do not try to compile Normaliz with static libraries for Mac OS X.

## 10.7. Windows

One can compile Windows executables with the Cygwin port of GCC. Unfortunately it is not compatible to OpenMP.

Using Visual Studio is a bit tricky. Microsoft's C++ compiler does not support OpenMP 3.0. Creating a Normaliz Visual Studio project via `cmake` is currently not fully supported. The executables that are offered in the Normaliz distribution have been compiled with `icpc` and a manually created project. Please contact us if you want to build Normaliz on Windows.

Note that the statically linked Linux binaries run in the Linux subsystem of Windows 10. We have not yet tried to build Normaliz in it.

## 11. Copyright and how to cite

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Please refer to Normaliz in any publication for which it has been used:

W. Bruns, B. Ichim, T. Römer, R. Sieg and C. Söger: Normaliz. Algorithms for rational cones and affine monoids. Available at <http://normaliz.uos.de>

The corresponding `\bibitem`:

```
\bibitem{Normaliz} W. Bruns, B. Ichim, T. Römer, R. Sieg and C. Söger:  
Normaliz. Algorithms for rational cones and affine monoids.  
Available at \url{http://normaliz.uos.de}.
```

A BibTeX entry:

```
@Misc{Normaliz,  
author = {W. Bruns and B. Ichim and T. Römer and R. Sieg and C. Söger},  
title = Normaliz. Algorithms for rational cones and affine monoids,  
howpublished = {Available at \url{http://normaliz.uos.de}}}
```

It is now customary to evaluate mathematicians by such data as numbers of publications, citations and impact factors. The data bases on which such dubious evaluations are based do not list mathematical software. Therefore we ask you to cite the article [8] in addition. This is very helpful for the younger members of the team.

## A. Mathematical background and terminology

For a coherent and thorough treatment of the mathematical background we refer the reader to [4].

### A.1. Polyhedra, polytopes and cones

An *affine halfspace* of  $\mathbb{R}^d$  is a subset given as

$$H_\lambda^+ = \{x : \lambda(x) \geq 0\},$$

where  $\lambda$  is an affine form, i.e., a non-constant map  $\lambda : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $\lambda(x) = \alpha_1 x_1 + \dots + \alpha_d x_d + \beta$  with  $\alpha_1, \dots, \alpha_d, \beta \in \mathbb{R}$ . If  $\beta = 0$  and  $\lambda$  is therefore linear, then the halfspace is called *linear*. The halfspace is *rational* if  $\lambda$  is *rational*, i.e., has rational coordinates. If  $\lambda$  is rational, we can assume that it is even *integral*, i.e., has integral coordinates, and, moreover, that these are coprime. Then  $\lambda$  is uniquely determined by  $H_\lambda^+$ . Such integral forms are called *primitive*, and the same terminology applies to vectors.

**Definition 1.** A (rational) *polyhedron*  $P$  is the intersection of finitely many (rational) halfspaces. If it is bounded, then it is called a *polytope*. If all the halfspaces are linear, then  $P$  is a *cone*.

The *dimension* of  $P$  is the dimension of the smallest affine subspace  $\text{aff}(P)$  containing  $P$ .

A support hyperplane of  $P$  is an affine hyperplane  $H$  that intersects  $P$ , but only in such a way that  $H$  is contained in one of the two halfspaces determined by  $H$ . The intersection  $H \cap P$  is called a *face* of  $P$ . It is a polyhedron (polytope, cone) itself. Faces of dimension 0 are called *vertices*, those of dimension 1 are called *edges* (in the case of cones *extreme rays*), and those of dimension  $\dim(P) - 1$  are *facets*.

When we speak of *the* support hyperplanes of  $P$ , then we mean those intersecting  $P$  in a facet. Their halfspaces containing  $P$  cut out  $P$  from  $\text{aff}(P)$ . If  $\dim(P) = d$ , then they are uniquely determined (up to a positive scalar).

The constraints by which Normaliz describes polyhedra are

- (1) linear equations for  $\text{aff}(P)$  and
- (2) linear inequalities (simply called support hyperplanes) cutting out  $P$  from  $\text{aff}(P)$ .

In other words, the constraints are given by a linear system of equations and inequalities, and a polyhedron is nothing else than the solution set of a linear system of inequalities and equations. It can always be represented in the form

$$Ax \geq b, \quad A \in \mathbb{R}^{m \times d}, b \in \mathbb{R}^m,$$

if we replace an equation by two inequalities.



## A.2. Cones

The definition describes a cone by constraints. One can equivalently describe it by generators:

**Theorem 2** (Minkowski-Weyl). *The following are equivalent for  $C \subset \mathbb{R}^d$ :*

1.  *$C$  is a (rational) cone;*
2. *there exist finitely many (rational) vectors  $x_1, \dots, x_n$  such that*

$$C = \{a_1x_1 + \dots + a_nx_n : a_1, \dots, a_n \in \mathbb{R}_+\}.$$

By  $\mathbb{R}_+$  we denote the set of nonnegative real numbers;  $\mathbb{Q}_+$  and  $\mathbb{Z}_+$  are defined in the same way.

The conversion between the description by constraints and that by generators is one of the basic tasks of Normaliz. It uses the *Fourier-Motzkin elimination*.

Let  $C_0$  be the set of those  $x \in C$  for which  $-x \in C$  as well. It is the largest vector subspace contained in  $C$ . A cone is *pointed* if  $C_0 = 0$ . If a rational cone is pointed, then it has uniquely determined *extreme integral generators*. These are the primitive integral vectors spanning the extreme rays. These can also be defined with respect to a sublattice  $L$  of  $\mathbb{Z}^d$ , provided  $C$  is contained in  $\mathbb{R}L$ . If a cone is not pointed, then Normaliz computes the extreme rays of the pointed  $C/C_0$  and lifts them to  $C$ . (Therefore they are only unique modulo  $C_0$ .)

The *dual cone*  $C^*$  is given by

$$C^* = \{\lambda \in (\mathbb{R}^d)^* : \lambda(x) \geq 0 \text{ for all } x \in C\}.$$

Under the identification  $\mathbb{R}^d = (\mathbb{R}^d)^{**}$  one has  $C^{**} = C$ . Then one has

$$\dim C_0 + \dim C^* = d.$$

In particular,  $C$  is pointed if and only if  $C^*$  is full dimensional, and this is the criterion for pointedness used by Normaliz. Linear forms  $\lambda_1, \dots, \lambda_n$  generate  $C^*$  if and only if  $C$  is the intersection of the halfspaces  $H_{\lambda_i}^+$ . Therefore the conversion from constraints to generators and its converse are the same task, except for the exchange of  $\mathbb{R}^d$  and its dual space.

## A.3. Polyhedra

In order to transfer the Minkowski-Weyl theorem to polyhedra it is useful to homogenize coordinates by embedding  $\mathbb{R}^d$  as a hyperplane in  $\mathbb{R}^{d+1}$ , namely via

$$\kappa : \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}, \quad \kappa(x) = (x, 1).$$

If  $P$  is a (rational) polyhedron, then the closure of the union of the rays from 0 through the points of  $\kappa(P)$  is a (rational) cone  $C(P)$ , called the *cone over  $P$* . The intersection  $C(P) \cap (\mathbb{R}^d \times \{0\})$  can be identified with the *recession* (or *tail*) *cone*

$$\text{rec}(P) = \{x \in \mathbb{R}^d : y + x \in P \text{ for all } y \in P\}.$$

It is the cone of unbounded directions in  $P$ . The recession cone is pointed if and only if  $P$  has at least one bounded face, and this is the case if and only if it has a vertex.

The theorem of Minkowski-Weyl can then be generalized as follows:

**Theorem 3** (Motzkin). *The following are equivalent for a subset  $P \neq \emptyset$  of  $\mathbb{R}^d$ :*

1.  $P$  is a (rational) polyhedron;
2.  $P = Q + C$  where  $Q$  is a (rational) polytope and  $C$  is a (rational) cone.

*If  $P$  has a vertex, then the smallest choice for  $Q$  is the convex hull of its vertices, and  $C = \text{rec}(P)$  is uniquely determined.*

The *convex hull* of a subset  $X \in \mathbb{R}^d$  is

$$\text{conv}(X) = \{a_1x_1 + \cdots + a_nx_n : n \geq 1, x_1, \dots, x_n \in X, a_1, \dots, a_n \in \mathbb{R}_+, a_1 + \cdots + a_n = 1\}.$$

Clearly,  $P$  is a polytope if and only if  $\text{rec}(P) = \{0\}$ , and the specialization to this case one obtains Minkowski's theorem: a subset  $P$  of  $\mathbb{R}^d$  is a polytope if and only if it is the convex hull of a finite set. A *lattice polytope* is distinguished by having integral points as vertices.

Normaliz computes the recession cone and the polytope  $Q$  if  $P$  is defined by constraints. Conversely it finds the constraints if the vertices of  $Q$  and the generators of  $C$  are specified.

Suppose that  $P$  is given by a system

$$Ax \geq b, \quad A \in \mathbb{R}^{m \times d}, \quad b \in \mathbb{R}^m,$$

of linear inequalities (equations are replaced by two inequalities). Then  $C(P)$  is defined by the *homogenized system*

$$Ax - x_{d+1}b \geq 0$$

whereas the  $\text{rec}(P)$  is given by the *associated homogeneous system*

$$Ax \geq 0.$$

It is of course possible that  $P$  is empty if it is given by constraints since inhomogeneous systems of linear equations and inequalities may be unsolvable. By abuse of language we call the solution set of the associated homogeneous system the recession cone of the system.

Via the concept of dehomogenization, Normaliz allows for a more general approach. The *dehomogenization* is a linear form  $\delta$  on  $\mathbb{R}^{d+1}$ . For a cone  $\tilde{C}$  in  $\mathbb{R}^{d+1}$  and a dehomogenization  $\delta$ , Normaliz computes the polyhedron  $P = \{x \in \tilde{C} : \delta(x) = 1\}$  and the recession cone  $C = \{x \in \tilde{C} : \delta(x) = 0\}$ . In particular, this allows other choices of the homogenizing coordinate. (Often one chooses  $x_0$ , the first coordinate then.)

In the language of projective geometry,  $\delta(x) = 0$  defines the hyperplane at infinity.

## A.4. Affine monoids

An *affine monoid*  $M$  is a finitely generated submonoid of  $\mathbb{Z}^d$  for some  $d \geq 0$ . This means:  $0 \in M$ ,  $M + M \subset M$ , and there exist  $x_1, \dots, x_n$  such that

$$M = \{a_1x_1 + \dots + a_nx_n : a_1, \dots, a_n \in \mathbb{Z}_+\}.$$

We say that  $x_1, \dots, x_n$  is a *system of generators* of  $M$ . A monoid  $M$  is *positive* if  $x \in M$  and  $-x \in M$  implies  $x = 0$ . An element  $x$  in a positive monoid  $M$  is called *irreducible* if it has no decomposition  $x = y + z$  with  $y, z \in M$ ,  $y, z \neq 0$ . The *rank* of  $M$  is the rank of the subgroup  $\text{gp}(M)$  of  $\mathbb{Z}^d$  generated by  $M$ . (Subgroups of  $\mathbb{Z}^d$  are also called sublattices.) For certain aspects of monoid theory it is very useful (or even necessary) to introduce coefficients from a field  $K$  (or a more general commutative ring) and consider the monoid algebra  $K[M]$ .

**Theorem 4** (van der Corput). *Every positive affine monoid  $M$  has a unique minimal system of generators, given by its irreducible elements.*

We call the minimal system of generators the *Hilbert basis* of  $M$ . Normaliz computes Hilbert bases of a special type of affine monoid:

**Theorem 5** (Gordan's lemma). *Let  $C \subset \mathbb{R}^d$  be a (pointed) rational cone and let  $L \subset \mathbb{Z}^d$  be a sublattice. Then  $C \cap L$  is a (positive) affine monoid.*

The monoids  $M = C \cap L$  of the theorem have the pleasant property that the group of units  $M_0$  (i.e., elements whose inverse also belongs to  $M$ ) splits off as a direct summand. Therefore  $M/M_0$  is a well-defined affine monoid. If  $M$  is not positive, then Normaliz computes a Hilbert basis of  $M/M_0$  and lifts it to  $M$ .

Let  $M \subset \mathbb{Z}^d$  be an affine monoid, and let  $N \supset M$  be an overmonoid (not necessarily affine), for example a sublattice  $L$  of  $\mathbb{Z}^d$  containing  $M$ .

**Definition 6.** The *integral closure* (or *saturation*) of  $M$  in  $N$  is the set

$$\widehat{M}_N = \{x \in N : kx \in M \text{ for some } k \in \mathbb{Z}, k > 0\}.$$

If  $\widehat{M}_N = M$ , one calls  $M$  *integrally closed* in  $N$ .

The integral closure  $\overline{M}$  of  $M$  in  $\text{gp}(M)$  is its *normalization*.  $M$  is *normal* if  $\overline{M} = M$ .

The integral closure has a geometric description:

**Theorem 7.**

$$\widehat{M}_N = \text{cone}(M) \cap N.$$

Combining the theorems, we can say that Normaliz computes integral closures of affine monoids in lattices, and the integral closures are themselves affine monoids as well. (More generally,  $\widehat{M}_N$  is affine if  $M$  and  $N$  are affine.)

In order to specify the intersection  $C \cap L$  by constraints we need a system of homogeneous inequalities for  $C$ . Every sublattice of  $\mathbb{Z}^d$  can be written as the solution set of a combined system of homogeneous linear diophantine equations and a homogeneous system of congruences (this follows from the elementary divisor theorem). Thus  $C \cap L$  is the solution set of a homogeneous linear diophantine system of inequalities, equations and congruences. Conversely, the solution set of every such system is a monoid of type  $C \cap L$ .

In the situation of Theorem 7, if  $\text{gp}(N)$  has finite rank as a  $\text{gp}(M)$ -module,  $\widehat{M}_N$  is even a finitely generated module over  $M$ . I.e., there exist finitely many elements  $y_1, \dots, y_m \in \widehat{M}_N$  such that  $\widehat{M}_N = \bigcup_{i=1}^m M + y_i$ . Normaliz computes a minimal system  $y_1, \dots, y_m$  and lists the nonzero  $y_i$  as a system of module generators of  $\widehat{M}_N$  modulo  $M$ . We must introduce coefficients to make this precise: Normaliz computes a minimal system of generators of the  $K[M]$ -module  $K[\widehat{M}_N]/K[M]$ .

## A.5. Affine monoids from binomial ideals

Let  $U$  be a subgroup of  $\mathbb{Z}^n$ . Then the natural image  $M$  of  $\mathbb{Z}_+^n \subset \mathbb{Z}^n$  in the abelian group  $G = \mathbb{Z}^n/U$  is a submonoid of  $G$ . In general,  $G$  is not torsionfree, and therefore  $M$  may not be an affine monoid. However, the image  $N$  of  $M$  in the lattice  $L = G/\text{torsion}(G)$  is an affine monoid. Given  $U$ , Normaliz chooses an embedding  $L \hookrightarrow \mathbb{Z}^r$ ,  $r = n - \text{rank } U$ , such that  $N$  becomes a submonoid of  $\mathbb{Z}_+^r$ . In general there is no canonical choice for such an embedding, but one can always find one, provided  $N$  has no invertible element except 0.

The typical starting point is an ideal  $J \subset K[X_1, \dots, X_n]$  generated by binomials

$$X_1^{a_1} \dots X_n^{a_n} - X_1^{b_1} \dots X_n^{b_n}.$$

The image of  $K[X_1, \dots, X_n]$  in the residue class ring of the Laurent polynomial ring  $S = K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  modulo the ideal  $JS$  is exactly the monoid algebra  $K[M]$  of the monoid  $M$  above if we let  $U$  be the subgroup of  $\mathbb{Z}^n$  generated by the differences

$$(a_1, \dots, a_n) - (b_1, \dots, b_n).$$

Ideals of type  $JS$  are called lattice ideals if they are prime. Since Normaliz automatically passes to  $G/\text{torsion}(G)$ , it replaces  $JS$  by the smallest lattice ideal containing it.

## A.6. Lattice points in polyhedra

Let  $P \subset \mathbb{R}^d$  be a rational polyhedron and  $L \subset \mathbb{Z}^d$  be an *affine sublattice*, i.e., a subset  $w + L_0$  where  $w \in \mathbb{Z}^d$  and  $L_0 \subset \mathbb{Z}^d$  is a sublattice. In order to investigate (and compute)  $P \cap L$  one again uses homogenization:  $P$  is extended to  $C(P)$  and  $L$  is extended to  $\mathcal{L} = L_0 + \mathbb{Z}(w, 1)$ . Then one computes  $C(P) \cap \mathcal{L}$ . Via this “bridge” one obtains the following inhomogeneous version of Gordan’s lemma:

**Theorem 8.** *Let  $P$  be a rational polyhedron with vertices and  $L = w + L_0$  an affine lattice as above. Set  $\text{rec}_L(P) = \text{rec}(P) \cap L_0$ . Then there exist  $x_1, \dots, x_m \in P \cap L$  such that*

$$P \cap L = \{(x_1 + \text{rec}_L(P)) \cap \dots \cap (x_m + \text{rec}_L(P))\}.$$

*If the union is irredundant, then  $x_1, \dots, x_m$  are uniquely determined.*

The Hilbert basis of  $\text{rec}_L(P)$  is given by  $\{x : (x, 0) \in \text{Hilb}(C(P) \cap \mathcal{L})\}$  and the minimal system of generators can also be read off the Hilbert basis of  $C(P) \cap \mathcal{L}$ : it is given by those  $x$  for which  $(x, 1)$  belongs to  $\text{Hilb}(C(P) \cap \mathcal{L})$ . (Normaliz computes the Hilbert basis of  $C(P) \cap L$  only at “levels” 0 and 1.)

We call  $\text{rec}_L(P)$  the *recession monoid* of  $P$  with respect to  $L$  (or  $L_0$ ). It is justified to call  $P \cap L$  a *module* over  $\text{rec}_L(P)$ . In the light of the theorem, it is a finitely generated module, and it has a unique minimal system of generators.

After the introduction of coefficients from a field  $K$ ,  $\text{rec}_L(P)$  is turned into an affine monoid algebra, and  $N = P \cap L$  into a finitely generated torsionfree module over it. As such it has a well-defined *module rank*  $\text{mrnk}(N)$ , which is computed by Normaliz via the following combinatorial description: Let  $x_1, \dots, x_m$  be a system of generators of  $N$  as above; then  $\text{mrnk}(N)$  is the cardinality of the set of residue classes of  $x_1, \dots, x_m$  modulo  $\text{rec}_L(P)$ .

Clearly, to model  $P \cap L$  we need linear diophantine systems of inequalities, equations and congruences which now will be inhomogeneous in general. Conversely, the set of solutions of such a system is of type  $P \cap L$ .

## A.7. Hilbert series

Normaliz can compute the Hilbert series and the Hilbert (quasi)polynomial of a graded monoid. A *grading* of a monoid  $M$  is simply a homomorphism  $\deg : M \rightarrow \mathbb{Z}^g$  where  $\mathbb{Z}^g$  contains the degrees. The *Hilbert series* of  $M$  with respect to the grading is the formal Laurent series

$$H(t) = \sum_{u \in \mathbb{Z}^g} \#\{x \in M : \deg x = u\} t_1^{u_1} \dots t_g^{u_g} = \sum_{x \in M} t^{\deg x},$$

provided all sets  $\{x \in M : \deg x = u\}$  are finite. At the moment, Normaliz can only handle the case  $g = 1$ , and therefore we restrict ourselves to this case. We assume in the following that  $\deg x > 0$  for all nonzero  $x \in M$  and that there exists an  $x \in \text{gp}(M)$  such that  $\deg x = 1$ . (Normaliz always rescales the grading accordingly.) In the case of a nonpositive monoid, these conditions must hold for  $M/M_0$ , and its Hilbert series is considered as the Hilbert series of  $M$ .

The basic fact about  $H(t)$  in the  $\mathbb{Z}$ -graded case is that it is the Laurent expansion of a rational function at the origin:

**Theorem 9** (Hilbert, Serre; Ehrhart). *Suppose that  $M$  is a normal positive affine monoid. Then*

$$H(t) = \frac{R(t)}{(1 - t^e)^r}, \quad R(t) \in \mathbb{Z}[t],$$

*where  $r$  is the rank of  $M$  and  $e$  is the least common multiple of the degrees of the extreme integral generators of  $\text{cone}(M)$ . As a rational function,  $H(t)$  has negative degree.*

The statement about the rationality of  $H(t)$  holds under much more general hypotheses.

Usually one can find denominators for  $H(t)$  of much lower degree than that in the theorem, and Normaliz tries to give a more economical presentation of  $H(t)$  as a quotient of two polynomials. One should note that it is not clear what the most natural presentation of  $H(t)$  is in general (when  $e > 1$ ). We discuss this problem in [8, Section 4]. The examples 2.5 and 2.6.1, may serve as an illustration.

A rational cone  $C$  and a grading together define the rational polytope  $Q = C \cap A_1$  where  $A_1 = \{x : \deg x = 1\}$ . In this sense the Hilbert series is nothing but the Ehrhart series of  $Q$ . The following description of the Hilbert function  $H(M, k) = \#\{x \in M : \deg x = k\}$  is equivalent to the previous theorem:

**Theorem 10.** *There exists a quasipolynomial  $q$  with rational coefficients, degree  $\text{rank } M - 1$  and period  $\pi$  dividing  $e$  such that  $H(M, k) = q(k)$  for all  $k \geq 0$ .*

The statement about the quasipolynomial means that there exist polynomials  $q^{(j)}$ ,  $j = 0, \dots, \pi - 1$ , of degree  $\text{rank } M - 1$  such that

$$q(k) = q^{(j)}(k), \quad j \equiv k \pmod{\pi},$$

and

$$q^{(j)}(k) = q_0^{(j)} + q_1^{(j)}k + \dots + q_{r-1}^{(j)}k^{r-1}, \quad r = \text{rank } M,$$

with coefficients  $q_i^{(j)} \in \mathbb{Q}$ . It is not hard to show that in the case of affine monoids all components have the same degree  $r - 1$  and the same leading coefficient:

$$q_{r-1} = \frac{\text{vol}(Q)}{(r-1)!},$$

where  $\text{vol}$  is the lattice normalized volume of  $Q$  (a lattice simplex of smallest possible volume has volume 1). The *multiplicity* of  $M$ , denoted by  $e(M)$  is  $(r-1)!q_{r-1} = \text{vol}(Q)$ .

Suppose now that  $P$  is a rational polyhedron in  $\mathbb{R}^d$ ,  $L \subset \mathbb{Z}^d$  is an affine lattice, and we consider  $N = P \cap L$  as a module over  $M = \text{rec}_L(P)$ . Then we must give up the condition that  $\deg$  takes the value 1 on  $\text{gp}(M)$  (see Section 6.12 for an example). But the Hilbert series

$$H_N(t) = \sum_{x \in N} t^{\deg x}$$

is well-defined, and the qualitative statement above about rationality remain valid. However, in general the quasipolynomial gives the correct value of the Hilbert function only for  $k > r$  where  $r$  is the degree of the Hilbert series as a rational function.

Let  $m$  be the gcd of the numbers  $\deg x$ ,  $x \in M$ . (For  $M = \{0\}$  we set  $m = 1$ .) Then we must define  $e(M) = e'(M)/m$  where  $e'(M)$  is the multiplicity of  $M$  with respect to the normalized grading  $\deg/m$ . The multiplicity of  $N$  is given by

$$e(N) = m \text{rank}(N) e(M).$$

Since  $N$  may have generators in negative degrees, Normaliz shifts the degrees into  $\mathbb{Z}_+$  by subtracting a constant, called the *shift*. (The shift may also be positive.)

## A.8. The class group

A normal affine monoid  $M$  has a well-defined divisor class group. It is naturally isomorphic to the divisor class group of  $K[M]$  where  $K$  is a field (or any unique factorization domain); see [4, 4.F], and especially [4, 4.56]. The class group classifies the divisorial ideals up to isomorphism. It can be computed from the standard embedding that sends an element  $x$  of  $\text{gp}(M)$  to the vector  $\sigma(x)$  where  $\sigma$  is the collection of support forms  $\sigma_1, \dots, \sigma_s$  of  $M$ :  $\text{Cl}(M) = \mathbb{Z}^s / \sigma(\text{gp}(M))$ . Finding this quotient amounts to an application of the Smith normal form to the matrix of  $\sigma$ .

## B. Annotated console output

### B.1. Primal mode

With

```
./normaliz -ch example/A443
```

we get the following terminal output.

```

                                     \.....|
                                Normaliz 3.2.0      \....|
                                     \...|
          (C) The Normaliz Team, University of Osnabrueck  \..|
                                January 2017      \.|
                                     \|
*****
Command line: -ch example/A443
Compute: HilbertBasis HilbertSeries
*****
starting primal algorithm with full triangulation ...
Roughness 1
Generators sorted by degree and lexicographically
Generators per degree:
1: 48
```

Self explanatory so far (see Section 6.2 for the definition of roughness). Now the generators are inserted.

```
Start simplex 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 19 22 25 26 27 28 31 34
37 38 39 40 43 46
```

Normaliz starts by searching linearly independent generators with indices as small as possible. They span the start simplex in the triangulation. The remaining generators are inserted successively. (If a generator does not increase the cone spanned by the previous ones, it is not listed, but this does not happen for A443.)

```
gen=17, 39 hyp, 4 simpl
```

We have now reached a cone with 39 support hyperplanes and the triangulation has 4 simplices so far. We omit some generators until something interesting happens:

```
gen=35, 667 hyp, 85 pyr, 13977 simpl
```

In view of the number of simplices in the triangulation and the number of support hyperplanes, Normaliz has decided to build pyramids and to store them for later triangulation.

```
gen=36, 723 hyp, 234 pyr, 14025 simpl
...
gen=48, 4948 hyp, 3541 pyr, 14856 simpl
```

All generators have been processed now. Fortunately our cone is pointed:

```
Pointed since graded
Select extreme rays via comparison ... done.
```

Normaliz knows two methods for finding the extreme rays. Instead of “comparison” you may see “rank”. Now the stored pyramids must be triangulated. They may produce not only simplices, but also pyramids of higher level, and indeed they do so:

```
*****
level 0 pyramids remaining: 3541
*****
*****
all pyramids on level 0 done!
*****
level 1 pyramids remaining: 5935
*****
*****
all pyramids on level 1 done!
*****
level 2 pyramids remaining: 1567
*****
1180 pyramids remaining on level 2, evaluating 2503294 simplices
```

At this point the preset size of the evaluation buffer for simplices has been exceeded. Normaliz stops the processing of pyramids, and empties the buffer by evaluating the simplices.

```
|||||
2503294 simplices, 0 HB candidates accumulated.
*****
all pyramids on level 2 done!
*****
level 3 pyramids remaining: 100
*****
```



```
*****
all pyramids on level 3 done!
```

This is a small computation, and the computation of pyramids goes level by level without the necessity to return to a lower level. But in larger examples the buffer for level  $n + 1$  may be filled before level  $n$  is finished. Then it becomes necessary to go back. Some simplices remaining in the buffer are now evaluated:

```
evaluating 150978 simplices
|||||||||||||||||||||||||||||||||||||||||
2654272 simplices, 0 HB candidates accumulated.
Adding 1 denominator classes... done.
```

Since our generators form the Hilbert basis, we do not collect any further candidates. If all generators are in degree 1, we have only one denominator class in the Hilbert series, but otherwise there may be many. The collection of the Hilbert series in denominator classes reduces the computations of common denominators to a minimum.

```
Total number of pyramids = 14137, among them simplicial 2994
```

Some statistics of the pyramid decomposition.

```
-----
transforming data... done.
```

Our computation is finished.

A typical pair of lines that you will see for other examples is

```
auto-reduce 539511 candidates, degrees <= 1 3 7
reducing 30 candidates by 73521 reducers
```

It tells you that Normaliz has found a list of 539511 new candidates for the Hilbert basis, and this list is reduced against itself (auto-reduce). Then the 30 old candidates are reduced against the 73521 survivors of the auto-reduction.

## B.2. Dual mode

Now we give an example of a computation in dual mode. It is started by the command

```
./normaliz -cid example/5x5
```

The option `i` is used to suppress the HSOP in the input file. The console output:

```

                                     \.....|
Normaliz 3.2.0                      \....|
                                     \...|
(C) The Normaliz Team, University of Osnabrueck \..|
January 2017                                \.|
                                             \|
```

```

*****
Command line: -cid example/5x5
Compute: DualMode
No inequalities specified in constraint mode, using non-negative orthant.
*****

```

Indeed, we have used only equations as the input.

```

*****
computing Hilbert basis ...
=====
cut with halfspace 1 ...
Final sizes: Pos 1 Neg 1 Neutral 0

```

The cone is cut out from the space of solutions of the system of equations (in this case) by successive intersections with halfspaces defined by the inequalities. After such an intersection we have the positive half space, the “neutral” hyperplane and the negative half space. The final sizes given are the numbers of Hilbert basis elements strictly in the positive half space, strictly in the negative half space, and in the hyperplane. This pattern is repeated until all hyperplanes have been used.

```

=====
cut with halfspace 2 ...
Final sizes: Pos 1 Neg 1 Neutral 1

```

We leave out some hyperplanes ...

```

=====
cut with halfspace 20 ...
auto-reduce 1159 candidates, degrees <= 13 27
Final sizes: Pos 138 Neg 239 Neutral 1592
=====
cut with halfspace 21 ...
Positive: 1027 Negative: 367
.....
Final sizes: Pos 1094 Neg 369 Neutral 1019

```

Sometimes reduction takes some time, and then Normaliz may issue a message on “auto-reduction” organized by degree (chosen for the algorithm, not defined by the given grading). The line of dots is printed is the computation of new Hilbert basis candidates takes time, and Normaliz wants to show you that it is not sleeping. Normaliz shows you the number of positive and negative partners that must be paired produce offspring.

```

=====
cut with halfspace 25 ...
Positive: 1856 Negative: 653
.....
auto-reduce 1899 candidates, degrees <= 19 39

```

```
Final sizes: Pos 1976 Neg 688 Neutral 2852
```

All hyperplanes have been taken care of.

```
Find extreme rays
Find relevant support hyperplanes
```

Well, in connection with the equations, some hyperplanes become superfluous. In the output file Normaliz will list a minimal set of support hyperplanes that together with the equations define the cone.

```
Hilbert basis 4828
```

The number of Hilbert basis elements computed is the sum of the last positive and neutral numbers.

```
Find degree 1 elements
```

The input file contains a grading.

```
transforming data... done.
```

Our example is finished.

The computation of the new Hilbert basis after the intersection with the new hyperplane proceeds in rounds, and there can be many rounds ... (not in the above example). then you can see terminal output like

```
Round 100
Round 200
Round 300
Round 400
Round 500
```

## C. Normaliz 2 input syntax

A Normaliz 2 input file contains a sequence of matrices. Comments or options are not allowed in it. A matrix has the format

```
<m>
<n>
<x_1>
...
<x_m>
<type>
```

where  $\langle m \rangle$  denotes the number of rows,  $\langle n \rangle$  is the number of columns and  $\langle x_1 \rangle \dots \langle x_n \rangle$  are the rows with  $\langle m \rangle$  entries each. All matrix types of Normaliz 3 are allowed (with Normaliz 3), also grading and dehomogenization. These vectors must be encoded as matrices with 1 row.

The optional output files of with suffix `cst` are still in this format. Just create one and inspect it.

## D. libnormaliz

The kernel of Normaliz is the C++ class library `libnormaliz`. It implements all the classes that are necessary for the computations. The central class is `Cone`. It realizes the communication with the calling program and starts the computations most of which are implemented in other classes. In the following we describe the class `Cone`; other classes of `libnormaliz` may follow in the future.

Of course, Normaliz itself is the prime example for the use of `libnormaliz`, but it is rather complicated because of the input and output it must handle. Therefore we have added a simple example program at the end of this introduction.

`libnormaliz` defines its own name space. In the following we assume that

```
using namespace std;
using namespace libnormaliz;
```

have been declared. It is clear that opening these name spaces is dangerous. In this documentation we only do it to avoid constant repetition of `std::` and `libnormaliz::`

### D.1. Integer type as a template parameter

A cone can be constructed for two integer types, `long long` and `mpz_class`. It is reasonable to choose `mpz_class` since the main computations will then be tried with `long long` and restarted with `mpz_class` if `long long` cannot store the results. This internal change of integer type is not possible if the cone is constructed for `long long`. (Nevertheless, the linear algebra routines can use `mpz_class` locally if intermediate results exceed `long long`; have a look into `matrix.cpp`.)

Internally the template parameter is called `Integer`. In the following we assume that the integer type has been fixed as follows:

```
typedef mpz_class Integer;
```

The internal passage from `mpz_class` to `long long` can be suppressed by

```
MyCone.deactivateChangeOfPrecision();
```

where we assume that `MyCone` has been constructed as described in the next section.

#### D.1.1. Alternative integer types

The predefined alternative to `mpz_class` is `long long`. It is possible to use `libnormaliz` with other integer types than `mpz_class` or `long long`, but we have tested only these types.

In an alternative configuration you have to include `libnormaliz-all.cpp`. (In this case you do not need to link `libnormaliz.a`). If you want to use other types, you probably have to implement some conversion functions which you can find in `integer.h`. Namely the functions

```
bool libnormaliz::try_convert(TypeA, TypeB);
    // converts TypeB to TypeA, returns false if not possible
```

where one type is your type and the other is long, long long or mpz\_class. Additionally, if your type uses infinite precision (for example, it is some wrapper for GMP), you must also implement

```
template<> inline bool libnormaliz::using_GMP<YourType>() { return true; }
```

## D.2. Construction of a cone

There is no default constructor for cones. The construction requires the specification of input data consisting of one or more matrices and the input types they represent.

The term “matrix” stands for

```
vector<vector<Integer> >
```

The available input types (from libnormaliz.h) are defined as follows:

```
namespace Type {
enum InputType {
integral_closure,
polyhedron,
normalization,
polytope,
rees_algebra,
inequalities,
strict_inequalities,
signs,
strict_signs,
equations,
congruences,
inhom_inequalities,
dehomogenization,
inhom_equations,
inhom_congruences,
lattice_ideal,
grading,
excluded_faces,
lattice,
saturation,
cone,
offset,
vertices,
support_hyperplanes,
cone_and_lattice,
subspace
```

```
};
} //end namespace Type
```

The input types are explained in Section 3. In certain environments it is not possible to use the enumeration. Therefore we provide a function that converts a string into the corresponding input type:

```
Type::InputType to_type(const string& type_string)
```

The types grading, dehomoneization and offset must be encoded as matrices with a single row. We come back to this point below.

The simplest constructor has the syntax

```
Cone<Integer>::Cone(InputType input_type, const vector< vector<Integer> >& Input)
```

and can be used as in the following example:

```
vector<vector<Integer> > Data = ...
Type::InputType type = cone;
Cone<Integer> MyCone = Cone<Integer>(type, Data);
```

For two and three pairs of type and matrix there are the constructors

```
Cone<Integer>::Cone(InputType type1, const vector< vector<Integer> >& Input1,
                  InputType type2, const vector< vector<Integer> >& Input2)

Cone<Integer>::Cone(InputType type1, const vector< vector<Integer> >& Input1,
                  InputType type2, const vector< vector<Integer> >& Input2,
                  InputType type3, const vector< vector<Integer> >& Input3)
```

If you have to combine more than three matrices, you can define a

```
map<InputType, vector< vector<Integer> > >
```

and use the constructor with syntax

```
Cone<Integer>::Cone(const map< InputType,
                  vector< vector<Integer> > >& multi_input_data)
```

The four constructors also exist in a variant that uses the libnormaliz type `Matrix<Integer>` instead of `vector< vector<Integer> >` (see `cone.h`).

For convenience we provide the function

```
vector<vector<Integer> > to_matrix<Integer>(vector<Integer> v)
```

It returns a matrix whose first row is `v`. A typical example:

```
size_t dim = ...
vector<vector<Integer> > Data = ...
Type::InputType type = cone;
vector<Integer> total_degree(dim,1);
Type::InputType grad = grading;
```

```
Cone<Integer> MyCone = Cone<Integer>(type, Data, grad, to_matix(total_degree));
```

### D.3. Computations in a cone

Before starting a computation in a (previously constructed) cone, one must decide what should be computed and in which way it should be computed. The computation goals and algorithmic variants (see Section 4) are defined as follows (cone\_property.h):

```
namespace ConeProperty {
    enum Enum {
        //
        // goals that can be computed (or are defined by input data)
        //
        // matrix valued
        Generators,
        ExtremeRays,
        VerticesOfPolyhedron,
        SupportHyperplanes,
        HilbertBasis,
        ModuleGenerators,
        DeglElements,
        ModuleGeneratorsOverOriginalMonoid,
        Sublattice,
        ExcludedFaces,
        OriginalMonoidGenerators,
        MaximalSubspace,
        Equations,
        Congruences,
        //vector valued
        Grading,
        Dehomogenization,
        WitnessNotIntegrallyClosed,
        // Cardinalities
        TriangulationSize,
        // Integer valued,
        TriangulationDetSum,
        ReesPrimaryMultiplicity,
        GradingDenom,
        UnitGroupIndex,
        InternalIndex,
        ExternalIndex,
        // rational valued
        Multiplicity,
        // dimensions
        RecessionRank,
        AffineDim,
```

```

ModuleRank,
Rank,
EmbeddingDim,
// boolean valued
IsPointed,
IsDeglExtremeRays,
IsDeglHilbertBasis,
IsIntegrallyClosed,
IsReesPrimary,
IsInhomogeneous,
// complex structures
Triangulation,
HilbertSeries,
InclusionExclusionData,
StanleyDec,
ClassGroup,
IntegerHull,
ConeDecomposition,
HilbertQuasiPolynomial,
//
// integer type for computations
//
BigInt,
//
// algorithmic variants
//
DefaultMode,
Approximate,
NoApproximation, // new
BottomDecomposition,
NoBottomDec,
DualMode,
PrimalMode, //new
Symmetrize, / new
NoSymmetrization, // new
NoSubdivision, // new
NoNestedTri, // new -- synonym for NoSubdivision
KeepOrder,
HSOP,
//
// checking properties of already computed data
// (cannot be used as a computation goal)
//
IsTriangulationNested, // new
IsTriangulationPartial, // new

```



```

        EnumSize // this has to be the last entry, to get the number of entries in the enum
    }; // remember to change also the string conversion function if you change this enum
}

```

The class `ConeProperties` is based on this enumeration. Its instantiation are essentially boolean vectors that can be accessed via the names in the enumeration. Instantiations of the class are used to set computation goals and algorithmic variants and to check whether the goals have been reached. The distinction between computation goals and algorithmic variants is not completely strict. See Section 4 for implications between some `ConeProperties`.

There exist versions of compute for up to 3 cone properties:

```

ConeProperties Cone<Integer>::compute(ConeProperty::Enum cp)

ConeProperties Cone<Integer>::compute(ConeProperty::Enum cp1,
                                     ConeProperty::Enum cp2)

ConeProperties Cone<Integer>::compute(ConeProperty::Enum cp1,
                                     ConeProperty::Enum cp2, ConeProperty::Enum cp3)

```

An example:

```

MyCone.compute(ConeProperty::HilbertBasis, ConeProperty::Multiplicity)

```

If you want to specify more than 3 cone properties, you can define an instance of `ConeProperties` yourself and call

```

ConeProperties Cone<Integer>::compute(ConeProperties ToCompute)

```

An example:

```

ConeProperties Wanted;
Wanted.set(ConeProperty::Triangulation, ConeProperty::HilbertBasis);
MyCone.compute(Wanted);

```

All `get...` functions that are listed in the next section, try to compute the data asked for if they have not yet been computed. Unless you are interested a single result, we recommend to use `compute` since the data asked for can then be computed in a single run. For example, if the Hilbert basis and the multiplicity are wanted, then it would be a bad idea to call `getHilbertBasis` and `getMultiplicity` consecutively. More importantly, however, is the lack of algorithmic variants if you use `get...` without `compute` beforehand.

If `DefaultMode` is not set, then `compute()` will throw a `NotComputableException` so that `compute()` cannot a value. In the presence of `DefaultMode`, the returned `ConeProperties` are those that have not been computed.

Please inspect `cone_property.cpp` for the full list of methods implemented in the class `ConeProperties`. Here we only mention the constructors

```

ConeProperties::ConeProperties(ConeProperty::Enum p1)

ConeProperties::ConeProperties(ConeProperty::Enum p1, ConeProperty::Enum p2)

```

```
ConeProperties::ConeProperties(ConeProperty::Enum p1, ConeProperty::Enum p2,
                             ConeProperty::Enum p3)
```

and the functions

```
ConeProperties& ConeProperties::set(ConeProperty::Enum p1, bool value)

ConeProperties& ConeProperties::set(ConeProperty::Enum p1, ConeProperty::Enum p2)

bool ConeProperties::test(ConeProperty::Enum Property) const
```

A string can be converted to a cone property and conversely:

```
ConeProperty::Enum toConeProperty(const string&)
const string& toString(ConeProperty::Enum)
```

## D.4. Retrieving results

As remarked above, all `get...` functions that are listed below, try to compute the data asked for if they have not yet been computed. As also remarked above, it is often better to use `compute` first.

The functions that return a matrix encoded as `vector<vector<Integer> >` have variants that return a matrix encoded in the `libnormaliz` class `Matrix<Integer>`. These are not listed below; see `cone.h`.

### D.4.1. Checking computations

In order to check whether a computation goal has been reached, one can use

```
bool Cone<Integer>::isComputed(ConeProperty::Enum prop) const
```

for example

```
bool done=MyCone.isComputed(ConeProperty::HilbertBasis)
```

### D.4.2. Rank, index and dimension

```
size_t Cone<Integer>::getEmbeddingDim()
size_t Cone<Integer>::getRank()
size_t Cone<Integer>::getRecessionRank()
long Cone<Integer>::getAffineDim()
size_t Cone<Integer>::getModuleRank()

Integer Cone<Integer>::getInternalIndex()
Integer Cone<Integer>::getUnitGroupIndex()
```

The *internal* index is only defined if original generators are defined. See Section D.4.13 for the external index.

The last three functions return values that are only well-defined after inhomogeneous computations.

#### D.4.3. Support hyperplanes and constraints

```
const vector< vector<Integer> >& Cone<Integer>::getSupportHyperplanes()  
size_t Cone<Integer>::getNrSupportHyperplanes()
```

The first function returns the support hyperplanes of the (homogenized) cone. The second function returns the number of support hyperplanes.

Together with the equations and congruences the support hyperplanes can also be accessed by

```
map< InputType , vector< vector<Integer> > > Cone<Integer>::getConstraints ()
```

The map returned contains three pairs whose keys are

```
Type::inequalities  
Type::equations  
Type::congruences
```

Note that equations congruences can also be accessed via the coordinate transformation (to which they belong internally). See Section D.4.13.

#### D.4.4. Extreme rays and vertices

```
const vector< vector<Integer> >& Cone<Integer>::getExtremeRays()  
size_t Cone<Integer>::getNrExtremeRays()  
const vector< vector<Integer> >& Cone<Integer>::getVerticesOfPolyhedron()  
size_t Cone<Integer>::getNrVerticesOfPolyhedron()
```

In the inhomogeneous case the first function returns the extreme rays of the recession cone, and the second the vertices of the polyhedron. (Together they form the extreme rays of the homogenized cone.)

#### D.4.5. Generators

```
const vector< vector<Integer> >& Cone<Integer>::getOriginalMonoidGenerators()  
size_t Cone<Integer>::getNrOriginalMonoidGenerators()
```

Note that original generators are not always defined. The system of generators of the cone that is used in the computations and its cardinality are returned by

```
const vector< vector<Integer> >& Cone<Integer>::getGenerators()  
size_t Cone<Integer>::getNrGenerators()
```

#### D.4.6. Lattice points in polytopes and elements of degree 1

```
const vector< vector<Integer> >& Cone<Integer>::getDeg1Elements()  
size_t Cone<Integer>::getNrDeg1Elements()
```

#### D.4.7. Hilbert basis

In the nonpointed case we need the maximal linear subspace of the cone:

```
const vector< vector<Integer> >& Cone<Integer>::getMaximalSubspace()  
size_t Cone<Integer>::getDimMaximalSubspace()
```

One of the prime results of Normaliz and its cardinality are returned by

```
const vector< vector<Integer> >& Cone<Integer>::getHilbertBasis()  
size_t Cone<Integer>::getNrHilbertBasis()
```

Inhomogeneous case the functions refer to the the Hilbert basis of the recession cone. The module generators of the lattice points in the polyhedron are accessed by

```
const vector< vector<Integer> >& Cone<Integer>::getModuleGenerators()  
size_t Cone<Integer>::getNrModuleGenerators()
```

#### D.4.8. Module generators over original monoid

```
const vector< vector<Integer> >&  
      Cone<Integer>::getModuleGeneratorsOverOriginalMonoid()  
size_t Cone<Integer>::getNrModuleGeneratorsOverOriginalMonoid()
```

#### D.4.9. Grading and dehomogenization

```
vector<Integer> Cone<Integer>::getGrading()  
Integer Cone<Integer>::getGradingDenom()
```

The second function returns the denominator of the grading.

```
vector<Integer> Cone<Integer>::getDehomogenization()
```

#### D.4.10. Enumerative data

```
mpq_class Cone<Integer>::getMultiplicity()
```

Don't forget that the multiplicity is measured by a rational, not necessarily integral polytope. Therefore it need not be an integer.

The Hilbert series is stored in a class of its own. It is retrieved by

```
const HilbertSeries& Cone<Integer>::getHilbertSeries()
```

It contains several data fields that can be accessed as follows (see `hilbert_series.h`):

```
const vector<mpz_class>& HilbertSeries::getNum() const;
const map<long, denom_t>& HilbertSeries::getDenom() const;

const vector<mpz_class>& HilbertSeries::getCyclotomicNum() const;
const map<long, denom_t>& HilbertSeries::getCyclotomicDenom() const;

long HilbertSeries::getDegreeAsRationalFunction() const;
long HilbertSeries::getShift() const;

bool HilbertSeries::isHilbertQuasiPolynomialComputed() const;
vector< vector<mpz_class> > HilbertSeries::getHilbertQuasiPolynomial() const;
long HilbertSeries::getPeriod() const;
mpz_class HilbertSeries::getHilbertQuasiPolynomialDenom() const;
```

The first four functions refer to two representations of the Hilbert series as a rational function in the variable  $t$ : the first has a denominator that is a product of polynomials  $(1 - t^g)^e$ , the second has a denominator that is a product of cyclotomic polynomials. In both cases the numerators are given by their coefficient vectors, and the denominators are lists of pairs  $(g, e)$  where in the second case  $g$  is the order of the cyclotomic polynomial.

The degree as a rational function is of course independent of the chosen representation, but may be negative, as well as the shift that indicates with which power of  $t$  the numerator starts. >since the denominator has a nonzero constant term in both cases, this is exactly the smallest degree in which the Hilbert function has a nonzero value.

The Hilbert quasipolynomial is represented by a vector whose length is the period and whose entries are itself vectors that represent the coefficients of the individual polynomials corresponding to the residue classes modulo the period. These integers must be divided by the common denominator that is returned by the last function.

For the input type `rees_algebra` we provide

```
Integer Cone<Integer>::getReesPrimaryMultiplicity()
```

#### D.4.11. Triangulation and disjoint decomposition

The triangulation, the size and the sum of the determinants are returned by

```
const vector< pair<vector<key_t>, Integer> >& Cone<Integer>::getTriangulation()
size_t Cone<Integer>::getTriangulationSize()
Integer Cone<Integer>::getTriangulationDetSum()
```

See Section 6.8 for the interpretation of these data. The first component of the pair is the vector of indices of the simplicial cones in the triangulation. Note that the indices are here counted from 0 (whereas they start from 1 in the `tri` file). The second component is the determinant.

The type of triangulation can be retrieved by

```
bool Cone<Integer>::isTriangulationNested()
bool Cone<Integer>::isTriangulationPartial()
```

If the disjoint decomposition has been computed, one gets the 0/1 vectors describing the facets to be removed

```
const vector<vector<bool> >& Cone<Integer>::getOpenFacets()
```

#### D.4.12. Stanley decomposition

The Stanley decomposition is stored in a list whose entries correspond to the simplicial cones in the triangulation:

```
const list< STANLEYDATA<Integer> >& Cone<Integer>::getStanleyDec()
```

Each entry is a record of type STANLEYDATA defined as follows:

```
struct STANLEYDATA {
vector<key_t> key;
Matrix<Integer> offsets;
};
```

The key has the same interpretation as for the triangulation, namely as the vector of indices of the generators of the simplicial cone (counted from 0). The matrix contains the coordinate vectors of the offsets of the components of the decomposition that belong to the simplicial cone defined by the key. See Section 6.9 for the interpretation. The format of the matrix can be accessed by the following functions of class Matrix<Integer>:

```
size_t nr_of_rows() const
size_t nr_of_columns() const
```

The entries are accessed in the same way as those of vector<vector<Integer> >.

#### D.4.13. Coordinate transformation

The coordinate transformation from the ambient lattice to the sublattice generated by the Hilbert basis and the basis of the maximal subspace can be returned as follows:

```
const Sublattice_Representation<Integer>& Cone<Integer>::getSublattice()
```

An object of type Sublattice\_Representation models a sequence of  $\mathbb{Z}$ -homomorphisms

$$\mathbb{Z}^r \xrightarrow{\varphi} \mathbb{Z}^n \xrightarrow{\pi} \mathbb{Z}^r$$

with the following property: there exists  $c \in \mathbb{Z}$ ,  $c \neq 0$ , such that  $\pi \circ \varphi = c \cdot \text{id}_{\mathbb{Z}^r}$ . In particular  $\varphi$  is injective. One should view the two maps as a pair of coordinate transformations:  $\varphi$  is determined by a choice of basis in the sublattice  $U = \varphi(\mathbb{Z}^r)$ , and it allows us to transfer vectors from  $U \cong \mathbb{Z}^r$  to the ambient lattice  $\mathbb{Z}^n$ . The map  $\pi$  is used to realize vectors from  $U$  as linear combinations of the given

basis of  $U \cong \mathbb{Z}^r$ : after the application of  $\pi$  one divides by  $c$ . (If  $U$  is a direct summand of  $\mathbb{Z}^n$ , one can choose  $c = 1$ , and conversely.) Normaliz considers vectors as rows of matrices. Therefore  $\varphi$  is given as an  $r \times n$ -matrix and  $\pi$  is given as an  $n \times r$  matrix.

The data just described can be accessed as follows (sublattice\_representation.h). For space reasons we omit the class specification Sublattice\_Representation<Integer>::

```
const vector<vector<Integer> >& getEmbedding() const
const vector<vector<Integer> >& getProjection() const
Integer getAnnihilator() const
```

Here “Embedding” refers to  $\varphi$  and “Projection” to  $\pi$  (though  $\pi$  is not always surjective). The “Annihilator” is the number  $c$  above. (It annihilates  $\mathbb{Z}^r$  modulo  $\pi(\mathbb{Z}^n)$ .)

The numbers  $n$  and  $r$  are accessed in this order by

```
size_t getDim() const
size_t getRank() const
```

The external index, namely the order of the torsion subgroup of  $\mathbb{Z}^n/U$ , is returned by

```
mpz_class getExternalIndex() const
```

Very often  $\varphi$  and  $\psi$  are identity maps, and this property can be tested by

```
bool IsIdentity()const
```

The constraints computed by Normaliz are “hidden” in the sublattice representation. They can be accessed by

```
const vector<vector<Integer> >& getEquations() const
const vector<vector<Integer> >& getCongruences() const
```

But see Section D.4.3 above for a more direct access.

#### D.4.14. Class group

```
vector<Integer> Cone<Integer>::getClassGroup()
```

The return value is to be interpreted as follows: The entry for index 0 is the rank of the class group. The remaining entries contain the orders of the summands in a direct sum decomposition of the torsion subgroup.

#### D.4.15. Integer hull

For the computation of the integer hull an auxiliary cone is constructed. A reference to it is returned by

```
Cone<Integer>& Cone<Integer>::getIntegerHullCone() const
```

For example, the support hyperplanes of the integer hull can be accessed by

```
MyCone.getIntegerHullCone().getSupportHyperplanes()
```

#### D.4.16. Excluded faces

Before using the excluded faces Normaliz makes the collection irredundant by discarding those that are contained in others. The irredundant collection (given by hyperplanes that intersect the cone in the faces) and its cardinality are returned by

```
const vector< vector<Integer> >& Cone<Integer>::getExcludedFaces()  
size_t Cone<Integer>::getNrExcludedFaces()
```

For the computation of the Hilbert series the all intersections of the excluded faces are computed, and for each resulting face the weight with which it must be counted is computed. These data can be accessed by

```
const vector< pair<vector<key_t>,long> >&  
Cone<Integer>::getInclusionExclusionData()
```

The first component of each pair contains the indices of the generators that lie in the face and the second component is the weight.

#### D.4.17. Boolean valued results

All the “questions” to the cone that can be asked by the boolean valued functions in this section start a computation if the answer is not yet known.

The first, The question

```
bool Cone<Integer>::isIntegrallyClosed()
```

does not trigger a computation of the full Hilbert basis. The computation stops as soon as the answer can be given, and this is the case when an element in the integral closure has been found that is not in the original monoid. Such an element is retrieved by

```
vector<Integer> Cone<Integer>::getWitnessNotIntegrallyClosed()
```

As discussed in Section 6.7.3 it can sometimes be useful to ask

```
bool Cone<Integer>::isPointed()
```

before a more complex computation is started.

The next two functions answer the question whether the Hilbert basis or at least the extreme rays live in degree 1.

```
bool Cone<Integer>::isDeg1ExtremeRays()  
bool Cone<Integer>::isDeg1HilbertBasis()
```

Finally we have

```
bool Cone<Integer>::isInhomogeneous()  
bool Cone<Integer>::isReesPrimary()
```

`isReesPrimary()` checks whether the ideal defining the Rees algebra is primary to the irrelevant maximal ideal.



## D.5. Exception handling

All exceptions that are thrown in `libnormaliz` are derived from the abstract class `NormalizException` that itself is derived from `std::exception`:

```
class NormalizException: public std::exception
```

The following exceptions must be caught by the calling program:

```
class ArithmeticException: public NormalizException
class BadInputException: public NormalizException
class NotComputableException: public NormalizException
class FatalException: public NormalizException
```

The `ArithmeticException` is leaves `libnormaliz` if a nonrecoverable overflow occurs (it is also used internally for the change of integer type). This should not happen for cones of integer type `mpz_class`, unless it is caused by a bug in the program. The `BadInputException` is thrown whenever the input is inconsistent; the reasons for this are manifold. The `NotComputableException` is thrown if a computation goal cannot be reached. The `FatalException` should never appear. It covers error situations that can only be caused by a bug in the program. At many places `libnormaliz` has `assert` verifications built in that serve the same purpose.

There are two more exceptions for the communication within `libnormaliz` that should not leave it:

```
class NonpointedException: public NormalizException
class NotIntegrallyClosedException: public NormalizException
```

## D.6. Control of output

By using

```
bool setVerboseDefault(bool v)
```

one can control the verbose output of `libnormaliz`. The default value is `false`. This is a global setting that effects all cones constructed afterwards. However, for every cone one can set an individual value of verbose by

```
bool Cone<Integer>::setVerbose(bool v)
```

Both functions return the previous value.

The default values of verbose output and error output are `std::cout` and `std::cerr`. These values can be changed by

```
void setVerboseOutput(std::ostream&)
void setErrorOutput(std::ostream&)
```

## D.7. A simple program

The example program is a simplified version of the program on which the experiments for the paper “Quantum jumps of normal polytopes” by W. Bruns, J. Gubeladze and M. Michałek, arXiv:1504.01036 are based. Its goal is to find a maximal normal lattice polytope  $P$  in the following sense: there is no normal lattice polytope  $Q \supset P$  that has exactly one more lattice point than  $P$ . ‘Normal’ means in this context that the Hilbert basis of the cone over  $P$  is given by the lattice points of  $P$ , considered as degree 1 elements in the cone.

The program generates normal lattice simplices and checks them for maximality. The dimension is set in the program, as well as the bound for the random coordinates of the vertices.

Let us have a look at source/maxsimplex/maxsimplex.cpp. First the more or less standard preamble:

```
#include <stdlib.h>
#include <vector>
#include <fstream>
#include <omp.h>
using namespace std;

#include "libnormaliz/libnormaliz.h"
#include "libnormaliz/cone.h"
#include "libnormaliz/vector_operations.h"
#include "libnormaliz/cone_property.h"
#include "libnormaliz/integer.h"
using namespace libnormaliz;
```

Since we want to perform a high speed experiment which is not expected to be arithmetically demanding, we choose 64 bit integers:

```
typedef long long Integer;
```

The first routine finds a random normal simplex of dimension `dim`. The coordinates of the vertices are integers between 0 and `bound`. We are optimistic that such a simplex can be found, and this is indeed no problem in dimension 4 or 5.

```
Cone<Integer> rand_simplex(size_t dim, long bound){

    vector<vector<Integer> > vertices(dim+1,vector<Integer> (dim));
    while(true){ // an eternal loop ...
        for(size_t i=0;i<=dim;++i){
            for(size_t j=0;j<dim;++j)
                vertices[i][j]=rand()%(bound+1);
        }

        Cone<Integer> Simplex(Type::polytope,vertices);
        // we must check the rank and normality
        if(Simplex.getRank()==dim+1 && Simplex.isDeg1HilbertBasis())
            return Simplex;
    }
}
```

```

vector<vector<Integer> > dummy_gen(1,vector<Integer>(1,1));
// to make the compiler happy
return Cone<Integer>(Type::cone,dummy_gen);
}

```

We are looking for a normal polytope  $Q \supset P$  with exactly one more lattice point. The potential extra lattice points  $z$  are contained in the matrix `jump_cands`. There are two obstructions for  $Q = \text{conv}(P, z)$  to be tested: (i)  $z$  is the only extra lattice point and (ii)  $Q$  is normal. It makes sense to test them in this order since most of the time condition (i) is already violated and it is much faster to test.

```

bool exists_jump_over(Cone<Integer>& Polytope,
                     const vector<vector<Integer> >& jump_cands){

    vector<vector<Integer> > test_polytope=Polytope.getExtremeRays();
    test_polytope.resize(test_polytope.size()+1);
    for(size_t i=0;i<jump_cands.size();++i){
        test_polytope[test_polytope.size()-1]=jump_cands[i];
        Cone<Integer> TestCone(Type::cone,test_polytope);
        if(TestCone.getNrDeg1Elements()!=Polytope.getNrDeg1Elements()+1)
            continue;
        if(TestCone.isDeg1HilbertBasis())
            return true;
    }
    return false;
}

```

In order to make the (final) list of candidates  $z$  as above we must compute the widths of  $P$  over its support hyperplanes.

```

vector<Integer> lattice_widths(Cone<Integer>& Polytope){

    if(!Polytope.isDeg1ExtremeRays()){
        cerr<< "Cone in lattice_widths is not defined by lattice polytope"<< endl;
        exit(1);
    }
    vector<Integer> widths(Polytope.getNrExtremeRays(),0);
    for(size_t i=0;i<Polytope.getNrSupportHyperplanes();++i){
        for(size_t j=0;j<Polytope.getNrExtremeRays();++j){
            // v_scalar_product is a useful function from vector_operations.h
            Integer test=v_scalar_product(Polytope.getSupportHyperplanes()[i],
                                           Polytope.getExtremeRays()[j]);

            if(test>widths[i])
                widths[i]=test;
        }
    }
    return widths;
}

```

```

int main(int argc, char* argv[]){

    time_t ticks;
    srand(time(&ticks));
    cout << "Seed " << ticks << endl; // we may want to reproduce the run

    size_t polytope_dim=4;
    size_t cone_dim=polytope_dim+1;
    long bound=6;
    vector<Integer> grading(cone_dim,0);
        // at some points we need the explicit grading
    grading[polytope_dim]=1;

    size_t nr_simplex=0; // for the progress report

```

Since the computations are rather small, we suppress parallelization (except for one step below).

```

    while(true){

#ifdef _OPENMP
        omp_set_num_threads(1);
#endif
        Cone<Integer> Candidate=rand_simplex(polytope_dim,bound);
        nr_simplex++;jordan
        if(nr_simplex%1000 ==0)
            cout << "simplex " << nr_simplex << endl;

```

Maximality is tested in 3 steps. Most often there exists a lattice point  $z$  of height 1 over  $P$ . If so, then  $\text{conv}(P, z)$  contains only  $z$  as an extra lattice point and it is automatically normal. In order to find such a point we must move the support hyperplanes outward by lattice distance 1.

```

    vector<vector<Integer> > supp_hyps_moved=Candidate.getSupportHyperplanes();
    for(size_t i=0;i<supp_hyps_moved.size();++i)
        supp_hyps_moved[i][polytope_dim]+=1;
    Cone<Integer> Candidate1(Type::inequalities,supp_hyps_moved,
                            Type::grading,to_matrix(grading));
    if(Candidate1.getNrDeg1Elements()>Candidate.getNrDeg1Elements())
        continue; // there exists a point of height 1

```

Among the polytopes that have survived the height 1 test, most nevertheless have suitable points  $z$  close to them, and it makes sense not to use the maximum possible height immediately. Note that we must now test normality explicitly.

```

    cout << "No ht 1 jump"<< " #latt " << Candidate.getNrDeg1Elements() << endl;
    // move the hyperplanes further outward
    for(size_t i=0;i<supp_hyps_moved.size();++i)

```

```

        supp_hyps_moved[i][polytope_dim]+=polytope_dim;
Cone<Integer> Candidate2(Type::inequalities,supp_hyps_moved,
                        Type::grading,to_matrix(grading));
cout << "Testing " << Candidate2.getNrDeg1Elements()
    << " jump candidates" << endl; // including the lattice points in P
if(exists_jump_over(Candidate,Candidate2.getDeg1Elements()))
    continue;

```

Now we can be optimistic that a maximal polytope  $P$  has been found, and we test all candidates  $z$  that satisfy the maximum possible bound on their lattice distance to  $P$ .

```

cout << "No ht <= 1+dim jump" << endl;
vector<Integer> widths=lattice_widths(Candidate);
for(size_t i=0;i<supp_hyps_moved.size();++i)
    supp_hyps_moved[i][polytope_dim]+=
        -polytope_dim+(widths[i])*(polytope_dim-2);

```

The computation may become arithmetically critical at this point. Therefore we use `mpz_class` for our cone. The conversion to and from `mpz_class` is done by routines contained in `convert.h`.

```

vector<vector<mpz_class> > mpz_supp_hyps;
convert(mpz_supp_hyps,supp_hyps_moved);
vector<mpz_class> mpz_grading=convertTo<vector<mpz_class> >(grading);

```

The computation may need some time now. Therefore we allow a little bit of parallelization.

```

#ifdef _OPENMP
    omp_set_num_threads(4);
#endif

```

Since  $P$  doesn't have many vertices (even if we use these routines for more general polytopes than simplices), we don't expect too many vertices for the enlarged polytope. In this situation it makes sense to set the algorithmic variant Approximate.

```

Cone<mpz_class> Candidate3(Type::inequalities,mpz_supp_hyps,
                        Type::grading,to_matrix(mpz_grading));
Candidate3.compute(ConeProperty::Deg1Elements,ConeProperty::Approximate);
vector<vector<Integer> > jumps_cand; // for conversion from mpz_class
convert(jumps_cand,Candidate3.getDeg1Elements());
cout << "Testing " << jumps_cand.size() << " jump candidates" << endl;
if(exists_jump_over(Candidate, jumps_cand))
    continue;

```

Success!

```

cout << "Maximal simplex found" << endl;
cout << "Vertices" << endl;
Candidate.getExtremeRaysMatrix().pretty_print(cout); // a goody from matrix.h
cout << "Number of lattice points = " << Candidate.getNrDeg1Elements();
cout << " Multiplicity = " << Candidate.getMultiplicity() << endl;

```

```
    } // end while  
} // end main
```

For the compilation of `maxsimplex.cpp` we have added a Makefile. Running the program needs a little bit of patience. However, within a few hours a maximal simplex should have emerged. From a log file:

```
simplex 143000  
No ht 1 jump #latt 9  
Testing 22 jump candidates  
No ht 1 jump #latt 10  
Testing 30 jump candidates  
No ht 1 jump #latt 29  
Testing 39 jump candidates  
No ht <= 1+dim jump  
Testing 173339 jump candidates  
Maximal simplex found  
Vertices  
1 3 5 3 1  
2 3 0 3 1  
3 0 5 5 1  
5 2 2 1 1  
6 5 6 2 1  
Number of lattice points = 29 Multiplicity = 275
```

## E. PyNormaliz

Before you can install PyNormaliz (written by Sebastian Gutsche), you must install Normaliz with shared libraries via `make install` (using `cmake` or `autotools`). PyNormaliz expects the installed files in a standard location.

For PyNormaliz itself type

```
pip install PyNormaliz==1.1
```

at a command prompt.

For a brief introduction (and the time being) please consult the file `README` in the directory PyNormaliz of the distribution.

## F. QNormaliz

The variant QNormaliz of Normaliz can use coefficients from subfields of  $\mathbb{R}$ , for example from real algebraic extensions of  $\mathbb{Q}$ . It is clear that the computations are then restricted to those that do not depend on lattice data. At present QNormaliz is restricted to the computations of

support hyperplanes/extreme rays and to triangulations. Instead of the template name `Integer` (see Appendix D) it uses `Number` to emphasize the difference.

The included implementation specializes `Number` to `mpq_class`. This only serves as a demonstration example. From the practical viewpoint it is superfluous since `Normaliz` can process rational data (made integral). Nevertheless have a look at the examples in `Qexample`.

## F.1. Prerequisites

In order to specialize `Number` to your field of coefficients you must of course make it available via a suitable class library in the compilation process.

In the code of `QNormaliz` the following adaptations are necessary:

1. In `Qnormaliz.cpp` replace `mpq_class` in the definition of `Cone`.
2. In `libQnormaliz-templated.cpp` you must do the same replacement.
3. If your number type has a convenient conversion to `string`, modify the specialization of the templated function `toString` in `Qinteger.h`. Otherwise remove the specialization for `mpq_class`.
4. In `Qvector_operations.cpp` you must replace the definition of `v_simplify` by a version for your numbers.

The last point is extremely important since the coefficients of the linear forms representing the support hyperplanes or the vectors representing the extreme rays tend to explode rapidly in size if not kept under control. For `mpq_class` this means that we multiply vectors by the least common denominator of the components and then extract the gcd of the numerators. (If you really need a demonstration of what happens otherwise, switch off the function by an immediate return, and run `QNormaliz` on the example `small` in `example`.)

## F.2. Restrictions

The following input types are NOT allowed in `QNormaliz`:

<code>lattice</code>	<code>congruences</code>	<code>grading</code>
<code>cone_and_lattice</code>	<code>inhom_congruences</code>	<code>offset</code>

Inhomogeneous types are allowed. Also saturation is allowed. It must be interpreted as a generating set for a subspace that is intersected with all the objects defined by other input items.

The only computation goals and algorithmic variants allowed are:

<code>Generators</code>	<code>ExtremeRays</code>	<code>VerticesOfPolyhedron</code>	<code>MaximalSubspace</code>
<code>SupportHyperplanes</code>	<code>Equations</code>	<code>Triangulation</code>	<code>ConeDecomposition</code>
<code>Dehomogenization</code>	<code>Rank</code>	<code>EmbeddingDim</code>	<code>Sublattice</code>
<code>KeepOrder</code>	<code>IsPointed</code>	<code>IsInhomogeneous</code>	<code>DefaultMode</code>

It may seem paradoxical that `Sublattice` appears here. As in the true lattice case, the `Sublattice Representation` is the coordinate transformation used by `QNormaliz`. Over a field  $F$  there is no need for the annihilator  $c$ , and one simply has a pair of linear maps  $F^r \rightarrow F^d \rightarrow F^r$  whose composition is the identity of  $F^r$ . Of course, congruences and external index make no sense anymore.

Moreover, the original monoid and any data referring to it are not defined.

Implicit or explicit `DefaultMode` is interpreted as `SupportHyperplanes`.

### F.3. Results

In the output file we use the same terminology as in that of `Normaliz`, for example, “basis elements of sublattice”. In `QNormaliz` these vectors form a basis of the vector subspace spanned by the cone that is computed. No reference to a sublattice is implied.

As far as possible, we apply `v_simplify` to the computation results. Note that it is not necessarily applied to all results. For example the generators of the cone (contained in the file with suffix `tn` or returned by `getGenerators`) are not necessarily simplified. (In particular, they need not be identical to those computed by `Normaliz` (if applicable to the input), but can differ from them by order and rational factors.)

As for `Normaliz`, for dehomogenization one must divide a vector by its value under the linear form that we call dehomogenization. Since fractions can be formed now, `Normaliz` could divide by the dehomogenization, but it does not do it in favor of the output in simplified form.

Note that there is no way for `QNormaliz` to define an invariant volume function on subspaces. (In `Normaliz` the situation is different since determinants of invertible matrices over  $\mathbb{Z}$  are equal to  $\pm 1$ .) Therefore `QNormaliz` does not define multiplicities. Together with triangulations, it computes determinants of the vectors spanning the simplicial cones. They are defined relative to the choice of basis shown as “basis of sublattice” if the cone is not fulldimensional. In the fulldimensional case `QNormaliz` uses the basis of unit vectors.

### F.4. Installation

The standard autotools build system automatically builds and installs `QNormaliz` as well.

As for `Normaliz` and `QNormaliz` one can use the “classical” Makefile. See `INSTALL` in the directory `source` (without `Q`!).

Moreover, `QNormaliz` can also be built by `cmake`. Simply follow the steps in Section 10.5 after having made a directory `BUILD_QNMZ` (for example) in the `Normaliz` directory. Then go to it and say `cmake ../Qsource`, followed by `make` and `make install`. With the default settings, `libQnormaliz` will be installed in `/usr/local`, subdirectories `bin`, `lib` and `include/libQnormaliz`.



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