RATES OF CONVERGENCE IN W_p^2 -NORM FOR THE MONGE-AMPÈRE EQUATION

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Abstract. We develop discrete W_p^2 -norm error estimates for the Oliker-Prussner method applied to the Monge-Ampère equation. This is obtained by extending discrete Alexandroff estimates and showing that the contact set of a nodal function contains information on its second order difference. In addition, we show that the size of the complement of the contact set is controlled by the consistency of the method. Combining both observations, we show that the error estimate

$$\|u - u_h\|_{W_p^2(\mathbb{N}_h^I)} \le C \begin{cases} h^{1/p} & \text{if } p > d, \\ h^{1/d} \left(\ln\left(\frac{1}{h}\right)\right)^{1/d} & \text{if } p \le d, \end{cases}$$

where the constant C depends on $||u||_{C^{3,1}(\bar{\Omega})}$, the dimension d, and the constant p. Numerical examples are given in two space dimensions and confirm that the estimate is sharp in several cases.

1. Introduction. In this paper we develop discrete W_p^2 error estimates for numerical approximations of the Monge-Ampère equation with Dirichlet boundary conditions:

$$\det(D^2 u) = f \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = 0$$
 on $\partial \Omega$, (1.1b)

with given function $f \in C(\bar{\Omega})$ satisfying $\underline{f} \leq f \leq \bar{f}$ in $\bar{\Omega}$, for some positive constants \underline{f}, \bar{f} . Here, D^2u denotes the Hessian matrix of u. The domain $\Omega \subset \mathbb{R}^d$ is assumed to be bounded and uniformly convex. We seek a solutions to (1.1) in the class of convex functions, which ensures ellipticity of the problem and its unique solvability [11].

The method we analyze in this paper is due to Oliker and Prussner [17], which is based on a geometric notion of generalized solutions called Alexandroff solutions. In this setting, the determinant of the Hessian matrix of u in (1.1a) is interpreted as the measure of the sub-differential of u; see [11]. The method proposed in [17] simply poses this solution concept onto the space of nodal functions and enforces the geometric condition implicitly given in (1.1a) at a finite number of points. Namely, the method seeks a nodal function u_h satisfying the Dirichlet boundary conditions on boundary nodes, and

$$|\partial u_h(x_i)| = f_i$$

at all interior grid points x_i . Here, $\partial u_h(x_i)$ denotes the sub-differential of u_h at x_i , $|\cdot|$ is the d-dimensional Lebesgue measure, $f_i \approx h^d f(x_i)$, and h is the mesh parameter. Existence and uniqueness of the method, and convergence to the Alexandroof solution is shown in two dimensions in [17].

Recently, Nochetto and the second author derived pointwise error estimates of the Oliker-Prussner scheme [19]. There it is shown that, if the exact convex solution to (1.1) is sufficiently smooth, and if the nodes are translation invariant, then the error is of (optimal) order $\mathcal{O}(h^2)$ in the L_{∞} norm. Generalities of these results, depending on solution regularity, are also given. The main tools to develop these results include operator consistency estimates, the Brunn-Minkowski inequality, and discrete Alexandroff-Bakelman-Pucci estimates for continuous, piecewise linear functions [12, 18].

Our contribution in this paper is to extend these results and to develop discrete W_p^2 error estimates for all $p \in [1, \infty)$. To summarize this result, we first introduce a discrete W_p^2

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norm for discrete nodal functions. We define the second-order difference operator of a nodal or continuous function v in the direction $e \in \mathbb{Z}^d$ at a node x_i as

$$\delta_e v(x_i) := \frac{v(x_i + he) - 2v(x_i) + v(x_i - he)}{|e|^2 h^2},$$

where |e| denotes the Euclidean norm of e, and it is assumed that $x_i \pm he$ is also a node in the domain $\bar{\Omega}$. If either $x_i - he$ or $x_i + he$ is outside Ω , we define

$$\delta_e v(x_i) := \frac{\rho_2 v(x_i + \rho_1 h e) - (\rho_1 + \rho_2) v(x_i) + \rho_1 v(x_i - \rho_2 h e)}{\rho_1 \rho_2 (\rho_1 + \rho_2) |e|^2 h^2 / 2},$$

where ρ_1 and ρ_2 are the largest number in (0,1] such that $x_i + \rho_1 he$ and $x_i - \rho_2 he$ are in $\bar{\Omega}$, respectively. The (weighted) W_p^2 -norm of a nodal function v with respect to direction e on a set of nodes S is given by

$$||v||_{W_p^2(S)} := \Big(\sum_{x_i \in S} f_i |\delta_e v(x_i)|^p\Big)^{1/p}.$$

The main result of the paper, precisely given in Theorem 5.3, is the estimate

$$||N_h u - u_h||_{W_p^2(\mathcal{N}_h^I)} \le \begin{cases} Ch^{1/p} & \text{if } p > d, \\ Ch^{1/d} \ln\left(\frac{1}{h}\right)^{1/d} & \text{if } p \le d, \end{cases}$$

where $N_h u$ denotes the nodal interpolant of u. Similar to the arguments in [19], one of the tools we use is operator consistency of the method. In addition, we extend the discrete Alexandroff-Bakelman-Pucci estimates given in [12, 18], and show that the contact set also contains useful information about the second-order differences.

Because of its wide array of applications in e.g., differential geometry, optimal mass transport, and meteorology, several numerical methods have been developed for the Monge-Ampère problem. These include the monotone finite difference schemes [16, 10, 5, 13], the vanishing moment method [8], C^1 finite element methods [4, 2], C^0 penalty methods [6, 14, 1], and semi-Lagrangian schemes [9]. We also refer the interested reader to a review of numerical methods for fully nonlinear elliptic equations [15]. One application of our results is to feed the solution of the Oliker-Prussner method into a higher-order scheme. For example, the results given in [14] state that Newton's method converges to the discrete solution provided that difference between the initial guess and the exact solution is sufficiently small in a W_p^2 -norm. Therefore, we show that the solution of the Oliker-Prussner scheme can be used as an initial guess within a higher-order scheme. We will explore this idea in a coming paper.

The organization of the paper is as follows. In the next section, we state the Oliker-Prussner method and state some preliminary results. In Section 3 we give operator consistency results of the scheme. Section 4 gives stability results with respect to the second-order difference operators, and in Section 5 we provide W_p^2 error estimates. Finally, we end the paper with some numerical experiments in Section 6.

2. Preliminaries.

2.1. Nodal Set and Nodal Function. Let \mathcal{N}_h be a set of nodes in the domain Ω . We denote the set of interior nodes $\mathcal{N}_h^I := \mathcal{N}_h \cap \Omega$, the set of boundary nodes $\mathcal{N}_h^B := \mathcal{N}_h \cap \partial \Omega$, and the nodal set

$$\mathcal{N}_h = \mathcal{N}_h^I \cup \mathcal{N}_h^B.$$

To ensure that the interior node is not too close to the boundary $\partial\Omega$, we require that

$$\operatorname{dist}(z,\partial\Omega) \ge \frac{h}{2}$$
 for any nodes $z \in \mathcal{N}_h^I$ (2.1)

Such a nodal set can be obtained by removing the nodes whose distance to $\partial\Omega$ is less than h/2. We assume that the nodal set is translation invariant, i.e., there exist a point $b \in \mathbb{R}^d$ and a basis $\{e_i\}_{i=1}^d$ in \mathbb{R}^d such that any interior node $z \in \mathbb{N}_b^I$ can be written as

$$z = b + \sum_{i=1}^{d} h z_i e_i$$
 for some integers $z_i \in \mathbb{Z}$. (2.2)

Since the basis e_i can be transformed into the canonical basis in \mathbb{R}^d under a linear transformation, hereafter to simplify the presentation, we will assume that $\mathcal{N}_h^I = b + h\mathbb{Z}^d$. We also make the following additional assumption on the boundary nodal set \mathcal{N}_h^B :

$$\operatorname{dist}(x, \mathcal{N}_h^B) \le h, \quad \forall x \in \partial \Omega.$$
 (2.3)

We say the nodal spacing of \mathcal{N}_h is h. It is worth mentioning that one can construct a translation invariant \mathcal{N}_h on a curved domain Ω . In fact, for a nodal set \mathcal{N}_h to be translation invariant, we only require the interior nodal set \mathcal{N}_h^I satisfies (2.2), while no such requirement is made on the boundary nodes.

Associated with the nodes is a simplicial triangulation \mathcal{T}_h , with vertices \mathcal{N}_h . We denote by h_T the diameter of $T \in \mathcal{T}_h$, and by ρ_T the diameter of the largest inscribed ball in T. We assume that that the triangulation is shape-regular, i.e., there exists $\sigma > 0$ such that

$$\frac{h_T}{\rho_T} \le \sigma \qquad \forall T \in \mathfrak{T}_h.$$

We denote by $\{\phi_i\}_{i=1}^n$, with $n=\#\mathcal{N}_h^I$, the canonical piecewise linear hat functions associated with \mathcal{T}_h . Namely, the function $\phi_i \in C(\bar{\Omega})$ is a piecewise linear polynomial with respect to \mathcal{T}_h , and is uniquely determined by the condition $\phi_i(x_j) = \delta_{i,j}$ (Kronecker delta) for all $x_j \in \mathcal{N}_h^I$ and $\phi_i(x_j) = 0$ for all $x_j \in \mathcal{N}_h^B$. We denote by ω_i the support of ϕ_i , i.e., the patch of elements in \mathcal{T}_h that have x_i as a vertex.

A function defined on \mathbb{N}_h is called a nodal function, and we denote the space of nodal functions by \mathbb{M}_h . For a nodal function g with nodal value $\{g_i\}_{x_i\in\mathbb{N}_h}$, and for a subset of nodal points $\mathbb{C}\subset\mathbb{N}_h$, we set the discrete ℓ^d norm as

$$||g||_{\ell^d(\mathcal{C})} := \left(\sum_{x_i \in \mathcal{C}} |g_i|^d\right)^{1/d}.$$

We say that a nodal function $u_h \in \mathcal{M}_h$ is convex if, for all $x_i \in \mathcal{N}_h^I$, there exists a supporting hyperplane L of u_h , i.e.,

$$L(x_j) \le u_h(x_j) \quad \forall x_j \in \mathcal{N}_h \text{ and } L(x_i) = u(x_i).$$

The convex envelope of u_h is the function $\Gamma(u_h) \in C(\bar{\Omega})$ given by

$$\Gamma(u_h)(x) = \sup_{L} \{L(x) \text{ is affine} : L(x_i) \le u_h(x_i) \ \forall x_i \in \mathcal{N}_h\}.$$

Finally, we denote by $N_h: C(\bar{\Omega}) \to \mathcal{M}_h$ the nodal interpolant satisfying $N_h v(x_i) = v(x_i)$ for all $x_i \in \mathcal{N}_h$. It is easy to see that if v is a convex function on $\bar{\Omega}$, then $N_h v$ is a convex nodal function.

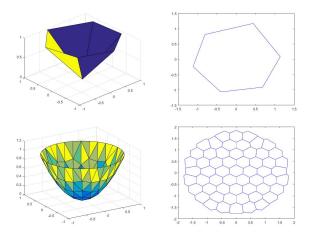


FIG. 2.1. A convex nodal function u_h induces a convex piecewise linear function $\gamma_h = \Gamma(u_h)$. The sub-differential $\partial u_h(0)$ of the convex nodal function u_h at node 0 is the convex hull of the piecewise gradients $\nabla \gamma_h|_T$, which is the polygon in the second figure. Let the domain Ω be a unit ball centered at 0 and \mathbb{N}_h be a nodal set in Ω . A convex nodal function u_h defined on \mathbb{N}_h induces a piecewise linear function $\Gamma(u_h)$. For each node $x_i \in \mathbb{N}_h$, there is an associated subdifferential $\partial u_h(x_i)$ which corresponds to a polygon cell in the last figure. The piecewise gradient of u_h can be viewed as a map between the domain Ω and the diagram.

2.2. The Oliker-Prussner Method. To motivate the method introduced in [17], we first introduce the notion of an Alexandroff solution to the Monge-Ampère equation (1.1). To this end, note that if the solution to (1.1) is strictly convex, and if $u \in C^2(\Omega)$, then a change of variables reveals that

$$\int_E f\,dx = \int_E \det(D^2u)\,dx = \int_{\nabla u(E)} dx = |\nabla u(E)| \quad \text{for all Borel } E \subset \Omega,$$

where $|\nabla u(E)|$ denote the d-dimensional Lebesgue measure of $\nabla u(E) = {\nabla u(x) : x \in E}$. To extend this identity to a larger class of functions, we introduce the subdifferential of the function u at the point x_0 as

$$\partial u(x_0) = \{ p \in \mathbb{R}^d : \ u(x) \ge u(x_0) + p \cdot (x - x_0) \quad \forall x \in \Omega \}.$$

Thus, $\partial u(x_0)$ is the set of supporting hyperplanes of the graph of u at x_0 . If u is strictly convex and smooth then $\partial u(x_0) = {\nabla u(x_0)}$, and the same calculation as above shows that

$$\int_{E} f \, dx = |\partial u(E)| \quad \text{for all Borel } E \subset \Omega. \tag{2.4}$$

DEFINITION 2.1. A convex function $u \in C(\bar{\Omega})$ is an Alexandroff solution to (1.1) provided that u = 0 on $\partial\Omega$ and (2.4) is satisfied.

The method introduced in [17] simply poses this solution concept onto the space of nodal functions. To do so, the definition of the subdifferential is extended to the spaces of nodal functions in the natural way:

$$\partial u_h(x_i) = \{ p \in \mathbb{R}^d : \ u(x_i) > u_h(x_i) + p \cdot (x_i - x_i) \ \forall x_i \in \mathcal{N}_h \}. \tag{2.5}$$

To characterize the sub-differential of a nodal function u_h , we note that the convex envelope of a convex nodal function u_h , which is a piecewise linear function defined in Ω , induces a mesh $\tilde{\mathfrak{T}}_h$; see Figure 2.1. Then the sub-differential of u_h at node x_i can be

characterized as the convex hull of the constant gradients $\nabla\Gamma(u_h)|_T$ for all $T\in \mathcal{T}_h$ which contain x_i ; see Figure 2.1.

The discrete method is to find a convex nodal function u_h with $u_h = 0$ on \mathcal{N}_h^B and

$$|\partial u_h(x_i)| = f_i \qquad \forall x_i \in \mathcal{N}_h^I, \tag{2.6}$$

where

$$f_i = \int_{\Omega} f(x)\phi_i(x) dx = \int_{\omega_i} f(x)\phi_i(x) dx.$$
 (2.7)

Remark 2.1. Existence and uniqueness of a solution to (2.6) is given in [17, 19].

2.3. Brunn Minkowski inequality and subdifferential of convex functions. In this subsection, we develop a few techniques which will be useful in establishing the error estimate. We start with the celebrated Brunn Minkowski inequality which relates the volumes of compact sets of \mathbb{R}^d .

PROPOSITION 2.1 (Brunn Minkowski inequality). Let A and B be two nonempty compact subsets of \mathbb{R}^d for $d \geq 1$. Then the following inequality holds:

$$|A+B|^{1/d} \ge |A|^{1/d} + |B|^{1/d},$$

where A + B denotes the Minkowski sum:

$$A + B := \{ v + w \in \mathbb{R}^d : v \in A \text{ and } w \in B \}.$$

Next, we make the following observation on the sum of two subdifferential sets.

LEMMA 2.2 (Lemma 2.3 in [19]). Let u_h and v_h be two convex nodal functions. Then there holds

$$\partial u_h(x_i) + \partial v_h(x_i) \subset \partial (u_h + v_h)(x_i)$$

for all $x_i \in \mathcal{N}_h^I$.

Proof. Let p_1 and p_2 be in $\partial u_h(x_i)$ and $\partial v_h(x_i)$, respectively. By the definition of subdifferential (2.5), we have

$$p_1 \cdot (x_j - x_i) \le u_h(x_j) - u_h(x_i) \quad \forall x_j \in \mathcal{N}_h,$$

$$p_2 \cdot (x_j - x_i) \le v_h(x_j) - v_h(x_i) \quad \forall x_j \in \mathcal{N}_h.$$

Adding both inequalities, we obtain

$$(p_1 + p_2) \cdot (x_i - x_i) \le (u_h + v_h)(x_i) - (u_h + v_h)(x_i) \quad \forall x_i \in \mathcal{N}_h.$$

This shows that $p_1 + p_2 \in \partial(u_h + v_h)(x_i)$. \square

Combining both estimates, we derive the following result.

LEMMA 2.3. Let u_h and v_h be two convex nodal functions defined on \mathcal{N}_h and \mathcal{C}_h be the lower contact set of $(u_h - v_h)$:

$$\mathcal{C}_h := \{ x_i \in \mathcal{N}_h^I : \ \Gamma(u_h - v_h)(x_i) = (u_h - v_h)(x_i) \}.$$

Then for any node $x_i \in \mathcal{C}_h$,

$$|\partial \Gamma(u_h - v_h)(x_i)|^{1/d} \le |\partial u_h(x_i)|^{1/d} - |\partial v_h(x_i)|^{1/d}.$$
 (2.8)

Proof. The proof of this result is implicitly given in [19, Proposition 4.3], but we give it here for completeness.

The definition of the convex envelope and the subdifferential shows that

$$\partial \Gamma(u_h - v_h)(x_i) \subset \partial (u_h - v_h)(x_i)$$

for all $x_i \in \mathcal{C}_{\epsilon}$. Applying Lemma 2.2 then yields

$$\partial v_h(x_i) + \partial \Gamma(u_h - v_h)(x_i) \subset \partial v_h(x_i) + \partial (u_h - v_h)(x_i) \subset \partial u_h(x_i).$$

An application of the Brunn-Minkowski inequality (cf. Lemma 2.1) gets

$$|\partial v_h(x_i)|^{1/d} + |\partial \Gamma(u_h - v_h)(x_i)|^{1/d} \le |\partial v_h(x_i) + \partial \Gamma(u_h - v_h)(x_i)|^{1/d}$$

$$\le |\partial u_h(x_i)|^{1/d}.$$

Rearranging terms we obtain (2.8). \square

We also note that the numerical method (2.6) has a discrete comparison principle. Here, we refer to [19] for a proof.

LEMMA 2.4 (discrete comparison principle, Corollary 4.4 in [19]). Let $v_h, w_h \in \mathcal{M}_h$ satisfy $v_h(x_i) \geq w_h(x_i)$ for all $x_i \in \mathcal{N}_h^B$ and $|\partial v_h(x_i)| \leq |\partial w_h(x_i)|$ for all $x_i \in \mathcal{N}_h^I$. Then

$$v_h(x_i) \ge w_h(x_i) \qquad \forall x_i \in \mathcal{N}_h.$$

3. Consistency of the Oliker-Prussner method. In this section, we state the consistency of the method (2.6) given in [19, Lemma 5.3, Proposition 5.4]. The result shows that the relative consistency error is of order $\mathcal{O}(h^2)$ away from the boundary and of order $\mathcal{O}(1)$ in a $\mathcal{O}(h)$ region of the boundary.

LEMMA 3.1. Let \mathcal{N}_h be translation invariant nodal set defined on the domain Ω . If $u \in C^{k,\alpha}(\bar{\Omega})$ is a convex function with $0 < \lambda I \le D^2 u \le \Lambda I$ and $2 \le k + \alpha \le 4$, there holds, for $\operatorname{dist}(x_i, \partial \Omega) \ge Rh$,

$$\left| \left| \partial N_h u(x_i) \right| - f_i \right| \le C h^{k+\alpha+d-2},\tag{3.1}$$

where R depends on λ and Λ . Moreover, there holds for $\operatorname{dist}(x_i, \partial\Omega) \leq Rh$,

$$|\partial N_h u(x_i) - f_i| \le Ch^d$$
.

REMARK 3.1. The regularity of f and $\partial\Omega$, the strict convexity of Ω , and the positivity of f guarantees that the convex solution to (1.1) enjoys the regularity $u \in C^{k,\alpha}(\bar{\Omega})$. For example, if $f \in C^{k-2,\alpha}(\bar{\Omega})$ and Ω is smooth, then the solutions satisfies $u \in C^{k,\alpha}(\bar{\Omega})$ [11, 7, 20]

Thanks to the consistency error of the method, Lemma 3.1, an L_{∞} -error estimate is derived in [19] which states

PROPOSITION 3.2. Let Ω be uniformly convex and \mathbb{N}_h^I be translation invariant. Suppose further that the boundary nodes satisfy (2.1), that $f \geq \underline{f} > 0$, and that the convex solution to (1.1) satisfies $u \in C^{k,\alpha}(\bar{\Omega})$ for some $2 \leq k + \alpha \leq 4$ and $0 < \lambda I \leq D^2 u \leq \Lambda I$. Then the numerical solution to the discrete Monge-Ampère equation (2.6) satisfies

$$||u_h - N_h u||_{L_{\infty}(\mathcal{N}_h)} \le C h^{k+\alpha-2} ||u||_{C^{k,\alpha}(\bar{\Omega})},$$

where $||v_h||_{L_{\infty}(\mathcal{N}_h)} := \max_{x_i \in \mathcal{N}_h} |v_h(x_i)|$.

We note that if $u \in C^{3,1}(\Omega)$, then the optimal order of the L_{∞} error is $O(h^2)$. By this L_{∞} error estimate and the assumption (2.1) that the boundary node is at least h/2 away from the boundary, we immediately deduce that $|\delta_e(N_h u - u_h)(x_i)|$ is bounded. This observation will be useful in the following sections when we investigate the discrete W_p^2 error estimate.

4. Stability of the Oliker-Prussner method. To derive the discrete W_p^2 -estimate, we first make an observation that the contact set of a nodal function contains interesting information on its second order difference.

LEMMA 4.1 (estimate of second order difference). Given two convex nodal functions v_h and u_h defined on the nodal set \mathcal{N}_h , let

$$w_{\epsilon} = u_h - (1 - \epsilon)v_h$$
 and $w^{\epsilon} = v_h - (1 - \epsilon)u_h$

for some $0 < \epsilon \le 1$ and the contact sets

$$\mathcal{C}_{\epsilon} := \{ x_i \in \mathcal{N}_h, \quad w_{\epsilon}(x_i) = \Gamma w_{\epsilon}(x_i) \}, \tag{4.1}$$

$$\mathcal{C}^{\epsilon} := \{ x_i \in \mathcal{N}_h, \quad w^{\epsilon}(x_i) = \Gamma w^{\epsilon}(x_i) \}. \tag{4.2}$$

If a node $x_i \in \mathcal{C}_{\epsilon} \cap \mathcal{C}^{\epsilon}$, then

$$-\epsilon \delta_e v_h(x_i) \le \delta_e (u_h - v_h)(x_i) \le \frac{\epsilon}{1 - \epsilon} \delta_e v_h(x_i)$$
(4.3)

for any vector $e \in \mathbb{Z}^d$.

Proof. We observe that if a node is in the contact set $x_i \in \mathcal{C}_{\epsilon}$, then the second order difference of w_{ϵ} satisfies $\delta_e w_{\epsilon}(x_i) \geq \delta_e \Gamma w_{\epsilon}(x_i) \geq 0$ for any vector $e \in \mathbb{Z}^d$. Hence, for any node $x_i \in \mathcal{C}_{\epsilon}$, we have

$$\delta_{e}(u_{h} - v_{h})(x_{i}) \ge -\epsilon \delta_{e} v_{h}(x_{i}). \tag{4.4}$$

This inequality yields a lower bound of the second order difference.

To derive the upper bound, we apply a similar argument above to the function w^{ϵ} and derive

$$\delta_e(v_h - u_h)(x_i) \ge -\epsilon \delta_e u_h(x_i)$$

for any node $x_i \in \mathcal{C}^{\epsilon}$. A simple algebraic manipulation yields

$$\delta_e(u_h - v_h)(x_i) \le \frac{\epsilon}{1 - \epsilon} \delta_e v_h(x_i). \tag{4.5}$$

Combining both the lower bound (4.4) and upper bound (4.5), we obtain the desired estimate. \square

REMARK 4.1. The lemma above shows that we have control of the error $\delta_e(u_h - v_h)$ on the contact sets \mathcal{C}_{ϵ} and \mathcal{C}^{ϵ} . Define the set E_{τ} to be

$$E_{\tau} = \left\{ x_i \in \mathcal{N}_h, \ \delta_e(v_h - u_h)(x_i) \ge \tau \delta_e v_h(x_i) \quad \text{for some vector } e \in \mathbb{Z}^d \right\}, \tag{4.6}$$

where $\tau = \epsilon/(1-\epsilon)$. Then the proof of Lemma 4.1 shows that E_{τ} is contained in the non-contact set

$$S_{\epsilon} := \mathcal{N}_h \setminus \mathcal{C}_{\epsilon}. \tag{4.7}$$

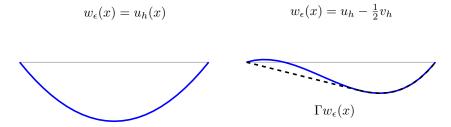


Fig. 4.1. A pictorial description of Remark 4.1

Analogously,

$$E^{\tau} := \left\{ x_i \in \mathcal{N}_h, \ \delta_e(u_h - v_h)(x_i) \ge \tau \delta_e v_h(x_i) \quad \text{for some vector } e \in \mathbb{Z}^d \right\}$$
$$\subset S^{\epsilon} := \mathcal{N}_h \setminus \mathcal{C}^{\epsilon}.$$

In the next step, we estimate the cardinality of S_{ϵ} . Heuristically, if $\epsilon = 1$, then $w_{\epsilon} = u_h$ which is a convex nodal function, and so we have $S_{\epsilon} = \emptyset$. As ϵ decreases to zero, the function w_{ϵ} becomes 'less convex', and the cardinality $\#(S_{\epsilon})$ increases; see Figure 4.1. Therefore, our next goal is to estimate how fast $\#(S_{\epsilon})$ increases as $\epsilon \to 0$. The following lemma shows that this is controlled by the consistency error of the method.

PROPOSITION 4.1. Let u_h and v_h be two convex nodal functions satisfying $u_h = v_h$ on \mathbb{N}_h^B , $u_h \leq v_h$ in \mathbb{N}_h^I , and

$$|\partial u_h(x_i)| = f_i, \quad and \quad |\partial v_h(x_i)| = g_i$$
 (4.8)

for all $x_i \in \mathbb{N}_h^I$. For any subset $S \subset \mathbb{N}_h^I$, let

$$\mu(S) = \sum_{x_i \in S} f_i \quad and \quad \nu_{\tau}(S) = \sum_{x_i \in S} \left(f_i^{1/d} + \frac{1}{\tau} e_i^{1/d} \right)^d, \tag{4.9}$$

where $e_i^{1/d} = g_i^{1/d} - f_i^{1/d}$. Then

$$\mu(S_{\epsilon}) \le \nu_{\tau}(\mathcal{C}_{\epsilon}) - \mu(\mathcal{C}_{\epsilon}), \tag{4.10}$$

where C_{ϵ} is given by (4.1), S_{ϵ} is given by (4.7), and $\tau = \epsilon/(1-\epsilon)$. Consequently, there holds

$$\mu(S_{\epsilon}) \le \tau^{-1} C_f \|e^{1/d}\|_{\ell^d(\mathcal{C}_{\epsilon})},\tag{4.11}$$

with $C_f = d \| f^{1/d} \|_{\ell^d(\mathbb{N}_h^I)}^{d-1}$.

Proof. We first show that

$$\sum_{x_i \in \mathcal{N}_h^I} \epsilon \partial u_h(x_i) \subset \sum_{x_i \in \mathcal{N}_h^I} \partial \Gamma w_{\epsilon}(x_i), \tag{4.12}$$

where $w_{\epsilon} = u_h - (1 - \epsilon)v_h$. Since $u_h \leq v_h$ in \mathcal{N}_h^I and $u_h = v_h$ on \mathcal{N}_h^B , we get

$$w_{\epsilon} \le \epsilon u_h \text{ in } \mathcal{N}_h^I, \quad \text{ and } \quad w_{\epsilon} = \epsilon u_h \text{ on } \mathcal{N}_h^B.$$

Taking convex envelope on both side of the inequality, we obtain

$$\Gamma w_{\epsilon}(x) \le \epsilon \Gamma u_h(x)$$
 in Ω and $\Gamma w_{\epsilon}(x) = \epsilon \Gamma u_h(x)$ on $\partial \Omega$. (4.13)

Since $u_h = \Gamma u_h$ on \mathcal{N}_h due to the convexity of u_h , the inequality (4.13) implies (4.12). Taking measure on both sides of (4.12) and substituting (4.8) yields

$$\epsilon^d \sum_{x_i \in \mathcal{N}_h^I} f_i = \epsilon^d \sum_{x_i \in \mathcal{N}_h^I} |\partial u_h(x_i)| \le \sum_{x_i \in \mathcal{C}_\epsilon} |\partial \Gamma w_\epsilon(x_i)|.$$

In view of the convexity of the measure of the subidfferential (2.8),

$$|\partial \Gamma w_{\epsilon}(x_i)|^{1/d} \le |f_i^{1/d} - (1 - \epsilon)g_i^{1/d}|.$$

Therefore, we infer that

$$\epsilon^d \mu(\mathbb{N}_h^I) = \epsilon^d \sum_{x_i \in \mathbb{N}_h^I} f_i \leq \sum_{x_i \in \mathbb{C}_\epsilon} |f_i^{1/d} - (1-\epsilon)g_i^{1/d}|^d.$$

Thus, subtracting $\epsilon^d \mu(\mathcal{C}_{\epsilon})$, we obtain

$$\epsilon^d \mu(S_{\epsilon}) = \epsilon^d \sum_{x_i \in S_{\epsilon}} f_i \le \sum_{x_i \in C_{\epsilon}} \left(|\epsilon f_i^{1/d} + (1 - \epsilon) e_i^{1/d}|^d - \epsilon^d f_i \right).$$

Therefore, dividing ϵ^d , we obtain

$$\mu(S_{\epsilon}) \leq \nu_{\tau}(\mathcal{C}_{\epsilon}) - \mu(\mathcal{C}_{\epsilon}).$$

To derive the estimate (4.11), we first see that (4.10) is equivalent to

$$||f^{1/d}||_{\ell^d(\mathcal{N}_h^I)} \le ||f^{1/d} + \tau^{-1}e^{1/d}||_{\ell^d(\mathcal{C}_{\epsilon})},$$

and therefore $||f^{1/d}||_{\ell^d(N_h^I)} - ||f^{1/d}||_{\ell^d(\mathcal{C}_{\epsilon})} \le \tau^{-1} ||e^{1/d}||_{\ell^d(\mathcal{C}_{\epsilon})}$ by the Minkowski inequality. From this estimate and the inequality $a^d - b^d \le da^{d-1}(a-b)$ for $a \ge b$, we derive

$$\mu(S_{\epsilon}) = \|f^{1/d}\|_{\ell^{d}(\mathbb{N}_{h}^{I})}^{d} - \|f^{1/d}\|_{\ell^{d}(\mathbb{C}_{\epsilon})}^{d}$$

$$\leq d\|f^{1/d}\|_{\ell^{d}(\mathbb{N}_{h}^{I})}^{d-1} (\|f^{1/d}\|_{\ell^{d}(\mathbb{N}_{h}^{I})} - \|f^{1/d}\|_{\ell^{d}(\mathbb{C}_{\epsilon})})$$

$$\leq C_{f}\tau^{-1}\|e^{1/d}\|_{\ell^{d}(\mathbb{C}_{\epsilon})}.$$

5. W_p^2 -estimate of the method. To establish W_p^2 -estimates of the method, we first introduce an estimate of the discrete L_1 norm of a nodal function in terms of its level sets.

LEMMA 5.1. Let s_h be a bounded nodal function with $|s_h(x_i)| \leq M$ for some M > 0. Then, for any $\sigma > 0$,

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |s_h(x_i)| \le \sigma \sum_{k=0}^N \mu(A_k),$$

where

$$A_k := \{ x_i \in \mathcal{N}_h^I : |s_h(x_i)| \ge k\sigma \},$$

 $\mu(\cdot)$ is given by (4.9), and $N = \lceil M/\sigma \rceil$.

Proof. The estimate is illustrated in the Figure 5. Here, we give a rigorous proof. Set

$$P_k := \{ x_i \in \mathcal{N}_h^I : k\sigma \le |s_h(x_i)| < (k+1)\sigma \}.$$

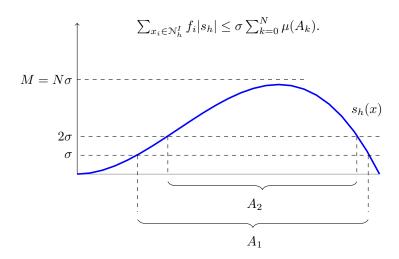


Fig. 5.1. A pictorial illustration of Lemma 5.1. Here, the measure $\mu(A_k) := \sum_{x_i \in A_k} f_i$.

Then we clearly have

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |s_h(x_i)| = \sum_{k=0}^N \sum_{x_i \in P_k} f_i |s_h(x_i)| \le \sum_{k=0}^N (k+1) \sigma \mu(P_k).$$

We also have

$$A_k = \bigcup_{m \ge k}^N P_m,$$

and so, since the sets $\{P_k\}$ are disjoint,

$$\mu(A_k) = \sum_{m=k}^{N} \mu(P_m).$$

Therefore

$$\sigma \sum_{k=0}^{N} \mu(A_k) = \sigma \sum_{k=0}^{N} \sum_{m=k}^{N} \mu(P_m) = \sigma \sum_{k=0}^{N} (k+1)\mu(P_k) \ge \sum_{x_i \in \mathcal{N}_h^I} f_i |s_h(x_i)|.$$

5.1. Ideal Case. Now we are ready to prove the estimate in the case that the consistency error (3.1) holds for all interior grid points.

Theorem 5.2. Let u be the solution of the Monge-Ampère equation (1.1). Assume that

$$|\partial N_h u(x_i)| - f_i| \le Ch^{2+d}$$
 for every node $x_i \in \mathcal{N}_h^I$, (5.1)

where $N_h u$ is the interpolation of u on the nodal set N_h . Assume further that f is uniformly positive on Ω . Then the error in the weighted W_p^2 -norm satisfies

$$||N_h u - u_h||_{W_p^2(\mathcal{N}_h^I)} \le C \begin{cases} h^2 |\ln h| & \text{if } p = 1, \\ h^{2/p} & \text{if } p > 1 \end{cases}$$

provided that h is sufficiently small.

Proof. We start by setting $v_h = (1 - Ch^2)^{1/d} N_h u$, where the constant C > 0 is large enough, but independent of h, to ensure that (cf. (5.1))

$$g_i := |\partial v_h(x_i)| = (1 - Ch^2)|\partial N_h u(x_i)| \le f_i.$$

By a comparison principle (cf. Lemma 2.4), we have $u_h \leq v_h$ on \mathcal{N}_h^I , and we see that

$$|f_i - g_i| \le Ch^{2+d} \qquad \forall x_i \in \mathcal{N}_h^I \tag{5.2}$$

due to the assumption (5.1). We also have $g_i \geq Ch^d$ provided h is sufficiently small, and $|(v_h - N_h u)(x_i)| \leq Ch^2$.

Note that

$$||N_h u - u_h||_{W_p^2(\mathcal{N}_h^I)} \le ||v_h - u_h||_{W_p^2(\mathcal{N}_h^I)} + Ch^2 ||N_h u||_{W_p^2(\mathcal{N}_h^I)}$$

Thus, to prove the theorem, it suffices to show that

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e(v_h - u_h)(x_i)|^p \le C \left\{ \begin{array}{ll} h^2 |\ln h| & \text{if } p = 1, \\ h^2 & \text{if } p > 1. \end{array} \right.$$

Define the positive and negative parts of $\delta_e(v_h - u_h)(x_i)$, respectively, as

$$\delta_e^+(v_h - u_h)(x_i) = \max\{\delta_e(v_h - u_h)(x_i), 0\},\$$

$$\delta_e^-(v_h - u_h)(x_i) = \max\{-\delta_e(v_h - u_h)(x_i), 0\}.$$

We shall prove

$$\sum_{x_i \in N_i^I} f_i |\delta_e^+(v_h - u_h)(x_i)|^p \le C \begin{cases} h^2 |\ln h| & \text{if } p = 1, \\ h^2 & \text{if } p > 1. \end{cases}$$

The estimate for the negative part can be proved in a similar fashion.

Due to the regularity assumption of u, a Taylor expansion shows that $|\delta_e v_h(x_i)| \leq C_2$ for all $x_i \in \mathbb{N}_h^I$, where $C_2 > 0$ depends on $||u||_{C^{1,1}(\bar{\Omega})}$. Moreover, from the L_{∞} error estimate, Proposition 3.2 and the assumption (2.1) that interior nodes are at least h/2 away from the boundary, we deduce that

$$\delta_e^+(v_h - u_h)(x_i) \le C_\infty \qquad \forall x_i \in \mathcal{N}_h^I,$$

where the constant $C_{\infty} > 0$ depends on $||u||_{C^{3,1}(\bar{\Omega})}$.

Let $\tau_k = C_2 k^{1/p} h^2$, and define the set

$$A_k := \{ x_i \in \mathcal{N}_h^I, \quad \delta_e^+(v_h - u_h)(x_i) \ge \tau_k \}.$$

By Lemma 5.1 with $s_h(x_i) = |\delta_e^+(v_h - u_h)(x_i)|^p$, $\sigma = C_2^p h^{2p}$, and $M = C_\infty^p$, we obtain

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e^+(v_h - u_h)(x_i)|^p \le C h^{2p} \left(\mu(\mathcal{N}_h^I) + \sum_{k=1}^{Ch^{-2p}} \mu(A_k) \right).$$
 (5.3)

We aim to estimate the measure of set $\mu(A_k)$. Due to the relations of the second order difference and contact set given in Remark 4.1, we have $A_k \subset S_{\epsilon_k} = \mathcal{N}_h^I \setminus \mathcal{C}_{\epsilon_k}$ with $\epsilon_k \in (0,1)$ satisfying $\tau_k = \epsilon_k/(1-\epsilon_k)$. Therefore, by the estimate (4.11) given in Proposition 4.1,

$$\mu(A_k) \leq \mu(S_{\epsilon_k}) \leq \frac{C_f}{\tau_k} \|g^{1/d} - f^{1/d}\|_{\ell^d(\mathcal{C}_{\epsilon_k})} = \frac{C_f}{k^{1/p} h^2} \|g^{1/d} - f^{1/d}\|_{\ell^d(\mathcal{C}_{\epsilon_k})}.$$

From the concavity of $t \to t^{1/d}$, we have $(t + \epsilon)^{1/d} - t^{1/d} \le d^{-1}t^{1/d-1}\epsilon$. Setting $t = g_i$ and $\epsilon = f_i - g_i \ge 0$, we get

$$|f_i^{1/d} - g_i^{1/d}| = f_i^{1/d} - g_i^{1/d} \le d^{-1}g_i^{1/d-1}(f_i - g_i) \le Ch^3$$

due to the consistency error (5.2) and the lower bound $g_i \geq Ch^d$. Consequently, we find that

$$||f^{1/d} - g^{1/d}||_{\ell^d(\mathcal{C}_{\epsilon_h})} \le Ch^2,$$

and therefore $\mu(A_k) \leq \frac{C}{k^{1/p}}$. Applying this bound in (5.3), we derive the estimate

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e^+(u_h - v_h)(x_i)|^p \le C h^{2p} \sum_{k=1}^{Ch^{-2p}} \frac{1}{k^{1/p}} \le C \begin{cases} h^2 |\ln h| & \text{if } p = 1, \\ h^2 & \text{if } p > 1. \end{cases}$$

This completes the proof. \Box

REMARK 5.1. It is worth mentioning that the assumption on the consistency error (5.1) holds for nodes bounded away from the boundary $\partial\Omega$ provided that $u \in C^{3,1}(\Omega)$. However, for nodes close to the boundary $\partial\Omega$, such an estimate holds only for structured domain, such as a rectangle domain; see the first numerical experiment in Section 6. In general, this estimate may not be true. In fact, Lemma 3.1 shows that the (relative) consistency error, O(h) away from the boundary, is of order O(1). In the following subsection, we take into account the lack of consistency in the boundary layer.

5.2. Estimate on general domain. To this end, we define the barrier nodal function

$$b_h(x_i) = \begin{cases} -h^2 & \text{if } x_i \in \mathcal{N}_h^I, \\ 0 & \text{if } x_i \in \mathcal{N}_h^B, \end{cases}$$

which will be used to "push down" the graph of the nodal interpolant of u and as such, develop error estimates in a general setting.

THEOREM 5.3. Let $u \in C^{3,1}(\bar{\Omega})$ be the solution of the Monge-Ampère equation (1.1) with $0 < \lambda I \le D^2 u \le \Lambda I$, and assume that the nodal set \mathcal{N}_h^I translation invariant and that f is uniformly positive on Ω . Then the error in the weighted $W^{2,p}$ -norm satisfies

$$||N_h u - u_h||_{W_p^2(\mathcal{N}_h^I)} \le C \begin{cases} h^{1/p} & \text{if } p > d, \\ h^{1/d} \left(\ln\left(\frac{1}{h}\right)\right)^{1/d} & \text{if } p \le d, \end{cases}$$

where $N_h u$ is the interpolation of u on the nodal set \mathbb{N}_h and the constant C depends on $||u||_{C^{3,1}(\bar{\Omega})}$, the dimension d, and the constant p.

Proof. We define the boundary layer:

$$\Omega_h := \{x_i \in \mathbb{N}_h^I, \operatorname{dist}(x_i, \partial \Omega) \leq Rh\},\$$

where the constant R is the constant in the consistency error, Lemma 3.1, which depends on the ellipticity constants λ and Λ of D^2u . We set

$$v_h = N_h u - C b_h, \qquad g_i = |\partial v_h(x_i)|,$$

where the constant C > 0 is sufficiently large so that $u_h \leq v_h$; see Proposition 3.2. It is clear from the definition of b_h that

$$|\partial v_h(x_i)| = |\partial N_h u(x_i)|$$
 for any $x_i \in \mathcal{N}_h^I \setminus \Omega_h$

and

$$|\partial N_h u(x_i)| \ge |\partial v_h(x_i)| \ge 0$$
 for any $x_i \in \Omega_h$.

This implies that $|f_i - g_i| \leq Ch^{2+d}$ in $\mathcal{N}_h^I \setminus \Omega_h$ and $|f_i - g_i| \leq Ch^d$ in Ω_h . We have that $|\delta_e v_h(x_i)| \leq C_2$ and $|\delta_e (v_h - u_h)(x_i)| \leq C_{\infty}$ for all $x_i \in \mathcal{N}_h^I$. As in Theorem 5.2, we shall prove the estimate for the positive part:

$$\sum_{x_i \in \mathcal{N}_i^I} f_i \left(\delta_e^+(v_h - u_h)(x_i) \right)^p \le \begin{cases} Ch & \text{if } p > d, \\ Ch \ln \left(\frac{1}{h} \right) & \text{if } p = d. \end{cases}$$

The estimate for the negative part can be proved in a similar fashion. Also note that the estimate for p < d follows from the estimate of p = d and Hölder's inequality:

$$||N_h u - u_h||_{W^2(\mathbb{N}_+^I)} \le C_\mu ||N_h u - u_h||_{W^2(\mathbb{N}_+^I)}$$
 where $C_\mu := \mu(\mathbb{N}_h^I)^{1/p-1/d}$.

We set $\tau_k = C_2 k^{1/p} h$ and define the set

$$A_k := \{ x_i \in \mathcal{N}_h^I, \quad \left(\delta_e^+(v_h - u_h)(x_i) \right) \ge \tau_k \}.$$

Then, by similar arguments as in Theorem 5.2, we find by Lemma 5.1 that

$$\sum_{x_i \in \mathcal{N}_h^I} f_i \left(\delta_e^+(v_h - u_h)(x_i) \right)^p \le C_2 h^p \left(\mu(\mathcal{N}_h^I) + \sum_{k=1}^{h^{-p}} \mu(A_k) \right).$$
 (5.4)

To estimate the measure of set $\mu(A_k)$, we note that $A_k \subset S_{\epsilon_k} = \mathbb{N}_h^I \setminus \mathcal{C}_{\epsilon_k}$ with $\tau_k = \epsilon_k/(1-\epsilon_k)$. Invoking the estimate of the measure of the non-contact set S_{ϵ} stated in Proposition 4.1, we obtain

$$\mu(A_k) \leq \mu(S_{\epsilon_k}) \leq \nu_{\tau_k}(\mathcal{C}_{\epsilon_k}) - \mu(\mathcal{C}_{\epsilon_k}).$$

We then divide the estimate of $\nu_{\tau_k}(\mathcal{C}_{\epsilon_k}) - \mu(\mathcal{C}_{\epsilon_k})$ into two parts:

$$\nu_{\tau_k}(\mathcal{C}_{\epsilon_k}) - \mu(\mathcal{C}_{\epsilon_k}) = \sum_{x_i \in \mathcal{C}_{\epsilon_k}} \left[\left(f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d} \right)^d - f_i \right]$$

$$= \left(\sum_{x_i \in \mathcal{C}_{\epsilon_k} \cap \Omega_h} + \sum_{x_i \in \mathcal{C}_{\epsilon_k} \setminus \Omega_h} \right) \left[\left(f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d} \right)^d - f_i \right],$$

where we recall that $e_i^{1/d} = f_i^{1/d} - g_i^{1/d}$. Since $f_i^{1/d} = \mathcal{O}(h)$ and $g_i^{1/d} = \mathcal{O}(h)$, we have

$$\begin{split} \left| \left(f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d} \right)^d - f_i \right| &\leq \frac{d}{\tau_k} \max\{ \left| f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d} \right|, f_i^{1/d} \}^{d-1} |e_i^{1/d}| \\ &\leq \frac{C h^{d-1}}{\tau_k^d} |e_i^{1/d}|. \end{split}$$

In the set $\mathcal{C}_{\epsilon_k} \cap \Omega_h$, the consistency error satisfies $|e_i^{1/d}| = \mathcal{O}(h)$; see Lemma 3.1. Therefore, we have

$$\left| \left(f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d} \right)^d - f_i \right| \le \frac{Ch^d}{\tau_k^d} \qquad \forall x_i \in \mathcal{C}_{\epsilon_k} \cap \Omega_h.$$

On the other hand, in the set $\mathcal{C}_{\epsilon_k} \setminus \Omega_h$, we conclude as in Theorem 5.2, that $|e_i^{1/d}| = \mathcal{O}(h^3)$, and

$$\left| \left(f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d} \right)^d - f_i \right| \le \frac{Ch^{2+d}}{\tau_k^d}.$$

Combining both estimate and applying the fact that $\#(\mathcal{C}_{\epsilon_k} \cap \Omega_h) \leq Ch^{1-d}$ and $\#(\mathcal{C}_{\epsilon_k} \setminus \Omega_h) \leq Ch^{-d}$, we obtain

$$\nu_{\tau_k}(\mathcal{C}_{\epsilon}) - \mu(\mathcal{C}_{\epsilon}) \le \frac{Ch}{\tau_k^d} + \frac{Ch^2}{\tau_k^d} \le \frac{Ch}{\tau_k^d}$$

because $h \leq 1$. Hence, we conclude that

$$\mu(A_k) \le \frac{Ch}{\tau_k^d}.$$

Applying this estimate to (5.4), we arrive at

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e^+(v_h - u_h)(x_i)|^p \le C_2 h^p \sum_{k=1}^{h^{-p}} \frac{h}{h^d k^{d/p}}.$$

Since

$$\sum_{k=1}^{h^{-p}} \frac{1}{k^{d/p}} \le \begin{cases} C(d,p)h^{d-p} & \text{if } p > d, \\ C\ln\left(\frac{1}{h}\right) & \text{if } p = d, \end{cases}$$

we conclude that

$$\sum_{x_i \in \mathcal{N}_{\epsilon}^I} f_i |\delta_e^+(v_h - u_h)(x_i)|^p \le \begin{cases} Ch & \text{if } p > d, \\ Ch \ln\left(\frac{1}{h}\right) & \text{if } p = d. \end{cases}$$

Finally we note that by Hölder's inequality, there holds for p < d,

$$||v_h||_{W_p^2(\mathcal{N}_h^I)} = \left(\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e v_h(x_i)|^p\right)^{1/p}$$

$$\leq \left(\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e v_h(x_i)|^d\right)^{1/d} \left(\sum_{x_i \in \mathcal{N}_h^I} f_i\right)^{(d-p)/(dp)} \leq C||v_h||_{W_d^2(\mathcal{N}_h^I)}.$$

This completes the proof. \Box

6. Numerical experiments. In this section, we perform numerical examples to illustrate the accuracy of the method, and to compare the results with the theory. In the tests, we replace the homogeneous boundary condition (1.1b) with u = g on $\partial\Omega$. The theoretical results developed in the previous sections can be applied to this slightly more general problem with minor modifications.

We consider three different test problems, each reflecting different scenarios of regularity. Each set of problems is performed in two dimensions (d=2), and errors are reported in the (discrete) L_{∞} , H^1 , W_1^2 , and W_2^2 norms. Here, a nine-point stencil is used in the definition

of the W_p^2 norms with $e_1 = (1,0)$, $e_2 = (0,1)$, $e_3 = (1,1)$ and $e_4 = (1,-1)$. That is, with an abuse of notation, we set

$$||v||_{W_p^2(\mathbb{N}_h^I)}^p = \sum_{j=1}^4 \sum_{x_i \in \mathbb{N}_h^I} |\delta_{e_j} v(x_i)|^p.$$

As explained in [19] and in Section 2.2, a convex nodal function induces a triangulation of Ω whose set of vertices corresponds to \mathcal{N}_h . For a computed solution u_h , we associate with it a piecewise linear polynomial on the induced mesh, which we still denote by u_h , and use the quantity $||u - u_h||_{H^1(\Omega)}$ to denote the H^1 error in the experiments below.

A summary of the theoretical results in Sections 2.3 and 5 when d=2 are

$$||N_h u - u_h||_{L_{\infty}(\mathbb{N}_h^I)} = \mathcal{O}(h^2), \qquad ||N_h u - u_h||_{W_{\pi}^2(\mathbb{N}_h^I)} = \mathcal{O}(h^{1/2 - \epsilon}), \ p = 1, 2$$

for any $\epsilon > 0$, provided that $u \in C^{3,1}(\bar{\Omega})$.

Example I: Smooth Solution $u \in C^{\infty}(\bar{\Omega})$. We consider the example

$$u(x,y) = e^{\frac{x^2 + y^2}{2}}, \quad f(x,y) = (1 + x^2 + y^2)e^{x^2 + y^2}, \quad \text{and } \Omega = (-1,1)^2,$$
 (6.1)

and list the resulting errors and rates of the scheme in Table 6. The Table clearly shows that the errors decay with rate $\mathcal{O}(h^2)$ in all norms. This behavior matches the theoretical results of Proposition 3.2, but indicates that the W_p^2 estimates stated in Theorem 5.3 are not sharp.

h	L_{∞}	rate	H^1	rate	W_{1}^{2}	rate	W_{2}^{2}	rate
1	1.12e-01	0.00	2.24e-01		4.49e-01		1.44e+01	
1/2	4.78e-02	1.23	1.35e-01	0.73	6.02e-01	-0.42	4.24e-01	5.08
1/4	1.37e-02	1.80	4.35e-02	1.63	2.94e-01	1.03	1.93e-01	1.13
1/8	3.55e-03	1.95	1.16e-02	1.91	9.93e-02	1.57	6.34e-02	1.61
1/16	8.96e-04	1.99	2.94e-03	1.98	2.86e-02	1.80	1.80e-02	1.82
1/32	2.24e-04	2.00	7.39e-04	1.99	7.66e-03	1.90	4.79e-03	1.91
1/64	5.61e-05	2.00	1.85e-04	2.00	1.98e-03	1.95	1.24e-03	1.95

Table 6.1

Rate of convergence for a smooth solution (Example I).

Example II: Piecewise Smooth Solution $u \in W^2_{\infty}$. In this example, the domain is $\Omega = (-1, 1)^2$, and the exact solution and data are taken to be

$$u(x) = \begin{cases} 2|x|^2 & \text{in } |x| \le 1/2, \\ 2(|x| - 1/2)^2 + 2|x|^2 & \text{in } 1/2 \le |x|, \end{cases}$$

$$f(x) = \begin{cases} 16 & \text{in } |x| \le 1/2, \\ 64 - 16|x|^{-1} & \text{in } |x| \le 1/2, \\ \text{in } 1/2 \le |x|. \end{cases}$$

A simple calculation shows that $u \in C^{1,1}(\bar{\Omega})$ and $u \in C^4(\Omega \setminus \partial B_1)$, but $u \notin C^2(\bar{\Omega})$. The errors and rates of convergence are given in Table 6. The table shows that, while all errors tend to zero as the mesh is refined, the rates of convergence in the L_{∞} and W_1^2 norms are less obvious than the previous set of experiments. Nonetheless, while Theorem 5.3 assumes more regularity of the exact solution, we do observe a convergence rate of approximately $\mathcal{O}(h^{1/2})$ in the W_2^2 as stated in the theorem.

h	L_{∞}	rate	H^1	rate	W_{1}^{2}	rate	W_{2}^{2}	rate
1	4.02e-01	0.00	8.04e-01	0.00	1.61	0.00	1.61	0.00
1/2	4.19e-02	3.26	1.30e-01	2.63	6.08e-01	1.40	5.39e-01	1.58
1/4	2.89e-02	0.53	6.84 e-02	0.92	6.46e-01	-0.09	5.54e-01	-0.04
1/8	1.27e-02	1.18	3.50e-02	0.97	5.14e-01	0.33	4.54e-01	0.29
1/16	4.58e-03	1.47	1.38e-02	1.34	2.76e-01	0.90	3.15e-01	0.53
1/32	8.02e-04	2.51	3.59e-03	1.94	1.08e-01	1.35	2.08e-01	0.60
1/64	4.33e-04	0.89	1.50e-03	1.26	6.36e-02	0.77	1.56e-01	0.42

Table 6.2

Rate of convergence of piecewise smooth viscosity solution (Example II).

Example III: Singular Solution $u \in W_p^2$ with p < 2. In the last series of experiments, the domain is $\Omega = (-1,1)^2$, and the solution and data are

$$u(x) = \begin{cases} x^4 + \frac{3}{2}y^2/x^2 & \text{in } |y| \le |x|^3, \\ \frac{1}{2}x^2y^{2/3} + 2y^{4/3} & \text{in } |y| \ge |x|^3, \end{cases}$$

$$f(x) = \begin{cases} 36 - 9y^2/x^6 & \text{in } |y| \le |x|^3, \\ \frac{8}{9} - \frac{5}{9}x^2/y^{2/3} & \text{in } |y| > |x|^3. \end{cases}$$

This example is constructed in [21] to show that $D^2u(x)$ may not be in W_p^2 for large p for discontinuous f. The errors of the method for this problem are listed in Table 6. Because the exact solution does not enjoy W_2^2 regularity, it is not expected that the discrete solution will converge in the discrete W_2^2 norm, and this is observed in the table. However, we do observe convergence in the L_{∞} , H^1 , and W_1^2 norms with approximate rates $||N_h u - u_h||_{L_{\infty}(\mathcal{N}_h^I)} =$ $\mathbb{O}(h^{4/3}), \|N_h u - u_h\|_{H^1(\mathbb{N}_h^I)} = \mathbb{O}(h), \text{ and } \|N_h u - u_h\|_{W_1^2(\mathbb{N}_h^I)} = \mathbb{O}(h^{1/2}).$

h	L_{∞}	rate	H^1	rate	W_{1}^{2}	rate	W_2^2	rate
1	8.36e-01	0.00	1.67	0.00	3.35	0.00	3.35	0.00
1/2	2.34e-01	1.84	9.11e-01	0.88	5.48	-0.71	3.94	-0.24
1/4	1.86e-01	0.33	4.80e-01	0.92	4.90	0.16	4.02	-0.03
1/8	8.52e-02	1.13	2.41e-01	1.00	4.00	0.29	3.94	0.03
1/16	3.41e-02	1.32	1.02e-01	1.24	2.38	0.75	3.33	0.24
1/32	1.35e-02	1.34	4.79e-02	1.09	1.59	0.58	3.17	0.07

Table 6.3
Rate of convergence of W_p^2 solution with p < 2 (Example III).

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