

# Global regularity for the Navier-Stokes equations with application to global solvability for the Euler equations

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## Abstract

We show that any Leray-Hopf weak solution to the  $d$ -dimensional Navier-Stokes equations ( $d \geq 3$ ) with initial values  $u_0 \in H^s(\mathbb{R}^d)$ ,  $s \geq -1 + \frac{d}{2}$ , belongs to  $L^\infty(0, \infty; H^s(\mathbb{R}^d))$  and thus it is globally regular. For the proof, first, we construct a supercritical space which has very sparse inverse logarithmic weight in the frequency domain, compared to the critical homogeneous Sobolev  $\dot{H}^{-1+d/2}$ -norm. Then we obtain the energy estimates of high frequency parts of the solution which involve the supercritical norm as a factor of the upper bounds. Finally, we superpose the energy norm of high frequency parts of the solution to get estimates of the critical and subcritical norms independent of the viscosity coefficient for the weak solution via the re-scaling argument.

As a direct application via the argument of vanishing viscosity, we obtain global existence of a solution for the incompressible Euler equations with initial values in  $H^s(\mathbb{R}^d)$ ,  $s \geq -1 + \frac{d}{2}$ , which is unique when  $s > 1 + \frac{d}{2}$ .

**Keywords:** Navier-Stokes equations; Euler equations; global existence; global regularity; supercritical space

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## 1 Introduction and main results

Let us consider the Cauchy problem for the Navier-Stokes equations:

$$\begin{aligned} u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= 0 && \text{in } (0, \infty) \times \mathbb{R}^d, \\ \operatorname{div} u &= 0 && \text{in } (0, \infty) \times \mathbb{R}^d, \\ u(0, x) &= u_0 && \text{in } \mathbb{R}^d. \end{aligned} \tag{1.1}$$

Since Leray [17] proved existence of a global weak solution (Leray-Hopf weak solution)  $u$  to (1.1) such that

$$u \in L^2(0, \infty; H^1(\mathbb{R}^d)) \cap L^\infty(0, \infty; L^2(\mathbb{R}^d))$$

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satisfies (1.1) in a weak sense and the *energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2 + \nu \int_0^t \|\nabla u(\tau)\|_2^2 d\tau \leq \frac{1}{2}\|u_0\|_2^2, \quad \forall t \in (0, \infty), \quad (1.2)$$

the problem of global regularity of the weak solution for  $d \geq 3$  has been a consistent key issue.

There is a great number of articles on scaling-invariant regularity criteria for weak solutions to (1.1). We recall that the Navier-Stokes equations are invariant by the scaling  $u_\lambda(t, x) \equiv \lambda u(\lambda^2 t, \lambda x)$ ,  $\lambda > 0$ , and a critical space for the equations is the space whose norm is invariant with respect to the scaling

$$u(x) \mapsto \lambda u(\lambda x), \quad \lambda > 0.$$

The following embedding holds between critical spaces for the  $d$ -dimensional Navier-Stokes equations:

$$\dot{H}^{-1+d/2} \hookrightarrow L^d \hookrightarrow \dot{B}_{p,\infty}^{-1+d/p} \hookrightarrow BMO^{-1} \hookrightarrow \dot{B}_{\infty,\infty}^{-1}, \quad d \leq p < \infty.$$

Eskauriaza, Seregin and Šverák in [7](2003) proved that a 3D Leray-Hopf weak solution  $u$  to (1.1) is regular in  $(0, T]$  if

$$u \in L^\infty(0, T; L^3(\mathbb{R}^3)),$$

employing the backward uniqueness property of parabolic equations. By developing a profile decomposition technique and using the method of “critical elements” developed in [14]-[16], Gallagher, Koch and Planchon proved for a mild solution  $u$  to (1.1) with initial values in  $L^3(\mathbb{R}^3)$  that a potential singularity at  $t = T$  implies  $\lim_{t \rightarrow T^-} \|u(t)\|_3 = \infty$  in [9](2013), and extended the result from  $L^3(\mathbb{R}^3)$  to wider critical Besov spaces  $\dot{B}_{p,q}^{-1+3/p}(\mathbb{R}^3)$ ,  $3 < p, q < \infty$  in [10](2016). In [19](2023), the author proved that the above results can be extended to the largest critical space  $\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^d)$  for the case of general  $d \geq 3$ , that is, a Leray-Hopf weak solution to (1.1) satisfying

$$u \in L^\infty(0, T; \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^d))$$

is regular in  $(0, T]$  by developing a superposition method of energy norms of high frequency parts for the weak solution.

Concerning supercritical regularity criteria, we mention recent results [3] and [4] by Barker and Prange that a 3D Leray-Hopf weak solution  $u$  satisfying the local energy inequality is regular in  $(0, T]$  if

$$\sup_{0 < t < T} \int_{\mathbb{R}^3} \frac{|u(x, t)|^3}{\left( \log \log \log \left( (\log(e^{e^{3e}} + |u(x, t)|))^{1/3} \right) \right)^\theta} dx < \infty, \quad \forall \theta \in (0, 1),$$

based on a Tao’s result [21, 22] on quantitative bound of the critical  $L^3$ -norm near a possible blow-up epoch for a 3D weak solution to (1.1); we also mention

Pan [18] and Seregin [20] where logarithmically supercritical regularity criteria for axisymmetric suitable weak solutions to (1.1) are obtained.

In this paper, we prove global regularity of the Leray-Hopf weak solutions to (1.1) for initial values in  $H^s(\mathbb{R}^d)$ ,  $d \geq 3$ ,  $s \geq -1 + d/2$ . More precisely, we have:

**Theorem 1.1** Let  $u$  be a Leray-Hopf weak solution to (1.1) with  $u_0 \in H^s(\mathbb{R}^d)$ ,  $d \geq 3$ ,  $s \geq -1 + d/2$ , and  $\operatorname{div} u_0 = 0$ . Then,

$$u \in L^\infty(0, \infty; H^s(\mathbb{R}^d)) \cap L^2(0, \infty; H^{s+1}(\mathbb{R}^d))$$

and the estimate

$$\|u(t)\|_{H^s(\mathbb{R}^d)}^2 + \nu \int_0^t \|u(\tau)\|_{H^{s+1}(\mathbb{R}^d)}^2 d\tau \leq C(s) \|u_0\|_{H^s(\mathbb{R}^d)}^2, \quad \forall t \in (0, \infty), \quad (1.3)$$

holds true with a constant  $C(s) > 0$  depending only on  $s$  and independent of  $\nu$  and  $d$ . In particular,  $u$  is globally regular and unique.

**Remark 1.2** (i) Combining the result of Theorem 1.1 with already known regularity theory of the Navier-Stokes equations, cf. e.g. [1], one can conclude the following statement:

If  $u_0 \in H^s(\mathbb{R}^d)$ ,  $s \geq -1 + d/2$ ,  $\operatorname{div} u_0 = 0$ , then the problem (1.1) has a unique global strong solution

$$u \in C([0, \infty), H^s(\mathbb{R}^d)) \cap L^2(0, \infty; H^{s+1}(\mathbb{R}^d))$$

satisfying the estimate (1.3).

(ii) The statement of Theorem 1.1 that the estimate constants of higher-order norms are independent of the viscosity is very important. This fact yields, by vanishing viscosity, a global existence of solutions for Euler equations of ideal incompressible fluids, see Theorem 4.2.

We use the following notations. The sets of all natural numbers and all integers are denoted by  $\mathbb{N}$  and  $\mathbb{Z}$ , respectively. The usual  $L^p$ -norm for  $1 \leq p \leq \infty$  is denoted by  $\|\cdot\|_p$ . The Fourier transform of a function  $u$  is given by  $\mathcal{F}u \equiv \hat{u} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} u(t, x) e^{ix \cdot \xi} dx$  and  $\mathcal{F}^{-1}u \equiv \check{u}$ . For  $s \in \mathbb{R}$ ,  $\dot{H}^s(\mathbb{R}^d)$  and  $H^s(\mathbb{R}^d)$  stand for the homogeneous and inhomogeneous Sobolev spaces with norms

$$\|u\|_{\dot{H}^s(\mathbb{R}^d)} \equiv \| |\xi|^s \hat{u}(\xi) \|_2 \quad \text{and} \quad \|u\|_{H^s(\mathbb{R}^d)} \equiv \|(1 + |\xi|^2)^{s/2} \hat{u}(\xi)\|_2,$$

respectively. We use the notation

$$\begin{aligned} u^k(t, x) &:= (2\pi)^{-d/2} \int_{|\xi| \geq k} \mathcal{F}u(t, \xi) e^{ix \cdot \xi} d\xi, \\ u_k &:= u - u^k, \quad u_{h,k} := u^h - u^k \quad \text{for } 0 \leq h < k. \end{aligned} \quad (1.4)$$

We do not distinguish between the spaces of vectorial functions and scalar functions.

Let us roughly explain the ideas for the proof of Theorem 1.1 for the particular case  $d = 3$  and  $s = 1/2$ ; the idea is exactly the same for general  $d \geq 3$  and  $s$ . As is well-known, the cancelation property  $((v \cdot \nabla)u, u) = 0$  for suitably smooth  $u, v$  with  $\operatorname{div} v = 0$  is the key to obtain global existence of the Leray-Hopf weak solutions to (1.1). However, this innermost property was not used, so long as the author knows, in most previous works for regularity. Indeed, as long as the convection term  $(u \cdot \nabla)u$  is treated as merely a quadratic nonlinear term, whatever improvement of its estimate could be made, global regularity of the weak solution is obtained under a smallness condition of a critical norm of initial values or under additional scaling-invariant conditions on the weak solution itself.

In order to circumvent such situations, we use essentially the cancelation property  $((u \cdot \nabla)u^k, u^k) = 0$  by testing the momentum equation with high frequency parts  $u^k$ ,  $k \in \mathbb{N}$ . Then, in the (local) time interval where the weak solution is regular we have

$$\frac{d}{2dt} \|u^k\|_2^2 + \nu \|\nabla u^k\|_2^2 \leq c \|u\|_X \|\nabla u^{k/2}\|_2^2, \quad \forall k \in \mathbb{N},$$

usually with a critical space  $X$  or, through a more refined observation,

$$\frac{d}{2dt} \|u^k\|_2^2 + \nu \|\nabla u^k\|_2^2 \leq c(k) \|u\|_{X_1} \|\nabla u^{k/2}\|_2^2, \quad \forall k \in \mathbb{N},$$

with a supercritical space  $X_1$ , where  $c(k)$  depends on the topology of  $X_1$ . Though this estimate, at the first glance, still seems of no special use due to the factor  $c(k)$ , we pay attention to the fact that

$$\sum_{k \in \mathbb{N}} \|u^k\|_2^2 \sim \|u^1\|_{\dot{H}^{1/2}}^2, \quad \sum_{k \in \mathbb{N}} \|\nabla u^k\|_2^2 \sim \|\nabla u^1\|_{\dot{H}^{1/2}}^2$$

and

$$\sum_{k \in \mathbb{N}} c(k) \|\nabla u^{k/2}\|_2^2 \lesssim \|\nabla u\|_2^2 + \sum_{n \in \mathbb{N}} \left( \sum_{k=1}^n c(k) \right) \|\nabla u_{n,n+1}\|_2^2.$$

Thus, if we can construct a supercritical space  $X_1$  such that the averaging condition  $\sum_{k=1}^n c(k) \lesssim n$  is satisfied, then we are led to

$$\frac{d}{2dt} \|u^1\|_{\dot{H}^{1/2}}^2 + \nu \|\nabla u^1\|_{\dot{H}^{1/2}}^2 \lesssim \|u\|_{X_1} (\|\nabla u\|_2^2 + \|\nabla u^1\|_{\dot{H}^{1/2}}^2),$$

yielding by energy inequality that

$$\frac{1}{2} \|u(t)\|_{\dot{H}^{1/2}}^2 + \nu \int_0^t \|\nabla u\|_{\dot{H}^{1/2}}^2 d\tau \leq \frac{1}{2} \|u_0\|_{\dot{H}^{1/2}}^2 + c \int_0^t \|u\|_{X_1} (\|\nabla u\|_2^2 + \|\nabla u\|_{\dot{H}^{1/2}}^2) d\tau$$

for each fixed  $t$ . Hence, if additionally uniform smallness of  $\|u(\tau)\|_{X_1}$  in  $\tau \in [0, t]$  can be guaranteed by a suitable re-scaling given for fixed  $t$ , then in the last term of the right-hand side the part  $c \int_0^t \|u\|_{X_1} \|\nabla u\|_{\dot{H}^{1/2}}^2 d\tau$  can be absorbed into

the left-hand side and the remaining part can be estimated using the energy inequality. Thus  $\|u(t)\|_{\dot{H}^{1/2}} \lesssim \|u_0\|_{H^{1/2}}$  is obtained irrespective of the viscosity and  $t$ . Remark here that the re-scaling parameter should depend on  $t$ , but it does not matter.

We successfully construct a supercritical space  $X_1$  with the above-mentioned averaging condition and the uniform smallness property by re-scaling, which has a very sparse inverse logarithmic weight in the frequency domain compared to the critical  $\dot{H}^{-1+d/2}$ -norm, Section 2.

The proof of Theorem 1.1 is given in Section 3.

The regularity estimate for the Navier-Stokes equations, see (1.3), which are uniform with respect to the viscosity coefficient  $\nu$  enables us to obtain easily global solvability for the incompressible Euler equations, Section 4.

## 2 A frequency-weighted scaling-variant space

In this section, we introduce a supercritical space, the norm of which is quite slightly weaker than that of  $\dot{H}^{-1+d/2}(\mathbb{R}^d)$ .

Let

$$\Delta_j u = (\chi_j(\xi) \hat{u}(\xi))^\vee \text{ for } j \in \mathbb{Z}, u \in \mathcal{S}'(\mathbb{R}^d) \text{ with } \hat{u} \in L^1_{loc}(\mathbb{R}_\xi^d),$$

where  $\chi_j$  is the characteristic function of the set  $\{\xi \in \mathbb{R}^d : 2^{j-1} \leq |\xi| < 2^j\}$  and  $\mathcal{S}'(\mathbb{R}^d)$  is the space of tempered distributions.

It immediately follows by Plancherel's theorem that for  $s \in \mathbb{R}$  the homogeneous Sobolev norm  $\|u\|_{\dot{H}^s(\mathbb{R}^d)}$  is equivalent to the norm

$$\left( \sum_{j \in \mathbb{Z}} \|2^{sj} \Delta_j u\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2}. \quad (2.1)$$

Let an infinite sequence  $\{a(j)\}_{j \in \mathbb{Z}} \subset \mathbb{R}$  be such that

$$a(j) := \begin{cases} \log_2 j & \text{if } j = i + 2^{2^k} \text{ for some } k \in \mathbb{N}, i \in \mathbb{Z}, -k \leq i \leq k, \\ 1 & \text{else.} \end{cases} \quad (2.2)$$

Let us define the space  $X_1$  by

$$\begin{aligned} X_1 &:= \{v \in \mathcal{S}'(\mathbb{R}^d) : \hat{v}(\xi) \in L^1_{loc}(\mathbb{R}_\xi^d), \{2^{j(-1+d/2)} a^{-1}(j) \|\Delta_j v\|_2\}_{j \in \mathbb{Z}} \in l_2\}, \\ \|v\|_{X_1} &:= \left\| \{2^{j(-1+d/2)} a^{-1}(j) \|\Delta_j v\|_2\}_{j \in \mathbb{Z}} \right\|_{l_2}, \end{aligned} \quad (2.3)$$

where the sequence  $\{a(j)\}$  is given by (2.2).

Obviously,  $a(j) = 1$  for  $j \leq 0$  and  $\dot{H}^{-1+d/2}(\mathbb{R}^d)$  is densely embedded into  $X_1$ , and

$$\|v\|_{X_1} \leq \|v\|_{\dot{H}^{-1+d/2}(\mathbb{R}^d)}, \forall v \in \dot{H}^{-1+d/2}(\mathbb{R}^d).$$

We pursue supercritical properties of the space  $X_1$  via the next lemma.

**Lemma 2.1** (i) Let  $v \in \dot{H}^{-1+d/2}(\mathbb{R}^d)$ . Then,

$$\begin{aligned} \forall \varepsilon > 0, \exists l_0 := \max\{M + 1, \log_2 \left[ \frac{\log_2 M \|v\|_{X_1}}{\varepsilon} \right] + 1\} > 0, \\ \|\lambda v(\lambda \cdot)\|_{X_1} \leq \varepsilon, \quad \forall \lambda = 2^{2^l} (l \geq l_0), \end{aligned}$$

where  $M > 0$  is such that

$$\sum_{|j| \geq M} 2^{2j(-1+d/2)} \|\Delta_j v\|_2^2 \leq \frac{\varepsilon^2}{2}. \quad (2.4)$$

(ii) Let  $u \in C([0, T], \dot{H}^{-1+d/2}(\mathbb{R}^d))$  with  $0 < T < \infty$ . Then, for any sufficiently small  $\varepsilon > 0$  there exists  $l_0 > 0$  such that

$$\|\lambda u(t, \lambda \cdot)\|_{X_1} \leq \varepsilon, \quad \forall t \in [0, T], \forall \lambda = 2^{2^l} (l \geq l_0).$$

**Proof:** – *Proof of (i):* Let  $v \in \dot{H}^{-1+d/2}(\mathbb{R}^d)$  and fix  $\varepsilon > 0$  arbitrarily. Then, in view of

$$\|v\|_{\dot{H}^{-1+d/2}} \sim \left\| \{2^{j(-1+d/2)} \|\Delta_j v\|_2\}_{j \in \mathbb{Z}} \right\|_{l_2},$$

there is  $M = M(\varepsilon, v) \in \mathbb{N}$  satisfying (2.4).

In the proof, for the moment, we use a short notation  $v_\lambda := \lambda v(\lambda \cdot)$ . By definition of the space  $X_1$  we have

$$\|v_\lambda\|_{X_1} = \left\| \{2^{j(-1+d/2)} a^{-1}(j) \|\Delta_j v_\lambda\|_2\}_{j \in \mathbb{Z}} \right\|_{l_2}.$$

Observe that  $(\mathcal{F}v_\lambda)(\xi) = \lambda(\mathcal{F}v(\lambda \cdot))(\xi) = \lambda^{1-d}(\mathcal{F}v)(\frac{\xi}{\lambda})$  for all  $\lambda > 0$ . Then, for all  $j \in \mathbb{Z}$

$$\begin{aligned} \Delta_j v_\lambda(x) &= \frac{1}{(2\pi)^{d/2}} \int_{2^{j-1} \leq |\xi| < 2^j} \mathcal{F}v_\lambda(\xi) e^{ix \cdot \xi} d\xi \\ &= \frac{\lambda^{1-d}}{(2\pi)^{d/2}} \int_{2^{j-1} \leq |\xi| < 2^j} \mathcal{F}v\left(\frac{\xi}{\lambda}\right) e^{ix \cdot \xi} d\xi \\ &= \frac{\lambda}{(2\pi)^{d/2}} \int_{\lambda^{-1} 2^{j-1} \leq |\eta| < \lambda^{-1} 2^j} \mathcal{F}v(\eta) e^{i\lambda x \cdot \eta} d\eta. \end{aligned}$$

Hence, if  $\lambda \equiv 2^{2^l}$ ,  $l \in \mathbb{N}$ , then

$$\begin{aligned} \Delta_j v_\lambda(x) &= \frac{\lambda}{(2\pi)^{d/2}} \int_{2^{j-2^{2^l}-1} \leq |\eta| < 2^{j-2^{2^l}}} \mathcal{F}v(\eta) e^{i\lambda_0 x \cdot \eta} d\eta \\ &= \lambda (\Delta_{j-2^{2^l}} v)(\lambda x), \end{aligned} \quad (2.5)$$

yielding

$$2^{j(-1+d/2)} \Delta_j v_\lambda(x) = 2^{j(-1+d/2)+2^{2^l}} (\Delta_{j-2^{2^l}} v)(\lambda x), \quad \forall j \in \mathbb{Z}.$$

Note that

$$\|(\Delta_{j-2^{2^l}} v)(\lambda \cdot)\|_2 = \lambda^{-d/2} \|\Delta_{j-2^{2^l}} v\|_2.$$

Therefore, by (2.5) we have

$$\begin{aligned} 2^{j(-1+d/2)} \|\Delta_j v_\lambda\|_2 &= 2^{j(-1+d/2)+2^{2^l}} \|(\Delta_{j-2^{2^l}} v)(\lambda \cdot)\|_2 \\ &= 2^{(-1+d/2)(j-2^{2^l})} \|\Delta_{j-2^{2^l}} v\|_2, \quad \forall j \in \mathbb{Z}, \end{aligned} \quad (2.6)$$

and, in view of (2.4),

$$\begin{aligned} \|v_\lambda\|_{X_1}^2 &= \left\| \left\{ 2^{j(-1+d/2)} a^{-1}(j) \|\Delta_j v_\lambda\|_2 \right\}_{j \in \mathbb{Z}} \right\|_{l_2}^2 \\ &= \left\| \left\{ 2^{(-1+d/2)(j-2^{2^l})} a^{-1}(j) \|\Delta_{j-2^{2^l}} v\|_2 \right\}_{j \in \mathbb{Z}} \right\|_{l_2}^2 \\ &= \left\| \left\{ \frac{1}{a(j+2^{2^l})} 2^{j(-1+d/2)} \|\Delta_j v\|_2 \right\}_{j \in \mathbb{Z}} \right\|_{l_2}^2 \\ &\leq \left\| \left\{ \frac{1}{a(j+2^{2^l})} 2^{j(-1+d/2)} \|\Delta_j v\|_2 \right\}_{|j| \leq M} \right\|_{l_2}^2 + \left\| \left\{ 2^{j(-1+d/2)} \|\Delta_j v\|_2 \right\}_{|j| > M} \right\|_{l_2}^2 \\ &\leq \frac{\varepsilon^2}{2} + \left\| \left\{ \frac{a(j)}{a(j+2^{2^l})} 2^{j(-1+d/2)} a^{-1}(j) \|\Delta_j v\|_2 \right\}_{|j| \leq M} \right\|_{l_2}^2. \end{aligned}$$

Here, for  $l \geq M+1$ , by definition of  $a(j)$  we have

$$\begin{aligned} &\left\| \left\{ \frac{a(j)}{a(j+2^{2^l})} 2^{j(-1+d/2)} a^{-1}(j) \|\Delta_j v\|_2 \right\}_{|j| \leq M} \right\|_{l_2} \\ &= \left\| \left\{ \frac{a(j)}{\log_2(j+2^{2^l})} 2^{j(-1+d/2)} a^{-1}(j) \|\Delta_j v\|_2 \right\}_{|j| \leq M} \right\|_{l_2} \\ &\leq \sup_{|j| \leq M} \frac{\log_2 M}{\log_2(j+2^{2^l})} \cdot \left\| \left\{ 2^{j(-1+d/2)} a^{-1}(j) \|\Delta_j v\|_2 \right\}_{|j| \leq M} \right\|_{l_2} \\ &\leq \frac{\log_2 M}{2^{l-1}} \|v\|_{X_1}. \end{aligned}$$

Therefore, for all

$$l \geq l_0 := \max\{M+1, \log_2 \left[ \frac{\log_2 M \|v\|_{X_1}}{\varepsilon} \right] + 1\}, \quad \lambda = 2^{2^{2^l}}, \quad (2.7)$$

we have

$$\|v_\lambda\|_{X_1}^2 \leq \frac{\varepsilon^2}{2} + \left( \frac{\log_2 M}{2^{l-1}} \|v\|_{X_1} \right)^2 \leq \varepsilon^2.$$

- *Proof of (ii):* Let  $u \in C([0, T]; \dot{H}^{-1+d/2}(\mathbb{R}^d))$  and put

$$f(t, s) := \|u_{1/s}(t)\|_{\dot{H}^{-1+d/2}} + \|u^s(t)\|_{\dot{H}^{-1+d/2}}, \quad (t, s) \in [0, T] \times [2, \infty).$$

Then,  $u \in C([0, T]; \dot{H}^{-1+d/2}(\mathbb{R}^d))$  implies that  $f \in BC([0, T] \times [2, \infty))$  and for all  $t \in [0, T]$  the mapping  $s \rightarrow f(t, s)$  is nonincreasing in  $s$  and  $f(t, s) \rightarrow 0$  as

$s \rightarrow \infty$ . Therefore, for any sufficiently small  $\varepsilon > 0$  and  $t \in [0, T]$  there exists  $s = s(t)$  satisfying  $f(t, s(t)) = \frac{\varepsilon}{2^{(d-1)/2}}$ , where one may choose  $s(t)$  such that

$$\exists N > 0, \forall t \in [0, T] : |s(t)| \leq N \quad (2.8)$$

holds. In fact, if (2.8) does not hold true, there is a sequence  $\{t_n\} \subset [0, T]$  satisfying  $f(t_n, s(t_n)) = \frac{\varepsilon}{2^{(d-1)/2}}$  and  $s(t_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Without loss of generality, we may assume  $t_n \rightarrow \tilde{t}$  for some  $\tilde{t} \in [0, T]$  for  $n \rightarrow \infty$ . Then, in view of  $u \in C([0, T], \dot{H}^{-1+d/2}(\mathbb{R}^d))$ , we are led to

$$\begin{aligned} \frac{\varepsilon}{2^{(d-1)/2}} &= f(t_n, s(t_n)) \leq |f(t_n, s(t_n)) - f(\tilde{t}, s(t_n))| + f(\tilde{t}, s(t_n)) \\ &\leq \| [u(t_n)]_{1/s(t_n)} \|_{\dot{H}^{-1+d/2}} - \| [u(\tilde{t})]_{1/s(t_n)} \|_{\dot{H}^{-1+d/2}} \\ &\quad + \| [u(t_n)]^{s(t_n)} \|_{\dot{H}^{-1+d/2}} - \| [u(\tilde{t})]^{s(t_n)} \|_{\dot{H}^{-1+d/2}} + f(\tilde{t}, s(t_n)) \\ &\leq \| [u(t_n) - u(\tilde{t})]_{1/s(t_n)} \|_{\dot{H}^{-1+d/2}} + \| [u(t_n) - u(\tilde{t})]^{s(t_n)} \|_{\dot{H}^{-1+d/2}} + f(\tilde{t}, s(t_n)) \\ &\leq \sqrt{2} \| u(t_n) - u(\tilde{t}) \|_{\dot{H}^{-1+d/2}} + f(\tilde{t}, s(t_n)) \\ &\rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

which is a contradiction.

Consequently, if we put

$$M := [\log_2 N] + 2 \quad (\text{with } N \text{ in (2.8)})$$

then, in view of  $N \leq 2^{M-1}$ , we have

$$\begin{aligned} \sum_{|j| \geq M} 2^{2j(-1+d/2)} \|\Delta_j u(t)\|_2^2 &= 2^{d-2} \sum_{|j| \geq M} 2^{2(j-1)(-1+d/2)} \|\Delta_j u(t)\|_2^2 \\ &\leq 2^{d-2} (\| [u(t)]_{2^{-M+1}} \|_{\dot{H}^{-1+d/2}}^2 + \| [u(t)]^{2^{M-1}} \|_{\dot{H}^{-1+d/2}}^2) \\ &\leq 2^{d-2} (\| [u(t)]_{1/N} \|_{\dot{H}^{-1+d/2}}^2 + \| [u(t)]^N \|_{\dot{H}^{-1+d/2}}^2) \\ &\leq 2^{d-2} (\| [u(t)]_{1/N} \|_{\dot{H}^{-1+d/2}} + \| [u(t)]^N \|_{\dot{H}^{-1+d/2}})^2 \\ &\leq 2^{d-2} 2^{1-d} \varepsilon^2 = \frac{\varepsilon^2}{2} \end{aligned}$$

and thus (2.4) holds true for  $u(t)$  uniformly for all  $t \in [0, T]$ .

Thus, for

$$l \geq l_0 := \max\{M + 1, \log_2 \left[ \frac{\log_2 M \|u\|_{C([0, T], \dot{H}^{-1+d/2})}}{\varepsilon} \right] + 1\},$$

see (2.7), we have

$$\|(u(t))_\lambda\|_{X_1}^2 \leq \frac{\varepsilon^2}{2} + \left( \frac{\log_2 M}{2^{l-1}} \|u(t)\|_{X_1} \right)^2 \leq \varepsilon^2, \quad \forall t \in [0, T],$$

by the above proved fact (i) and  $\|u(t)\|_{X_1} \leq \|u(t)\|_{\dot{H}^{-1+d/2}}$ . ■

### 3 Proof of Theorem 1.1

**Lemma 3.1** Let

$$b(j) \equiv 2^{-j-1} \sum_{i=1}^j 2^i a(i), \quad j \in \mathbb{N}. \quad (3.1)$$

Then, for all  $j \in \mathbb{N}$

$$b(j) \leq \begin{cases} \log_2 j & \text{if } -k + 2^{2^k} \leq j \leq 2^{2^k} + 2k \text{ for some } k \in \mathbb{N}, \\ 2 & \text{else.} \end{cases}$$

**Proof:** Note that  $a(j) \geq 1$  and  $b(j+1) = \frac{1}{2}(b(j) + a(j+1))$  for all  $j \in \mathbb{N}$ .

First we prove the lemma for  $j = 1 \sim 2^{2^1} = 4$ . We have

$$\begin{aligned} b(1) &= 2^{-2} \cdot 2 \cdot a(1) = \frac{1}{2} < a(1) = 1, \\ b(2) &= 2^{-3} \cdot (2 + 2^2) = \frac{3}{4} < a(2), \\ b(3) &= \frac{b(2)+a(3)}{2} = \frac{3+4\log_2 3}{8} \leq \log_2 3 = a(3), \\ b(4) &= \frac{b(3)+a(4)}{2} \leq 2 = a(4). \end{aligned} \quad (3.2)$$

Next, let us fix any  $k \in \mathbb{N}$  and prove the lemma for  $2^{2^k} < j \leq 2^{2^{k+1}}$ . Let  $j_1 = 2^{2^k}$  and assume in view of (3.2) that  $b(j_1) \leq a(j_1)$  which, at the moment, is satisfied for  $k = 1$ . Then, for all  $l = 1, 2, \dots, k$  we have

$$\begin{aligned} b(j_1 + 1) &= \frac{b(j_1)}{2} + \frac{a(j_1+1)}{2} \leq \frac{a(j_1)}{2} + \frac{a(j_1+1)}{2} \leq a(j_1 + 1) = \log_2(j_1 + 1), \\ b(j_1 + 2) &= \frac{b(j_1+1)}{2} + \frac{a(j_1+2)}{2} \leq \frac{a(j_1+1)}{2} + \frac{a(j_1+2)}{2} \leq a(j_1 + 2) = \log_2(j_1 + 2), \\ &\dots \quad \dots \quad \dots \\ b(j_1 + l) &= \frac{b(j_1+l-1)}{2} + \frac{a(j_1+l)}{2} \leq a(j_1 + l) = \log_2(j_1 + l). \end{aligned}$$

Moreover, for  $l \in \mathbb{N}$  with  $1 \leq l \leq 2^{2^{k+1}} - (2k + 2^{2^k})$  we have

$$\begin{aligned} b(j_1 + k + 1) &= \frac{b(k+2^{2^k})}{2} + \frac{a(k+2^{2^k}+1)}{2} \leq \frac{a(k+2^{2^k})}{2} + \frac{1}{2} \leq \frac{\log_2(k+2^{2^k})}{2} + \frac{1}{2}, \\ b(j_1 + k + 2) &= \frac{b(k+2^{2^k}+1)}{2} + \frac{a(k+2^{2^k}+2)}{2} \leq \frac{\log_2(k+2^{2^k})}{4} + \frac{1}{4} + \frac{1}{2}, \\ &\dots \quad \dots \quad \dots \\ b(j_1 + k + l) &= \frac{b(k+2^{2^k}+l-1)}{2} + \frac{a(k+2^{2^k}+l)}{2} \leq \frac{\log_2(k+2^{2^k})}{2^l} + \frac{1}{2^l} + \dots + \frac{1}{4} + \frac{1}{2}. \end{aligned}$$

Hence, if  $1 \leq l < \log_2(k + 2^{2^k})$ , then we have

$$b(j_1 + k + l) \leq \log_2(k + 2^{2^k})$$

and, if  $\log_2(k + 2^{2^k}) \leq l \leq 2^{2^{k+1}} - (2k + 2^{2^k})$ , then we have

$$b(j_1 + k + l) \leq 2.$$

Note that  $k < \log_2 \log_2(k + 2^{2^k}) < k + 1$  for any  $k \in \mathbb{N}$ . Consequently,

$$\begin{aligned} b(j_1 + k + l) &\leq \log_2(k + 2^{2^k}), \quad \text{for } l = 1, \dots, k, \\ b(j_1 + k + l) &\leq 2, \quad \text{for } l = k + 1, \dots, 2^{2^{k+1}} - (2k + 2^{2^k}). \end{aligned} \quad (3.3)$$

Finally, we shall prove

$$b(j) \leq \log_2 j \quad \text{for } 2^{2^{k+1}} - k + 1 \leq j \leq 2^{2^{k+1}}. \quad (3.4)$$

In fact, we have

$$\begin{aligned} b(2^{2^{k+1}} - k + 1) &= \frac{b(2^{2^{k+1}} - k)}{2} + \frac{a(2^{2^{k+1}} - k + 1)}{2} \\ &\leq 1 + \frac{\log_2(2^{2^{k+1}} - k + 1)}{2} \leq \log_2(2^{2^{k+1}} - k + 1), \\ b(2^{2^{k+1}} - k + 2) &= \frac{b(2^{2^{k+1}} - k + 1)}{2} + \frac{a(2^{2^{k+1}} - k + 2)}{2} \\ &\leq \frac{\log_2(2^{2^{k+1}} - k + 1)}{2} + \frac{\log_2(2^{2^{k+1}} - k + 2)}{2} \leq \log_2(2^{2^{k+1}} - k + 2), \\ &\dots \quad \dots \quad \dots \\ b(2^{2^{k+1}}) &= \frac{b(2^{2^{k+1}} - 1)}{2} + \frac{a(2^{2^{k+1}})}{2} \\ &\leq \frac{\log_2(2^{2^{k+1}} - 1)}{2} + \frac{\log_2 2^{2^{k+1}}}{2} \leq 2^{k+1} = \log_2 2^{2^{k+1}} = a(2^{2^{k+1}}). \end{aligned}$$

Here, in particular,  $b(2^{2^{k+1}}) \leq a(2^{2^{k+1}})$  together with  $b(4) \leq a(4)$  implies by induction that the assumption  $b(j_1) \leq a(j_1)$  is satisfied for all  $k \in \mathbb{N}$ .

Thus, by (3.2) – (3.4), the proof of the lemma is complete. ■

In the below we use the notation

$$S(n) = \{l \in \mathbb{N} : l \leq n, -k + 2^{2^k} \leq l \leq 2^{2^k} + 2k, \exists k \in \mathbb{N}\}, \quad n \in \mathbb{N}. \quad (3.5)$$

**Lemma 3.2** The cardinal number  $|S(n)|$  of the set  $S(n)$  is estimated by

$$|S(n)| \leq \frac{(3 \log_2 \log_2 n + 5) \log_2 \log_2 n}{2}.$$

**Proof:** Suppose that  $2^{2^{k-1}} \leq n < 2^{2^k}$  for some  $k \in \mathbb{N}$ . Then, by definition of  $S(n)$ , we have

$$\begin{aligned} |S(n)| &\leq \sum_{l=1}^{k-1} (3l + 1) = \frac{(3k + 2)(k - 1)}{2} \\ &< \frac{(3 \log_2 \log_2 n + 5) \log_2 \log_2 n}{2}. \end{aligned}$$

Thus, the lemma is proved. ■

**Proof of Theorem 1.1:** Since  $u_0 \in H^{-1+d/2}(\mathbb{R}^d)$ , the problem (1.1) has a unique local strong (mild) solution  $u \in C([0, T), \dot{H}^{-1+d/2}(\mathbb{R}^d))$  (This fact follows by Kato's approach just changing  $L^d(\mathbb{R}^d)$  in [12] with  $\dot{H}^{-1+d/2}(\mathbb{R}^d)$ , cf. [13]) and it coincides in its existence interval with the corresponding Leray-Hopf weak solution ([1], Theorem 0.8). Moreover, it is smooth in  $(0, T)$ , as is well known (cf. e.g. [8]). Let  $(0, T)$  be the maximal interval where a Leray-Hopf weak solution  $u$  to (1.1) with  $u_0 \in H^{-1+d/2}(\mathbb{R}^3)$  is smooth. It is well-known that energy inequality (1.2) (even energy equation) is satisfied in  $(0, T)$ .

The remaining proof is divided into two steps.

**Step 1. Energy estimate of high frequency parts:**

Recall the notation (1.4). By  $L^2$ -scalar product to the momentum equation of (1.1) with  $u^k(t)$ ,  $k \geq 1$ ,  $t \in (0, T)$ , we have

$$\begin{aligned} \frac{d}{dt} \|u^k\|_2^2 + \nu \|\nabla u^k\|_2^2 &= -((u \cdot \nabla) u_k, u^k) \\ &= -((u_k \cdot \nabla) u_k + (u^k \cdot \nabla) u_k, u^k) \\ &= -((u_k \cdot \nabla) u_{k/2,k}, u^k) - ((u_{k/2,k} \cdot \nabla) u_{k/2}, u^k) - ((u^k \cdot \nabla) u_k, u^k) \text{ in } (0, T) \end{aligned} \quad (3.6)$$

in view of  $u_k = u_{k/2} + u_{k/2,k}$  and

$$(u_k, u^l)_2 = 0, \quad \text{supp } \widehat{(u_k u_l)} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq k+l\}, \quad k \leq l < \infty.$$

In the right-hand side of (3.6) we have

$$|((u_{k/2,k} \cdot \nabla) u_{k/2}, u^k) + (u^k \cdot \nabla) u_k, u^k| \leq (\|\nabla u_{k/2}\|_\infty + \|\nabla u_k\|_\infty) \|u^{k/2}\|_2^2$$

by Hölder's inequality. On the other hand, we have

$$\begin{aligned} |((u_k \cdot \nabla) u_{k/2,k}, u^k)| &\leq \|u_k\|_\infty \|\nabla u_{k/2,k}\|_2 \|u^k\|_2 \\ &\leq k \|u_k\|_\infty \|u^{k/2}\|_2^2. \end{aligned}$$

Therefore, we get from (3.6) that

$$\frac{d}{dt} \|u^k\|_2^2 + \nu \|\nabla u^k\|_2^2 \leq (\|\nabla u_{k/2}\|_\infty + \|\nabla u_k\|_\infty + k \|u_k\|_\infty) \|u^{k/2}\|_2^2 \quad \text{in } (0, T). \quad (3.7)$$

Let

$$j_0 \equiv j_0(k) = \lceil \log_2 k \rceil + 1. \quad (3.8)$$

Note that

$$\begin{aligned} \|\widehat{\Delta_j u_k}\|_1 &\leq \|\widehat{\Delta_j u_k}\|_2 \cdot |\{2^{j-1} \leq |\xi| \leq 2^j\}|^{1/2} \\ &\leq c_0 2^{dj/2} \|\Delta_j u_k\|_2 \\ &\leq c_0 2^{dj/2} \|\Delta_j u\|_2, \quad \forall j \in \mathbb{Z}, \end{aligned}$$

where  $c_0 = \sigma^{1/2}$  and  $\sigma$  is the volume of the unit ball of  $\mathbb{R}^d$ . Hence, we get by

the definition of the space  $X_1$  that

$$\begin{aligned}
\|u_k\|_\infty &\leq (2\pi)^{-d/2} \|\widehat{u_k}\|_1 = (2\pi)^{-d/2} \left\| \sum_{j=-\infty}^{j_0} \widehat{\Delta_j u_k} \right\|_1 \\
&\leq (2\pi)^{-d/2} \sum_{j=-\infty}^{j_0} \|\widehat{\Delta_j u_k}\|_1 \\
&\leq c_0 (2\pi)^{-d/2} \sum_{j=-\infty}^{j_0} 2^j a(j) (a^{-1}(j) 2^{j(-1+d/2)} \|\Delta_j u\|_2) \\
&\leq c_0 (2\pi)^{-d/2} \left( \sum_{j=-\infty}^{j_0} 2^j a(j) \right) \cdot \sup_{j \leq j_0} \{a^{-1}(j) 2^{j(-1+d/2)} \|\Delta_j u\|_2\} \\
&\leq c_0 (2\pi)^{-d/2} (2 + b(j_0(k)) 2^{j_0+1}) \|u\|_{X_1} \\
&\leq 8c_0 (2\pi)^{-d/2} b(j_0(k)) k \|u\|_{X_1}
\end{aligned}$$

due to  $k \geq 1$  and  $b(j) \geq 1/2$  for all  $j \in \mathbb{N}$ , see (2.2) and (3.1). In the same way, we have

$$\begin{aligned}
\|\nabla u_k\|_\infty &\leq c_0 (2\pi)^{-d/2} \left( \sum_{j=-\infty}^{j_0} 2^j a(j) \right) \cdot \sup_{j \leq j_0} \{a^{-1}(j) 2^{j(-1+d/2)} \|\Delta_j \nabla u\|_2\} \\
&\leq c_0 (2\pi)^{-d/2} (2 + b(j_0(k)) 2^{j_0+1}) \cdot 2^{j_0} \sup_{j \leq j_0} \{a^{-1}(j) 2^{j(-1+d/2)} \|\Delta_j u\|_2\} \\
&\leq 8c_0 (2\pi)^{-d/2} b(j_0(k)) k^2 \|u\|_{X_1}
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla u_{k/2}\|_\infty &\leq c_0 (2\pi)^{-d/2} \left( \sum_{j=-\infty}^{j_0-1} 2^j a(j) \right) \cdot \sup_{j \leq j_0-1} \{a^{-1}(j) 2^{j(-1+d/2)} \|\Delta_j \nabla u\|_2\} \\
&\leq c_0 (2\pi)^{-d/2} \left( \sum_{j=-\infty}^{j_0} 2^j a(j) \right) \cdot 2^{j_0-1} \sup_{j \leq j_0-1} \{a^{-1}(j) 2^{j(-1+d/2)} \|\Delta_j u\|_2\} \\
&\leq 4c_0 (2\pi)^{-d/2} b(j_0(k)) k^2 \|u\|_{X_1}.
\end{aligned}$$

Thus, we have by (3.7) that

$$\begin{aligned}
\frac{d}{dt} \|u^k\|_2^2 + \nu \|\nabla u^k\|_2^2 &\leq C_1 b(j_0(k)) k^2 \|u\|_{X_1} \|u^{k/2}\|_2^2 \\
&\leq 4C_1 b(j_0(k)) \|u\|_{X_1} \|\nabla u^{k/2}\|_2^2, \quad \forall k \geq 1, \quad \text{in } (0, T),
\end{aligned} \tag{3.9}$$

with a generic constant  $C_1 > 0$  depending on  $d$  and independent of  $\nu$ .

### Step 2. Estimates of higher order norms for the solution:

Adding up (3.9) multiplied by  $k^s$ ,  $s \geq d - 3$ , over all  $k \in \mathbb{N}$ , we have

$$\frac{d}{dt} \sum_{k \in \mathbb{N}} k^s \|u^k\|_2^2 + \nu \sum_{k \in \mathbb{N}} k^s \|\nabla u^k\|_2^2 \leq 4C_1 \|u(t)\|_{X_1} \sum_{k \in \mathbb{N}} b(j_0(k)) k^s \|\nabla u^{k/2}\|_2^2 \tag{3.10}$$

in  $(0, T)$ . With the notation  $2\mathbb{N} := \{2n : n \in \mathbb{N}\}$  we have

$$\begin{aligned}
& \sum_{k \in \mathbb{N}} b(j_0(k)) k^s \|\nabla u^{k/2}\|_2^2 \\
&= \frac{1}{2} \|\nabla u^{1/2}\|_2^2 + \sum_{k \in 2\mathbb{N}} (b(j_0(k)) k^s \|\nabla u^{k/2}\|_2^2 + b(j_0(k+1)) (k+1)^s \|\nabla u^{(k+1)/2}\|_2^2) \\
&\leq \|\nabla u^{1/2}\|_2^2 + \sum_{k \in 2\mathbb{N}} (b(j_0(k)) k^s + b(j_0(k+1)) (k+1)^s) \|\nabla u^{k/2}\|_2^2 \\
&= \|\nabla u^{1/2}\|_2^2 + \sum_{k \in \mathbb{N}} (b(j_0(2k)) (2k)^s + b(j_0(2k+1))) (2k+1)^s \|\nabla u^k\|_2^2 \\
&= \|\nabla u^{1/2}\|_2^2 + \sum_{k \in \mathbb{N}} (b(j_0(2k)) (2k)^s + b(j_0(2k+1)) (2k+1)^s) \sum_{n \geq k} \|\nabla u_{n,n+1}\|_2^2 \\
&= \|\nabla u^{1/2}\|_2^2 + \sum_{n \in \mathbb{N}} \|\nabla u_{n,n+1}\|_2^2 \sum_{k=1}^n (b(j_0(2k)) (2k)^s + b(j_0(2k+1)) (2k+1)^s) \\
&\leq \|\nabla u^{1/2}\|_2^2 + \sum_{n \in \mathbb{N}} (3n)^s \|\nabla u_{n,n+1}\|_2^2 \sum_{k=1}^n (b(j_0(2k)) + b(j_0(2k+1))). \tag{3.11}
\end{aligned}$$

Note that  $j_0(2k) = \lceil \log_2 k \rceil + 2$ , see (3.8), and

$$\begin{aligned}
& \sum_{k=1}^n b(j_0(2k)) \\
&= \sum_{k \leq n, j_0(2k) \notin S(\lceil \log_2 n \rceil + 2)} b(j_0(2k)) + \sum_{k \leq n, j_0(2k) \in S(\lceil \log_2 n \rceil + 2)} b(j_0(2k)),
\end{aligned}$$

where, by Lemma 3.1,

$$\sum_{k \leq n, j_0(2k) \notin S(\lceil \log_2 n \rceil + 2)} b(j_0(2k)) \leq 2n.$$

Moreover, we have by (3.5) and Lemma 3.2 that

$$\begin{aligned}
& \sum_{k \leq n, j_0(2k) \in S(\lceil \log_2 n \rceil + 2)} b(j_0(2k)) \\
&\leq |S(\lceil \log_2 n \rceil + 2)| \cdot \max_{k \leq n, j_0(2k) \in S(\lceil \log_2 n \rceil + 2)} b(j_0(2k)) \\
&\leq |S(\lceil \log_2 n \rceil + 2)| \cdot \log_2(\lceil \log_2 n \rceil + 2) \\
&\leq \frac{3 \log_2 \log_2(\lceil \log_2 n \rceil + 2) + 5}{2} \cdot \log_2 \log_2(\lceil \log_2 n \rceil + 2) \cdot \log_2(\lceil \log_2 n \rceil + 2) \\
&\leq n, \quad \forall n \geq n_0,
\end{aligned}$$

for some generic number  $n_0 \in \mathbb{N}$ . Therefore, for all  $n \geq n_0$  we have

$$\sum_{k=1}^n b(j_0(2k)) \leq 2n + n \leq 3n. \tag{3.12}$$

In the same way, we have

$$\sum_{k=1}^n b(j_0(2k+1)) \leq 3n, \quad \forall n \in \mathbb{N}, n \geq n_0, \quad (3.13)$$

with a generic number  $n_0 \in \mathbb{N}$  possibly larger than the former  $n_0$ .

Thus we get by (3.11)-(3.13) that

$$\begin{aligned} & \sum_{k \in \mathbb{N}} b(j_0(k)) k^s \|\nabla u^{k/2}\|_2^2 \leq \|\nabla u^{1/2}\|_2^2 \\ & + \sum_{n=1}^{n_0-1} (3n)^s \|\nabla u_{n,n+1}\|_2^2 \sum_{k=1}^n (b(j_0(2k)) + b(j_0(2k+1))) \\ & + \sum_{n \geq n_0} (3n)^s \|\nabla u_{n,n+1}\|_2^2 \sum_{k=1}^n (b(j_0(2k)) + b(j_0(2k+1))) \\ & \leq \|\nabla u^{1/2}\|_2^2 + (c(n_0, s) - 1) \|\nabla u_{n_0}\|_2^2 + 2 \cdot 3^{s+1} \sum_{n \geq n_0} n^{s+1} \|\nabla u_{n,n+1}\|_2^2 \\ & \leq c(n_0, s) \|\nabla u\|_2^2 + 2 \cdot 3^{s+1} \sum_{n \geq n_0} n^{s+1} \|\nabla u_{n,n+1}\|_2^2, \end{aligned} \quad (3.14)$$

where  $c(n_0, s)$  is a generic constant depending only on  $n_0$  and  $s$ .

Therefore, we get from (3.10) – (3.14) that

$$\begin{aligned} & \frac{d}{dt} \sum_{k \in \mathbb{N}} k^s \|u^k\|_2^2 + \nu \sum_{k \in \mathbb{N}} k^s \|\nabla u^k\|_2^2 \\ & \leq 4C_1 \|u(t)\|_{X_1} (c(n_0, s) \|\nabla u(t)\|_2^2 + 2 \cdot 3^{s+1} \sum_{n \geq n_0} n^{s+1} \|\nabla u_{n,n+1}(t)\|_2^2) \quad (3.15) \\ & \leq C_2 \|u(t)\|_{X_1} (\|\nabla u(t)\|_2^2 + \sum_{n \in \mathbb{N}} n^{s+1} \|\nabla u_{n,n+1}(t)\|_2^2), \quad \text{in } (0, T), \end{aligned}$$

where  $C_2$  is a generic constant depending only on  $s$  and  $d$ .

By integrating (3.15) from 0 to  $t \in (0, T)$  we have

$$\begin{aligned} & \frac{1}{2} \sum_{k \in \mathbb{N}} k^s \|u^k(t)\|_2^2 + \nu \int_0^t \sum_{k \in \mathbb{N}} k^s \|\nabla u^k(\tau)\|_2^2 d\tau \leq \frac{1}{2} \sum_{k \in \mathbb{N}} k^s \|u_0^k\|_2^2 \\ & + C_2 \int_0^t \|u(\tau)\|_{X_1} (\|\nabla u(\tau)\|_2^2 + \sum_{n \in \mathbb{N}} n^{s+1} \|\nabla u_{n,n+1}(\tau)\|_2^2) d\tau, \quad \forall t \in (0, T). \end{aligned} \quad (3.16)$$

Note that

$$\begin{aligned} \sum_{k \in \mathbb{N}} k^s \|u^k\|_2^2 &= \sum_{n \in \mathbb{N}} \left( \sum_{k=1}^n k^s \right) \|u_{n,n+1}\|_2^2, \\ \sum_{k \in \mathbb{N}} \|\nabla u^k\|_2^2 &= \sum_{n \in \mathbb{N}} \left( \sum_{k=1}^n k^s \right) \|\nabla u_{n,n+1}\|_2^2, \end{aligned} \quad (3.17)$$

where

$$\frac{n^{s+1}}{s+1} \leq \sum_{k=1}^n k^s \leq \frac{(n+1)^{s+1}}{s+1}. \quad (3.18)$$

By (3.16)–(3.18) we have

$$\begin{aligned} & \frac{1}{2} \sum_{n \in \mathbb{N}} n^{s+1} \|u_{n,n+1}(t)\|_2^2 + \nu \int_0^t \sum_{n \in \mathbb{N}} n^{s+1} \|\nabla u_{n,n+1}(\tau)\|_2^2 d\tau \\ & \leq \frac{1}{2} \sum_{n \in \mathbb{N}} (n+1)^{s+1} \|[u_0]_{n,n+1}\|_2^2 \\ & + C_2(s+1) \int_0^t \|u(\tau)\|_{X_1} (\|\nabla u(\tau)\|_2^2 + \sum_{n \in \mathbb{N}} n^{s+1} \|\nabla u_{n,n+1}(\tau)\|_2^2) d\tau, \forall t \in (0, T). \end{aligned} \quad (3.19)$$

Note that, in view of  $|\xi|^{s+1} - n^{s+1} \leq (s+1)|\xi|^s$  for  $n < |\xi| \leq n+1$ ,

$$\frac{1}{2} \|u^1(t)\|_{\dot{H}^{(s+1)/2}}^2 - \frac{1}{2} \sum_{n \in \mathbb{N}} n^{s+1} \|u_{n,n+1}(t)\|_2^2 \leq \frac{s+1}{2} \|u^1(t)\|_{\dot{H}^{s/2}}^2, \quad (3.20)$$

$$\begin{aligned} & \nu \int_0^t \left( \|\nabla u^1(\tau)\|_{\dot{H}^{(s+1)/2}}^2 - \sum_{n \in \mathbb{N}} n^{s+1} \|\nabla u_{n,n+1}(\tau)\|_2^2 \right) d\tau \\ & \leq (s+1)\nu \int_0^t \|\nabla u^1(\tau)\|_{\dot{H}^{s/2}}^2 d\tau. \end{aligned} \quad (3.21)$$

Thus, summing up (3.19), (3.20), (3.21) and taking into account

$$\begin{aligned} & \sum_{n \in \mathbb{N}} (n+1)^{s+1} \|[u_0]_{n,n+1}\|_2^2 \leq 2^{s+1} \|u_0\|_{\dot{H}^{(s+1)/2}}^2, \\ & \sum_{n \in \mathbb{N}} n^{s+1} \|\nabla u_{n,n+1}\|_2^2 \leq \|u^1\|_{\dot{H}^{(s+3)/2}}^2, \end{aligned}$$

we obtain

$$\begin{aligned} & \frac{1}{2} \|u^1(t)\|_{\dot{H}^{(s+1)/2}}^2 + \nu \int_0^t \|\nabla u^1(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau \\ & \leq (s+1) \left( \frac{1}{2} \|u^1(t)\|_{\dot{H}^{s/2}}^2 + \nu \int_0^t \|\nabla u^1(\tau)\|_{\dot{H}^{s/2}}^2 d\tau \right) + 2^s \|u_0\|_{\dot{H}^{(s+1)/2}}^2 \\ & + C_3 \int_0^t \|u(\tau)\|_{X_1} (\|\nabla u(\tau)\|_2^2 + \|\nabla u^1(\tau)\|_{\dot{H}^{(s+1)/2}}^2) d\tau, \forall t \in (0, T), \end{aligned} \quad (3.22)$$

where  $C_3 := C_2(s+1)$ . Since the inequality

$$\frac{1}{2} \|u_1(t)\|_{\dot{H}^{(s+1)/2}}^2 + \nu \int_0^t \|\nabla u_1(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau \leq \frac{1}{2} \|u_0\|_2^2, \forall t \in (0, T),$$

holds thanks to  $\|v_1\|_{\dot{H}^{(s+1)/2}} \leq \|v_1\|_2 \leq \|v\|_2, \forall v \in H^{(s+1)/2}(\mathbb{R}^d)$ , and the energy

inequality (1.2) in  $(0, T)$ , we get by (3.22) that

$$\begin{aligned}
& \frac{1}{2} \|u(t)\|_{\dot{H}^{(s+1)/2}}^2 + \nu \int_0^t \|\nabla u(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau \\
& \leq \frac{1}{2} \|u_1(t)\|_{\dot{H}^{(s+1)/2}}^2 + \nu \int_0^t \|\nabla u_1(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau \\
& \quad + (s+1) \left( \frac{1}{2} \|u^1(t)\|_{\dot{H}^{s/2}}^2 + \nu \int_0^t \|\nabla u^1(\tau)\|_{\dot{H}^{s/2}}^2 d\tau \right) + 2^s \|u_0\|_{\dot{H}^{(s+1)/2}}^2 \\
& \quad + C_3 \int_0^t \|u(\tau)\|_{X_1} (\|\nabla u(\tau)\|_2^2 + \|\nabla u^1(\tau)\|_{\dot{H}^{(s+1)/2}}^2) d\tau \\
& \leq \frac{1}{2} \|u_0\|_2^2 + 2^s \|u_0\|_{\dot{H}^{(s+1)/2}}^2 + (s+1) \left( \frac{1}{2} \|u(t)\|_{\dot{H}^{s/2}}^2 + \nu \int_0^t \|\nabla u(\tau)\|_{\dot{H}^{s/2}}^2 d\tau \right) \\
& \quad + C_3 \int_0^t \|u(\tau)\|_{X_1} (\|\nabla u(\tau)\|_2^2 + \|\nabla u(\tau)\|_{\dot{H}^{(s+1)/2}}^2) d\tau, \quad \forall t \in (0, T).
\end{aligned} \tag{3.23}$$

Using

$$(s+1) \|v\|_{\dot{H}^{s/2}}^2 \leq c(s) \|v\|_2^2 + \frac{1}{2} \|v\|_{\dot{H}^{(s+1)/2}}^2, \quad \forall v \in H^{(s+1)/2}(\mathbb{R}^d),$$

due to complex interpolation  $\dot{H}^{s/2} = [L^2, \dot{H}^{(s+1)/2}]_{s/(s+1)}$ , we get from (3.23) that

$$\begin{aligned}
& \frac{1}{4} \|u(t)\|_{\dot{H}^{(s+1)/2}}^2 + \frac{\nu}{2} \int_0^t \|\nabla u(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau \\
& \leq \frac{1}{2} \|u_0\|_2^2 + 2^s \|u_0\|_{\dot{H}^{(s+1)/2}}^2 + c(s) \left( \frac{1}{2} \|u(t)\|_2^2 + \nu \int_0^t \|\nabla u(\tau)\|_2^2 d\tau \right) \\
& \quad + C_3 \int_0^t \|u(\tau)\|_{X_1} (\|\nabla u(\tau)\|_2^2 + \|\nabla u(\tau)\|_{\dot{H}^{(s+1)/2}}^2) d\tau \\
& \leq \frac{1}{2} (1 + c(s)) \|u_0\|_2^2 + 2^s \|u_0\|_{\dot{H}^{(s+1)/2}}^2 \\
& \quad + C_3 \int_0^t \|u(\tau)\|_{X_1} (\|\nabla u(\tau)\|_2^2 + \|\nabla u(\tau)\|_{\dot{H}^{(s+1)/2}}^2) d\tau, \quad \forall t \in (0, T).
\end{aligned} \tag{3.24}$$

Now, put  $\varepsilon = \frac{\nu}{4C_3}$  and fix  $t \in (0, T)$ . By Lemma 2.1 (ii) in view of  $u \in C([0, t], \dot{H}^{-1+d/2}(\mathbb{R}^d))$  there exists  $l_0(t) > 0$  such that

$$\|\lambda u(\tau, \lambda \cdot)\|_{X_1} \leq \varepsilon, \quad \forall \tau \in [0, t], \quad \forall \lambda = 2^{2^l} (l \geq l_0(t)). \tag{3.25}$$

Put  $\lambda = 2^{2^l}$  with fixed  $l \geq l_0(t)$  and observe the re-scaled function

$$(u)_\lambda(\tau, x) := \lambda u(\lambda^2 \tau, \lambda x), \quad \tau \in (0, \lambda^{-2}t]. \tag{3.26}$$

Then, (3.25) implies that

$$\|(u)_\lambda(\tau)\|_{X_1} \leq \varepsilon, \quad \forall \tau \in [0, \lambda^{-2}t]. \tag{3.27}$$

Since  $(u)_\lambda$  is the Leray-Hopf weak solution to (1.1) with initial value  $(u_0)_\lambda$  and the first blow-up epoch  $\lambda^{-2}T$ , we have by (3.24) that

$$\begin{aligned} & \frac{1}{4}\|(u)_\lambda(\lambda^{-2}t)\|_{\dot{H}^{(s+1)/2}}^2 + \frac{\nu}{2} \int_0^{\lambda^{-2}t} \|\nabla(u)_\lambda(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau \\ & \leq \frac{1}{2}(1+c(s))\|(u_0)_\lambda\|_2^2 + 2^s\|(u_0)_\lambda\|_{\dot{H}^{(s+1)/2}}^2 \\ & \quad + C_3 \int_0^{\lambda^{-2}t} \|(u)_\lambda(\tau)\|_{X_1} (\|\nabla(u)_\lambda(\tau)\|_2^2 + \|\nabla(u)_\lambda(\tau)\|_{\dot{H}^{(s+1)/2}}^2) d\tau, \end{aligned} \quad (3.28)$$

which yields by (3.27) and energy inequality for  $(u)_\lambda$  in  $(0, \lambda^{-2}T)$  that

$$\begin{aligned} & \frac{1}{4}\|(u)_\lambda(\lambda^{-2}t)\|_{\dot{H}^{(s+1)/2}}^2 + \frac{\nu}{4} \int_0^{\lambda^{-2}t} \|\nabla(u)_\lambda(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau \\ & \leq \frac{1}{2}(1+c(s))\|(u_0)_\lambda\|_2^2 + 2^s\|(u_0)_\lambda\|_{\dot{H}^{(s+1)/2}}^2 + \frac{\nu}{4} \int_0^{\lambda^{-2}t} \|\nabla(u)_\lambda(\tau)\|_2^2 d\tau \\ & \leq (1+c(s))\|(u_0)_\lambda\|_2^2 + 2^s\|(u_0)_\lambda\|_{\dot{H}^{(s+1)/2}}^2. \end{aligned} \quad (3.29)$$

Thanks to the scaling-variant property

$$\begin{aligned} & \|(u)_\lambda(\lambda^{-2}t)\|_{\dot{H}^{(s+1)/2}}^2 = \lambda^{3+s-d}\|u(t)\|_{\dot{H}^{(s+1)/2}}^2, \\ & \int_0^{\lambda^{-2}t} \|\nabla(u)_\lambda(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau = \lambda^{3+s-d} \int_0^t \|\nabla u(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau, \\ & \|(u_0)_\lambda\|_{\dot{H}^{(s+1)/2}}^2 = \lambda^{3+s-d}\|u_0\|_{\dot{H}^{(s+1)/2}}^2, \|(u_0)_\lambda\|_2^2 = \lambda^{2-d}\|u_0\|_2^2, \end{aligned}$$

we get by (3.29) that

$$\begin{aligned} & \|u(t)\|_{\dot{H}^{(s+1)/2}}^2 + \nu \int_0^t \|\nabla u(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau \\ & \leq 4(1+c(s))\lambda^{-s-1}\|u_0\|_2^2 + 2^{s+2}\|u_0\|_{\dot{H}^{(s+1)/2}}^2. \end{aligned}$$

Consequently, in view of  $\lambda \geq 4$ , we have

$$\|u(t)\|_{\dot{H}^{(s+1)/2}}^2 + \nu \int_0^t \|\nabla u(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau \leq (1+c(s))\|u_0\|_2^2 + 2^{s+2}\|u_0\|_{\dot{H}^{(s+1)/2}}^2,$$

which together with the energy inequality (1.2) in  $(0, T)$  yields that

$$\|u(t)\|_{\dot{H}^{(s+1)/2}}^2 + \nu \int_0^t \|\nabla u(\tau)\|_{\dot{H}^{(s+1)/2}}^2 d\tau \leq C(s)\|u_0\|_{\dot{H}^{(s+1)/2}}^2. \quad (3.30)$$

Note here that  $C(s)$  is irrespective of  $d$  and  $t$ , of course.

Thus we have obtained (3.30) for any fixed  $t \in (0, T)$ . Since  $\frac{s+1}{2} \geq -1 + \frac{d}{2}$  for  $s \geq d - 3$ , we can conclude from (3.30) that  $T = \infty$ . Moreover, (3.30) with  $s \geq d - 3$  implies (1.3) with  $s \geq -1 + d/2$ .

The proof of the theorem is complete. ■

**Remark 3.3** The above argument essentially requires that the initial value  $u_0$  should be taken, at least, in  $H^{-1+d/2}(\mathbb{R}^d)$  so that a local smooth solution can exist and the uniform smallness condition (3.27) for the re-scaled solution can be satisfied.

## 4 Application to global solvability for incompressible Euler equations

The above regularity result for the Navier-Stokes system can be directly applied to showing the global existence of solutions to the Euler equations of ideal incompressible fluids:

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= 0 && \text{in } (0, \infty) \times \mathbb{R}^d, \\ \operatorname{div} u &= 0 && \text{in } (0, \infty) \times \mathbb{R}^d, \\ u(0, x) &= u_0 && \text{in } \mathbb{R}^d (d \geq 3). \end{aligned} \quad (4.1)$$

Let us denote by  $a \otimes b := (a_i b_j)$  for vectors  $a, b \in \mathbb{R}^d$  and  $A : B = \sum_{i,j} a_{ij} b_{ij}$  for matrices  $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{d \times d}$ . A  $d$ -dimensional divergence-free vector field  $u \in L^2_{\text{loc}}([0, \infty) \times \mathbb{R}^d)$  is called a weak solution to (4.1) if it satisfies the equations in a weak sense, i.e.,

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} (u \cdot \varphi_t + (u \otimes u) : \nabla \varphi) dx dt + \int_{\mathbb{R}^d} u_0(x) \cdot \varphi(0, x) dx &= 0, \\ \forall \varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^d) (\operatorname{div} \varphi = 0). \end{aligned} \quad (4.2)$$

**Remark 4.1** (i) If  $u$  is a weak solution to (4.1), then by De Rham's theorem there is a distribution  $p$  such that  $\{u, p\}$  satisfies the equations of (4.1) in the distribution sense.

(ii) Suppose a weak solution  $u$  to (4.1) belongs to  $L^\infty(0, \infty; H^s(\mathbb{R}^d))$  for  $s > 1 + \frac{d}{2}$ . It then follows by the Sobolev embedding  $H^{s-1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$  that  $(u \cdot \nabla)u \in L^\infty(0, \infty; L^2(\mathbb{R}^d))$  and

$$-\Delta p = \operatorname{div}(u \cdot \nabla)u \in L^\infty(0, \infty; H^{-1}(\mathbb{R}^d))$$

holds for the corresponding associated pressure  $p$ . Consequently,  $\nabla p$  and hence  $u_t$  belong to  $L^\infty(0, \infty; L^2(\mathbb{R}^d))$ . Therefore,  $u$  becomes a strong solution to (4.1).

For (4.1) in 3D setting, nonuniqueness of weak solutions for some initial values is proved (e.g. in [6], [11]) and local existence of a unique strong solution for  $u_0 \in H^s(\mathbb{R}^3)$ ,  $s > 5/2$ , is known (cf. [2]). However, there seems no general establishment asserting global existence of a solution under a certain regularity assumption on the initial values.

As a corollary of Theorem 1.1, we can obtain the following global solvability for (4.1).

**Theorem 4.2** Let  $u_0 \in H^s(\mathbb{R}^d)$ ,  $d \geq 3$ ,  $s \geq -1 + d/2$ ,  $\operatorname{div} u_0 = 0$ . Then the Cauchy problem (4.1) has a global weak solution such that

$$u \in L^\infty(0, \infty; H^s(\mathbb{R}^d))$$

and

$$\|u\|_{L^\infty(0, \infty; H^s(\mathbb{R}^d))} \leq C \|u_0\|_{H^s(\mathbb{R}^d)} \quad (4.3)$$

with some constant  $C > 0$ . This solution is unique if  $s > 1 + d/2$ .

**Proof:** The proof is based on the argument of Navier-Stokes regularization of the problem (4.1) and vanishing viscosity. Observe the Navier-Stokes problem (1.1) with the same initial value  $u_0$ .

By Theorem 1.1, the Navier-Stokes problem (1.1) has a unique solution  $u_\nu \in L^\infty(0, \infty; H^s(\mathbb{R}^d))$  satisfying

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} (u_\nu \cdot \varphi_t - \nu \nabla u_\nu \cdot \nabla \varphi + (u_\nu \otimes u_\nu) : \nabla \varphi) dx dt + \int_{\mathbb{R}^d} u_0(x) \cdot \varphi(0, x) dx = 0, \\ \forall \varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^d) \text{ (div } \varphi = 0\text{)}, \end{aligned} \quad (4.4)$$

and

$$\|u_\nu\|_{L^\infty(0, \infty; H^s(\mathbb{R}^d))} \leq C(s) \|u_0\|_{H^s(\mathbb{R}^d)}, \quad (4.5)$$

where  $C(s)$  is independent of  $\nu$ .

Let  $u \in L^\infty(0, \infty; H^s(\mathbb{R}^d))$  be the weak-\* limit of a subsequence of  $\{u_\nu\}_{\nu>0}$  (say,  $\{u_\nu\}_{\nu>0}$  again). Since

$$\begin{aligned} \nu \left| \int_0^\infty \int_{\mathbb{R}^d} \nabla u_\nu \cdot \nabla \varphi dx dt \right| &\leq \nu \|u_\nu\|_{L^\infty(0, \infty; H^s(\mathbb{R}^d))} \|\Delta \varphi\|_{L^1(0, \infty; H^{-s}(\mathbb{R}^d))} \\ &\leq \nu C \|u_0\|_{H^s(\mathbb{R}^d)} \|\Delta \varphi\|_{L^1(0, \infty; H^{-s}(\mathbb{R}^d))} \rightarrow 0 \quad (\nu \rightarrow 0) \end{aligned}$$

for all  $\varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^d)$ , the integral identity (4.2) will be satisfied by  $u$  provided

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} (u_\nu \otimes u_\nu) : \nabla \varphi dx dt &\rightarrow \int_0^\infty \int_{\mathbb{R}^d} (u \otimes u) : \nabla \varphi dx dt \quad (\nu \rightarrow 0), \\ \forall \varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^d) \text{ (div } \varphi = 0\text{)}. \end{aligned} \quad (4.6)$$

Let a distribution  $p_\nu$  be an associated pressure for (1.1) corresponding to  $u_\nu$ , that is, the pair  $\{u_\nu, p_\nu\}$  satisfies the momentum equation of (1.1) in the distribution sense. Since

$$H^{-1+d/2}(\mathbb{R}^d) \cdot H^{-1+d/2}(\mathbb{R}^d) \hookrightarrow H^{-1/2}(\mathbb{R}^d), \quad d \geq 3, \quad (4.7)$$

it follows by (4.5) that  $\{u_\nu \otimes u_\nu\}_{\nu>0}$  is uniformly bounded in  $L^\infty(0, \infty; H^{-1/2}(\mathbb{R}^d))$ . On the other hand, since the associated pressure  $p_\nu$  satisfies

$$-\Delta p_\nu = \operatorname{div} \operatorname{div} (u_\nu \otimes u_\nu),$$

we have, in view of (4.7),

$$\begin{aligned} \|p_\nu\|_{L^\infty(0, \infty; H^{-1/2}(\mathbb{R}^d))} &\leq C \|u_\nu \otimes u_\nu\|_{L^\infty(0, \infty; H^{-1/2}(\mathbb{R}^d))} \\ &\leq C \|u_\nu\|_{L^\infty(0, \infty; H^{-1+d/2}(\mathbb{R}^d))}^2 \leq C \|u_0\|_{H^{-1+d/2}(\mathbb{R}^d)}^2 \end{aligned}$$

with  $C > 0$  independent of  $\nu > 0$ . Therefore, in view of

$$u_{\nu t} = \Delta u_\nu - \operatorname{div}(u_\nu \otimes u_\nu) - \nabla p_\nu,$$

it follows that  $\{u_{\nu t}\}_{\nu>0}$  is uniformly bounded in  $L^\infty(0, \infty; H^{-3/2}(\mathbb{R}^d))$ .

Since  $\{u_\nu\}_{\nu>0}$  and  $\{u_{\nu t}\}_{\nu>0}$  are uniformly bounded in  $L^\infty(0, \infty; H^s(\mathbb{R}^d))$  and in  $L^\infty(0, \infty; H^{-3/2}(\mathbb{R}^d))$ , respectively, the restriction of both sequences on  $(0, T) \times G$  for any finite  $T > 0$  and smooth bounded domain  $G \subset \mathbb{R}^d$  are uniformly bounded in  $L^2(0, T; H^s(G))$  and in  $L^2(0, T; H^{-3/2}(G))$ , respectively. Therefore, it follows by Aubin's lemma on compactness that, possibly with a new subsequence,

$$u_\nu \rightarrow u \quad \text{in } L^2(0, T; L^2(G)) \quad \text{as } \nu \rightarrow 0 \quad (\text{strongly}).$$

Consequently, (4.6) holds and  $u$  is a weak solution satisfying (4.2) and (4.3).

Let us prove the uniqueness of the solution in  $L^\infty(0, \infty; H^s(\mathbb{R}^d))$  for  $s > 1 + \frac{d}{2}$ . Let  $u_1, u_2 \in L^\infty(0, \infty; H^s(\mathbb{R}^d))$ ,  $s > 1 + \frac{d}{2}$ , be two solutions to (4.1), and let  $u \equiv u_1 - u_2$  and  $p \equiv p_1 - p_2$  with corresponding associated pressure  $p_1, p_2$ . Then,

$$u_t + (u \cdot \nabla)u_1 + (u_2 \cdot \nabla)u + \nabla p = 0 \quad \text{in } (0, \infty).$$

Hence, in view of Remark 4.1 (ii), we have

$$(u_t, u)_{L^2} + ((u \cdot \nabla)u_1, u)_{L^2} = 0 \quad \text{in } (0, \infty),$$

which yields

$$\frac{d}{dt} \|u(t)\|_2^2 \leq \|\nabla u_1(t)\|_\infty \|u(t)\|_2^2 \leq c \|u_1(t)\|_{H^s(\mathbb{R}^d)} \|u(t)\|_2^2, \quad \forall t \in (0, \infty).$$

Thus, by Gronwall's inequality we have  $u(t) \equiv 0, \forall t \in (0, \infty)$ .

The proof of the theorem is complete. ■

**Conflict of interest:** The author declares that there is no potential conflict of interest with other people or organization associated with this manuscript.

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