

A forward-only scheme for online learning of proposal distributions in particle filters

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Abstract

We introduce a new online approach for constructing proposal distributions in particle filters using a forward scheme. Our method progressively incorporates future observations to refine proposals. This is in contrast to backward-scheme algorithms that require access to the entire dataset, such as the iterated auxiliary particle filters (Guarniero et al., 2017) and controlled sequential Monte Carlo (Heng et al., 2020), which leverage all future observations through backward recursion. In comparison, our forward scheme achieves a gradual improvement of proposals that converges toward the proposal targeted by these backward methods. We show that backward approaches can be numerically unstable even in simple settings. Our forward method, however, offers significantly greater robustness with only a minor trade-off in performance, measured by the variance of the marginal likelihood estimator. Numerical experiments on both simulated and real data illustrate the enhanced stability of our forward approach.

1 Introduction

Sequential Monte Carlo (SMC) methods (Doucet and Johansen, 2011; Chopin and Papaspiliopoulos, 2020) are a class of particle algorithms designed to approximate sequences of probability measures known up to a normalizing constant with a sequence of weighted particles. They are widely used for inference in *state-space models* (SSMs), a class of models describing the evolution of an unobserved latent process through time, combined with a noisy observation process (Durbin and Koopman, 2012). An SSM typically consists of a latent Markov process and an observation process conditionally independent given the states. These models are particularly useful when observations are highly noisy, but the underlying dynamics provide essential structure to link observations together and share information across time,

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thereby reducing uncertainty and improving estimation. SSMs are ubiquitous in statistics, with applications in econometrics, signal processing, epidemiology, and robotics (Cappé et al., 2005). However, inference in nonlinear and non-Gaussian SSMs is challenging because the likelihood, the filtering, and the smoothing distributions lack closed-form expressions and are high-dimensional. While exact solutions exist for linear Gaussian cases via the Kalman filter (Kalman, 1960), Monte Carlo methods, such as particle filters and sequential Monte Carlo methods were introduced to handle more general settings.

SMC algorithms may be naturally derived for most SSMs from the sequential nature of the model. The simplest and canonical SMC algorithm used to infer an SSM is the bootstrap particle filter (Gordon et al., 1993). This algorithm is often subject to particle degeneracy, resulting in a poor approximation of the target distribution when one or a few particles carry most of the weight. Designing efficient and robust filtering algorithms is crucial, as poor approximations and particle degeneracy lead to high variances for Monte Carlo estimates and unreliable inference, particularly in complex models. There have been many attempts to develop improved SMC algorithms that adapt to the target distribution, i.e., to the model and the observed data. For example, some authors have proposed guided particle filters (Doucet et al., 2000) or auxiliary particle filters (Pitt and Shephard, 1999). Most of these methods involve looking one step (one observation) ahead to sample and weight particles at each time step. There are various ways to construct such SMC algorithms manually (Chopin and Papaspiliopoulos, 2020), but they require a deep understanding of the target distributions (for example, using Taylor expansions). Automatic and adaptive particle filters aim to reduce manual tuning and improve reliability across diverse applications, making them an active area of research.

One approach to automatically construct better proposals is to iterate forward and backward passes over the data in order to progressively refine the proposals, as the iterated auxiliary particle filter (IAPF) of Guarniero et al. (2017), and the controlled sequential Monte Carlo (cSMC) of Heng et al. (2020). We observe that, in moderately challenging cases, these methods may fail to produce low-variance estimates of the marginal likelihood of the SSM. One explanation is that it is often difficult to design the initial forward pass that provides a good initialization to the next backward pass. This happens, for instance, if one uses the bootstrap filter for this initial forward pass.

In this article, we propose a forward scheme that produces an SMC algorithm without requiring a good initialization. The difference with backward schemes can be understood as follows: the backward schemes start from the last time step of the SSM and go backward in time, essentially using all future observations to construct the proposals, whereas our forward scheme starts one step ahead and proceeds forward in time, gradually incorporating future

observations to refine the proposals. Thus, our forward scheme iterates over easier steps than the backward schemes and is much more robust to difficult initializations. Although our scheme can be less efficient in easy cases — it often converges more slowly than backward schemes that incorporate all available information — we find this loss is modest. In Markovian models, observations far ahead typically carry little information about the current state, which limits the performance gap. Through extensive illustrations on simulated and real data, we observe that our algorithm is able to produce lower variance estimates than backward algorithms, which suffer from a lack of robustness.

A recent proposal (Xue et al., 2025) uses the same forward decomposition to construct an online smoothing algorithm, relying on two simultaneous particle systems. The implementation and goals differ from ours: they focus on estimating smoothing distributions rather than the marginal likelihood and parameter inference. Nonetheless, this work demonstrates that the forward decomposition can be used fruitfully to construct adaptive particle algorithms for SSMs.

Building on these ideas, we now present our contribution in detail. The remainder of the article is organized as follows. In the next section, we introduce notation, state-space models, and the Feynman-Kac formalism underlying SMC algorithms. Next, we introduce auxiliary Feynman-Kac models and characterize globally and locally optimal proposals. Then, we derive our forward-only iterative scheme and discuss its practical implementation. The last section reports numerical experiments on simulated and real data, comparing our method to existing backward-scheme algorithms and to a reference bootstrap particle filter. We conclude with a brief discussion of limitations and possible extensions.

2 Context

2.1 Notations

Given two measurable spaces $(\mathcal{X}_1, \mathcal{A}_1)$ and $(\mathcal{X}_2, \mathcal{A}_2)$, we define an unnormalized kernel (resp. a Markov kernel) K from \mathcal{X}_1 to \mathcal{X}_2 as a function $\mathcal{X}_1 \times \mathcal{A}_2 \rightarrow [0, \infty]$ such that for any set $A \in \mathcal{A}_2$ the function $x \in \mathcal{X}_1 \mapsto K(x, A)$ is measurable and for any $x \in \mathcal{X}_1$ the function $A \in \mathcal{A}_2 \mapsto K(x, A)$ is a finite measure (resp. a probability measure). Given some positive measurable function $f: \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow [0, \infty]$, we define the function $K(f): \mathcal{X}_1 \rightarrow [0, \infty]$ as follows:

$$K(f)(x_0) = \int f(x_0, x)K(x_0, dx).$$

Similarly, given some measure $\mathbb{K}(dx)$ and some positive measurable function $f: \mathcal{X} \rightarrow [0, \infty]$ we define the quantity $\mathbb{K}(f) \in [0, \infty]$ as follows:

$$\mathbb{K}(f) = \int f(x) \mathbb{K}(dx).$$

The notation $t:s$ denotes the sequence $(t, t+1, \dots, s)$, $x_{t:s}$ denotes a sequence (x_t, \dots, x_s) , $x^{n:m}$ denotes a sequence (x^n, \dots, x^m) and $x_{t:s}^{n:m}$ denotes the collection of all x_i^j with $t \leq i \leq s$ and $n \leq j \leq m$. For spaces, $\mathcal{X}_{t:s}$ denotes the product space $\mathcal{X}_t \times \dots \times \mathcal{X}_s$. Given a sequence of kernels $(K_t)_{t \geq 2}$ from \mathcal{X}_{t-1} to \mathcal{X}_t and a measure \mathbb{K}_1 on \mathcal{X}_1 , $K_{t:s}$ denotes the joint kernel from \mathcal{X}_{t-1} to $\mathcal{X}_{t:s}$ as follows:

$$K_{t:s}(x_{t-1}, dx_{t:s}) = K_t(x_{t-1}, dx_t) \dots K_s(x_{s-1}, dx_s)$$

and the notation $\mathbb{K}_{1:t}$ denote the joint measure on $\mathcal{X}_{1:t}$ as follows:

$$\mathbb{K}_{1:t}(dx_{1:t}) = \mathbb{K}_1(dx_1) \dots K_t(x_{t-1}, dx_t).$$

Given a sequence of positive measurable functions $(G_t)_{t \geq 2}$ on $\mathcal{X}_{t-1} \times \mathcal{X}_t$ and G_1 on \mathcal{X}_1 , $G_{t:s}$ (resp. $G_{1:t}$) denotes the product function on $\mathcal{X}_{t-1:s}$ (resp $\mathcal{X}_{1:t}$):

$$\begin{aligned} G_{t:s}(x_{t-1:s}) &= G_t(x_{t-1}, x_t) \dots G_s(x_{s-1}, x_s), \\ G_{1:s}(x_{1:s}) &= G_1(x_1) \dots G_s(x_{s-1}, x_s). \end{aligned}$$

For two kernels K and U from \mathcal{X}_1 to \mathcal{X}_2 such that, for each $x_1 \in \mathcal{X}_1$, the measure $K(x_1, dx_2)$ is absolutely continuous with respect to $U(x_1, dx_2)$, we denote by $\frac{dK}{dU}$ the Radon-Nikodym derivative between these two measures:

$$\frac{dK}{dU}(x_1, x_2) = \frac{K(x_1, dx_2)}{U(x_1, dx_2)}(x_2)$$

We denote by $\mathbf{1}$ any constant unit function. We denote by $\mathcal{N}(m, \Sigma)$ or $\mathcal{N}(dx, m, \Sigma)$ the multivariate Gaussian distribution with mean m and covariance matrix Σ . We denote $\text{Categorical}(w_1, \dots, w_n)$ the categorical distribution that gives probability $w_i / \sum_{j=1}^n w_j$ to $i \in 1:n$.

2.2 State-space models

Our main motivation comes from the problem of performing inference for state-space models (SSMs), which are a class of models that link a sequence of observations $y_{1:T}$ to a latent variable $x_{1:T}$ that takes values in \mathcal{X} and typically follows a Markovian dependency structure.

The first component of an SSM is the emission density $\pi(y_t | x_t)$ (with respect to, typically, the Lebesgue measure) of an individual observation given the latent value at time t . The second component is a latent Markov

chain, with an initial distribution denoted by $\pi(dx_1)$, and Markov transition kernels denoted by $\pi(dx_t | x_{t-1})$. In an SSM, the standard tasks are to infer the filtering distributions:

$$\pi(dx_t | y_{1:t}), \quad t \in 1:T$$

the smoothing distribution:

$$\pi(dx_{1:T} | y_{1:T})$$

and the marginal likelihood:

$$\pi(y_{1:T}).$$

In general, closed-form expressions for these tasks are not available, and the typical solution is to construct Monte Carlo estimates of these targets.

Obtaining the smoothing distribution is an inference problem on a latent space \mathcal{X}^T , which can be very high-dimensional, but there is a lower intrinsic dimension, thanks to the sequential dependency structure of the latent space. To exploit this sequential dependency, we define a sequence of growing target probability measures ν_t on the space \mathcal{X}^t as:

$$\nu_t(dx_{1:t}) = \pi(dx_{1:t} | y_{1:t}).$$

The smoothing distribution corresponds to the terminal target ν_T and the filtering distributions correspond to marginals of each ν_t .

To infer our targets ν_t sequentially (from ν_1 to ν_T), Bayes' rule gives the following relations:

$$\begin{aligned} \pi(dx_1 | y_1) &= \frac{\pi(y_1 | x_1)\pi(dx_1)}{\pi(y_1)} & (1) \\ \pi(dx_{1:t+1} | y_{1:t+1}) \\ &= \frac{\pi(y_{t+1} | x_{t+1})\pi(dx_{1:t+1} | y_{1:t})}{\pi(y_{t+1} | y_{1:t})} & (2) \\ &= \frac{\pi(y_{t+1} | x_{t+1})\pi(dx_{t+1} | x_t)\pi(dx_{1:t} | y_{1:t})}{\pi(y_{t+1} | y_{1:t})} \end{aligned}$$

which can be seen as a recursive relation between ν_{t+1} and ν_t .

Let us assume that we know how to sample from the transition kernels $\pi(dx_{t+1} | x_t)$ (as well as form the initial distribution $\pi(dx_1)$) and that we can compute the emission densities $\pi(y_t | x_t)$. In this setting, equations (1) and (2) describe the sequential construction of $\nu_{1:T}$ and define a Feynman–Kac model. We will detail this notion in the following sections and explain how it can be used to construct importance sampling approximations of the targets $\nu_{1:T}$ within a sequential Monte Carlo algorithm.

2.3 The Feynman-Kac formalism

We are interested in approximating the targets $\nu_{1:T}$ sequentially. To this end, we begin by introducing Feynman-Kac models (Del Moral, 2004), which provide a useful framework for describing sequentially constructed measures and sequential Monte Carlo algorithms.

A Feynman-Kac model $(\mathbb{M}_{1:T}, G_{1:T})$ up to time T is composed of a Markov chain $\mathbb{M}_{1:T}$ from which we can sample, and non-negative weight functions $G_t(x_{t-1}, x_t)$ for $t \in 1:T$.

We define the target path measures $\nu_{1:T}$ and the normalizing constants Z_t associated with a Feynman-Kac model as:

$$\begin{aligned}\nu_t(dx_{1:t}) &= \frac{1}{Z_t} G_{1:t}(x_{1:t}) \mathbb{M}_{1:t}(dx_{1:t}), \\ Z_t &= \int G_{1:t}(x_{1:t}) \mathbb{M}_{1:t}(dx_{1:t}).\end{aligned}$$

These measures satisfy a relation similar to (2):

$$\nu_{t+1} = \frac{Z_t}{Z_{t+1}} G_{t+1} \nu_t M_{t+1}. \quad (3)$$

For an SSM, this leads to a canonical Feynman-Kac model $(\mathbb{M}_{1:t}, G_{1:t})$ defined as:

$$\begin{aligned}\mathbb{M}_1(dx_1) &= \pi(dx_1) \\ M_t(x_{t-1}, dx_t) &= \pi(dx_t | x_{t-1}) \\ G_t(x_{t-1}, x_t) &= \pi(y_t | x_t).\end{aligned} \quad (4)$$

With this definition, the target path measures coincide with the targets defined in the previous section, that is, $\nu_t(dx_{1:t}) = \pi(dx_{1:t} | y_{1:t})$ and the normalizing constants correspond to the marginal likelihoods $Z_t = \pi(y_{1:t})$.

For a given Feynman-Kac model, we define the cost-to-go function $H_{t \rightarrow s}$ (from t to s) as:

$$H_{t \rightarrow s}(x_t) = M_{t+1:s}(G_{t+1:s})(x_t) \quad (5)$$

which is, up to a multiplicative constant, the Radon-Nikodym derivative between the target path measure ν_s and the marginal of ν_t over $x_{1:t}$. By convention, we set $H_{s \rightarrow s} \equiv 1$ and $H_{0 \rightarrow t} \equiv Z_t$.

For the canonical Feynman-Kac model associated with an SSM, a cost-to-go function is the likelihood of future observations given the current latent state: $H_{t \rightarrow s}(x_t) = \pi(y_{t+1:s} | x_t)$.

2.4 Sequential Monte Carlo algorithms

A sequential Monte Carlo (SMC) algorithm is a method for sampling from a sequence of target measures ν_1, \dots, ν_T , known up to normalizing constants Z_1, \dots, Z_T .

Given a Feynman-Kac model $(\mathbb{M}_{1:T}, G_{1:T})$, since each target ν_t is available in closed form from the previous target ν_{t-1} (see (3)), we can construct Monte Carlo approximations sequentially, from ν_1 to ν_T .

A standard sequential Monte Carlo algorithm (see Algorithm 1) follows the evolution of N particles $x_t^{1:N}$ with associated weights $w_t^{1:N}$ over discrete time t , starting at 1 and ending at T . These particles are sampled and weighted according to the Feynman-Kac model to construct an importance sampling approximation of the targets $\nu_t(dx_{1:t})$. We refer to $\mathbb{M}_{1:T}$ as the proposal Markov chain and $G_{1:T}$ the weight functions.

If we were to reduce the algorithm to a sequence of importance sampling steps (i.e., without resampling), we would observe particle degeneracy, that is, a few particles would receive most of the weight, resulting in a poor approximation. One of the key features of an SMC algorithm that mitigates particle degeneracy is the resampling step: particles with low weights are discarded, while particles with large weights are selected to produce offspring. Formally, this can be described as selecting ancestors indices $a_t^{1:N} \in \mathbb{N}^N$ for the next generation of N particles at $t+1$, according to the weights $w_t^{1:N}$.

To simplify notation, when the context creates no ambiguity, we denote by $\xi_{1:t}^n$ the trajectory of a particle, taking into account its ancestors:

$$\xi_{1:t}^n = (\xi_{1:t-1}^{a_{t-1}^n}, x_t^n).$$

From the output $(x_{1:T}^{1:N}, a_{1:T}^{1:N}, w_{1:T}^{1:N})$ of an SMC algorithm, we construct the following estimators of $\nu_t(\varphi)$, for a \mathbb{R} -valued function φ , and of Z_t :

$$\hat{\nu}_t^N(\varphi) = \frac{1}{\sum_{n=1}^N w_t^n} \sum_{n=1}^N w_t^n \varphi(\xi_{1:t}^n) \quad (6)$$

$$\hat{Z}_t^N = \prod_{s=1}^t \frac{1}{N} \sum_{n=1}^N w_s^n = \hat{Z}_{t-1}^N \frac{1}{N} \sum_{n=1}^N w_t^n. \quad (7)$$

When the targets ν_t correspond to the previously introduced inference targets $\pi(dx_{1:t} | y_{1:t})$ of an SSM, the resulting SMC algorithm is called a particle filter. When the Feynman-Kac model is the canonical Feynman-Kac model of an SSM defined by (4), an SMC algorithm is called a bootstrap particle filter. The bootstrap particle filter is the simplest particle filter to implement, and it will serve as a baseline reference algorithm throughout this article.

2.5 Objectives

The bootstrap particle filter is simple but often leads to inefficient algorithms due to particle degeneracy, arising from the use of $\pi(dx_t | x_{t-1})$ as the proposal kernels $M_t(x_{t-1}, dx_t)$. The goal of our work is to automatically refine

Algorithm 1: Standard Sequential Monte Carlo

Input: A Feynman-Kac model $(\mathbb{M}_{1:T}, G_{1:T})$, a number of particles N .

Output: Particles $x_{1:T}^{1:N}$, weights $w_{1:T}^{1:N}$, ancestors index $a_{1:T}^{1:N}$.

At time $t = 1$:

- | **Sample** $x_1^n \sim \mathbb{M}_1(dx_1)$ for $n \in 1:N$.
- | **Compute** weights $w_1^n = G_1(x_1^n)$ for $n \in 1:N$.
- | **Resample** to get ancestors $a_1^n \sim \text{Categorical}(w_1^{1:N})$ for $n \in 1:N$.

At time $t \in 2:T$:

- | **Sample** $x_t^n \sim M_t(x_{t-1}^{a_{t-1}^n}, dx_t)$ for $n \in 1:N$.
 - | **Compute** weights $w_t^n = G_t(x_{t-1}^{a_{t-1}^n}, x_t^n)$ for $n \in 1:N$.
 - | **Resample** to get ancestors $a_t^n \sim \text{Categorical}(w_t^{1:N})$ for $n \in 1:N$.
-

the bootstrap particle filter by proposing a method to iteratively construct better proposals, that is, better Feynman-Kac models.

More precisely, given a reference Feynman-Kac model $(\mathbb{M}_{1:T}^{\text{ref}}, G_{1:T}^{\text{ref}})$, the bootstrap filter in our case, we are interested in accurately estimating the full target ν_T^{ref} and its normalizing constant Z_T^{ref} . Within the constraint of preserving these two, we will try to find the best Feynman-Kac model.

3 Method

3.1 Auxiliary Feynman-Kac models with preserved final target

Our goal is to develop a method for constructing optimal Feynman-Kac models that sample from the full target distribution. We introduce auxiliary Feynman-Kac models, inspired by the auxiliary particle filter (Pitt and Shephard 1999, see also Chap. 10 of Chopin and Papaspiliopoulos 2020). They are defined by a proposal chain $\mathbb{M}_{1:T}$ and auxiliary weight functions $\eta_{1:T}$, with $\eta_0 \equiv \eta_T \equiv 1$, such that the auxiliary Feynman-Kac model $(\mathbb{M}_{1:T}, \eta_{1:T})$ given a reference model $(\mathbb{M}_{1:T}^{\text{ref}}, G_{1:T}^{\text{ref}})$, defines a Feynman-Kac model $(\mathbb{M}_{1:T}, G_{1:T})$ that satisfies:

$$\begin{aligned} G_1(x_1) &= G_1^{\text{ref}}(x_1) \frac{\mathbb{M}_1^{\text{ref}}(dx_1)}{\mathbb{M}_1(dx_1)} \eta_1(x_1) \\ G_t(x_{t-1}, x_t) &= G_t^{\text{ref}}(x_{t-1}, x_t) \\ &\quad \times \frac{M_t^{\text{ref}}(x_{t-1}, dx_t)}{M_t(x_{t-1}, dx_t)} \frac{\eta_t(x_t)}{\eta_{t-1}(x_{t-1})}. \end{aligned} \tag{8}$$

In this context, at time t , the target path measure ν_t is given by:

$$\nu_t = \frac{G_{1:t}}{\int G_{1:t} d\mathbb{M}_{1:t}} \mathbb{M}_{1:t} = \frac{\eta_t}{\int \eta_t d\nu_t^{\text{ref}}} \nu_t^{\text{ref}} \tag{9}$$

the normalizing constant Z_t is:

$$Z_t = \int G_{1:t} d\mathbb{M}_{1:t} = Z_t^{\text{ref}} \int \eta_t d\nu_t^{\text{ref}}$$

and the cost-to-go function $H_{t \rightarrow T}$ is:

$$H_{t \rightarrow T} = M_{t+1:T}(G_{t+1:T}) = \frac{H_{t \rightarrow T}^{\text{ref}}}{\eta_t}.$$

Since we require that $\eta_T \equiv 1$ we have $\nu_T = \nu_T^{\text{ref}}$ and $Z_T = Z_T^{\text{ref}}$. The above relations describe a family of Feynman-Kac models with the same full target and final normalizing constant. In fact, we describe all the Feynman-Kac models satisfying these two properties. If we set $\mathbb{M}_{1:T} \equiv M_{1:T}^{\text{ref}}$ and $\eta_{1:T-1} \equiv 1$, we recover exactly the reference. In practice, as shown by (9), the auxiliary weights allow us to use a different sequence of measures that still lead to ν_T .

In the context of an SSM, it is crucial to emphasize that an SMC algorithm derived from an auxiliary Feynman-Kac model will not target the usual particle filtering distributions, since the intermediate targets are modified, although it remains possible to produce filtering estimates. Therefore, we do not expect to obtain optimal algorithms for the filtering task. Instead, our objective is to achieve optimality for the marginal likelihood, that is, for the estimator of Z_T , and, since we preserve the final target, this approach should also be efficient for the smoothing task.

3.2 Variance of the marginal likelihood estimator

We denote by \hat{Z}_T^N the Monte Carlo estimator of the normalizing constant Z_T obtained from the output of an auxiliary Feynman-Kac model and defined in (7).

This estimator is unbiased, $\mathbb{E}[\hat{Z}_T^N] = Z_T$, and its variance satisfies (Doucet and Johansen, 2011):

$$\frac{\mathbb{V}[\hat{Z}_T^N]}{Z_T^2} = \frac{1}{N} \sum_{t=1}^T \left(\int G_t H_{t \rightarrow T} d\nu_T \int \frac{1}{G_t H_{t \rightarrow T}} d\nu_T - 1 \right) + o\left(\frac{1}{N}\right). \quad (10)$$

The first integral term in (10) can be expressed, for an auxiliary Feynman-Kac model, as:

$$G_t(x_{t-1}, x_t) H_{t \rightarrow T}(x_t) = \frac{G_t^{\text{ref}}(x_{t-1}, x_t) H_{t \rightarrow T}^{\text{ref}}(x_t)}{\eta_{t-1}(x_{t-1})} \frac{M_t^{\text{ref}}(x_{t-1}, dx_t)}{M_t(x_{t-1}, dx_t)}. \quad (11)$$

The variance of this estimator is of particular interest for various reasons. First, it determines the performance of the popular PMMH algorithm (Andrieu et al., 2010), which may be used to sample from the posterior distribution of the parameters of the considered state-space model. Second, a

low variance also implies reduced weight degeneracy and suggests that the full proposal distribution is close to the full target distribution. For this reason, we use this variance to define optimality in the next section.

The summand in (10) is a (reverse) χ^2 divergence between the target ν_T and the proposal ($\frac{1}{G_t H_{t \rightarrow T}} \nu_T$ up to a constant). More precisely, given two probability measures ν and μ on \mathcal{X} , and a non-negative function f on \mathcal{X} , we define a chi-squared divergence between ν and μ , as follows:

$$\mathcal{D}\chi^2(\nu | \mu) = \int \frac{d\nu}{d\mu} d\mu \int \frac{d\mu}{d\nu} d\mu - 1$$

and to simplify notation, we define a chi-squared divergence of f relative to μ as the chi-squared divergence between μ and $\nu \propto f\mu$, as follows:

$$\mathcal{D}\chi_\mu^2(f) = \int \frac{1}{f} d\mu \int f d\mu - 1.$$

Then, using (11), (10) may be rewritten as:

$$\frac{\mathbb{V}[\hat{Z}_T^N]}{Z_T^2} = \frac{1}{N} \sum_{t=1}^T \mathcal{D}\chi_{\nu_T}^2 \left(\frac{G_t^{\text{ref}} H_{t \rightarrow T}^{\text{ref}}}{\eta_{t-1}} \frac{dM_t^{\text{ref}}}{dM_t} \right) + O\left(\frac{1}{N^2}\right). \quad (12)$$

See Appendix A for further details on how to derive (12) from Doucet and Johansen (2011).

In our context, we aim to minimize the leading term in (12) with respect to $\eta_{t-1} M_t$, while keeping $G_{1:T}^{\text{ref}}$ and $M_{1:T}^{\text{ref}}$ fixed.

Therefore, to simplify the expressions, we introduce the following quantities:

$$\mathbb{U}_1 = \eta_0 \mathbb{M}_1 \quad U_t = \eta_{t-1} M_t$$

that is, \mathbb{U}_1 is a finite measure and the U_t 's are unnormalized kernels. We can recover both η_{t-1} and M_t from U_t via:

$$\begin{aligned} \eta_{t-1}(x_{t-1}) &= \int U_t(x_{t-1}, dx_t), \\ M_t(x_{t-1}, dx_t) &= \frac{1}{\eta_{t-1}(x_{t-1})} U_t(x_{t-1}, dx_t). \end{aligned}$$

The auxiliary Feynman-Kac model notations allow, via (12), to precisely characterize which quantity, which discrepancy, and which measure should drive our learning algorithms, given that we want to minimize this variance. The main consequence of this formulation is that optimality can be understood from (12) on the unconstrained set of unnormalized measures $U_{1:T}$ more directly than from (10), which is defined on the set of Feynman-Kac models constrained by their full target distribution.

3.3 Optimality

3.3.1 Global optimality

We define optimality in our context as finding U_t (or equivalently M_t and η_t) that minimizes the leading term of (10). The minimizer U_t^* follows directly from (12) and can be written as:

$$\begin{aligned}\mathbb{U}_1^* &= G_1^{\text{ref}} H_{1 \rightarrow T}^{\text{ref}} M_1^{\text{ref}}, \\ U_t^* &= G_t^{\text{ref}} H_{t \rightarrow T}^{\text{ref}} M_t^{\text{ref}}, \quad t \geq 2.\end{aligned}\tag{13}$$

By integration, the normalizing constant is, by definition of the cost-to-go function (5):

$$\eta_{t-1}^* = M_t^{\text{ref}}(G_t^{\text{ref}} H_{t \rightarrow T}^{\text{ref}}) = H_{t-1 \rightarrow T}^{\text{ref}}\tag{14}$$

This leads to the following optimal proposal kernels:

$$\begin{aligned}\mathbb{M}_1^* &= \frac{G_1^{\text{ref}} H_{1 \rightarrow T}^{\text{ref}}}{Z_T} \mathbb{M}_1^{\text{ref}}, \\ M_t^* &= \frac{G_t^{\text{ref}} H_{t \rightarrow T}^{\text{ref}}}{H_{t-1 \rightarrow T}^{\text{ref}}} M_t^{\text{ref}}, \quad t \geq 2.\end{aligned}$$

If the reference is the bootstrap particle filter, this yields:

$$\begin{aligned}M_1^*(dx_1) &= \pi(dx_1 \mid y_{1:T}) \\ M_t^*(x_{t-1}, dx_t) &= \pi(dx_t \mid x_{t-1}, y_{t:T}) \\ \eta_{t-1}^*(x_{t-1}) &= \pi(y_{t:T} \mid x_{t-1}).\end{aligned}$$

This corresponds to sampling directly from the full target distribution, i.e., the smoother in this context (Guarniero et al., 2017; Heng et al., 2020):

$$\nu_t^*(dx_{1:t}) = \pi(dx_{1:t} \mid y_{1:T}).$$

Recast in our formalism, the approaches of Guarniero et al. (2017) and Heng et al. (2020) approximate the optimal sequence $\mathbb{U}_{1:T}$ by means of the following explicit backward recursion, for $t = T - 1, T - 2, \dots, 1$:

$$U_t^* = G_t^{\text{ref}} U_{t+1}^*(\mathbf{1}) M_t^{\text{ref}}\tag{15}$$

which follows from (13) and (14). This implies the following relations for η_{t-1}^* and M_t^* :

$$\begin{aligned}\mathbb{M}_1^* &= \frac{G_1^{\text{ref}} \eta_1^*}{\int G_1^{\text{ref}} \eta_1^* d\mathbb{M}_1^{\text{ref}}} \mathbb{M}_1^{\text{ref}} \\ M_t^* &= \frac{G_t^{\text{ref}} \eta_t^*}{M_t^{\text{ref}}(G_t^{\text{ref}} \eta_t^*)} M_t^{\text{ref}} \\ \eta_{t-1}^* &= M_t^{\text{ref}}(G_t^{\text{ref}} \eta_t^*).\end{aligned}$$

If the reference is the bootstrap particle filter, we recover the backward smoothing recursion (Doucet and Johansen, 2011):

$$\pi(x_t | x_{t-1}, y_{t:T}) = \frac{\pi(y_t | x_t) \pi(y_{t+1:T} | x_t) \pi(x_t | x_{t-1})}{\pi(y_{t:T} | x_{t-1})}.$$

3.3.2 Tractability issues and local optimality

Global optimality introduces some computational difficulties. The main difficulty is that this optimum is typically intractable, since the optimal target at time t , ν_t^* , is precisely the marginal of the full target $\nu_T^* = \nu_T$. This can be addressed by building approximations iteratively, alternating forward and backward passes over the data, as in Guarniero et al. (2017) and Heng et al. (2020). However, as we shall observe in our experiments, this approach requires a sufficiently good initialization (i.e., the first forward pass), which is non-trivial in cases where the reference Feynman-Kac model leads to a very inefficient particle filter (e.g., a bootstrap filter for a state-space model with highly informative data).

Instead, our method relies on the concept of local optimality, which can be intuitively understood as optimality with respect to a finite horizon of L steps ahead ($H_{t \rightarrow t+L}^{\text{ref}}$ in (13)), rather than with respect to the full remaining sequence $H_{t \rightarrow T}^{\text{ref}}$.

This allows us to follow a forward scheme, which can be used either as an initial proposal for backward-scheme algorithms or to construct an online algorithm — something not achievable with a purely backward scheme.

More precisely, local optimality is defined in the same sense as the local optimality for a guided particle filter, described in Doucet et al. (2000). The main idea underlying this notion of optimality is to ignore knowledge of the full target and only include weight functions up to time t . Namely, we aim to optimize the variance of the estimator \hat{Z}_t^N of Z_t instead of \hat{Z}_T^N of Z_T .

We look for an optimum with the target path $\nu_{1:T}$ fixed (i.e., at $\eta_{1:T}$ fixed). By reducing this problem to minimizing the variance of the importance weights w_t^n , it is well known (Doucet et al., 2000) that this leads to the following optimum:

$$\begin{aligned} \mathbb{M}_1^{\text{loc}} &= \frac{G_1^{\text{ref}} \eta_1}{\int G_1^{\text{ref}} \eta_1 d\mathbb{M}_1^{\text{ref}}} \mathbb{M}_1^{\text{ref}} \\ M_t^{\text{loc}} &= \frac{G_t^{\text{ref}} \eta_t}{M_t^{\text{ref}}(G_t^{\text{ref}} \eta_t)} M_t^{\text{ref}}. \end{aligned} \tag{16}$$

Local optimality provides insight into how to construct the best proposals for a fixed target path $\nu_{1:T}$. In the next section, we develop a scheme that connects this notion of local optimality to a global optimal path.

3.4 Iterated forward scheme

The optimal auxiliary weight functions $\eta_t^* = H_{t \rightarrow T}^{\text{ref}}$ correspond to the cost-to-go from time t to final time T . Therefore, if we fix a number L of future time points to take into account when constructing our proposals, we can define auxiliary functions $\eta_{0:T}^{(L)}$ as the cost-to-go from time t to time $t + L$:

$$\eta_t^{(L)} = \begin{cases} H_{t \rightarrow t+L}^{\text{ref}} & \text{if } t + L \leq T \\ H_{t \rightarrow T}^{\text{ref}} = \eta_t^* & \text{else.} \end{cases}$$

These auxiliary functions converge to η_t^* as L increases.

When the bootstrap particle filter is used as a reference, $\eta_t^{(L)}$ satisfies

$$\begin{aligned} \eta_t^{(L)}(x_t) &= \int G_{t+1:t+L}^{\text{ref}}(x_{t:t+L}) M_{t+1:t+L}^{\text{ref}}(x_t, dx_{t+1:t+L}) \\ &= \int \prod_{s=t+1}^{t+L} \pi(y_{t+s} | x_{t+s}) \pi(dx_{t+s} | x_{t+s-1}) \\ &= \pi(y_{t+1:t+L} | x_t) \end{aligned}$$

for $t + L \leq T$, so that $\eta_t^{(L)}$ weights x_t according to the information contained in L future observations.

This establishes a connection with the fixed-lag SMC literature (Doucet and Sénecal, 2004), which focuses on sampling from $\pi(dx_{1:t} | y_{1:t+L})$, which is precisely the target $\nu_t^{(L)}$ associated with the auxiliary weight $\eta_t^{(L)}$. Rapid convergence of this target toward the smoothing distribution was shown by Olsson et al. (2008), and by extension, the target $\nu_t^{(L)}$ should rapidly approximate the target induced by the backward scheme.

There are two main differences between our approach and fixed-lag SMC. First, instead of sampling blocks $x_{t:t+L}$ to estimate $\nu_t^{(L)}$, we sample from an approximation of its x_t -marginal. Second, we build our approximation iteratively over L . As a result, our algorithm uses an increasing value of L , compared to the standard block sampling method, which uses a fixed L .

These auxiliary weights may be computed recursively, using a backward scheme:

$$\eta_t^{(L+1)} = M_{t+1}^{\text{ref}}(G_{t+1}^{\text{ref}} \eta_{t+1}^{(L)})$$

starting from $\eta_t^{(0)} \equiv 1$, corresponding to the reference. We recall that $\eta_T \equiv 1$ is required to preserve the last target. One crucial difference compared to the backward scheme is that we only need to integrate up to L times to compute the auxiliary weight as opposed to T times under global optimality.

By applying local optimality (16) to fixed auxiliary weights $\eta_t^{(L)}$, we obtain an auxiliary model $(\mathbb{M}_{1:T}^{(L+1)}, \eta_{0:T}^{(L)})$ defined by:

$$\begin{aligned}\mathbb{M}_1^{(L+1)} &= \frac{G_1^{\text{ref}} \eta_1^{(L)}}{\int G_1^{\text{ref}} \eta_1^{(L)} d\mathbb{M}_1^{\text{ref}}} \mathbb{M}_1^{\text{ref}} \\ M_t^{(L+1)} &= \frac{G_t^{\text{ref}} \eta_t^{(L)}}{M_t^{\text{ref}} (G_t^{\text{ref}} \eta_t^{(L)})} M_t^{\text{ref}}.\end{aligned}$$

This leads to the following relation, similar to (15):

$$U_t^{(L+1)} = G_t^{\text{ref}} U_{t+1}^{(L)}(\mathbf{1}) M_t^{\text{ref}}. \quad (17)$$

Although this equation remains a backward recursion with respect to time t , it can also be interpreted as a forward recursion with respect to L . We use this latter interpretation to derive our practical, forward-only algorithm, as explained in the next section.

4 Implementation

4.1 Parametrization using twisted Feynman-Kac models

We use the same family of approximating proposal kernels as in Guarniero et al. (2017) and Heng et al. (2020), that is, we parametrize the unnormalized kernel in terms of a twisting function φ_t :

$$U_t(x_{t-1}, dx_t) = \varphi_t(x_{t-1}, x_t) M_t^{\text{ref}}(x_{t-1}, dx_t).$$

This choice is motivated by the fact that the optimal kernel satisfies

$$U_t^\star(x_{t-1}, dx_t) = G_t^{\text{ref}}(x_{t-1}, x_t) \eta_t^\star(x_t) M_t^{\text{ref}}(x_{t-1}, dx_t)$$

according to (13). Hence, the problem reduces to learning a strictly positive function φ_t of the form:

$$\begin{aligned}\varphi_t(x_{t-1}, x_t) &= \frac{U_t(x_{t-1}, dx_t)}{M_t^{\text{ref}}(x_{t-1}, dx_t)} \\ &\approx G_t^{\text{ref}}(x_{t-1}, x_t) \eta_t^\star(x_t).\end{aligned}$$

When the reference is derived from the bootstrap particle filter, G_t^{ref} depends only on x_t , and is constant with respect to x_{t-1} . In that case, we restrict φ_t to be constant in x_{t-1} .

This parametrization is especially useful when the transition kernels admit conjugacy properties, e.g., when $M_t^{\text{ref}}(x_{t-1}, dx_t)$ is Gaussian (conditional

on x_{t-1}). In that case, if φ_t is chosen in the space of log-quadratic functions, that is, provided

$$\begin{aligned}\mathbb{M}_1(dx_1) &= \mathcal{N}(m_1, \Sigma_1) \\ M_t(x_{t-1}, dx_t) &= \mathcal{N}(m_t(x_{t-1}), \Sigma_t(x_{t-1}))\end{aligned}$$

and

$$\varphi_t(x_t) = \exp\left(-\frac{1}{2}x'_t A_t x_t - x'_t b_t - \frac{1}{2}c_t\right) \quad (18)$$

under the condition $\Sigma_t(x_{t-1}) + A_t \gg 0$ for all t , all quantities of interest are available in closed form:

$$\begin{aligned}M_t &= \frac{1}{U_t(\mathbf{1})} U_t = \frac{\varphi_t}{M_t^{\text{ref}}(\varphi_t)} M_t^{\text{ref}} \\ \eta_t &= U_t(\mathbf{1}) = M_t^{\text{ref}}(\varphi_t).\end{aligned}$$

Function φ_t with nice conjugacy properties seems to be typical of Gaussian applications: we are not aware of non-Gaussian examples that admit such functions. The methods presented below use this parametrization, but could be adapted to optimize over a class of unnormalized kernels U_t instead of φ_t when necessary.

4.2 Recursively computing the twisting functions

The main idea in approximating the schemes is to substitute approximations $\hat{\varphi}_t$ (resp. $\hat{\varphi}_t^{(L)}$) of the optimal twisting functions $\varphi_t^\star = dU_t^\star/dM_t^{\text{ref}}$ (resp. $\varphi_t^{(L)} = dU_t^{(L)}/dM_t^{\text{ref}}$) into the recursions (15) (resp. (17)). Several methods exist to construct these approximations. In this section, we detail an adaptation to the forward scheme (17) of the approach proposed by Heng et al. (2020) for the backward scheme (15). See Appendix B for more details on the backward implementation of Heng et al. (2020).

Expressing the forward scheme (17) in terms of the twist functions $\varphi_t^{(L)} = dU_t^{(L)}/dM_t^{\text{ref}}$, we obtain:

$$\varphi_t^{(L+1)} M_t^{\text{ref}} = G_t^{\text{ref}} M_{t+1}^{\text{ref}}(\varphi_{t+1}^{(L)}) M_t^{\text{ref}}.$$

This gives the following recursion:

$$\varphi_t^{(L+1)} = G_t^{\text{ref}} M_{t+1}^{\text{ref}}(\varphi_{t+1}^{(L)}). \quad (19)$$

This equation expresses the scheme entirely in terms of the twist functions φ_t . Following (19), we iteratively construct $\hat{\varphi}_{1:T}^{(L+1)}$, an approximation of $\varphi_{1:T}^{(L+1)}$, using the previous approximation $\hat{\varphi}_{1:T}^{(L)}$ as input:

$$\begin{aligned}\hat{\varphi}_t^{(L+1)} &\approx G_t^{\text{ref}} M_{t+1}^{\text{ref}}(\hat{\varphi}_{t+1}^{(L)}) \\ &\dots \\ \hat{\varphi}_T^{(L+1)} &\approx G_T^{\text{ref}}\end{aligned}$$

Algorithm 2: Forward SMC Training

Input: A reference Feynman-Kac model $(M_{1:T}^{\text{ref}}, G_{1:T}^{\text{ref}})$, a number of particles N , a number of iterations L^{\max} and sets of strictly positive functions $\Psi_{1:T}$.

Output: Approximations $\hat{\varphi}_{1:t}^{(L)}$ for $1 \leq L \leq L^{\max}$ of the forward scheme.

Set $\hat{\varphi}_{1:T}^{(0)} \equiv 1$, $\hat{M}_1^{(0)} \sim M_1^{\text{ref}}$, $\hat{M}_{2:T}^{(0)} \equiv M_{2:T}^{\text{ref}}$ and $\hat{\eta}_{1:T}^{(0)} \equiv 1$.

For $L \in 0:L^{\max} - 1$:

For $t \in 1:T$:

- Approximate** $G_t^{\text{ref}} \hat{\eta}_t^{(L)}$ by $\hat{\varphi}_t^{(L+1)} \in \Psi_t$.
- Set** $\hat{\eta}_{t-1}^{(L+1)} = M_t^{\text{ref}}(\hat{\varphi}_t^{(L+1)})$.
- Set** $\hat{M}_t^{(L+1)} = \frac{\hat{\varphi}_t^{(L+1)}}{\hat{\eta}_{t-1}^{(L+1)}} M_t^{\text{ref}}$.
- // Sample particles according to trained proposal
- Sample and Compute** $x_t^{(L+1),n}$, $w_t^{(L+1),n}$ and $a_t^{(L+1),n}$ according to the t -th SMC step (Algorithm 1) with $M_t = \hat{M}_t^{(L+1)}$,
- $\eta_{t-1:t} = \eta_{t-1:t}^{(L)}$ and ancestor $\tilde{x}_{t-1}^{(L+1),n}$.
- Denote** $\tilde{x}_t^{(L+1),n} = x_t^{(L+1),a_t^{(L+1),n}}$.

The learned auxiliary Feynman-Kac model $(\hat{M}_{1:T}^{(L+1)}, \hat{\eta}_{1:T}^{(L+1)})$ follows:

$$\begin{aligned}\hat{M}_t^{(L+1)} &= \frac{\hat{\varphi}_t^{(L+1)}}{M_t^{\text{ref}}(\hat{\varphi}_t^{(L+1)})} M_t^{\text{ref}}, \\ \hat{\eta}_{t-1}^{(L+1)} &= M_t^{\text{ref}}(\hat{\varphi}_t^{(L+1)}).\end{aligned}$$

The simplified Algorithm 2 summarizes how functions $\hat{\varphi}_t^{(L)}$ are computed recursively. Since the scheme is forward, the target at each step is available directly from previously computed quantities. In particular, $\hat{\varphi}_t^{(L)}$ does not depend on any future approximation $\hat{\varphi}_{t+h}^{(L)}$, such that the algorithm is a succession of forward passes over the data. Furthermore, with fixed depth L of the training, we can implement an online version of this algorithm that does exactly the same computations, the construction and some implementation details can be found in Appendix C, including the online version Algorithm 5 of Algorithm 2. In practice, the approximations depend on a certain learning procedure, \mathcal{A}^{fwd} , which we describe in the next section.

4.3 Learning the twisting functions from particle samples

We use a learning procedure \mathcal{A}^{fwd} , in Algorithm 3, similar to Controlled SMC (see Appendix B), but the sampling part is more involved. Particles

Algorithm 3: Forward Iterated SMC

Input: A reference Feynman-Kac model $(\mathbb{M}_{1:T}^{\text{ref}}, G_{1:T}^{\text{ref}})$, a number of particles N , a number of iterations L^{\max} .

Output: Approximations $\hat{\varphi}_{1:t}^{(L)}$ for $1 \leq L \leq L^{\max}$ of the forward scheme.

Set $\hat{\varphi}_{1:T}^{(0)} \equiv 1$, $\hat{M}_1^{(0)} \sim \mathbb{M}_1^{\text{ref}}$, $\hat{M}_{2:T}^{(0)} \equiv M_{2:T}^{\text{ref}}$ and $\hat{\eta}_{1:T}^{(0)} \equiv 1$.

For $L \in 0:L^{\max} - 1$:

For $t \in 1:T$:

 // Sample training particles according to previous iterations

Sample $\bar{x}_t^{(L),n}$ and **Compute** $\bar{w}_t^{(L),n}$ according to the t -th SMC step (Algorithm 1) with $M_t = \hat{M}_t^{(L)}$ and $\eta_{t-1:t} = \eta_{t-1:t}^{(L)}$ and ancestors $\tilde{x}_{t-1}^{(L+1),1:N}$.

 // Compute the approximations

Set $f(x_{t-1}, x_t) = G_t^{\text{ref}}(x_{t-1}, x_t) \hat{\eta}_t^{(L)}(x_t)$

Set $\hat{\varphi}_t^{(L+1)} = \mathcal{A}_t^{\text{fwd}}(f, \tilde{x}_{t-1}^{(L+1),1:N}, \bar{x}_t^{(L),1:N}, \bar{w}_t^{(L),1:N})$.

Set $\hat{\eta}_{t-1}^{(L+1)} = M_t^{\text{ref}}(\hat{\varphi}_t^{(L+1)})$.

Set $\hat{M}_t^{(L+1)} = \frac{\hat{\varphi}_t^{(L+1)}}{\hat{\eta}_{t-1}^{(L+1)}} M_t^{\text{ref}}$.

 // Sample particles according to trained proposal

Sample and **Compute** $x_t^{(L+1),n}$, $w_t^{(L+1),n}$ and $a_t^{(L+1),n}$ according to the t -th SMC step (Algorithm 1) with $M_t = \hat{M}_t^{(L+1)}$, $\eta_{t-1:t} = \eta_{t-1:t}^{(L)}$ and ancestor $\tilde{x}_{t-1}^{(L+1),n}$.

Denote $\tilde{x}_t^{(L+1),n} = x_t^{(L+1),a_t^{(L+1),n}}$.

are sampled as we construct the approximation of the scheme: for each time t we sample and weight training particles $(\bar{x}_t^{(L),1:N}, \bar{w}_t^{(L),1:N})$ according to the current auxiliary Feynman-Kac model $(\hat{\mathbb{M}}_{1:T}^{(L)}, \hat{\eta}_{1:T}^{(L)})$. After computing an approximation $\hat{\varphi}_t^{(L+1)}$ from these particles, we sample new particles and ancestors $(x_t^{(L+1),1:N}, a_t^{(L+1),1:N}, w_t^{(L+1),1:N})$ with the trained proposal $\hat{M}_t^{(L+1)}$ and the same fixed auxiliary weights $\hat{\eta}_{1:T}^{(L)}$.

At each step t of iteration $L + 1$ of Algorithm 3, ancestor particles $\tilde{x}_{t-1}^{1:N}$ with indices $a_{t-1}^{1:N}$ and particles $\bar{x}_t^{1:N}$ weighted by \bar{w}_t^n from the previous iteration (L) are used to build approximations as:

$$\hat{\varphi}_t^{(L+1)} = \mathcal{A}_t^{\text{fwd}}(G_t^{\text{ref}} \hat{\eta}_t^{(L)}, \tilde{x}_{t-1}^{(L+1),1:N}, \bar{x}_t^{(L),1:N}, \bar{w}_t^{(L),1:N})$$

where, for a function f_t that will be approximated in the function class Ψ_t , particles $x_{t-1}^{1:N}$ and $x_t^{1:N}$, and weights $w_t^{1:N}$, $\mathcal{A}_t^{\text{fwd}}$ is defined as follows:

$$\begin{aligned} \mathcal{A}_t^{\text{fwd}}(f_t, x_{t-1}^{1:N}, x_t^{1:N}, w_t^{1:N}) \\ = \arg \min_{\hat{f} \in \Psi_t} \sum_{n=1}^N w_t^n \left(\log \hat{f}(x_{t-1}^n, x_t^n) * -\log f_t(x_{t-1}^n, x_t^n) \right)^2. \end{aligned}$$

When the function class Ψ_t consists of single log-quadratic functions (i.e., $\Psi_t = \{x \mapsto \exp(-\frac{1}{2}x'Ax - x'b - \frac{1}{2}c) \mid A \in \mathbb{S}_d, b \in \mathbb{R}^d, c \in \mathbb{R}\}$), the minimization above amounts to a simple linear regression problem (with a semi-definite constraint that is addressed by doing a projection of the unconstrained solution) and is relatively easy to compute.

The optimization problem described by $\mathcal{A}_t^{\text{fwd}}$ is ill-defined when the importance weights $w_t^{1:N}$ are too degenerate. A measure of degeneracy is the effective sample size (ESS):

$$\text{ESS}(w^{1:N}) = \frac{(\sum_n w^n)^2}{\sum_n (w^n)^2}.$$

It represents an effective number of different particles in the sample and ranges from 1 (highly degenerate) to N (optimal). Whenever the ESS falls below $N_0 = 2d$, where d is the dimension of the space of functions F , the following tempered update is used instead:

$$\mathcal{A}^{\text{fwd}^\alpha}(\dots) = \arg \min_{\hat{f} \in \Psi_t} \sum_{n=1}^N e^{\alpha \log(w_t^n)} \left(\log \hat{f}(x_{t-1}^n, x_t^n) - \log f_t(x_{t-1}^{A_t^n}, x_t^n) \right)^2$$

where $\alpha \in (0, 1)$ is a tempering weight. The exponent α is calibrated to ensure that the ESS of $(e^{\alpha \log(w_t^n)})^{1:N}$ is approximately N_0 .

The Algorithm 3 can be used with either $\mathcal{A}_{1:T}^{\text{fwd}}$ or $\mathcal{A}_{1:T}^{\text{fwd}^\alpha}$, but will definitely be less robust without the tempering weight.

The computational cost of Algorithm 3 is comparable to that of the backward Algorithm 4, although the Forward Iterated SMC algorithm samples roughly twice as many particles as controlled SMC. If the number of iterations is fixed in advance, or multiple iterations are performed at once, an online version of Algorithm 3 can be used, reusing particles for both sampling and training, achieving a computational cost comparable to controlled SMC. These computational cost considerations and online versions of the algorithm are detailed in Appendix D.

5 Numerical experiments

In this section, we compare our forward-only training algorithm with alternative methods.

We first consider a simple one-dimensional nonlinear model, illustrating the difficulties encountered by backward algorithms. Then, we study a real data application of the multivariate stochastic volatility used in Guarniero et al. (2017). Primarily, the methods used for comparison are controlled SMC (Heng et al., 2020) as a backward training algorithm and the bootstrap particle filter as a baseline reference.

Our primary performance criterion is an estimate of the variance of the log-normalizing-constant estimator, i.e., $\text{Var}[\log \hat{Z}_T]$ (marginal log-likelihood in the bootstrap case). This quantity serves as a proxy for the relative variance $\mathbb{V}[\hat{Z}_T]/\mathbb{E}[\hat{Z}_T]^2$, which motivates the methods but is harder to estimate directly. Moreover, it is a commonly used criterion for tuning the number of particles in particle Metropolis-Hastings algorithms (Doucet et al., 2015; Sherlock, 2016).

We also consider the variance of the importance weights as a measure of degeneracy. High weight variance indicates high variability in the estimators. More precisely, we consider the normalized quantity:

$$\frac{1}{T} \sum_{t=1}^T \frac{\frac{1}{N} \sum_{n=1}^N (w_t^n)^2 - (\frac{1}{N} \sum_{n=1}^N w_t^n)^2}{(\frac{1}{N} \sum_{n=1}^N w_t^n)^2}.$$

5.1 Nonlinear observation model

In this section, we are interested in a simple state space model with nonlinear observations, with latent space $\mathcal{X} = \mathbb{R}$ and observation space $\mathcal{Y} = \mathbb{R}$, and the following autoregressive transition model:

$$x_1 = \frac{\sigma_x}{\sqrt{1 - \alpha^2}} \varepsilon_{x,1}$$

$$x_t = \alpha x_{t-1} + \sigma_x \varepsilon_{x,t},$$

with $\varepsilon_{x,t} \sim \mathcal{N}(0, 1)$. The nonlinear observation model is defined as follows:

$$y_t = f(x_t) + \sigma_y \varepsilon_{y,t}$$

with $\varepsilon_{y,t} \sim \mathcal{N}(0, 1)$ and f is a non-decreasing function.

The choice of f controls the level of difficulty for SMC inference. For example, if f is a linear function, we get a linear Gaussian model for which the backward path can be easily and exactly computed.

We set $f(x) = \exp(x) + x/10$ as an example. With this function, the model is more informative with larger values of y . The intuition is that for small y 's, the optimal path will be more difficult to learn and to infer from.

The model is motivated by the observation that most studies in the literature consider non-linear transitions with linear (or quasi-linear) observations. Often, the class of functions approximates well the optimal twisting functions in this case. Non-linear observations, which are very common in applications, introduce additional difficulty due to potentially ill-behaved emission densities.

We try different scenarios with parameter values of $\alpha \in [0.9, 0.95, 0.98, 0.99, 0.995]$, 11 values for σ_x^2 ranging from 0.05 to 0.15 and 6 values for σ_y^2 ranging from 0.005 to 0.055. For each scenario, we generated 10 simulated datasets (over $T = 100$ time points). We ran each algorithm on each dataset and compared their performance. More precisely, we compare the first iterations of Algorithm 3 based on the forward scheme and controlled SMC (Algorithm 4) based on the backward scheme. We ran each algorithm 64 times to evaluate the variances of the estimator with $N = 1024$. For each iteration and each run of the algorithm, we get an SMC output $(x_{1:T}^{1:N}, a_{1:T}^{1:N}, w_{1:T}^{1:N})$ and an associated estimator \hat{Z}_T .

We measure "dataset difficulty" by the variance of the bootstrap particle filter weights. Large weight variance indicates that the dataset is challenging to process. The Figure 1 shows that the backward algorithm tends to fail (i.e., to have a high standard deviation) more often than our forward algorithm, especially in harder data and parameter sets. In easy cases, i.e., when the bootstrap particle filter does well, our forward algorithm does not improve on the backward algorithm. This is expected since low weight variances suggest less informative data and slower convergence of the forward scheme toward the optimal scheme. The Figure 2 shows that for a low number of iterations, the backward scheme will often return something that is worse than the bootstrap particle filter (for 80% of the datasets at iteration $L = 1$). The backward scheme is 10 times worse than the bootstrap particle filter in about 25% of the datasets across all iterations. In most of these cases, the actual trained Feynman-Kac model will end up being either highly degenerate or will provoke ill-defined inversions in the training. With the forward scheme, we also observe these issues, but for only 20 datasets (over 3300). This showcases the forward robustness to the dataset and robustness to the number of iterations: with a lower number of iterations, we do not make things worse than the bootstrap.

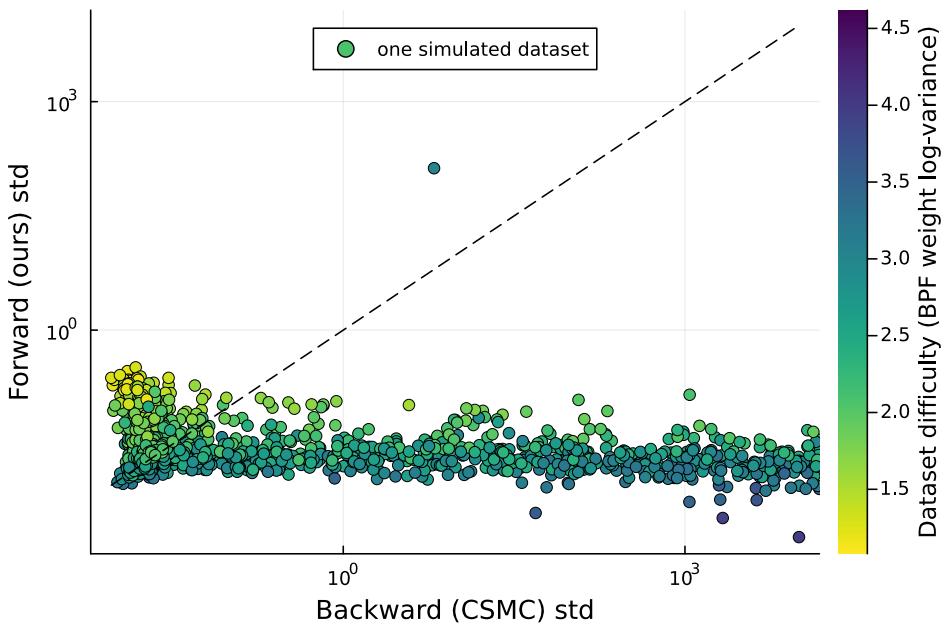


Figure 1: Comparison of the standard deviation of \hat{Z}_t over 64 runs after $L = 4$ training iterations between a forward algorithm (ours) and controlled SMC (Heng et al., 2020). Better performance is associated with lower standard deviation. Each point represents a single simulated dataset colored by the mean (over 64 iterations) of the sum of the relative empirical weight variances of the bootstrap particle filter (BPF). This variance is expected to be lower for an easier dataset.

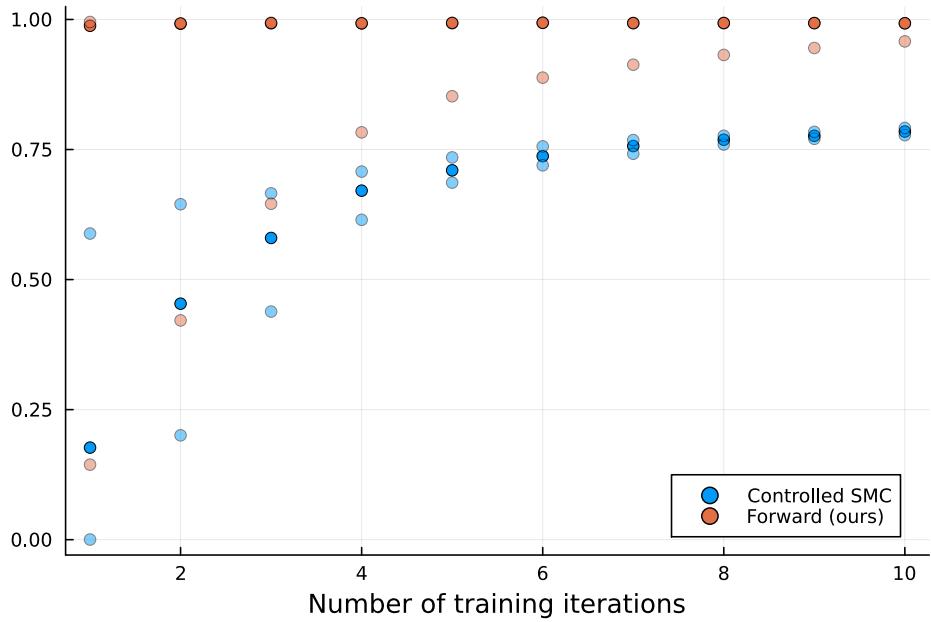


Figure 2: Proportion of simulated datasets on which the standard deviation of \hat{Z}_t over 64 runs is lower than with the bootstrap particle filter (BPF) after each iteration of the training algorithms (from 1 to 10). Denoting sd^{BPF} the standard deviation of \hat{Z}_t for the BPF, lighter dots show $P(sd^{\text{alg}} \leq 0.1sd^{BPF})$, the proportion of simulated datasets for which the standard deviation of alg is 10 time lower than the BPF standard deviation (below the main dots), and $P(sd^{\text{alg}} \leq 10sd^{BPF})$ (above the main dots).

5.2 Multivariate Stochastic Volatility model

In this section, we study the application of our forward-only approach to a multivariate stochastic volatility model, on a real dataset. We use the same model and data as Guarniero et al. (2017). We observe data $y_{1:T}$ consisting of monthly log returns on the exchange rate of d currencies. We infer the stochastic volatility $x_{1:T}$ of dimension d of this log-return as follows:

$$y_{t,i} \mid x_t \sim \mathcal{N}(0, \exp(x_{t,i})).$$

The latent space follows an autoregressive transition model:

$$\begin{aligned} x_1 &\sim \mathcal{N}(m, \Sigma_\infty) \\ x_t \mid x_{t-1} &\sim \mathcal{N}((I - \text{diag}(\alpha))m + \text{diag}(\alpha)x_{t-1}, \Sigma) \end{aligned}$$

with parameters $m \in \mathbb{R}^d$, $\alpha \in \mathbb{R}^{d-1}$ and $\Sigma \in \mathbb{R}^{d \times d}$ positive definite, and with Σ_∞ the covariance matrix such that we sample from the invariant measure at time 1.

To reduce the dimensionality of the parameter space, Guarniero et al. (2017) constrain Σ to have variances $\sigma^2 \in \mathbb{R}^d$ and zero correlations except for the sub- and super-diagonals, denoted $\rho \in \mathbb{R}^{d-1}$. We proceed similarly.

We use data from the Federal Reserve¹ from March 2000 to August 2008 ($T = 102$). Like in Guarniero et al. (2017), we set a uniform improper prior on m , $[0, 1]$ uniform priors on each coordinate of α , inverse gamma priors with mean 0.2 and variance 1 on each coordinate of σ^2 , and a $[-1, 1]$ triangular prior on each coordinate of ρ . To infer parameters, we do 1.2×10^5 iterations of particle marginal Metropolis-Hastings (PMMH) on this model that uses estimators \hat{Z}_t obtained from running an SMC algorithm.

The different methods of estimating \hat{Z}_t that we compare are: the bootstrap particle filter with N particles as a baseline and an SMC algorithm using N_{sample} particles with a twisted Feynman-Kac model as input, trained on the class of log-quadratic twisting functions (18) with a diagonal quadratic term matrix A . For the training, we use either the controlled SMC algorithm (Algorithm 4) or our forward iterated SMC algorithm (Algorithm 3), both algorithms use L iterations and N_{train} particles. We try to match the computational cost of the forward-based SMC algorithm and the bootstrap particle filter.

We use PMMH with Gaussian random-walk proposals tuned using the forward algorithm. Beforehand, we transform the model parameters to ensure that proposals respect the parameter constraints. We initialize the PMMH with the empirical variance of $y_{1:T}$ for m , 0.9 for α , 0.2 for σ^2 and 0.25 for ρ .

To assess the variability of the estimates, we perform 10 additional SMC runs every 100 PMMH steps (incurring an extra 10% computational cost) to estimate the empirical variance of Z_t .

¹<https://www.federalreserve.gov/releases/h10/hist/>

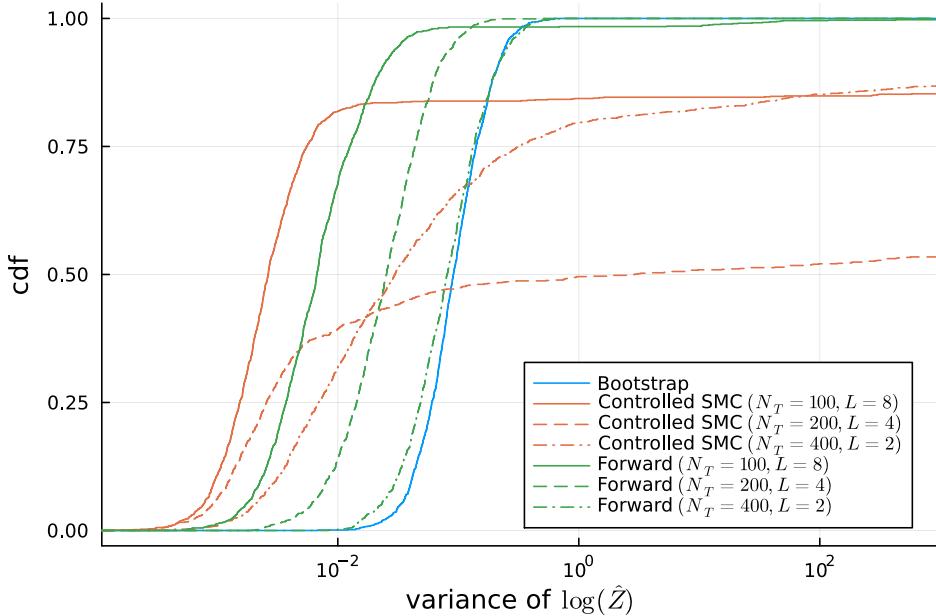


Figure 3: Empirical cumulative distribution of the empirical variances of $\log \hat{Z}_T$ for $d = 8$, measured every 100 steps of the PMMH algorithm over 10 evaluation.

In the experiment, we studied $d = 8$ currencies. We chose $N = 4500$ for the bootstrap particle filter, $N_{\text{sample}} = 600$, and either $(L = 8, N_{\text{train}} = 100)$, $(L = 4, N_{\text{train}} = 200)$, or $(L = 2, N_{\text{train}} = 400)$ such that we use the same number of training particles in each algorithm. Trace plots indicate that the PMMH chains have reached stationarity, and the autocorrelation function (ACF) suggests adequate tuning. In Figure 3 we observe the empirical CDF of the estimated variances of $\log \hat{Z}_T$, measured every 100 PMMH steps. We recall that, when we perform PMMH, we want $\mathbb{V}[\log \hat{Z}_T] \ll 1$.

Our forward iterated SMC algorithm achieves significantly lower variance than the bootstrap particle filter. Moreover, as observed on simulated data, the forward algorithm proves more robust than the backward algorithm: while the backward algorithm can potentially reach lower variances, it frequently produces very large variances, which can cause PMMH chains to become trapped in regions of parameter space where \hat{Z}_t estimates are unreliable. At a fixed training cost, the backward algorithm is also more sensitive to the number of iterations than the forward one. The behavior of controlled SMC for $(L = 2, N_{\text{train}} = 200)$ is explained by the chain getting stuck in a high variance region. A further comparison on the robustness of the forward and backward algorithms on a slightly different dataset is presented in Appendix E.

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A Expression of the variance

If we translate equations of Doucet and Johansen (2011) into our notations we get:

$$\frac{\mathbb{V}[\hat{Z}_T]}{Z_T^2} = \frac{1}{N} \sum_{t=1}^T \left(\int \frac{\nu_T(dx_{1:t})}{\nu_{t-1}M_t(dx_{1:t})} \nu_T(dx_{1:t}) - 1 \right) + O\left(\frac{1}{N^2}\right). \quad (20)$$

We define the chi-squared divergence between two probability measures ν and μ as:

$$\mathcal{D}\chi^2(\nu \mid \mu) = \int \left(\frac{d\mu}{d\nu} \right)^2 d\nu - 1 = \int \frac{d\mu}{d\nu} d\mu - 1.$$

We can express (20) with the chi-squared divergence as follows:

$$\frac{\mathbb{V}[\hat{Z}_T]}{Z_T^2} = \frac{1}{N} \sum_{t=1}^T \mathcal{D}\chi^2(\nu_{t-1}M_t \mid \nu_T) + O\left(\frac{1}{N^2}\right).$$

Let us introduce the chi-squared divergence of a non-negative function f relative to a measure μ as:

$$\mathcal{D}\chi_\mu^2(f) = \mathcal{D}\chi^2\left(\frac{f}{\mu(f)}\mu \middle| \mu\right) = \int \frac{1}{f} d\mu \int f d\mu - 1.$$

A crucial property of this quantity is that it is null if and only if f is constant, just like the chi-squared divergence is null when $\nu = \mu$.

Since, for a Feynman-Kac model $(\mathbb{M}_{1:T}, G_{1:T})$,

$$\begin{aligned}\nu_{t-1} M_t(dx_{1:t}) &= \frac{1}{Z_{t-1}} G_{1:t-1}(x_{1:t-1}) M_{1:t}(dx_{1:t}) \\ \nu_T(dx_{1:t}) &= \frac{1}{Z_T} G_{1:t}(x_{1:t}) H_{t \rightarrow T}(x_t) M_{1:t}(dx_{1:t})\end{aligned}$$

we have

$$\frac{\mathbb{V}[\hat{Z}_T]}{Z_T^2} = \frac{1}{N} \sum_{t=1}^T \mathcal{D}\chi_{\nu_T}^2(G_t H_{t \rightarrow T}) + O\left(\frac{1}{N^2}\right).$$

For an auxiliary Feynman-Kac model $(\mathbb{M}_1, M_{2:T}, \eta_{1:T})$, we have:

$$G_t H_{t \rightarrow T} = \frac{G_t^{\text{ref}} H_{t \rightarrow T}^{\text{ref}}}{\eta_{t-1}} \frac{dM_t^{\text{ref}}}{dM_t}$$

so we get the following expression:

$$\frac{\mathbb{V}[\hat{Z}_T]}{Z_T^2} = \frac{1}{N} \sum_{t=1}^T \mathcal{D}\chi_{\nu_T}^2\left(\frac{G_t^{\text{ref}} H_{t \rightarrow T}^{\text{ref}}}{\eta_{t-1}} \frac{dM_t^{\text{ref}}}{dM_t}\right) + O\left(\frac{1}{N^2}\right).$$

B Backward algorithm: Controlled SMC

In this section, we will explain the Controlled SMC algorithm developed by Heng et al. (2020).

The backward scheme (15), in terms of twisting functions, yields:

$$\begin{aligned}\varphi_T^\star &= G_T^{\text{ref}} \\ &\dots \\ \varphi_t^\star &= G_t^{\text{ref}} M_{t+1}^{\text{ref}}(\varphi_{t+1}^\star).\end{aligned}$$

The algorithm constructs an approximation with twisting functions $\hat{\varphi}_t^\star$. The successive approximations follow:

$$\begin{aligned}\hat{\varphi}_T^\star &\approx G_T^{\text{ref}} \\ &\dots \\ \hat{\varphi}_t^\star &\approx G_t^{\text{ref}} M_{t+1}^{\text{ref}}(\hat{\varphi}_{t+1}^\star).\end{aligned}$$

Algorithm 4: Controlled SMC (2020)

Input: A reference Feynman-Kac model $(\mathbb{M}_{1:T}^{\text{ref}}, G_{1:T}^{\text{ref}})$, a number of particles N , a number of iterations L^{\max} .

Output: Approximations $\hat{\varphi}_{1:t}^{*,(L)}$ for $1 \leq L \leq L^{\max}$ of the optimal twisting functions $\varphi_{1:t}^*$.

```

// Sample initial training particles
Sample  $(x_{1:t}^{0,1:N}, w_{1:T}^{0,1:N}, a_{1:T}^{0,1:N})$  from the reference Feynman-Kac model
 $(\mathbb{M}_{1:T}^{\text{ref}}, G_{1:T}^{\text{ref}})$  with Algorithm 1.
Denote  $\tilde{x}_t^{(L+1),n} = x_t^{(L+1),a_t^{(L+1),n}}$ .
For  $L$  from 1 to  $L^{\max}$  :
    // Compute the approximations
    Set  $\hat{\eta}_T^{*,(L)} \equiv 1$ .
    For  $t$  from  $T$  to 1 :
        Set  $f(x_{t-1}, x_t) = G_t^{\text{ref}}(x_{t-1}, x_t)\eta_t^{*,(L)}(x_t)$ .
        Set  $\hat{\varphi}_t^{*,(L)} = \mathcal{A}^{\text{bck}}(f, \tilde{x}_{t-1}^{(L-1),1:N}, x_t^{(L-1),1:N})$ .
        Set  $\hat{\eta}_{t-1}^{*,(L)} = M_t^{\text{ref}}(\hat{\varphi}_t^{*,(L)})$ .
        Set  $M_t^{*,(L)} = \frac{\hat{\varphi}_t^{*,(L)}}{\hat{\eta}_{t-1}^{*,(L)}} M_t^{\text{ref}}$ .
    // Sample particles according to the last iteration
    Sample  $(x_{1:t}^{(L),1:N}, w_{1:T}^{(L),1:N}, a_{1:T}^{(L),1:N})$  from the auxiliary
    Feynman-Kac model  $(\hat{\mathbb{M}}_{1:T}^{*,(L)}, \hat{\eta}_{1:T}^{*,(L)})$  with Algorithm 1.
    Denote  $\tilde{x}_t^{(L+1),n} = x_t^{(i+1),a_t^{(L+1),n}}$ .

```

The learned auxiliary Feynman-Kac model $(\hat{M}_{1:T}, \hat{\eta}_{1:T})$ follows:

$$\hat{M}_t = \frac{\hat{\varphi}_t}{M_t^{\text{ref}}(\hat{\varphi}_t)} M_t^{\text{ref}}, \quad \hat{\eta}_{t-1} = M_t^{\text{ref}}(\hat{\varphi}_t).$$

The backward scheme is not naturally iterative. The idea of Guarniero et al. (2017) and Heng et al. (2020) is that each iteration of the algorithm samples from an auxiliary Feynman-Kac model $(\hat{\mathbb{M}}_{1:T}^{\star,(L)}, \hat{\eta}_{1:T}^{\star,(L)})$ and gives us, applying the backward scheme, the next iteration $(\hat{\mathbb{M}}_{1:T}^{\star,(L+1)}, \hat{\eta}_{1:T}^{\star,(L+1)})$. The idea is that each of the iterations will bring U_t closer to the optimal U_t^* and the SMC output closer to being directly sampled from the optimal ν_T^* . The validity of this convergence was proved under strong assumptions in Heng et al. (2020).

Since the backward scheme starts from the end, the corresponding Algorithm 4 builds approximations from outputs of an SMC algorithm $(x_{1:T}^{1:N}, a_{1:T}^{1:N}, w_{1:T}^{1:N})$ obtained with Algorithm 1 on the auxiliary Feynman-Kac model $(\hat{\mathbb{M}}_{1:T}^{\star,(L+1)}, \hat{\eta}_{1:T}^{\star,(L+1)})$ returned by the previous iteration. Such outputs provide points used to make the approximations in Algorithm 4. Thus, if the initial SMC algorithm is degenerate, there is a good chance that the approximation will not learn anything useful from the target.

Approximations are built as follows:

$$\hat{\varphi}_t^{\star,(L)} = \mathcal{A}^{\text{bck}}(G_t \hat{\eta}_t^{\star,(L)}, x_{t-1}^{(L-1), a_{t-1}^{(L-1), 1:N}}, x_t^{(L-1), 1:N})$$

where, given a function f_t that is approximated in a set of functions Ψ_t , particles $x_{t-1}^{1:N}$ and $x_t^{1:N}$, \mathcal{A}^{bck} is defined as the minimizer of the following averaged loss over the particles:

$$\mathcal{A}^{\text{bck}}(f_t, x_{t-1}^{1:N}, x_t^{1:N}) = \arg \min_{\hat{f} \in F} \sum_{n=1}^N \left(\log \hat{f}(x_{t-1}^n, x_t^n) - \log f_t(x_{t-1}^n, x_t^n) \right)^2.$$

C Online version of the Forward Iterated Algorithm

The forward algorithm (Algorithm 2) can be transformed into Algorithm 5 by changing the order of the operations and iterating over t in the inner loop. This algorithm is online since at iteration t we can discard all previous $y_{1:t-L^{\max}}$ data.

One important distinction between the forward and the online versions of our method is that we need to fix the depth L^{\max} of our scheme before we run the online Algorithm 5. In the forward Algorithm 2, we can decide to increment L^{\max} at the end of each inner loop.

Algorithm 5: Online SMC Training

Input: A reference Feynman-Kac model $(M_{1:T}^{\text{ref}}, G_{1:T}^{\text{ref}})$, a number of particles N , a number of iterations L^{\max} and sets of strictly positive functions $\Psi_{1:T}$.

Output: Approximations $\hat{\varphi}_{1:t}^{(L)}$ for $1 \leq L \leq L^{\max}$ of the forward scheme.

Set $\hat{\varphi}_{1:T}^{(0)} \equiv 1$, $\hat{M}_1^{(0)} \sim M_1^{\text{ref}}$, $\hat{M}_{2:T}^{(0)} \equiv M_{2:T}^{\text{ref}}$ and $\hat{\eta}_{1:T}^{(0)} \equiv 1$.

For $t_{\text{end}} \in 1:T$ **with** $t_{\text{start}} = \min(1, t_{\text{end}} - L^{\max} + 1)$:

For t **from** t_{end} **to** t_{start} **with** $L = t_{\text{end}} - t$:

Approximate $G_t^{\text{ref}} \hat{\eta}_t^{(L)}$ **by** $\hat{\varphi}_t^{(L+1)} \in \Psi_t$.

Set $\hat{\eta}_{t-1}^{(L+1)} = M_t^{\text{ref}}(\hat{\varphi}_t^{(L+1)})$.

Set $\hat{M}_t^{(L+1)} = \frac{\hat{\varphi}_t^{(L+1)}}{\hat{\eta}_{t-1}^{(L+1)}} M_t^{\text{ref}}$.

// Sample particles according to trained proposal

Sample and **Compute** $x_t^{(L+1),n}$, $w_t^{(L+1),n}$ and $a_t^{(L+1),n}$ according to the t -th SMC step (Algorithm 1) with $M_t = \hat{M}_t^{(L+1)}$,

$\eta_{t-1:t} = \eta_{t-1:t}^{(L)}$ and ancestor $\tilde{x}_{t-1}^{(L+1),n}$.

Denote $\tilde{x}_t^{(L+1),n} = x_t^{(L+1),a_t^{(L+1),n}}$.

D Computational cost of the Forward Iterated Algorithm

The computational cost can be divided between two major operations: sampling particles and solving \mathcal{A}^{fwd} and \mathcal{A}^{bck} . Both the forward Algorithm 3 and the backward Algorithm 4 use the same number of minimizations per iteration (T minimizations), but these algorithms do not sample the same number of particles. At each iteration, the forward Algorithm 3 uses $N \times T$ sampling of particles for the training and $N \times T$ sampling of ancestors according to the trained proposal, whereas the backward Algorithm 4 uses the same particles as training particles and as ancestors in the SMC.

If we denote by C_{Sample} and C_{Solve} the cost per particle of these two main operations, the forward cost C_{fwd} and the backward cost C_{bck} are:

$$\begin{aligned}\frac{C_{\text{fwd}}}{NT} &= 2C_{\text{Sample}} + C_{\text{Solve}} \\ \frac{C_{\text{bck}}}{NT} &= C_{\text{Sample}} + C_{\text{Solve}}.\end{aligned}$$

This assumes that the cost of minimization is linear in N (or with a negligible constant part) and the same for \mathcal{A}^{fwd} and \mathcal{A}^{bck} , which is the case in our application.

Algorithm 6: Faster Online Forward Iterated SMC

Input: A reference Feynman-Kac model $(\mathbb{M}_1, M_{2:t}, G_{1:t})$, a number of particles N , a number of iterations L^{\max} .

Output: Approximations $\hat{\varphi}_{1:t}^{(L)}$ for $1 \leq L \leq L^{\max}$ of the forward scheme.

```

Set  $\hat{\varphi}_{1:T}^{(0)} \equiv 1$ ,  $\hat{\mathbb{M}}_1^{(0)} \sim \mathbb{M}_1$ ,  $\hat{M}_{2:T}^{(0)} \equiv M_{2:T}^{(0)}$  and  $\hat{\eta}_{1:T}^{(0)} \equiv 1$ .
For  $t_{\text{end}} \in 1:T$  with  $t_{\text{start}} = \max(1, t_{\text{end}} - L^{\max} + 1)$  :
    // Sample particles
    if  $t_{\text{start}} = 1$  with  $L = t_{\text{end}}$  :
        Sample  $x_1^{(L),n} \sim \hat{\mathbb{M}}_1^{(L)}(dx_t)$ .
        Compute  $\bar{w}_1^{(L),n} = \hat{G}_1^{(L)}(x_t^{(L),n})$ 
    else with  $t = t_{\text{start}}$  and  $L = L^{\max}$  :
        Resample  $x_{t-1}^{(L),1:N}$  and get ancestors  $a_{t-1}^n$  according to  $w_t^{(L),1:N}$ .
        Sample  $x_t^{(L),n} \sim \hat{M}_t^{(L)}(\bar{x}_{t-1}^{(L),a_{t-1}^n}, dx_t)$ .
        Compute  $\bar{w}_t^{(L),n} = \hat{G}_t^{(L)}(x_{t-1}^{(L),a_{t-1}^n}, x_t^{(L),n})$ .
    For  $t$  from  $t_{\text{start}} + 1$  to  $t_{\text{end}}$  with  $L = t_{\text{end}} - t$  :
        Resample  $x_{t-1}^{(L+1),1:N}$  to  $\bar{x}_{t-1}^{(L+1),1:N}$  according to  $\bar{w}_t^{(L),1:N}$ .
        Sample  $x_t^{(L),n} \sim \hat{M}_t^{(L)}(\bar{x}_{t-1}^{(L+1),n}, dx_t)$ .
        Compute  $\bar{w}_t^{(L),n} = \hat{G}_t^{(L)}(x_{t-1}^{(L+1),n}, x_t^{(L),n})$ .
    Set  $w_t^{(0),t_{\text{end}}} = \bar{w}_t^{(0),t_{\text{end}}}$ 
    // Compute the approximations
    For  $t$  from  $t_{\text{end}}$  to  $t_{\text{start}}$  with  $L = t_{\text{end}} - t$  :
        Set  $f(x_{t-1}, x_t) = G_t(x_{t-1}, x_t) \hat{\eta}_t^{(L)}(x_t)$ 
        Set  $\hat{\varphi}_t^{(L+1)} = \mathcal{A}^{\text{fwd}}(f, \bar{x}_{t-1}^{(L+1),1:N}, x_t^{(L),1:N}, \bar{w}_t^{(L),1:N})$ .
        Set  $\hat{\eta}_{t-1}^{(L+1)} = M_t(\hat{\varphi}_t^{(L+1)})$ .
        Set  $\hat{M}_t^{(L+1)} = \frac{\hat{\varphi}_t^{(L+1)}}{\hat{\eta}_{t-1}^{(L+1)}} M_t$ .
        Set  $\hat{G}_t^{(L+1)} = G_t \frac{dM_t}{dM_t^{(L+1)}} \frac{\eta_t^{(L)}}{\eta_{t-1}^{(L+1)}}$ .
        Compute  $w_{t-1}^{(L+1)} = \bar{w}_{t-1}^{(L+1)} \times \frac{\eta_{t-1}^{(L+1)}}{\eta_t^{(L)}}(x_t^{(L),n})$ .

```

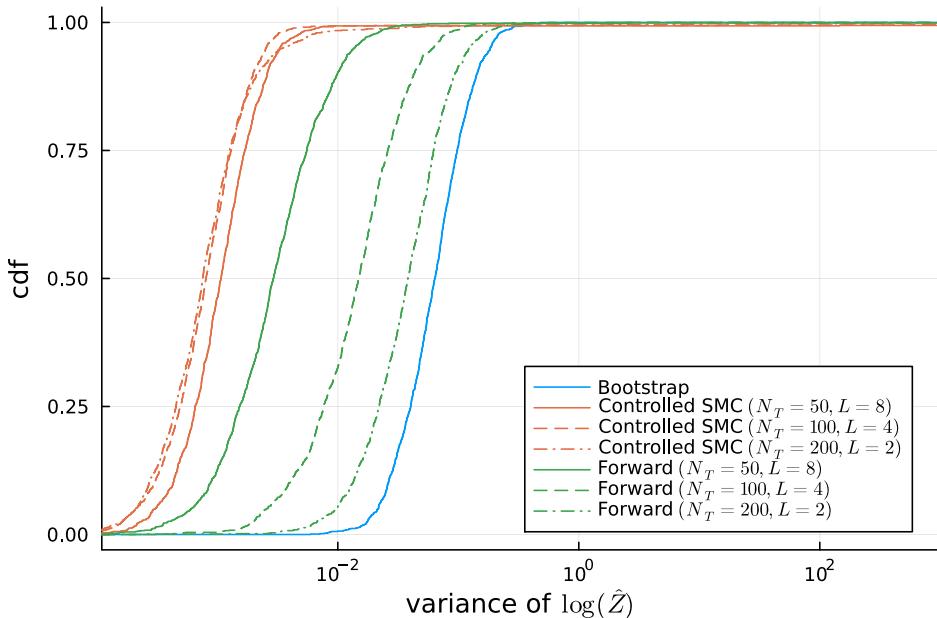


Figure 4: Empirical cumulative distribution of the empirical variances of $\log \hat{Z}_T$ $d = 7$ (bottom), measured every 100 steps of the PMMH algorithm over 10 evaluation.

With log-quadratic twisting functions, the sampling cost is linear in the dimension and the minimization cost is either $O(d^4)$ with a full quadratic term A or $O(d^2)$ with a diagonal quadratic term A . Then, with problems of higher dimension, the cost per iteration of the forward Algorithm 3 is nearly identical to the cost per iteration of the backward Algorithm 4. Without this log-quadratic setting, the cost per iteration is higher since it is more involved than the computation of an inverse. Therefore, the difference in computation time per iteration should also be small between the algorithms.

One way to reduce the computation time of Algorithm 3 is by doing things online. If one wants to run many iterations of the Forward Iterated Algorithm, one could recycle the sampled particles to use them for training. It is done in Algorithm 6, which requires half the number of sampling particles than the initial Algorithm 3 (exactly the same amount as the backward algorithm).

E Details on the robustness of the MSV experiment

In this section, we keep the model and settings of the multivariate stochastic volatility model defined in subsection 5.2.

We run the same experiment, but we remove one currency (INR – the

Indian rupee), which is especially difficult for the model. We end up with $d = 7$, we choose $N = 3000$ for the bootstrap particle filter, $N_{\text{sample}} = 800$, and either $(L = 8, N_{\text{train}} = 50)$, $(L = 4, N_{\text{train}} = 100)$, or $(L = 2, N_{\text{train}} = 200)$ for the iterated algorithms. In Figure 4, we observe that all versions of the controlled SMC algorithm achieve similar variances, better than the forward variances. By contrast with Figure 3, this shows that posterior sampling for this model is not necessarily challenging, but that real datasets can be more difficult than others. Controlled SMC, as shown in the simulated experiment (Figure 1), does well with easier datasets but struggles with more difficult ones. Conversely, the forward scheme is slower by nature (especially with easier datasets), but can work with more difficult datasets, and here gives similar performances in both cases. This is especially important with real data applications, since practitioners are not always able to choose their datasets, and our algorithm extends the sets of models that can be tackled with iterated SMC methods.