

Asymmetric super-Heston-rough volatility model with Zumbach effect as scaling limit of quadratic Hawkes processes

Priyanka Chudasama* and Srikanth Krishnan Iyer†

Department of Mathematics, Indian Institute of Science, Bangalore, India

Abstract

Hawkes processes were first introduced to obtain microscopic models for the rough volatility observed in asset prices. Scaling limits of such processes leads to the rough-Heston model that describes the macroscopic behavior. Blanc et al. (2017) show that Time-reversal asymmetry (TRA) or the Zumbach effect can be modeled using Quadratic Hawkes (QHawkes) processes. Dandapani et al. (2021) obtain a super-rough-Heston model as scaling limit of QHawkes processes in the case where the impact of buying and selling actions are symmetric. To model asymmetry in buying and selling actions, we propose a bivariate QHawkes process and derive a super-rough-Heston model as scaling limits for the price process in the stable and near-unstable regimes that preserves TRA. A new feature of the limiting process in the near-unstable regime is that the two driving Brownian motions exhibit a stochastic covariation that depends on the spot volatility.

Key words and phrases: Quadratic Hawkes processes, limit theorems, microstructure modeling, super rough Heston model, Zumbach effect, time reversal asymmetry.

1 Introduction

“The hunt for a “perfect” statistical model of financial markets is still going on,” wrote the authors in [2]. The goal is to derive models for macroscopic behavior of financial asset prices that arise as scaling limits of stochastic models that incorporate micro-structural features observed

*priyankac@iisc.ac.in

†Corresponding author: skiyer@iisc.ac.in

in time series of price processes (see [1, 2, 6, 12, 15, 24]). Properties of interest include volatility clustering, leverage effect, fat (power-law) tails of the return distribution, etc. Volatility clustering is the tendency in the price process where large changes are usually followed by large changes of either sign while small changes usually follow small changes. The leverage effect refers to the negative correlation between return and volatility. That is, the volatility drops as asset prices increase and vice-versa. Other stylized facts observed in price movements are that the markets are highly endogenous, there is an absence of statistical arbitrage and price movements are statistically asymmetric.

In 2013, Bacry et al. [1] proposed using Hawkes processes to study the macroscopic behavior of price processes. A Hawkes process $(N_t : t \geq 0)$ is a self-exciting point process that can be fully determined using the intensity (event arrival rate) function,

$$\lambda_t := \mu + \sum_{t_i < t} \phi(t - t_i) = \mu + \int_0^t \phi(t - s) dN_s.$$

Here, μ is the baseline intensity and ϕ is a nonnegative function that satisfies $\|\phi\|_1 := \int_0^\infty \phi(s) ds < 1$. The later condition guarantees stability, that is, the existence of a stationary distribution for the intensity process. $0 < t_1 < t_2 < \dots$ denote the jump times of the process $N(t)$. Each arrival results in a jump in the intensity and the impact of a jump on the intensity decays according to the kernel ϕ . This feedback from the process to its intensity is referred to as the self-exciting property and leads to clustering of arrivals.

Price processes can be modeled using Hawkes process in two ways. One approach, as introduced in [1], is to use a bivariate Hawkes process $(\mathbf{N}_t^1, \mathbf{N}_t^2 : t \geq 0)$ representing upward and downward movements of the price process. The price process is then given by $\mathbf{P}_t := \mathbf{N}_t^1 - \mathbf{N}_t^2$.

Alternately, one can use a univariate Hawkes process $N(t)$ to model jump events in the price process where the size of jumps in price takes values $+1$ or -1 with arbitrary probability independent of everything else. Thus,

$$P_t = \sum_{i=1}^{N(t)} \xi_i, \tag{1}$$

where $\xi_i \stackrel{iid}{\sim} \pm 1$ with probability $\mathbb{P}(\xi_1 = 1) = p$.

In the first approach, the intensity function of the bivariate process satisfies

$$\begin{pmatrix} \lambda_t^1 \\ \lambda_t^2 \end{pmatrix} = \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} + \int_0^t \Phi(t-s) \begin{pmatrix} dN_s^1 \\ dN_s^2 \end{pmatrix}.$$

where

$$\Phi = \begin{pmatrix} \phi_1 & c\phi_2 \\ \phi_2 & \phi_1 + (c-1)\phi_2 \end{pmatrix}. \quad (2)$$

In this case, the condition under which the intensity process has a stationary distribution, also known as the stability condition, is given by $\rho(K) < 1$ where $\rho(K)$ is the spectral norm of the matrix

$$K = \int_0^\infty \Phi(t) dt.$$

Scaling limits of such processes are Gaussian processes with fixed volatility. The precise scaling procedure will be described later. To motivate the model we wish to study, we discuss the following important observations made in [6]. The only way to obtain stochastic volatility in the above set-up is to consider a scaling regime where $\rho(K)$ approaches one at an appropriate rate. This is referred to as the near unstable regime. Secondly, to ensure absence of arbitrage one must have $\mu_1 = \mu_2$ and the row sums of Φ must be identical (which explains the particular choice of the matrix in (2)). Finally, negative correlations between price returns and volatility increments occur in this model only if $c > 1$. Under these conditions one obtains the Heston model in the scaling limit [6], that is, the price process satisfies,

$$dP_t = \frac{1}{1 - (\|\phi_1\|_1 - \|\phi_2\|_1)} \sqrt{\frac{2}{1+c}} \sqrt{V_t} dW_t, \quad (3)$$

where the process V follows the SDE

$$dV_t = \kappa(v_0 - V_t)dt + \eta \sqrt{V_t} dB_t. \quad (4)$$

The covariation between the standard Brownian motions W, B is given by

$$d\langle W, B \rangle_t = \frac{1-c}{\sqrt{2(1+c^2)}} dt,$$

which is negative only if $c > 1$.

Yet another stylized fact observed in the markets is the presence of long range correlations. Such a phenomenon could arise, for instance, due to large meta orders getting split by algorithms to avoid impact of orders on market prices. In [6] it is shown that long range dependence can be obtained in the above set-up of a bivariate Hawkes process by imposing the following tail-decay

condition on the largest eigenvalue of the excitation kernel Φ , that is,

$$\int_t^\infty (\phi_1 + c\phi_2)(s) ds \sim Ct^{-\alpha}, \quad \alpha \in \left(\frac{1}{2}, 1\right).$$

The resulting scaling limit is the rough-Heston model where the price process satisfies (3) but with (4) replaced by

$$V_t = V_0 + \int_0^t (t-s)^{\alpha-1} \left(\kappa(1+c-V_s) ds + \eta \sqrt{V_s} dB_s \right),$$

for suitable constants κ, η . The roughness here is a consequence of the singular factor in the integrand.

Another important but less discussed aspect of the price process, as observed by Zumbach in [26], is the statistical symmetry at high frequency on the macro scale when past and future are interchanged. This is referred to as time-reversal asymmetry (TRA) or the Zumbach effect. This asymmetry manifests itself in two ways. One is that the past returns negatively affect future volatility but not other way around. Secondly, past large scale realized volatility is more correlated with future small scale realized volatility than vice-versa.

In 2015, Blanc et al. [14] proposed using a Quadratic Hawkes (QHawkes) process to model TRA. It turns out that a scaling limit of this model exhibits rough volatility behavior even in the stable regime. A QHawkes process is a self-exciting process similar to the Hawkes process with an additional quadratic term in its intensity function. Here one follows the second approach to modeling a price process using a univariate Hawkes process as described in (1) with $p = \frac{1}{2}$. (we will return to this point later). The QHawkes counting process $(\mathbf{N}_t)_{T>0}$ has intensity

$$\lambda_t = \mu + \int_0^t \phi(t-s) d\mathbf{N}_s + \mathbf{Z}_t^2, \text{ with } \mathbf{Z}_t := \int_0^t k(t-s) d\mathbf{P}_s. \quad (5)$$

The quadratic term in the intensity process leads to TRA. Increases in the intensity of the arrival of future events occur whenever price movements of the same sign are seen in succession. In other words, price momentum leads to an increase in volatility. The QHawkes price process exhibits volatility clustering, leverage effect, and fat tails of return distribution on the microscale. It was shown in [12] that the scaling limit of such processes yields the super-Heston-rough process. The limiting price process P_t satisfies $dP_t = \sqrt{V_t} dW_t$, where W is a standard Brownian motion. The

volatility process takes the form

$$V_t = V_0 + \frac{1}{\alpha} \int_0^t (t-s)^{\alpha-1} \lambda (\theta - V_s) ds + \frac{\lambda \nu}{\alpha} \left(\int_0^t (t-s)^{\alpha-1} \sqrt{V_s} dB_s \right)^\beta,$$

where B_t is a standard Brownian motion independent of W , $\beta = 2$ and $\alpha \in (\frac{1}{2}, 1)$. Note that $\beta = 1$ yields the rough-Heston model. A super-Heston model would have powers higher than the classical square-root term in the spot volatility evolution, represented here by the fact that $\beta > 1$.

The choice of $p = \frac{1}{2}$ yields a martingale inside the quadratic term in (5) which plays an important role in deriving the scaling limit. However, this leads to the restriction that the impact of buying and selling actions is symmetric. This is the main issue we address in this paper. We shall keep notations closely aligned with that in [12] for ease of comparison.

To model asymmetry in the impact of buy and sell actions, we go back to the first modeling approach of using a bivariate process, but do so with QHawkes processes instead. The price process $\mathbf{P}_t := \mathbf{N}_t^1 - \mathbf{N}_t^2$ where \mathbf{N}_t^1 and \mathbf{N}_t^2 are two QHawkes processes with respective intensities given by

$$\begin{aligned} \lambda_t^1 &:= \mu_1 + \int_0^t \phi_1(t-s) d\mathbf{N}_s^1 + \int_0^t \phi_2(t-s) d\mathbf{N}_s^2 + \left(\int_0^t k_1(t-s) d\mathbf{M}_s^1 - \int_0^t k_2(t-s) d\mathbf{M}_s^2 \right)^2 \\ \lambda_t^2 &:= \mu_2 + \int_0^t \phi_2(t-s) d\mathbf{N}_s^1 + \int_0^t \phi_1(t-s) d\mathbf{N}_s^2 + \left(\int_0^t k_2(t-s) d\mathbf{M}_s^1 - \int_0^t k_1(t-s) d\mathbf{M}_s^2 \right)^2, \end{aligned} \quad (6)$$

where $\mathbf{M}_t^1 = \mathbf{N}_t^1 - \int_0^t \lambda_s^1 ds$ and $\mathbf{M}_t^2 = \mathbf{N}_t^2 - \int_0^t \lambda_s^2 ds$ are martingales. If $k_1 > k_2$ then a larger-than-expected upward movement in prices will result in an increase in λ^1 and likewise for the downward movement.

We show that the scaling limit of the above price process is a super-Heston-rough volatility model that preserves the Zumbach effect at the macroscale and the asymmetry in price movements at microscopic level. In comparison with (2), note that we are setting $c = 1$ in the linear term. We show that it is possible to obtain a negative correlation between returns and volatility increments without having to take $c > 1$ in the nearly-unstable regime. In fact we show that the cross variation between the two Brownian motions is itself stochastic. Depending on the parameters and the price trends, this could take both positive and negative values, thus allowing for a broader range of possibilities.

2 Stability of the Price Process

The structure specified in (6) lends itself to a nice simplification of the intensity function of the price process. The intensity function $\lambda_t := \lambda_t^1 - \lambda_t^2$ of the price process satisfies

$$\lambda_t = \mu + \int_0^t \phi(t-s) d\mathbf{P}_s + \left[\int_0^t (k_1 + k_2)(t-s) d\mathbf{M}_s \right] \left[\int_0^t (k_1 - k_2)(t-s) d\mathbf{M}_s \right],$$

where $\mu := \mu_1 - \mu_2$ and $\phi := \phi_1 - \phi_2$. $\mathbf{M}_t := \mathbf{M}_t^1 - \mathbf{M}_t^2$ is a martingale as the counting processes \mathbf{N}_t^1 and \mathbf{N}_t^2 do not have simultaneous jumps. Since we shall assume that $k_1 > k_2$, the model can be simplified by taking $k_1 + k_2 := \sqrt{\alpha_1}k$ and $k_1 - k_2 := \sqrt{\alpha_2}k$ where $\alpha_1, \alpha_2 > 0$. Setting $\alpha := \sqrt{\alpha_1 \alpha_2}$, λ_t can be rewritten as

$$\lambda_t = \mu + \int_0^t \phi(t-s) d\mathbf{P}_s + \alpha \left(\int_0^t k(t-s) d\mathbf{M}_s \right)^2. \quad (7)$$

Thus the price process \mathbf{P}_t has intensity given by (7) with associated martingale

$$\mathbf{M}_t = \mathbf{P}_t - \int_0^t \lambda_s ds.$$

We now examine conditions under which the price process is stable. This follows (see [18, 19]) provided the mean intensity $\mathbb{E}(\lambda_t)$ converges. Taking expectations in (7) and using the fact that

$$\langle \mathbf{M} \rangle_t = \langle M^1 \rangle_t + \langle M^2 \rangle_t = \langle N^1 \rangle_t + \langle N^2 \rangle_t = \int_0^t (\lambda_s^1 + \lambda_s^2) ds,$$

we obtain

$$\begin{aligned} \mathbb{E}[\lambda_t] &= \mu + \int_0^t \phi(t-s) \mathbb{E}[\lambda_s] ds + \alpha \mathbb{E} \left[\left\langle \int_0^t k(t-s) d\mathbf{M}_s \right\rangle \right] \\ &= \mu + \int_0^t \phi(t-s) \mathbb{E}[\lambda_s] ds + \alpha \mathbb{E} \left[\int_0^t (k(t-s))^2 d\langle \mathbf{M} \rangle_s \right] \\ &= \mu + \int_0^t \phi(t-s) \mathbb{E}[\lambda_s] ds + \alpha \mathbb{E} \left[\int_0^t k^2(t-s) (\lambda_s^1 + \lambda_s^2) ds \right]. \end{aligned}$$

Let $\bar{\lambda}_t = \lambda_t^1 + \lambda_t^2$, $\bar{\mu} := \mu_1 + \mu_2$ and $\bar{\phi} := \phi_1 + \phi_2$. Proceeding as above we obtain

$$\begin{aligned}\mathbb{E}[\bar{\lambda}_t] &= \bar{\mu} + \int_0^t \bar{\phi}(t-s) \mathbb{E}[\bar{\lambda}_s] ds + \frac{(\alpha_1 + \alpha_2)}{2} \mathbb{E} \left[\left\langle \int_0^t k(t-s) d\mathbf{M}_s \right\rangle \right] \\ &= \bar{\mu} + \int_0^t \bar{\phi}(t-s) \mathbb{E}[\bar{\lambda}_s] ds + \frac{(\alpha_1 + \alpha_2)}{2} \mathbb{E} \left[\int_0^t (k(t-s))^2 d\langle \mathbf{M} \rangle_s \right] \\ &= \bar{\mu} + \int_0^t \left(\bar{\phi}(t-s) + \frac{(\alpha_1 + \alpha_2)}{2} k^2(t-s) \right) \mathbb{E}[\bar{\lambda}_s] ds.\end{aligned}\tag{8}$$

Convergence of $\mathbb{E}[\bar{\lambda}_t]$ will follow by monotonicity provided we can show that it is bounded. This in turn will imply the convergence of $\mathbb{E}[\lambda_t]$ (again by boundedness and monotonicity of $\mathbb{E}[\lambda_t^i]$). From (8) it is clear that $\mathbb{E}[\bar{\lambda}_s] \leq \mathbb{E}[\bar{\lambda}_t]$ for any $s < t$. Using this on right-hand side of (8) and simplifying, we obtain

$$\mathbb{E}[\bar{\lambda}_t] \leq \frac{\bar{\mu}}{1 - \left(\|\bar{\phi}\|_1 + \frac{(\alpha_1 + \alpha_2)}{2} \|k\|_2^2 \right)}.$$

We thus obtain the following sufficient condition for stability.

$$\|\bar{\phi}\|_1 + \frac{(\alpha_1 + \alpha_2)}{2} \|k\|_2^2 < 1.$$

This condition is also required to show existence of subsequential limits for the sequence of scaled processes which we shall describe next.

3 Main Results

3.1 The Stable Regime

To obtain a description for the macroscopic behavior, we consider a sequence of processes indexed by a parameter $T > 0$. The price processes $\mathbf{P}_t^T = \mathbf{N}_t^{(1,T)} - \mathbf{N}_t^{(2,T)}$, $t \in [0, T]$ has intensity

$$\lambda_t^T = \lambda_t^{(1,T)} - \lambda_t^{(2,T)},$$

where following (6) we have

$$\begin{aligned}\lambda_t^{(1,T)} &:= \mu_1^T + \int_0^t \phi_1^T(t-s) d\mathbf{N}_s^{(1,T)} + \int_0^t \phi_2^T(t-s) d\mathbf{N}_s^{(2,T)} + \left(\int_0^t k_1^T(t-s) d\mathbf{M}_s^{(1,T)} - \int_0^t k_2^T(t-s) d\mathbf{M}_s^{(2,T)} \right)^2, \\ \lambda_t^{(2,T)} &:= \mu_2^T + \int_0^t \phi_2^T(t-s) d\mathbf{N}_s^{(1,T)} + \int_0^t \phi_1^T(t-s) d\mathbf{N}_s^{(2,T)} + \left(\int_0^t k_2^T(t-s) d\mathbf{M}_s^{(1,T)} - \int_0^t k_1^T(t-s) d\mathbf{M}_s^{(2,T)} \right)^2.\end{aligned}$$

Analogous to (7) the intensity of the price process $\lambda_t^T = \lambda_t^{(1,T)} - \lambda_t^{(2,T)}$ can then be written as

$$\lambda_t^T = \mu + \int_0^t \phi^T(t-s) d\mathbf{P}_s^T + \alpha \left(\int_0^t k^T(t-s) d\mathbf{M}_s^T \right)^2, \quad (9)$$

where $\phi^T := \phi_1^T - \phi_2^T$, $k_1^T + k_2^T := \sqrt{\alpha_1} k^T$, $k_1^T - k_2^T := \sqrt{\alpha_2} k^T$ with $\alpha_1, \alpha_2 > 0$, and $\alpha := \sqrt{\alpha_1 \alpha_2}$. \mathbf{M}_t^T is the martingale $\mathbf{P}_t^T - \int_0^t \lambda_s^T ds$ and we have set $\mu^T := \mu_1^T - \mu_2^T = \mu$. Rescaling $t \rightarrow tT$ and $s \rightarrow sT$ in (9) we obtain

$$\lambda_{tT}^T = \mu + \int_0^t \phi^T(T(t-s)) d\mathbf{P}_{Ts}^T + \alpha \left(\int_0^t k^T(T(t-s)) d\mathbf{M}_{Ts}^T \right)^2, \quad t \in [0, 1],$$

which can be rewritten as

$$\begin{aligned} \lambda_{tT}^T = & \mu + \int_0^t \phi^T(T(t-s)) T \lambda_{Ts}^T ds + \int_0^t \phi^T(T(t-s)) d\mathbf{M}_{Ts}^T \\ & + \alpha \left(\int_0^t k^T(T(t-s)) d\mathbf{M}_{Ts}^T \right)^2, \quad t \in [0, 1]. \end{aligned} \quad (10)$$

Similarly, the intensity $\bar{\lambda}_t^T = \lambda_t^{(1,T)} + \lambda_t^{(2,T)}$ of the processes $\bar{\mathbf{P}}_t^T := \mathbf{N}_t^{(1,T)} + \mathbf{N}_t^{(2,T)}$ can be written as

$$\bar{\lambda}_t^T = \bar{\mu} + \int_0^t \bar{\phi}^T(t-s) d\bar{\mathbf{P}}_s^T + \frac{\alpha_1 + \alpha_2}{2} \left(\int_0^t k^T(t-s) d\mathbf{M}_s^T \right)^2, \quad t \in [0, T],$$

where $\bar{\phi}^T := \phi_1^T + \phi_2^T$ and we have set $\bar{\mu}^T := \mu_1^T + \mu_2^T = \bar{\mu}$. Rescaling as above we get

$$\begin{aligned} \bar{\lambda}_{tT}^T = & \bar{\mu} + \int_0^t \bar{\phi}^T(T(t-s)) T \bar{\lambda}_{Ts}^T ds + \int_0^t \bar{\phi}^T(T(t-s)) d\bar{\mathbf{M}}_{Ts}^T \\ & + \frac{\alpha_1 + \alpha_2}{2} \left(\int_0^t k^T(T(t-s)) d\mathbf{M}_{Ts}^T \right)^2, \quad t \in [0, 1], \end{aligned}$$

where $\bar{\mathbf{M}}_t^T$ is the martingale $\bar{\mathbf{P}}_t^T - \int_0^t \bar{\lambda}_s^T$.

It is natural to try and show convergence of the sequence of processes $\frac{1}{\sqrt{T}} \mathbf{M}_{Tt}^T$. The appropriate scaling assumptions for the parameters given below are then evident. We shall add a technical condition that will be required to prove tightness.

3.1.1 Assumptions

- i. The sequence of kernels satisfy

$$\phi^T(\cdot) = \phi\left(\frac{\cdot}{T}\right) \frac{\beta}{T}, \quad \bar{\phi}^T(\cdot) = \phi\left(\frac{\cdot}{T}\right) \frac{\bar{\beta}}{T} \quad \text{and} \quad k^T(\cdot) = k\left(\frac{\cdot}{T}\right) \frac{1}{\sqrt{T}}$$

for some constants $\beta, \bar{\beta}$ and kernels ϕ, k satisfying $|\beta| < \bar{\beta}$ and

$$0 < \bar{\beta} \|\phi\|_1 + \frac{(\alpha_1 + \alpha_2)}{2} \|k\|_2^2 < 1.$$

ii. There exists $\eta, \bar{\eta} \geq 0$ such that

$$\|\phi\|_1 + 2\alpha \|k\|_2 (\eta + \bar{\mu}) < \frac{\eta}{\eta + |\mu|}, \text{ and } \|\phi\|_1 + (\alpha_1 + \alpha_2) \|k\|_2 (\bar{\eta} + \bar{\mu}) < \frac{\bar{\eta}}{\bar{\eta} + \bar{\mu}}.$$

These conditions will be required to prove the uniqueness of the limiting processes (see section 5).

iii. The function $k \in L^{2+\epsilon}$ for some $\epsilon > 0$ and for any $0 \leq t \leq \hat{t} \leq 1$,

$$\int_0^t |k(\hat{t}-s) - k(t-s)|^2 ds < C |\hat{t}-t|^r$$

for some $r, C > 0$. Further, for some $\eta \in (0, 1)$.

$$\int_0^1 |k(t)|^2 t^{-2\eta} dt + \int_0^1 \int_0^1 \frac{|k(t) - k(s)|^2}{|t-s|^{1+2\eta}} ds dt < \infty.$$

3.1.2 Scaling Limit in the Stable Regime

Define the sequence of scaled processes $\mathbf{X}_t^T = \frac{\mathbf{P}_{tT}^T}{T}$, $\mathbf{M}_t^{*T} = \frac{1}{\sqrt{T}} \mathbf{M}_{Tt}^T$, $t \in [0, 1]$, $T > 0$. Consider the Mittag-Leffler function

$$f^{\tilde{\alpha}, \sigma}(s) := \sum_{n=0}^{\infty} \frac{s^n}{\Gamma(\tilde{\alpha}n + \sigma)}. \quad (11)$$

For $\tilde{\alpha} \in (\frac{1}{2}, 1)$ and $\sigma > 0$, $f^{\tilde{\alpha}, \sigma}$ satisfies assumption 3.1.1 (iii) for any $\epsilon \in (0, \frac{2\tilde{\alpha}-1}{1-\tilde{\alpha}})$, $\eta \in (0, \tilde{\alpha} - \frac{1}{2})$ and $r = 2\tilde{\alpha} - 1$. We are now ready to state the first main result of this paper.

Theorem 3.1. *Under the assumptions 3.1.1, the sequence $(\mathbf{X}^T, \mathbf{M}^{*T})_{T>0}$ is C-tight for the Skorokhod topology on $[0, 1]$, and as $T \rightarrow \infty$ converges to a process (X, M^*) with the following properties.*

1. $X_t = \int_0^t V_s ds$, where V is the **unique solution** of the equation

$$V_t = \mu + \beta \int_0^t \phi(t-s) V_s ds + \alpha (Z_t^*)^2, \text{ with } Z_t^* = \int_0^t k(t-s) dM_s^*.$$

2. There exists a Brownian motion B such that

$$M_t^* = \int_0^t \sqrt{V_s} dB_s,$$

where \bar{V} is a solution of the equation

$$\bar{V}_t = \bar{\mu} + \bar{\beta} \int_0^t \phi(t-s) \bar{V}_s ds + \frac{(\alpha_1 + \alpha_2)}{2} (Z_t^*)^2.$$

3. For any $\epsilon > 0$, if $k = f^{\tilde{\alpha}, \sigma}$ with $\tilde{\alpha} \in (\frac{1}{2}, 1)$ and $\sigma > 0$, then V, \bar{V} have almost surely $\tilde{\alpha} - \frac{1}{2} - \epsilon$ Hölder regularity.

3.2 The Near Unstable Regime

A more interesting limit process arises when we consider the sequence of scaled processes that approaches the stability barrier.

Let

$$a^T = \left\| \bar{\phi}^T \right\|_1 + \frac{(\alpha_1 + \alpha_2)}{2} \|k_T\|_2^2. \quad (12)$$

Recall that in deriving the scaling limit in the stable regime we had assumed that $\sup_{T>0} a^T < 1$. We now state the assumptions required to prove the scaling limit in the near unstable regime, that is $1 > a^T \rightarrow 1$ as $T \rightarrow \infty$.

3.2.1 Assumptions

i. Fix $\phi \geq 0$ such that $\|\phi\|_1 = 1$ and let

$$\phi^T(\cdot) = \phi\left(\frac{\cdot}{T}\right) \frac{\beta^T}{T}, \quad \bar{\phi}^T(\cdot) = \phi\left(\frac{\cdot}{T}\right) \frac{\bar{\beta}^T}{T},$$

where $|\beta^T| = \frac{1}{1+c_1(1-a^T)}$ and $\bar{\beta}^T = \frac{1}{1+c_2(1-a^T)}$, where the constants c_1, c_2 satisfy $c_1 > c_2 > \frac{(\alpha_1 + \alpha_2)}{2}$ and $1 > a^T \rightarrow 1$ as $T \rightarrow \infty$.

ii. There exists constants $K > 0$ and $\tilde{\alpha} \in (\frac{1}{2}, 1)$ such that

$$\tilde{\alpha} x^{\tilde{\alpha}} \int_x^{+\infty} \phi(s) ds \rightarrow K \text{ as } x \rightarrow +\infty.$$

iii. Let $\delta = K \frac{\Gamma(1-\tilde{\alpha})}{\tilde{\alpha}}$. There exists $\mu^*, \bar{\mu}^*, \sigma > 0$ and such that

$$(1 - a^T) T^{\tilde{\alpha}} \rightarrow \sigma \delta, \quad \mu^T T^{(1-\tilde{\alpha})} \rightarrow \frac{\mu^*}{\delta} \text{ and } \bar{\mu}^T T^{(1-\tilde{\alpha})} \rightarrow \frac{\bar{\mu}^*}{\delta} \text{ as } T \rightarrow \infty.$$

iv. $k_T(\cdot) = k(\cdot/T) \sqrt{\frac{1-a^T}{T}}$, where k is a non-negative continuously differentiable function such that $\|k\|_2^2 = 1$.

Remark: Observe that under the assumption 3.2.1 (i) the right hand side of equation (12) reduces to

$$0 < \frac{1}{1+c_2(1-a^T)} + \frac{(\alpha_1 + \alpha_2)}{2} (1 - a^T) \rightarrow 1 \text{ as } T \rightarrow \infty.$$

3.2.2 Scaling Limit in the Near Unstable Regime

Consider the sequence of scaled processes $\mathbf{X}_t^T = \frac{1-a^T}{\bar{\mu}^T} \frac{\mathbf{P}_{tT}^T}{\sqrt{T}}$, $\mathbf{M}_t^{*T} = \sqrt{\frac{1-a^T}{\bar{\mu}^T}} \frac{1}{\sqrt{T}} \mathbf{M}_{Tt}^T$, $t \in [0, 1]$, $T > 0$. Let $f^{\tilde{\alpha}, \sigma}(s) = \sigma s^{\tilde{\alpha}-1} \sum_{n=0}^{\infty} \frac{(-\sigma s^{\tilde{\alpha}})^n}{\Gamma(\tilde{\alpha}n + \tilde{\alpha})}$ where the sum is the Mittag-Leffler function as defined in (11). Further, let $F^{\tilde{\alpha}, \sigma}(t) := \int_0^t f^{\tilde{\alpha}, \sigma}(s) ds$.

Theorem 3.2. *Under the assumptions 3.2.1 (i)-(iv), the sequence $(\mathbf{X}^T, \mathbf{M}^{*T}, t \in [0, 1])_{T>0}$ is C-tight for the Skorokhod topology on $[0, 1]$ and any subsequential limit process $(X_t, M_t^*, t \in [0, 1])$ will satisfy the following evolution equations.*

$$X_t = \int_0^t \frac{1}{c_1} f^{\tilde{\alpha}, \sigma}(t-s) \frac{1}{\sqrt{\sigma \bar{\mu}^*}} M_s^* ds + \int_0^t \frac{1}{c_1} F^{\tilde{\alpha}, \sigma}(t-s) \left(\frac{\mu^*}{\bar{\mu}^*} + \alpha(Z_s^*)^2 \right) ds, \quad (13)$$

with $Z_t^* = \int_0^t k(t-s) dM_s^*$ and $\langle M^* \rangle = \bar{X}_t$ where,

$$\bar{X}_t = \int_0^t \frac{1}{c_2} f^{\tilde{\alpha}, \sigma}(t-s) \frac{1}{\sqrt{\sigma \bar{\mu}^*}} \bar{M}_s^* ds + \int_0^t \frac{1}{c_2} F^{\tilde{\alpha}, \sigma}(t-s) \left(1 + \frac{(\alpha_1 + \alpha_2)}{2} (Z_s^*)^2 \right) ds. \quad (14)$$

Further, if $\tilde{\alpha} \in (\frac{1}{2}, 1)$, we have

1. Upto an enlargement of the filtration, there exist Brownian motions B^1, B^2 such that

$$M_t^* = \int_0^t \sqrt{V_s} dB_s^1, \quad \bar{M}_t^* = \int_0^t \sqrt{\bar{V}_s} dB_s^2, \quad \langle B^1, B^2 \rangle_t = \frac{V_t}{\bar{V}_t} \quad (15)$$

where, V and \bar{V}_t are solutions of

$$V_t = \int_0^t \frac{1}{c_1} f^{\tilde{\alpha}, \sigma}(t-s) \left[\frac{1}{\sqrt{\sigma \bar{\mu}^*}} dM_s^* + \left(\frac{\mu^*}{\bar{\mu}^*} + \alpha(Z_s^*)^2 \right) ds \right],$$

$$\bar{V}_t = \int_0^t \frac{1}{c_2} f^{\tilde{\alpha}, \sigma}(t-s) \left[\frac{1}{\sqrt{\sigma \bar{\mu}^*}} d\bar{M}_s^* + \left(1 + \frac{(\alpha_1 + \alpha_2)}{2} (Z_s^*)^2 \right) ds \right]. \quad (16)$$

2. The processes (X_t, \bar{X}_t) are almost surely differentiable with derivatives (V_t, \bar{V}_t) .

3. For any $\epsilon > 0$, V, \bar{V} have almost surely $\tilde{\alpha} - \frac{1}{2} - \epsilon$ Hölder regularity.

4 Proofs

In this section, we will prove Theorem 3.2. The proof of tightness and the structure of the evolution equations in Theorem 3.1 is similar and is therefore omitted. The proof of uniqueness of the evolution equations uniqueness in Theorem 3.1 is given in Section 5. These proofs follow along similar lines to those in [12]. We prove Theorem 3.2 into two parts. We will first prove C-tightness of the family $(\mathbf{X}^T, \mathbf{M}_t^{*T}, t \in [0, 1])_{T>0}$.

This will ensure existence and path continuity of the subsequential limits. In the second part, we establish the dynamics of limit points to find the system of equations that any subsequential limit will satisfy.

Consider the process $\bar{\mathbf{P}}_t = \mathbf{N}_t^{(1,T)} + \mathbf{N}_t^{(2,T)}$, $t \in [0, T]$ with intensity

$$\bar{\lambda}_t^T = \lambda_t^{(1,T)} + \lambda_t^{(2,T)},$$

and the corresponding martingale $\bar{\mathbf{M}}_t^T = \bar{\mathbf{P}}_t - \int_0^t \bar{\lambda}_s^T ds$. Define the scaled processes

$$\bar{\mathbf{X}}_t^T := \frac{1-a^T}{\bar{\mu}^T} \frac{\bar{\mathbf{P}}_{tT}^T}{T}, \quad \bar{\mathbf{M}}_t^{*T} = \sqrt{\frac{1-a^T}{\bar{\mu}^T}} \frac{1}{\sqrt{T}} \bar{\mathbf{M}}_{Tt}^T.$$

We will also prove C-tightness of the family $((\bar{\mathbf{X}}_t^T, \bar{\mathbf{M}}_t^{*T}), t \in [0, 1])_{T>0}$ as their limits appear in (15), (16).

The scaled processes $\mathbf{X}_t^T, \bar{\mathbf{X}}_t^T$ can be written as

$$\mathbf{X}_t^T = \mathbf{X}_t^{(1,T)} - \mathbf{X}_t^{(2,T)}, \quad \bar{\mathbf{X}}_t^T = \mathbf{X}_t^{(1,T)} + \mathbf{X}_t^{(2,T)},$$

where

$$\mathbf{X}_t^{(i,T)} := \frac{(1-a^T)}{\bar{\mu}^T} \frac{\mathbf{N}_{tT}^{(i,T)}}{T}, \quad i = 1, 2.$$

The compensator of the process $\mathbf{X}_t^{(i,T)}$ is given by

$$\Lambda_t^{(i,T)} := \frac{(1-a^T)}{\bar{\mu}^T} \int_0^t \lambda_{sT}^{(i,T)} ds, \quad i = 1, 2. \quad (17)$$

Let

$$\Lambda_t^T = \Lambda_t^{(1,T)} - \Lambda_t^{(2,T)} = \frac{(1-a^T)}{\bar{\mu}^T} \int_0^t \lambda_{sT}^T ds, \quad \bar{\Lambda}_t^T = \Lambda_t^{(1,T)} + \Lambda_t^{(2,T)} = \frac{(1-a^T)}{\bar{\mu}^T} \int_0^t \bar{\lambda}_{sT}^T ds. \quad (18)$$

4.1 C-Tightness of the processes $((\mathbf{X}_t^T, \bar{\mathbf{X}}_t^T, \mathbf{M}_t^{*T}, \bar{\mathbf{M}}_t^{*T}), t \in [0, 1])_{T>0}$

For the scaled martingale processes $\mathbf{M}_t^{*T}, \bar{\mathbf{M}}_t^{*T}$ we have

$$\begin{aligned} \langle \mathbf{M}^{*T} \rangle_t &= \langle \bar{\mathbf{M}}^{*T} \rangle_t = \frac{1-a^T}{\bar{\mu}^T} \frac{1}{T} \left(\int_0^{tT} (\lambda_s^{(1,T)} + \lambda_s^{(2,T)}) ds \right) \\ &= \Lambda_t^{(1,T)} + \Lambda_t^{(2,T)}. \end{aligned}$$

By Theorem VI-4.13 in [17], tightness of the family $\{(\mathbf{M}_t^{*T}, \bar{\mathbf{M}}_t^{*T}) : t \in [0, 1]\}_{T>0}$ follows if we show that the corresponding family of sharp bracket processes is tight. Since linear combinations of tight processes are tight and joint tightness holds if the marginals are tight, it suffices to show C-tightness of the families of processes $\{\mathbf{X}_t^{(i,T)} : t \in [0, 1]\}_{T>0}, \{\Lambda_t^{(i,T)} : t \in [0, 1]\}_{T>0}, i = 1, 2$.

First we shall show that these two families of processes are tight. Since $\mathbf{X}_t^{(i,T)}$, $\Lambda_t^{(i,T)}$ are right continuous increasing processes, it suffices to show tightness of the families of random variables $\mathbf{X}_1^{(i,T)}$, $\Lambda_1^{(i,T)}$. This in turn follows if we show that $\mathbb{E}[\mathbf{X}_1^{(i,T)}]$ and $\mathbb{E}[\Lambda_1^{(i,T)}]$ are uniformly bounded in T . Clearly,

$$\mathbb{E}[\mathbf{X}_1^{(i,T)}] = \mathbb{E}[\Lambda_1^{(i,T)}] \leq \mathbb{E}[\bar{\Lambda}_1^T].$$

From (17), we have

$$\mathbb{E}[\bar{\Lambda}_1^T] = \frac{(1-a^T)}{\bar{\mu}^T} \int_0^1 \mathbb{E}[\bar{\lambda}_{sT}^T] ds. \quad (19)$$

Analogous to (8) we have

$$\mathbb{E}[\bar{\lambda}_t^T] = \bar{\mu}^T + \int_0^t \left(\bar{\phi}^T(t-s) + \frac{(\alpha_1 + \alpha_2)}{2} (k^T(t-s))^2 \right) \mathbb{E}[\bar{\lambda}_s^T] ds. \quad (20)$$

Using $\bar{\lambda}_s^T \leq \bar{\lambda}_t^T$ for $s \leq t$ and then setting $t = 1$ on the right in (20) we obtain

$$\mathbb{E}[\bar{\lambda}_1^T] \leq \frac{\bar{\mu}^T}{1 - \left(\|\bar{\phi}^T\|_1 + \frac{(\alpha_1 + \alpha_2)}{2} \|k^T\|_2^2 \right)} \quad (21)$$

Using the assumptions 3.2.1 to substitute for $\bar{\phi}^T$, k^T in (21) yields

$$\mathbb{E}[\bar{\lambda}_1^T] \leq \frac{\bar{\mu}^T}{1 - \left(\bar{\beta}^T \|\phi\|_1 + \frac{(\alpha_1 + \alpha_2)}{2} (1-a^T) \|k\|_2^2 \right)}.$$

Substituting the above bound in (19) and using the assumptions 3.2.1 (i), (iv), we obtain the bound

$$\begin{aligned} \mathbb{E}(\bar{\Lambda}_1^T) &\leq \frac{1-a^T}{1 - \left(\bar{\beta}^T \|\phi\|_1 + \frac{(\alpha_1 + \alpha_2)}{2} (1-a^T) \|k\|_2^2 \right)} \\ &\leq \frac{1-a^T}{1 - \left(\frac{1}{1+c_2(1-a^T)} + \frac{(\alpha_1 + \alpha_2)}{2} (1-a^T) \right)} \\ &\leq \frac{1}{\frac{c_2}{1+c_2(1-a^T)} - \frac{(\alpha_1 + \alpha_2)}{2}} \rightarrow \frac{1}{c_2 - \frac{(\alpha_1 + \alpha_2)}{2}} < \infty, \end{aligned} \quad (22)$$

as $1-a^T \rightarrow 0$ as $T \rightarrow \infty$ and $c_2 > \frac{(\alpha_1 + \alpha_2)}{2}$. This proves tightness. To verify that all convergent subsequences converge to processes with continuous paths, note that \mathbf{N}_t^1 and \mathbf{N}_t^2 are counting processes with jump size 1. Thus,

$$|\Delta \mathbf{X}_t^{(i,T)}| = \frac{1-a^T}{\bar{\mu}^T} \frac{|\Delta \mathbf{N}_t^{(i,T)}|}{T} \leq \frac{1-a^T}{\bar{\mu}^T T} \leq \frac{(1-a^T)^2}{T^{1-\tilde{\alpha}} \bar{\mu}^T (1-a^T) T^{\tilde{\alpha}}} \rightarrow 0$$

by 3.2.1 (iii) and the fact that $1-a^T \rightarrow 0$ as $T \rightarrow \infty$. Furthermore, $|\Delta \Lambda_1^T| = 0$ as Λ_1^T is continuous on

$[0, 1]$. Therefore, by proposition VI-3.35 (from (iii) \Rightarrow (i)) in [17] we have that $(\mathbf{X}_t^{(i,T)})_{T>0}$ and $(\Lambda_t^{(i,T)})_{T>0}$, $i \in \{1, 2\}$ are C-tight for the Skorokhod topology on $[0, 1]$.

4.2 Dynamics of the limit points

In this section we find limit points for the family of processes $\left((\mathbf{X}^T, \bar{\mathbf{X}}^T, \mathbf{M}_t^{*T}, \bar{\mathbf{M}}_t^{*T}), t \in [0, 1]\right)_{T>0}$. By the following Lemma, it is enough to study the dynamics of limit points for the family of processes $\left(\Lambda_t^T, \bar{\Lambda}_t^T, \mathbf{M}_t^{*T}, \bar{\mathbf{M}}_t^{*T}, t \in [0, 1]\right)_{T>0}$.

Lemma 4.1. *The sequence of martingales $\left(\mathbf{X}_t^T - \Lambda_t^T, t \in [0, 1]\right)_{T>0}$ and $\left(\bar{\mathbf{X}}_t^T - \bar{\Lambda}_t^T, t \in [0, 1]\right)_{T>0}$ converge uniformly to 0 in probability.*

Proof. Observe that $\mathbb{E} \left[\sup_{t \in [0,1]} (\bar{\mathbf{X}}_t^T - \bar{\Lambda}_t^T)^2 \right]$ and $\mathbb{E} \left[\sup_{t \in [0,1]} (\mathbf{X}_t^T - \Lambda_t^T)^2 \right]$ are respectively bounded by

$$\mathbb{E} \left[\sup_{t \in [0,1]} ((\mathbf{X}_t^{(1,T)} - \Lambda_t^{(1,T)}) \pm (\mathbf{X}_t^{(1,T)} - \Lambda_t^{(2,T)}))^2 \right] \leq 2 \sum_{i=1}^2 \mathbb{E} \left[\sup_{t \in [0,1]} (\mathbf{X}_t^{(i,T)} - \Lambda_t^{(i,T)})^2 \right].$$

For $i \in \{1, 2\}$, we have

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0,1]} (\mathbf{X}_t^{(i,T)} - \Lambda_t^{(i,T)})^2 \right] &= \mathbb{E} \left[\sup_{t \in [0,1]} \left(\frac{\mathbf{N}_{tT}^{(i,T)}}{T} - \int_0^t \lambda_s^{(i,T)} ds \right)^2 \right] \\ &= \mathbb{E} \left[\sup_{t \in [0,1]} \left(\frac{\mathbf{N}_{tT}^{(i,T)}}{T} - \frac{1}{T} \int_0^{tT} \lambda_s^{(i,T)} ds \right)^2 \right] \\ &= \frac{1}{T^2} \mathbb{E} \left[\sup_{t \in [0,1]} \left(\mathbf{N}_{tT}^{(i,T)} - \int_0^{tT} \lambda_s^{(i,T)} ds \right)^2 \right] \\ &\leq \frac{4}{T^2} \mathbb{E} \left[(\mathbf{M}_T^{(i,T)})^2 \right] = \frac{4}{T^2} \mathbb{E} \left([\mathbf{M}^{(i,T)}]_T \right), \end{aligned} \tag{23}$$

where the second-last inequality follows by the Doob's martingale inequality. Further,

$$\mathbb{E} \left([\mathbf{M}^{(i,T)}]_T \right) = \mathbb{E} \left(\mathbf{N}_T^{(i,T)} \right) = \mathbb{E} \left(\int_0^T \lambda_s^{(i,T)} ds \right) = \mathbb{E} \left(T \int_0^1 \lambda_{Ts}^{(i,T)} ds \right) = E \left(T \Lambda_1^{(i,T)} \right) \leq \mathbb{E}(T \bar{\Lambda}_1^T). \tag{24}$$

Substituting the bounds from (22), (24) in (23) we get

$$\mathbb{E} \left[\sup_{t \in [0,1]} (\mathbf{X}_t^{(i,T)} - \Lambda_t^{(i,T)})^2 \right] \leq \frac{4}{T} \mathbb{E}(\bar{\Lambda}_1^T) \leq \frac{4}{T} \frac{1}{\frac{c_2}{1+c_2(1-a^T)} - \frac{(\alpha_1+\alpha_2)}{2}} \rightarrow 0.$$

as $T \rightarrow \infty$ for $i \in \{1, 2\}$ since $1 - a^T \rightarrow 0$ and $c_2 > \frac{(\alpha_1+\alpha_2)}{2}$. ■

Let $(X, \bar{X}, M^*, \bar{M}^*)$ be a limit point of $(\Lambda^T, \bar{\Lambda}^T, \mathbf{M}^{*T}, \bar{\mathbf{M}}^{*T})_{T>0}$. By C-tightness, the process $(X, M^*, \bar{X}, \bar{M}^*)$ is continuous. By the Skhorohod representation theorem we may suppose that the convergence is almost surely uniform on $[0, 1]$.

To derive the limit points of $(\Lambda^T, \bar{\Lambda}^T)$ as given in (18) we begin by analysing the intensities $(\lambda^T, \bar{\lambda}^T)$. From (10) (with μ replaced by μ^T) the intensity of the price process P_{tT}^T satisfies

$$\begin{aligned}\lambda_{tT}^T &= \mu^T + \int_0^t \phi^T(T(t-s)) T \lambda_{Ts}^T ds + \int_0^t \phi^T(T(t-s)) d\mathbf{M}_{Ts}^T + \alpha \left[\int_0^t k^T(T(t-s)) d\mathbf{M}_{Ts}^T \right]^2 \\ &= \mu^T + \alpha \left(\mathbf{Z}_{tT}^T \right)^2 + \int_0^t \phi^T(T(t-s)) T \lambda_{Ts}^T ds + \int_0^t \phi^T(T(t-s)) d\mathbf{M}_{Ts}^T,\end{aligned}\quad (25)$$

where

$$\mathbf{Z}_{tT}^T := \int_0^{tT} k_T(Tt-s) \mathbf{M}_s^T = \int_0^t k_T(T(t-s)) \mathbf{M}_{sT}^T.$$

Proposition 2.1 from [24] now yields

$$\lambda_{tT}^T = \mu^T + \alpha \left(\mathbf{Z}_{tT}^T \right)^2 + \int_0^t T \psi^T(T(t-s)) \left(\mu^T + \alpha \left(\mathbf{Z}_{tT}^T \right)^2 \right) ds + \int_0^t \psi^T(T(t-s)) d\mathbf{M}_{Ts}^T,$$

where $\psi^T := \sum_{i \geq 0} (\beta^T \phi^T)^{\circledast i}$, with \circledast denoting the convolution operator.

Therefore, $\lambda_t^{*T} := \frac{(1-a^T)}{\bar{\mu}^T} \lambda_t^T$ is given by

$$\lambda_{tT}^{*T} = \frac{(1-a^T)\mu^T}{\bar{\mu}^T} + \frac{1-a^T}{\bar{\mu}^T} \alpha (\mathbf{Z}_{tT}^T)^2 + \int_0^t (1-a^T) T \psi^T(T(t-s)) \left(\frac{\mu^T}{\bar{\mu}^T} + \frac{\alpha}{\bar{\mu}^T} (\mathbf{Z}_{sT}^T)^2 \right) ds. \quad (26)$$

Recall that $\mathbf{M}_t^{*T} := \sqrt{\frac{1-a^T}{T\bar{\mu}^T}} \mathbf{M}_{tT}^T$. Let

$$\mathbf{Z}_t^{*T} := \frac{\mathbf{Z}_{tT}^T}{\sqrt{\frac{1-a^T}{T\bar{\mu}^T}}} = \int_0^t k(t-s) d\mathbf{M}_s^{*T},$$

where the last equality follows from assumption 3.2.1 (iv). We can thus rewrite (26) as

$$\begin{aligned}\lambda_{tT}^{*T} &= \frac{(1-a^T)\mu^T}{\bar{\mu}^T} + (1-a^T)\alpha (\mathbf{Z}_t^{*T})^2 + \int_0^t f^T(t-s) \left(\frac{\mu^T}{\bar{\mu}^T} + \frac{\alpha}{\bar{\mu}^T} (\mathbf{Z}_{sT}^T)^2 \right) ds \\ &\quad + \int_0^t \frac{f^T(t-s)}{\sqrt{\bar{\mu}^T(1-a^T)T}} d\mathbf{M}_s^{*T},\end{aligned}\quad (27)$$

where $f^T(s) := (1-a^T)T\psi^T(Ts)$. Note that $\Lambda_t^T = \int_0^t \lambda_{sT}^{*T} ds$. Let $F^T(t) := \int_0^t f^T(s) ds$. Substituting from

(27) and interchanging the order of the integrals we obtain

$$\begin{aligned}\Lambda_t^T &= \frac{(1-a^T)\mu^T t}{\bar{\mu}^T} + \int_0^t (1-a^T)\alpha(\mathbf{Z}_s^{*T})^2 ds + \int_0^t \frac{F^T(t-s)}{\sqrt{\bar{\mu}^T(1-a^T)T}} d\mathbf{M}_s^{*T} \\ &\quad + \int_0^t F^T(t-s) \left(\frac{\mu^T}{\bar{\mu}^T} + \alpha(\mathbf{Z}_s^{*T})^2 \right) ds.\end{aligned}\quad (28)$$

Similarly, for the process $\bar{\lambda}_t^{*T} = \frac{(1-a^T)}{\bar{\mu}^T} \bar{\lambda}_t^T$ we have

$$\begin{aligned}\bar{\lambda}_{tT}^{*T} &= (1-a^T) + \frac{1-a^T}{\bar{\mu}^T} \left(\frac{\alpha_1 + \alpha_2}{2} \right) (\mathbf{Z}_{tT}^T)^2 + \int_0^t \frac{(1-a^T)T\bar{\psi}^T(T(t-s))}{\sqrt{\bar{\mu}^T(1-a^T)T}} d\bar{\mathbf{M}}_{Ts}^{*T} \\ &\quad + \int_0^t (1-a^T)T\bar{\psi}^T(T(t-s)) \left(1 + \frac{(\alpha_1 + \alpha_2)}{2\bar{\mu}^T} (\mathbf{Z}_{sT}^T)^2 \right) ds,\end{aligned}$$

where $\bar{\psi}^T := \sum_{i \geq 0} (\beta^T \bar{\phi}^T)^{\otimes i}$. Further, if we define

$$\bar{F}^T(t) := \int_0^t \bar{f}^T(s) ds, \text{ with } \bar{f}^T(s) := (1-a^T)T\bar{\psi}^T(Ts),$$

we obtain

$$\begin{aligned}\bar{\Lambda}_t^T &= \int_0^t \bar{\lambda}_{sT}^{*T} = (1-a^T)t + \int_0^t (1-a^T) \left(\frac{\alpha_1 + \alpha_2}{2} \right) (\mathbf{Z}_s^{*T})^2 ds + \int_0^t \frac{\bar{F}^T(t-s)}{\sqrt{\bar{\mu}^T(1-a^T)T}} d\bar{\mathbf{M}}_{Ts}^{*T} \\ &\quad + \int_0^t \bar{F}^T(t-s) \left(1 + \frac{(\alpha_1 + \alpha_2)}{2} (\mathbf{Z}_s^{*T})^2 \right) ds.\end{aligned}\quad (29)$$

From (28), (29) we see that to find the dynamics of the limit points for $(\Lambda_t^T, \bar{\Lambda}_t^T)_{T>0}$, we need to find the limit points of $(\mathbf{M}_t^{*T}, \bar{\mathbf{M}}_t^{*T}, \mathbf{Z}_s^{*T})_{T>0}$.

Since $|\Delta \mathbf{M}^{*T}|, |\Delta \bar{\mathbf{M}}^{*T}| \leq \frac{2(1-a^T)}{\bar{\mu}^T \sqrt{T}}$, we have from Corollary IX-1.19 in [17] that

$$[\mathbf{M}^{*T}]_t = [\bar{\mathbf{M}}^{*T}]_t = \bar{\mathbf{X}}_t^T.$$

Further, M_t^*, \bar{M}_t^* are continuous as they are limits of C-tight processes $\{(\mathbf{M}_t^{*T}, \bar{\mathbf{M}}_t^{*T}) : t \in [0,1]\}_{T>0}$. Consequently, we have

$$[M^*]_t = \langle M^* \rangle_t = [\bar{M}^*]_t = \langle \bar{M}^* \rangle_t = \bar{X}_t.$$

Moreover, $\mathbb{E}[\bar{\mathbf{X}}_1^T]$ is uniformly bounded as $\mathbb{E}[\mathbf{X}_1^{(1,T)}]$ and $\mathbb{E}[\mathbf{X}_1^{(2,T)}]$ are uniformly bounded in T by (22). By Fatou's Lemma we have that \bar{X}_1 is integrable. Hence, we have $\mathbb{E}([M^*]_t) = \mathbb{E}([\bar{M}^*]_t) = \mathbb{E}[\bar{X}_t] \leq \mathbb{E}[\bar{X}_1]$ is bounded, and thus M^*, \bar{M}^* are true martingales.

We are now ready to derive the limit of each term on the right in (28). The first term on the RHS of (28) converges to 0 as $T \rightarrow \infty$ as $(1 - a^T) \rightarrow 0$ and $\frac{\mu^T}{\bar{\mu}^T} \rightarrow \frac{\mu^*}{\bar{\mu}^*}$ (from assumption 3.2.1 (iii)).

Consider the second term on RHS of (28). Recall that $\mathbf{Z}_t^{*T} := \frac{\mathbf{Z}_{iT}^T}{\sqrt{\bar{\mu}}} = \int_0^t k(t-s)d\mathbf{M}_s^{*T}$. Using integration by parts we obtain

$$\mathbf{Z}_t^{*T} = \int_0^t k(t-s)d\mathbf{M}_s^{*T} = k(0)\mathbf{M}_t^{*T} + \int_0^t k'(t-s)\mathbf{M}_s^{*T}ds,$$

where k' is the derivative of k . By assumption 3.2.1 (iv) the subsequential limit Z_t^* of \mathbf{Z}_t^{*T} exists and satisfies

$$Z_t^* = \int_0^t k(t-s)dM_s^* = k(0)M_t^* + \int_0^t k'(t-s)M_s^*ds,$$

which is continuous since k' , M^* are continuous. Thus, the second term on the right in (28) converges to 0.

Using Lemma 4.3 in [15] we have for the fourth term on the right in (28)

$$F^T(t) := T \int_0^t T(1 - a^T)\psi^T(Ts)ds \rightarrow \frac{1}{c_1} F^{\tilde{\alpha}, \sigma}(t), \quad \text{as } T \rightarrow \infty, \quad (30)$$

where the convergence is uniform in t . Here, $F^{\tilde{\alpha}, \sigma}(t) := \int_0^t f^{\tilde{\alpha}, \sigma}(s)ds$, and $f^{\tilde{\alpha}, \sigma}(s) := \sigma s^{\tilde{\alpha}-1} \sum_{n=0}^{\infty} \frac{(-\sigma s^{\tilde{\alpha}})^n}{\Gamma(\tilde{\alpha}n+\tilde{\alpha})}$ with the summation term in $f^{\tilde{\alpha}, \sigma}(s)$ being the Mittag-Leffler function as defined in (11). It follows that

$$\int_0^t F^T(t-s)(\mathbf{Z}_s^{*T})^2 ds \xrightarrow{T \rightarrow \infty} \int_0^t \frac{1}{c_1} F^{\tilde{\alpha}, \sigma}(t-s)(Z_s^*)^2 ds.$$

Consequently, the fourth term on the right in (28) converges to $\int_0^t \frac{1}{c_1} F^{\tilde{\alpha}, \sigma}(t-s) \left(\frac{\mu^*}{\bar{\mu}^*} + \alpha(Z_s^*)^2 \right) ds$.

It remains to analyze the third term on the right in (28). Applying integration by parts we get

$$\int_0^t \frac{F^T(t-s)}{\sqrt{\bar{\mu}^T(1-a^T)T}} d\mathbf{M}_s^{*T} = \int_0^t \frac{f^T(t-s)}{\sqrt{\bar{\mu}^T(1-a^T)T}} \mathbf{M}_s^{*T} ds. \quad (31)$$

To obtain the limit for the expression on the right in (31) we write

$$\begin{aligned} \int_0^t f^T(t-s)\mathbf{M}_s^{*T} ds &= \int_0^t \frac{1}{c_1} f^{\tilde{\alpha}, \sigma}(t-s)M_s^* ds + \int_0^t \frac{1}{c_1} f^{\tilde{\alpha}, \sigma}(t-s)(\mathbf{M}_s^{*T} - M_s^*) ds \\ &\quad + \int_0^t \left(f^T(t-s) - \frac{1}{c_1} f^{\tilde{\alpha}, \sigma}(t-s) \right) \mathbf{M}_s^{*T} ds. \end{aligned} \quad (32)$$

The second term on the right in the above equation converges uniformly to 0 almost surely by the remark above (25). Applying integration by parts to the third term on the right in (32), we get

$$\int_0^t \left(f^T(t-s) - \frac{1}{c_1} f^{\tilde{\alpha}, \sigma}(t-s) \right) \mathbf{M}_s^{*T} ds = \int_0^t \left(F^T(t-s) - \frac{1}{c_1} F^{\tilde{\alpha}, \sigma}(t-s) \right) d\mathbf{M}_s^{*T}.$$

By the Burkholder-Davis-Gundy inequality we obtain (C_i below are positive constants)

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in [0,1]} \left(\int_0^t \left(F^T(t-s) - \frac{1}{c_1} F^{\tilde{\alpha},\sigma}(t-s) \right) d\mathbf{M}_s^{*T} \right)^2 \right] \\
& \leq C_1 \mathbb{E} \left[\int_0^t \left(F^T(t-s) - \frac{1}{c_1} F^{\tilde{\alpha},\sigma}(t-s) \right)^2 d\bar{\mathbf{X}}_s^T \right] \\
& \leq C_2 \int_0^t \left(F^T(t-s) - \frac{1}{c_1} F^{\tilde{\alpha},\sigma}(t-s) \right)^2 \frac{1-a^T}{\mu^T} \mathbb{E}[\bar{\lambda}_{Ts}^T] ds \\
& \leq C_3 \int_0^t \left(F^T(t-s) - \frac{1}{c_1} F^{\tilde{\alpha},\sigma}(t-s) \right)^2 ds \rightarrow 0,
\end{aligned}$$

by (30). It follows that the third term on the right in (28) converges to $\int_0^t \frac{1}{c_1} f^{\tilde{\alpha},\sigma}(t-s) \frac{1}{\sqrt{\sigma\mu^*}} M_s^* ds$.

This completes the proof for the dynamics of \mathbf{X} in the Theorem 3.2. The proof for the dynamics of $\bar{\mathbf{X}}$ in the Theorem 3.2 follows from (29) using similar arguments and we shall omit it.

4.3 Regularity property for $\tilde{\alpha} \in (1/2, 1)$

Regularity follows along the same lines as in section 4.3 and 4.4 of [15] and so we only provide a sketch. Recall from (13) and (14) that

$$X_t = \int_0^t \frac{1}{c_1} f^{\tilde{\alpha},\sigma}(t-s) \frac{1}{\sqrt{\sigma\mu^*}} M_s^* ds + \int_0^t \frac{1}{c_1} F^{\tilde{\alpha},\sigma}(t-s) \left(\frac{\mu^*}{\mu^*} + \alpha(Z_s^*)^2 \right) ds,$$

and

$$\bar{X}_t = \int_0^t \frac{1}{c_2} f^{\tilde{\alpha},\sigma}(t-s) \frac{1}{\sqrt{\sigma\mu^*}} \bar{M}_s^* ds + \int_0^t \frac{1}{c_2} F^{\tilde{\alpha},\sigma}(t-s) \left(1 + \frac{(\alpha_1 + \alpha_2)}{2} (Z_s^*)^2 \right) ds.$$

Note that $\int_0^t Z_s^* ds$ is continuously differentiable as Z^* is continuous. Using arguments similar to those in Sections 4.3 and 4.4 of [15], we conclude that X and \bar{X} are almost surely differentiable with respective derivatives V and \bar{V} satisfying

$$V_t = \int_0^t \frac{1}{c_1} f^{\tilde{\alpha},\sigma}(t-s) \left[\frac{1}{\sqrt{\sigma\mu^*}} dM_s^* + \left(\frac{\mu^*}{\mu^*} + \alpha(Z_s^*)^2 \right) ds \right],$$

and

$$\bar{V}_t = \int_0^t \frac{1}{c_2} f^{\tilde{\alpha},\sigma}(t-s) \left[\frac{1}{\sqrt{\sigma\mu^*}} d\bar{M}_s^* + \left(1 + \frac{(\alpha_1 + \alpha_2)}{2} (Z_s^*)^2 \right) ds \right].$$

Moreover, by theorem V-3.9 in [9], there exists two Brownian motions B^1 and B^2 such that

$$M_t^* = \int_0^t \sqrt{V_s} dB_s^1, \quad \bar{M}_t^* = \int_0^t \sqrt{\bar{V}_s} dB_s^2$$

as $[M^*]_t = [\bar{M}^*]_t = \bar{X}_t$ and \bar{X} is almost surely differentiable with derivative \bar{V}_t . Consequently,

$$Z_t^* = \int_0^t k(t-s) dM_s^* = \int_0^t k(t-s) \sqrt{\bar{V}_s} dB_s^1,$$

from which it follows (see section 4.3 and section 4.4 of [15]) that V_t and \bar{V}_t are almost surely $\alpha - \frac{1}{2} - \epsilon$ Hölder regular.

4.4 Correlation between the Brownian motions

In order to compute $\langle B^1, B^2 \rangle_t$ observe that

$$\mathbf{M}_t^{*T} = \mathbf{M}_t^{*(1,T)} - \mathbf{M}_t^{*(2,T)} \text{ and } \bar{\mathbf{M}}_t^{*T} = \mathbf{M}_t^{*(1,T)} + \mathbf{M}_t^{*(2,T)}$$

where $\mathbf{M}_t^{*(i,T)} = \sqrt{\frac{1-a^T}{T\mu^T}} \mathbf{M}_{iT}^{(i,T)}$ for $i \in \{1, 2\}$. Thus $\mathbf{M}_t^{*T} \bar{\mathbf{M}}_t^{*T} = (\mathbf{M}_t^{*(1,T)})^2 - (\mathbf{M}_t^{*(2,T)})^2$ from which it follows that $\mathbf{M}_t^{*T} \bar{\mathbf{M}}_t^{*T} - \Lambda_t^T$ is a martingale and hence $\langle \mathbf{M}_t^{*T}, \bar{\mathbf{M}}_t^{*T} \rangle_t = \Lambda_t^T$. Moreover, by Theorem 3.2 we have

$$\mathbf{M}_t^{*T} \rightarrow M_t^* = \int_0^t \sqrt{\bar{V}_s} dB_s^1, \quad \bar{\mathbf{M}}_t^{*T} \rightarrow \bar{M}_t^* = \int_0^t \sqrt{\bar{V}_s} dB_s^2 \text{ and } \Lambda_t^T \rightarrow \int_0^t V_s ds.$$

It follows that

$$\langle M^*, \bar{M}^* \rangle_t = \int_0^t \bar{V}_s d\langle B^1, B^2 \rangle_s \text{ and } \langle M^*, \bar{M}^* \rangle_t = \int_0^t V_s ds.$$

Comparing the two expressions for $\langle M^*, \bar{M}^* \rangle_t$ we conclude that

$$\langle B^1, B^2 \rangle_t = \frac{V_t}{\bar{V}_t}.$$

This completes the proof of Theorem 3.2. ■

5 Appendix: Uniqueness of limiting process for the Stable Regime

We will prove the uniqueness of the limit in Theorem 3.1 using the Banach fixed point theorem for operators (see [27], page-4). We will prove uniqueness for V_t . Uniqueness for \bar{V}_t can be proved similarly.

5.1 Uniqueness for V_t :

$$V_t = \mu + \int_0^t \phi(t-s) V_s ds + \alpha \left(\int_0^t k(t-s) V_s ds \right)^2, \quad t \in [0, 1]. \quad (33)$$

We define an operator $T : S \rightarrow S$ as

$$T[g](t) = \mu + \int_0^t \phi(t-s) g(s) ds + \alpha \left(\int_0^t k(t-s) g(s) ds \right)^2,$$

where $S = \{g \in \mathcal{C}[0, 1] \mid \|g - v_0\|_{sup} \leq \eta\}$, $v_0 = V(0) = \mu$ and η to be determined later. We wish to choose η appropriately so that T satisfies the following two conditions required for applying the Banach fixed point theorem:

- i. T maps S to S ,
- ii. T is a contraction.

Step-1: To find conditions on $\|\phi\|_1$, $\|k\|_2$, and η so that T maps S to S . For any $g \in S$ we have

$$\begin{aligned} |T[g](t) - v_0(t)| &= \left| \int_0^t \phi(t-s)g(s)ds + \alpha \left(\int_0^t k(t-s)g(s)ds \right)^2 \right| \\ &\leq \|g\|_{sup} \int_0^1 \phi(t-s)ds + \alpha \|g\|_{sup}^2 \left(\int_0^t k(t-s)ds \right)^2 \\ &\leq \|g\|_{sup} \|\phi\|_1 + \alpha \|g\|_{sup}^2 \|k\|_2^2 \end{aligned}$$

Thus we have $\|T[g] - v_0\|_{sup} \leq \|g\|_{sup} \|\phi\|_1 + \alpha \|g\|_{sup}^2 \|k\|_2^2$. As $g \in S$ we have $\|g\|_{sup} \leq \eta + |\mu|$ and thus $T[g] \in S$ provided

$$(\eta + |\mu|) \|\phi\|_1 + \alpha (\eta + |\mu|)^2 \|k\|_2^2 \leq \eta. \quad (34)$$

Step-2: To find conditions on $\|\phi\|_1$, $\|k\|_2$, and η so that T is a contraction. For $w, u \in S$ we have

$$\begin{aligned} |T[w](t) - T[u](t)| &\leq \left| \int_0^t \phi(t-s)(w(s) - u(s))ds \right| + \alpha \left(\left(\int_0^t k(t-s)w(s)ds \right)^2 - \left(\int_0^t k(t-s)u(s)ds \right)^2 \right) \\ &\leq \|w - u\|_{sup} \|\phi\|_1 + \alpha \left(\left| \int_0^t k(t-s)(w(s) - u(s))ds \right| \right) \left(\left| \int_0^t k(t-s)(w(s) + u(s))ds \right| \right) \\ &\leq \|w - u\|_{sup} \left(\|\phi\|_1 + \alpha \|k\|_2^2 \|u + w\|_{sup} \right) \\ &\leq \|w - u\|_{sup} \left(\|\phi\|_1 + 2\alpha \|k\|_2^2 (\eta + |\mu|) \right) \end{aligned}$$

Thus T is a contraction if

$$\|\phi\|_1 + 2\alpha \|k\|_2^2 (\eta + |\mu|) < 1. \quad (35)$$

It follows from (34) and (35) that for T to satisfy the conditions of Banach fixed point theorem it is enough to have

$$\|\phi\|_1 + 2\alpha \|k\|_2^2 (\eta + |\mu|) < \frac{\eta}{\eta + |\mu|}. \quad (36)$$

This proves the uniqueness for the process V_t .

5.2 Uniqueness for \bar{V}_t :

$$\bar{V}_t = \bar{\mu} + \int_0^t \bar{\phi}(t-s) \bar{V}_s ds + \frac{(\alpha_1 + \alpha_2)}{2} \left(\int_0^t k(t-s) \bar{V}_s ds \right)^2, \text{ for } t \in [0, 1].$$

Uniqueness for process \bar{V}_t follows along similar lines as above if there exists a positive constant $\bar{\eta}$ such that

$$\|\phi\|_1 + (\alpha_1 + \alpha_2) \|k^2\|_1 (\bar{\eta} + \bar{\mu}) < \frac{\bar{\eta}}{\bar{\eta} + \bar{\mu}}.$$

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