MATH3013 Final Project

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Introduction

Maintaining a comfortable and consistent water temperature is crucial for various water environments. By precise temperature control, it is possible to ensure that the human body is always in a thermally comfortable water environment. This project focuses on the development of mathematical models to simulate the thermal behavior when hot water is introduced into water environments, with a particular emphasis on home bathtubs. Two models are developed in this project. The first model is based on Newton's cooling law and specific heat equations. The second model incorporates spatial factors using the heat conduction equation with Neumann boundary conditions. Both models aim to provide insights of improving the process of temperature control by understanding the dynamics of heat transfer and distribution.

Problem

In home bathtubs, users typically start with water at a desired temperature, but the water cools down as time passes. Then hot water is added to deal with the cooling.

1. How can we maintain the temperature of bathtub water to keep comfort?
2. How does the temperature of bathtub water change when persistantly adding hot water?
3. What impact do different environmental of physical parameters have on the temperature change of bathtub water?
4. What is the spatial temperature distribution over time in the bathtub?

Assumptions

- All the physical properties of air, water, and bathtub are stable.
- Any addition of water results in an immediate drainage of the same volume.
- The user is considered as an extension of the bathtub water.
- All radiation effects are neglected.

Geometric Consideration

The faucet is placed at $x = 0$. L, W, and H represent length, width, and height of the bathtub respectively. Here, the bathtub has dimensions of 1.7m × 1.0m × 0.7m. In the PDE model We will assume that the flow and heat transfer of the water occur only in the x-direction.

Model I

By Newton's cooling law and the specific heat equation, we can derive

- Cooling:

$$\frac{dT}{dt} = -\frac{hS}{cm_{tub}}(T - T_\infty) \tag{1}$$

- Heating:

$$\frac{dT}{dt} = \frac{dm}{dt}\frac{(T_{in} - T)}{m_{tub}} \tag{2}$$

- Complete Model:

$$\frac{dT}{dt} = \left(\frac{-hS - \dot{m}c}{cm_{tub}}\right)T + \left(\frac{hST_\infty + \dot{m}cT_{in}}{cm_{tub}}\right) \tag{3}$$

where T/u is temperature, t is time, h is heat transfer coefficient, S is surface area, $\dot{m} = 0.4$ kg/s is mass flow rate of water flowing in, m_{tub} is mass of tub water, c is specific heat capacity of water, $T_\infty/u_\infty = 20°$C is air temperature, and $T_{in}/u_{in} = 45°$C is temepratur of water flowing in.

Results

We notice that after turning on hot water, $T \to -\frac{\xi}{\eta}$ as $t \to \infty$, where $\xi > 0$ and $\eta < 0$.

$$\begin{cases} T(t) = -\dfrac{\xi}{\eta} + Ce^{\eta t}, \quad t \geq 0 \\ \xi = \left(\dfrac{hST_\infty + \dot{m}cT_{in}}{cm_{tub}}\right) \\ \eta = \left(\dfrac{-hS - \dot{m}c}{cm_{tub}}\right) \end{cases} \tag{4}$$

To maintain the water temperature around 37°C without wasting too much water, a strategy is to keep a slow, steady trickle of water flowing. We noticed that in Figure 2, turning on a 45°C water flow of around 0.1 kg/s can keep the water temperature in the bathtub stable at 37°C.

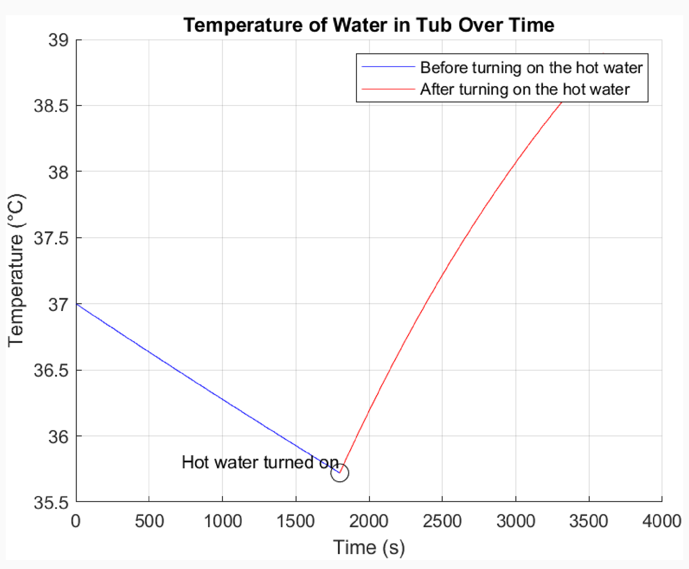

Figure 1: Temperature change within an hour

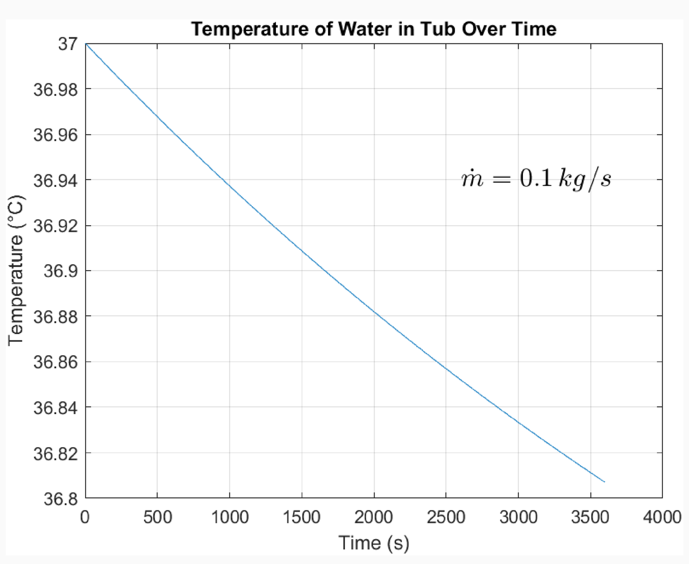

Figure 2: Trickling hot water flowing in

Model II

$$\begin{aligned} \frac{\partial u}{\partial t} &= \alpha\frac{\partial^2 u}{\partial x^2} + \tilde{Q}(x,t), \quad 0 < x < L, \quad t > 0, \quad \alpha \in \mathbb{R}, \\ u_x(x=0,t) &= \hat{h}(u_{in} - u(0,t)), \quad t > 0, \\ u_x(x=L,t) &= 0, \quad t > 0, \\ u(x,t=0) &= 37°\text{C}, \quad 0 < x < L, \\ \tilde{Q}(x,t) &= \tilde{h}(u_\infty - u(x,t)), \quad 0 < x < L, \quad t > 0. \end{aligned} \tag{5}$$

$$\tilde{Q}(x,t) = \tilde{h}(u_\infty - u(x,t)), \quad 0 < x < L, \quad t > 0.$$

$$\alpha = \frac{k}{\rho c},\ \hat{h} = \frac{\dot{m}c}{kWH},\ \text{and}\ \tilde{h} = \frac{1}{\rho c}\left(\frac{2h_w}{W} + \frac{h_b}{H} + \frac{h_t}{H}\right)$$

Results

Figure 3 illustates the temperature distribution in the bathtub. With two insulated ends, water temperature drops uniformly due to temperature differences.The results corresponding to Equation 4 are shown in Figure 4. Due to low thermal diffusivity of water, the result is expected.

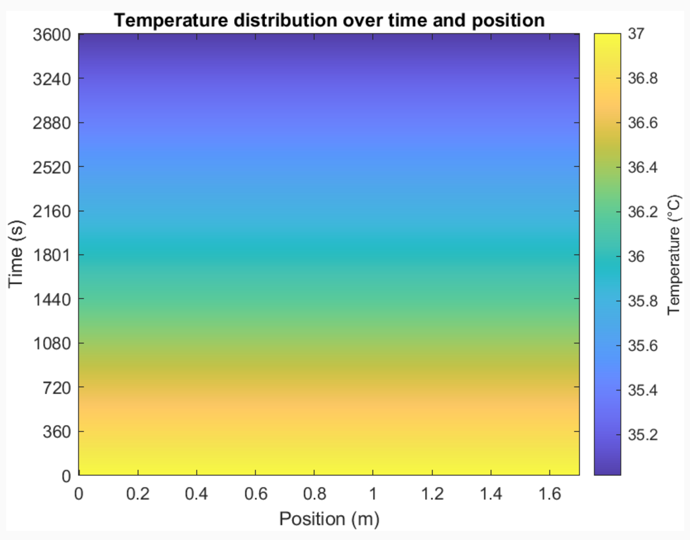

Figure 3: Temperature change within an hour

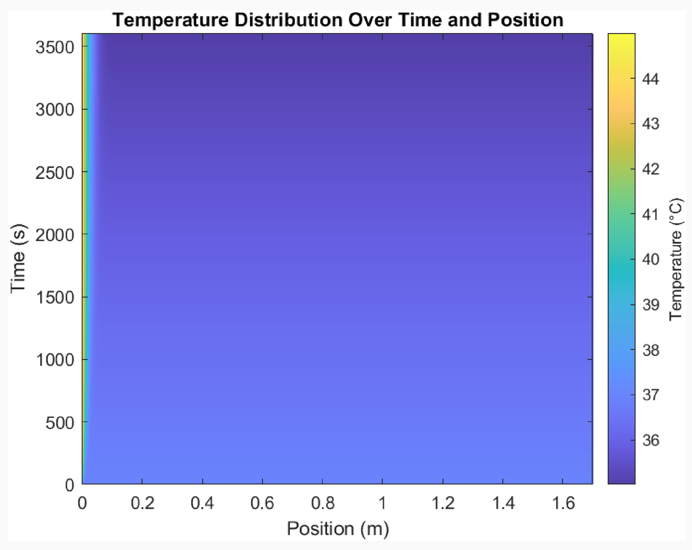

Figure 4: Hot water flowing in

Comments

- Model can be implemented sensitivity analysis of various parameters.
- Model offers a comprehensive view of the thermal dynamics in the system.
- Model's accuracy is highly dependent on the precise values of parameters.
- Model does not include the advection term, which is crucial for accurately representing the movement of hot water within the bathtub.

Further Development

It can be seen that the result is unsatisfactory in Figure 4. Equation 5 considers the heat diffusion with a source term representing heat loss. However, this model assumes only diffusion and heat transfer at the boundaries.

An alternative model incorporates advection, representing the movement of water and heat due to the inflow of hot water. The advection-diffusion equation is

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} - \epsilon v\frac{\partial u}{\partial x} + \hat{Q}(x,t) \tag{6}$$

Modelling in this way, we can better simulate realistic scenarios and provide more accrate suggestion for maintaining water temperature.

References

References

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