

# Convergence of Heavy-Tailed Hawkes Processes and the Microstructure of Rough Volatility\*

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## Abstract

We establish the weak convergence of the intensity of a nearly-unstable Hawkes process with heavy-tailed kernel. Our result is used to derive a scaling limit for a financial market model where orders to buy or sell an asset arrive according to a Hawkes process with power-law kernel. After suitable rescaling the price-volatility process converges weakly to a rough Heston model. Our convergence result is stronger than previously established ones that have either focused on light-tailed kernels or the convergence of integrated volatility process. The key is to establish the tightness of the family of rescaled volatility processes. This is achieved by introducing a new methods to establish the  $C$ -tightness of càdlàg processes based on the classical Kolmogorov-Chentsov tightness criterion for continuous processes.

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## 1 Introduction and overview

First introduced by Hawkes in [34, 35] to model cross-dependencies between earthquakes and their aftershocks, Hawkes processes have long become a powerful tool to model a variety of phenomena in the sciences, humanities, economics and finance.

A Hawkes process is a random point process  $\{N(t) : t \geq 0\}$  that models self-exciting arrivals of random events. In such settings, events arrive at random points in time  $\tau_1 < \tau_2 < \tau_3 < \dots$  according to an *intensity* process  $\{V(t) : t \geq 0\}$  that is usually of the form

$$V(t) := \mu(t) + \sum_{0 < \tau_i < t} \phi(t - \tau_i) = \mu(t) + \int_{(0,t)} \phi(t - s) N(ds), \quad t \geq 0, \quad (1.1)$$

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where the *immigration density*  $\mu(\cdot)$  captures the arrival of exogenous events and the *kernel*  $\phi(\cdot)$  captures the self-exciting impact of past events on the arrivals of future events.

Applications of Hawkes processes in finance include intraday transaction dynamics [9, 15], asset price and limit order book dynamics [4, 38], financial contagion [2, 30, 47] and - in particular - stochastic volatility modeling [19, 40, 44, 45, 55]. We consider a stochastic volatility model where orders to buy or sell an asset arrive according to a Hawkes process with a *heavy-tailed kernel* of the form

$$\phi(t) = \alpha\sigma \cdot (1 + \sigma \cdot t)^{-\alpha-1} \quad \text{for some constants } \alpha \in (1/2, 1) \text{ and } \sigma > 0$$

and prove the weak convergence of a sequence of suitably rescaled volatility process to a fractional diffusion and the joint convergence of the price-volatility process to a rough Heston model.

Our model is strongly inspired by the ones studied in [19, 45, 55] but our convergence results are much more refined. While the aforementioned works focused on the *integrated volatility* processes we establish a weak convergence result for the volatility process itself. This provides a strong microstructure foundation for a class of rough volatility models.

Microstructure models of financial markets have been extensively studied in the financial mathematics and economics literature in the last decades. This literature often provides economic foundations for the use of specific market models by linking important model assumptions and features to investor and/or order arrival characteristics. As pointed out by O'Hara in her influential book *Market Microstructure Theory* [52], it was Garman's 1976 paper [27] that inaugurated the explicit study of market microstructure. He argued that "market agents can be treated as a statistical ensemble [and that] their market activities [can be] depicted as the stochastic generation of market orders according to a Poisson process". In this point of view, one models right away the aggregate order flow rather than characterizing agents' investment decisions as solutions to individual utility maximization problems. Garman's approach was later extended beyond the benchmark case of Poisson dynamics by many authors including [21, 22, 37] to provide microstructure foundation for the emergence of financial bubbles and, more recently, microstructure foundations of stochastic volatility models [19, 40, 44, 45].

## 1.1 Hawkes process and (rough) volatility

Hawkes processes with light-tailed kernels have been used in Horst and Xu [40] to establish scaling limits for a class of continuous-time stochastic volatility models with self-exciting jump dynamics. Many of the existing jump diffusion stochastic volatility models including the classical Heston model [36], the Heston model with jumps [7, 8, 53], the OU-type volatility model [6], the multi-factor model with self-exciting volatility spikes [8] and the alpha Heston model [46] were obtained as scaling limits under different scaling regimes.

Nearly unstable Hawkes processes with light-tailed kernels were first analyzed by Jaisson and Rosenbaum [44]. They proved the weak convergence of the rescaled intensity to a Feller diffusion - also known as CIR-model in finance - and the convergence of the rescaled point process to the integrated diffusion; their result was extended to multi-variate processes in [58]. Under a heavy-tailed condition on the kernel, Jaisson and Rosenbaum [45] later considered the weak convergence of the rescaled point process to the integral of a rough fractional diffusion; a corresponding convergence result for the characteristic function of the rescaled Hawkes process has been considered in [20]. Analogous scaling limits in the multivariate case were established in [19, 55].

The aforementioned works provide microstructural foundations for large classes of stochastic volatility models, including the standard Heston model. The Heston model assumes that the volatility process follows a Brownian semi-martingale. However, the analysis in [28] suggests that historical volatility time series are much rougher than those of Brownian martingales and that log-volatility is better modeled by a fractional Brownian motion with a Hurst parameter  $H < 1/2$ .

This observation spurred substantial research on the properties and the microstructural foundations of rough volatility models. For instance, a rough Heston model of the form

$$dS(t) = S(t)\sqrt{V(t)}dW(t),$$

$$V(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \cdot b(\theta - V(s))ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \cdot \gamma \sqrt{V(s)}dB(s),$$

has been introduced [19, 20]. The rough Heston model is an affine Volterra process [1] and admits a semi-explicit representation of the characteristic function in terms of a fractional Riccati equation. Other popular rough volatility models include the quadratic rough Heston model [56], the (mixed) rough Bergomi model [43, 50] and the rough SABR model [26].

Although estimating the precise degree of roughness of volatility is subtle and challenging empirically (see [16, 17, 18, 25, 32] and references therein for a detailed discussion) the use of rough volatility models is by now an established paradigm for modeling equity markets. Rough volatility provides excellent fits to market data; in particular, it reproduces very well the behaviour of the implied volatility surface, in particular the at-the-money skew that it is often observed in practice as shown in, e.g. [12]. At the same time, the lack of Markovianity and regularity of the sample paths leads to significant challenges for sample path simulation and option pricing. Markovian approximations and asymptotic expansions as established in, e.g. [10, 11, 23, 24] or the recently introduced rough PDE approach for local stochastic volatility models [5] provide partial remedies to these practical problems.

## 1.2 Our contribution

Our analysis is inspired by and complements the earlier work on microstructure foundations of rough volatility models by Rosenbaum and co-workers, especially [19, 45, 55]. We first introduce a family of order driven financial market models where orders to buy and sell an asset arrive according to a Hawkes process, and each order changes the logarithmic price by a random amount. In other words, our logarithmic price process is driven by a Hawkes point measure as introduced in [40]; price processes driven by multi-variate Hawkes processes are contained as a special case.

Subsequently, we consider a family of rescaled volatility/intensity processes  $\{V^{(n)}\}_{n \geq 1}$  where the intensity of order arrivals tends to infinity, the impact of an individual order on the price process tends to zero and the average number of child orders triggered by each mother order tends to one. We prove the weak convergence of the sequence  $\{V^{(n)}\}_{n \geq 1}$  in the usual Skorokhod space  $\mathbf{D}(\mathbb{R}_+; \mathbb{R})$  of all  $\mathbb{R}$ -valued càdlàg functions on  $\mathbb{R}_+$  endowed with the Skorokhod topology to a fractional diffusion.

Previous work on microstrucual foundation of rough volatility models focused on the weak convergence of the integrated volatility processes

$$\mathcal{I}_{V^{(n)}}(t) := \int_0^t V^{(n)}(s) ds, \quad t \geq 0.$$

The weak convergence of the sequence of integrated volatility processes does in general not imply the convergence of the volatility processes. Even if the sequence of integrated processes would converge, one cannot obtain the limit of the sequence  $\{V^{(n)}\}_{n \geq 1}$  by differentiating the limit of the sequence  $\{\mathcal{I}_{V^{(n)}}\}_{n \geq 1}$  unless the  $C$ -tightness of the former sequence has been established.<sup>1</sup> This calls for more refined convergence results for the volatility process that we establish in this paper.

As already argued in [44] the main challenge is to prove the  $C$ -tightness of the sequence of rescaled volatility processes; the increments of the volatility process do not satisfy the standard moment condition that is usually required to establish the  $C$ -tightness of a sequence of stochastic processes. To overcome this challenge we introduce a novel technique to verify the  $C$ -tightness of a sequence càdlàg processes based on the classical Kolmogorov-Chentsov tightness criterion for continuous processes.

A family of càdlàg stochastic processes is  $C$ -tight if it can be approximated in probability by  $C$ -tight processes. Our key observation is that a family of càdlàg processes  $\{X^{(n)}\}_{n \geq 1}$  admits a  $C$ -tight approximation by piecewise linear processes defined on the time grids

$$\{k/n^\theta : k = 0, \dots, [Tn^\theta]\} \quad \text{for some } \theta > 2,$$

if (i) the initial states satisfy a uniform moment condition; (ii) the jumps sizes at the grid points converge to zero uniformly in probability,

$$\sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h X^{(n)}(k/n^\theta)| \xrightarrow{\text{P}} 0 \quad (1.2)$$

as  $n \rightarrow \infty$ ; (iii) the increments at arbitrary time points satisfy the polynomial moment condition

$$\sup_{t \in [0,T]} \mathbf{E} \left[ |\Delta_h X^{(n)}(t)|^p \right] \leq C \cdot \sum_{i=1}^m \frac{h^{b_i}}{n^{a_i}} \quad (1.3)$$

for all  $h \in (0, 1)$ , some integer  $m$  and suitable constants  $(a_i, b_i)$ ,  $i = 1, \dots, m$ . Our non-standard moment condition resembles the classical Kolmogorov-Chentsov tightness criterion for continuous processes but it is much weaker and applies to general càdlàg processes.<sup>2</sup>

The uniform moment condition on the initial states of stochastic integral terms is not difficult to establish. Establishing the convergence of jumps to zero along the chosen sequence of grid points is more subtle. The key is to increase the number of grid-points at the correct rate, and then to apply a suitable moment estimate for stochastic integrals driven by Poisson random measures. The added difficulty in our setting is that one of the upper integral boundaries is specified by a stochastic process (the volatility process). A powerful moment estimate for such integrals has recently been established in [59]. Applying this same result again, we then prove that the discontinuities of the stochastic integral term satisfies the non-standard moment condition.

With the  $C$ -tightness of the sequence  $\{V^{(n)}\}_{n \geq 1}$  in hand, we then proceed to prove the weak convergence of the rescaled volatility processes. An application of Skorokhod's representation theorem shows that the weak convergence of the rescaled volatility processes implies the weak convergence of the sequence  $\{\mathcal{I}_{V^{(n)}}\}_{n \geq 1}$  - as pointed out above, the converse is *not* true in general. We then utilize

<sup>1</sup>A tight sequence of processes is called  $C$ -tight if any weak accumulation point is continuous; see Definition 3.25 in [42, p.351].

<sup>2</sup>One way to prove the Kolmogorov-Chentsov tightness condition is by approximation on dyadic time grids. We work with coarser time yet very specific grids  $\{t_i^{(n)}\}_{i=1}^{M^{(n)}}$ .

the weak convergence of the integrated processes along a subsequence to prove the uniqueness of the weak accumulation points of the sequence  $\{V^{(n)}\}_{n \geq 1}$  and hence the convergence of the rescaled volatility processes.

We strongly emphasize that our analysis utilizes the convergence of the integrated volatility process only to identify the weak limits of the sequence  $\{V^{(n)}\}_{n \geq 1}$ . Furthermore, our approach strongly hinges on the  $C$ -tightness of the sequence of the rescaled volatility processes. Without the  $C$ -tightness of the volatility processes we can neither prove the convergence of the integrated processes, nor utilize the limit of the integrated processes to identify the limiting volatility process.

Having established the weak convergence of the volatility process, the joint convergence of the rescaled price-volatility process to a rough Heston-type model is then easily obtained. As a byproduct we also obtain the weak convergence of the rescaled Hawkes process.

The remainder of this paper is organized as follows. In Section 2, we introduce our benchmark model financial market model and the state the main results of this paper. All proofs are given in Section 3.

**Notation.** We denote by  $[x]$  the integer part of the real number  $x \in \mathbb{R}$  and put  $\mathbb{R}_+ = [0, \infty)$ .

For two real-valued functions  $f, g$  on  $\mathbb{R}_+$ , we define their convolution  $f * g$  by

$$f * g(t) := \int_0^t f(t-s)g(s)ds = \int_0^t f(s)g(t-s)ds, \quad t \geq 0,$$

and write  $f^{*n}$  for the  $n$ -th convolution of  $f$ . For any  $p \in (0, \infty]$ , we denote by  $L^p(\mathbb{R}; \mathbb{R})$  the space of measurable functions  $f$  on  $\mathbb{R}$  the space of all functions  $f$  on  $\mathbb{R}$  that satisfy  $\|f\|_{L^p}^p := \int_{\mathbb{R}} |f(s)|^p ds < \infty$  and by  $L_{\text{loc}}^p(\mathbb{R}; \mathbb{R})$  the space of all functions  $f$  on  $\mathbb{R}$  such that  $\int_{|s| \leq T} |f(s)|^p ds < \infty$  for any  $T \geq 0$ .

Almost sure convergence, convergence in distribution and convergence in probability is denoted by  $\xrightarrow{\text{a.s.}}$ ,  $\xrightarrow{\text{d}}$  and  $\xrightarrow{\text{P}}$ , respectively. We write  $\stackrel{\text{a.s.}}{=}$ ,  $\stackrel{\text{d}}{=}$  and  $\stackrel{\text{P}}{=}$  to denote almost sure equality, equality in distribution and equality in probability.

Throughout this paper, we denote by  $C$  a generic constant that may vary from line to line.

## 2 Heavy-tailed Hawkes market model

In this section we introduce a benchmark asset price model for which we drive a scaling limit in a later section. As pointed out above, the model is similar to the one studied in [19, 45, 55] but we derive more refined convergence results.

We assume throughout that all random variables and stochastic processes are defined on a common probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  endowed with a filtration  $\{\mathcal{F}_t : t \geq 0\}$  that satisfies the usual hypotheses. The convergence concept for stochastic processes we use will be weak convergence in the space  $\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$  of all  $\mathbb{R}^d$ -valued continuous functions on  $\mathbb{R}_+$  endowed with the uniform topology or in the space  $\mathbf{D}(\mathbb{R}_+; \mathbb{R}^d)$  of all  $\mathbb{R}^d$ -valued càdlàg functions on  $\mathbb{R}_+$  endowed with the Skorokhod topology; see [13, 42].

### 2.1 The benchmark model

We consider an order-driven model where asset prices are driven by incoming orders to buy or sell the asset. The order arrivals times are described by an increasing sequence of  $(\mathcal{F}_t)$ -adaptable

random times  $\{\tau_k\}_{k \geq 1}$ . The impact of each order on the price is described by an independent sequence of i.i.d.  $\mathbb{R}$ -valued random variables  $\{\xi_k\}_{k \geq 1}$  with distribution  $\nu(du)$ . In terms of these sequences we define the random point measure

$$N(ds, du) := \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau_k \in ds, \xi_k \in du\}}$$

on  $(0, \infty) \times \mathbb{R}$  and assume that the logarithmic price process  $\{P(t) : t \geq 0\}$  satisfies the dynamics

$$P(t) = P(0) + \sum_{\tau_k \leq t} \xi_k = P(0) + \int_0^t \int_{\mathbb{R}} u N(ds, du), \quad (2.1)$$

where  $P(0)$  is the initial price at time zero.

**Remark 2.1** *The special case  $\nu(du) = \frac{1}{2} \cdot \delta_{-1}(du) + \frac{1}{2} \cdot \delta_1(du)$  corresponds to the special case where each order increases or decreases the log price by one tick with equal probability. Here  $\delta_x(du)$  is the Dirac measure at point  $x$ .*

We assume that  $N(ds, du)$  is a *marked Hawkes point measure*. More precisely, we assume that the embedded point process

$$N(t) := N((0, t], \mathbb{R}), \quad t \geq 0,$$

is a Hawkes process with intensity process

$$V(t) := \Lambda(t) + \mu + \sum_{k=1}^{N(t)} \zeta \cdot \phi(t - \tau_k) = \Lambda(t) + \mu + \int_0^t \zeta \cdot \phi(t - s) dN(s), \quad t \geq 0. \quad (2.2)$$

Explicit constructions of this process have been provided in [39] using Poisson random measures on  $(0, \infty)^3$  and in [58] using Crump-Mode-Jagers branching processes with immigration. The function  $\Lambda \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R}_+)$  represents the combined impact of all the events that arrived prior to time zero on future arrivals; the positive constant intensity  $\mu$  describes the arrival rate of “exogenous” orders; the kernel  $\phi$  specifies the self-exciting impact of past order arrivals on future arrivals; the positive constant  $\zeta$  measures the impact of each child order on the overall order arrival dynamics. We call the vectors

$$(\Lambda, \mu, \zeta, \phi, \nu) \quad \text{and} \quad (\Lambda, \mu, \zeta, \phi)$$

the *characteristic* of the point measure  $N(ds, du)$  and the embedded Hawkes process  $\{N(t) : t \geq 0\}$ , respectively.

Accounting for the empirically well-documented long-range dependencies in order arrivals we consider a power-law kernel of the form

$$\phi(t) = \alpha \sigma \cdot (1 + \sigma \cdot t)^{-\alpha-1} \quad \text{for some constants } \alpha \in (1/2, 1) \text{ and } \sigma > 0. \quad (2.3)$$

The function  $\phi$  is a probability density function on  $\mathbb{R}_+$  with tail-distribution

$$\bar{\Phi}(t) := \int_t^{\infty} \phi(s) ds = (1 + \sigma \cdot t)^{-\alpha}, \quad t \geq 0. \quad (2.4)$$

Associated with the kernel  $\phi$  we define the *resolvent*  $R$  by the unique solution of the resolvent equation

$$R(t) = \zeta \cdot \phi(t) + \zeta \cdot \phi * R(t), \quad t \geq 0. \quad (2.5)$$

The resolvent plays an important role when analyzing the dynamics of the volatility process.

**Remark 2.2** *In our benchmark model each order triggers an average number*

$$\zeta \cdot \|\phi\|_{L_1} = \zeta$$

*of child orders. Following [19, 45, 55] we assume that  $\zeta < 1$  and then consider a family of rescaled models where the number of child orders tends to one.*

To give an alternative and more convenient representation of the benchmark model, we recall that Theorem 7.4 in [41, p.93] provides sufficient conditions under which a random point measure such as our Hawkes point measure can be represented in terms of a Poisson random measure. Specifically, the theorem states that given a measurable space  $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$  and a random point measure  $N_p(dt, dx)$  on  $(0, \infty) \times \mathbf{X}$  with intensity  $dt q(t, dx)$  where  $q(t, dx)$  is a progressive measure-valued process on  $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ , then if there exists a  $\sigma$ -finite measure  $m(dz)$  on a standard measurable space  $(\mathbf{Z}, \mathcal{B}(\mathbf{Z}))$  and a predictable process  $\theta(t, z) : \mathbb{R}_+ \times \mathbf{Z} \mapsto \mathbf{X}$  such that

$$m(\{z : \theta(t, z) \in E\}) \stackrel{\text{a.s.}}{=} q(t, E), \quad E \in \mathcal{B}(\mathbf{X}),$$

then on an extension of the original probability space we can define a time-homogeneous Poisson random measure  $N_q(dt, dz)$  on  $(0, \infty) \times \mathbf{Z}$  with intensity  $dt m(dz)$  such that

$$N_p((0, t], E) = \int_0^t \int_{\mathbf{Z}} \mathbf{1}_{\{\theta(s, z) \in E\}} N_q(ds, dz), \quad E \in \mathcal{B}(\mathbf{X}).$$

Since our marked Hawkes point measure  $N(ds, du)$  has intensity  $V(s-)ds\nu(du)$ , applying this theorem with  $\mathbf{X} = \mathbb{R}$ ,  $\mathbf{Z} = \mathbb{R} \times \mathbb{R}_+$ ,  $x = x$ ,  $z = (u, z)$  and

$$m(du, dz) = \nu(du)dz, \quad q(t, dx) = V(t-) \nu(dx), \quad \theta(t, (u, z)) = u \cdot \mathbf{1}_{\{0 < z \leq V(t-)\}},$$

we see that on an extension of the original probability space we can define a time-homogeneous Poisson random measure  $N_0(ds, du, dz)$  on  $(0, \infty) \times \mathbb{R} \times \mathbb{R}_+$  with intensity measure  $ds\nu(du)dz$  such that the price-volatility process can be represented as

$$\begin{aligned} P(t) &= P(0) + \int_0^t \int_{\mathbb{R}} \int_0^{V(s-)} u N_0(ds, du, dz), \quad t \geq 0, \\ V(t) &= \Lambda(t) + \mu + \int_0^t \int_0^{V(s-)} \zeta \cdot \phi(t-s) N_1(ds, dz), \quad t \geq 0, \end{aligned} \quad (2.6)$$

where  $N_1(ds, dz) := N_0(ds, \mathbb{R}, dz)$ . It is well known that the compensated Poisson random measures

$$\tilde{N}_0(ds, du, dz) := N_0(ds, du, dz) - ds\nu(du)dz \quad \text{and} \quad \tilde{N}_1(ds, dz) := N_1(ds, dz) - dsdz$$

are two  $(\mathcal{F}_t)$ -martingale measures. In terms of the compensated measures and the resolvent  $R$  the price-volatility process can be represented as the unique solution to the integral equation

$$\begin{aligned} P(t) &= P(0) + \int_{\mathbb{R}} u \nu(du) \cdot \int_0^t V(s) ds + \int_0^t \int_{\mathbb{R}} \int_0^{V(s-)} u \tilde{N}_0(ds, du, dz), \quad t \geq 0, \\ V(t) &= \Lambda(t) + \Lambda * R(t) + \mu + \mu \int_0^t R(s) ds + \int_0^t \int_0^{V(s-)} R(t-s) \tilde{N}_1(ds, dz), \quad t \geq 0. \end{aligned} \quad (2.7)$$

## 2.2 Scaling limits

In what follows we introduce a sequence of rescaled models where the intensity of order arrivals tends to infinity, the impact of an individual order tends to zero and the average number of child orders tends to one. The dynamics of the price-volatility process  $(P_n, V_n)$  in  $n$ -th market model is defined in terms of an underlying marked Hawkes point measure  $N_n(ds, du)$  with characteristic

$$(\Lambda_n, \mu_n, \zeta_n, \phi, \nu_n),$$

akin to the dynamics (2.1) and (2.2). The corresponding Hawkes process that specifies the order arrival dynamics is denoted  $\{N_n(t) : t \geq 0\}$ ; the corresponding Poisson random measures are denoted by, respectively

$$N_{n,0}(ds, du, dz) \quad \text{and} \quad N_{n,1}(ds, dz).$$

### 2.2.1 The volatility process

The main challenge is to establish the weak convergence of the rescaled volatility process on which we focus in this subsection. Specifically, we consider the sequence of rescaled volatility processes defined by

$$V^{(n)}(t) := \frac{V_n(nt)}{n^{2\alpha-1}}, \quad t \geq 0.$$

Applying a change of variables to (2.7), the rescaled process satisfies

$$\begin{aligned} V^{(n)}(t) &= \frac{\Lambda_n(nt)}{n^{2\alpha-1}} + \frac{R_n * \Lambda_n(nt)}{n^{2\alpha-1}} + \frac{\mu_n}{n^{2\alpha-1}} + \frac{\mu_n}{n^{2\alpha-1}} \int_0^{nt} R_n(s) ds \\ &\quad + \int_0^t \int_0^{V^{(n)}(s-)} \frac{R_n(n(t-s))}{n^{2\alpha-1}} \tilde{N}_{n,1}(n \cdot ds, n^{2\alpha-1} \cdot dz), \quad t \geq 0, \end{aligned} \quad (2.8)$$

where  $R_n$  is the resolvent defined as in (2.5) with  $\zeta = \zeta_n$ , and

$$\tilde{N}_{n,1}(n \cdot ds, n^{2\alpha-1} \cdot dz) := N_{n,1}(n \cdot ds, n^{2\alpha-1} \cdot dz) - n^{2\alpha} \cdot ds \cdot dz.$$

We assume throughout that the model parameters  $(\Lambda_n, \mu_n, \zeta_n)$  satisfy the following condition. In particular, we assume that the expected number of child orders increases to one and that the impact of orders that arrived prior to time zero is slowly decaying.

**Assumption 2.3** *Assume that the following hold.*

- (1) *On average, each order triggers less than one child order, that is,  $\zeta_n < 1$  for each  $n \geq 1$ .*
- (2) *The function  $\Lambda_n$  satisfies  $\Lambda_n(t) = V_{n,0} \cdot \bar{\Phi}(t)$  with  $V_{n,0} \in \mathbb{R}_+$  for each  $n \geq 1$  and  $t \geq 0$ .*
- (3) *There exist three constants  $V_*(0) \in \mathbb{R}$ ,  $a \geq 0$  and  $b > 0$  such that as  $n \rightarrow \infty$ ,*

$$\frac{V_{n,0}}{n^{2\alpha-1}} \rightarrow V_*(0), \quad \frac{\mu_n}{n^{\alpha-1}} \rightarrow a, \quad n^\alpha(1 - \zeta_n) \rightarrow b. \quad (2.9)$$

**Remark 2.4** Let us briefly comment on our initial state condition. It is well known that a standard Hawkes process  $N$  defined on the whole real line with intensity

$$\lambda(t) = \mu + \int_{-\infty}^t \phi(t-s)dN(s) = \mu + \int_{-\infty}^0 \phi(t-s)dN(s) + \int_0^t \phi(t-s)dN(s), \quad t \in \mathbb{R}$$

is stationary if each event triggers less than one child event on average, i.e.,  $\|\phi\|_{L^1} < 1$ . The first integral on the right side of the second equality is the impact of all past events prior to time zero at time  $t > 0$  with mean impact

$$\begin{aligned} \mathbf{E}\left[\int_{-\infty}^0 \phi(t-s)dN(s)\right] &= \frac{\mu}{1 - \|\phi\|_{L^1}} \int_{-\infty}^0 \phi(t-s)ds \\ &= \frac{\mu}{1 - \|\phi\|_{L^1}} \int_t^\infty \phi(s)ds \\ &= \frac{\mu}{1 - \|\phi\|_{L^1}} \cdot \bar{\Phi}(t). \end{aligned}$$

Here  $\frac{\mu}{1 - \|\phi\|_{L^1}}$  is the mean residual impact of all events that happened prior to time zero at time zero. It decreases according to the function  $\bar{\Phi}$ . It is hence reasonable to assume that conditioned on the residual impact  $V_{n,0}$ , the impact of the events that happened prior to time zero on the arrival of future events at any time  $t > 0$  equals

$$V_{n,0} \cdot \int_{-\infty}^0 \phi(t-s)ds = V_{n,0} \cdot \int_t^\infty \phi(s)ds = V_{n,0} \cdot \bar{\Phi}(t).$$

This explains our choice of  $\Lambda_n$ . It is worth noting that our initial condition is different from that in [20, Definition 2.3] where for technical reasons the authors had to assume that

$$V_{n,0} \cdot \left(\bar{\Phi}(t) - n^{\alpha-1} \int_0^t \phi(s)ds\right).$$

We are now ready to state our first main result. It states that our sequence of rescaled volatility processes converges weakly to a unique limiting process; the proof is given in Section 3.6.

To formulate our result we denote by  $F^{\alpha,\gamma}$  and  $f^{\alpha,\gamma}$  the Mittag-Leffler probability distribution and density function<sup>3</sup> respectively with parameters  $(\alpha, \gamma)$ , where

$$\gamma := \frac{b\sigma^\alpha}{\Gamma(1-\alpha)}$$

and  $\Gamma(\cdot)$  is the gamma function.

**Theorem 2.5** Under Assumption 2.3, any accumulation point  $\{V_*(t); t \geq 0\}$  of rescaled processes  $\{V^{(n)}\}_{n \geq 1}$  in  $\mathbf{D}([0, \infty); \mathbb{R}_+)$  satisfies the following stochastic Volterra equation:

$$V_*(t) = V_*(0) \cdot (1 - F^{\alpha,\gamma}(t)) + \int_0^t \frac{a}{b} \cdot f^{\alpha,\gamma}(s)ds + \int_0^t f^{\alpha,\gamma}(t-s) \cdot \frac{1}{b} \sqrt{V_*(s)} dB(s), \quad t \geq 0. \quad (2.10)$$

The solution to the above equation is unique in law. In particular, the sequence of rescaled volatility processes converges in law.

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<sup>3</sup>We recall the definition and basic properties of the Mittag-Leffler function and the Mittag-Leffler density function in Appendix A.

Existence and uniqueness in law of the stochastic Volterra equation (2.10) has been established in [1]. Our contribution is the proof of convergence of the rescaled volatility process. Any solution to this equation can be equivalently rewritten in a more familiar form. The proof can be carried out as in [20]. We provide an alternative and simpler proof in Section 3 by solving the corresponding Wiener-Hopf equation.

**Theorem 2.6** *Any solution to the stochastic Volterra equation (2.10) can be equivalently represented as*

$$V_*(t) = V_*(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \cdot \gamma \left( \frac{a}{b} - V_*(s) \right) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{\gamma}{b} \sqrt{V_*(s)} dB(s), \quad t \geq 0. \quad (2.11)$$

## 2.2.2 The price process

The preceding theorem shows that the intensity of our order arrivals process converges in law to the unique weak solution of a stochastic Volterra equation. In this section we show that the convergence of the intensity process implies the weak convergence of the sequence of rescaled price processes defined by

$$P^{(n)}(t) := \frac{P_n(nt)}{n^\alpha}, \quad t \geq 0.$$

In terms of the compensated Poisson random measure  $\tilde{N}_{n,0}(ds, du, dz) := N_{n,0}(ds, du, dz) - ds\nu_n(du)dz$ , the rescaled price process  $P^{(n)}$  satisfies the stochastic integral equation

$$\begin{aligned} P^{(n)}(t) &= \frac{P_{n,0}}{n^\alpha} + n^\alpha \int_{\mathbb{R}} u \nu_n(du) \cdot \int_0^t V^{(n)}(s) ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_0^{V^{(n)}(s-)} \frac{u}{n^\alpha} \tilde{N}_{n,0}(n \cdot ds, du, n^{2\alpha-1} \cdot dz), \quad t \geq 0. \end{aligned} \quad (2.12)$$

To obtain a non-degenerate limit for the sequence  $\{P^{(n)}\}_{n \geq 1}$ , it is natural to assume that the sequence of probability measures  $\{\nu_n\}_{n \geq 1}$  satisfies the following condition.

**Assumption 2.7** *There exist four constants  $P_*(0), \sigma_p \geq 0, b_p \in \mathbb{R}$  and  $\epsilon > 0$  such that*

$$\frac{P_{n,0}}{n^\alpha} \rightarrow P_*(0), \quad n^\alpha \int_{\mathbb{R}} u \nu_n(du) \rightarrow b_p, \quad \int_{\mathbb{R}} |u|^2 \nu_n(du) \rightarrow \sigma_p^2, \quad \sup_{n \geq 1} \int_{\mathbb{R}} |u|^{2+\epsilon} \nu_n(du) < \infty.$$

We are now ready to state our convergence result for the price process. The proof is given in Section 3.6 below.

**Theorem 2.8** *Let  $V_*$  be any solution to the stochastic integral equation (2.10). Under Assumption 2.7, the sequence of rescaled price processes  $\{P^{(n)}\}_{n \geq 1}$  converges in law to the unique solution of the stochastic differential equation*

$$P_*(t) = P_*(0) + \int_0^t b_p V_*(s) ds + \int_0^t \sigma_p \sqrt{V_*(s)} dW(s), \quad t \geq 0, \quad (2.13)$$

where the standard Brownian motion  $W$  is independent of  $B$ . In particular, under Assumption 2.3 and 2.7 the sequence  $\{(P^{(n)}, V^{(n)})\}_{n \geq 1}$  converges in law to the unique solution to the coupled system of stochastic equations (2.10) and (2.13).

The preceding theorem shows that the sequence of rescaled price-volatility processes converges in law to a rough Heston model driven by two *independent* Brownian motions. In the light-tailed cases considered in [40, 44] and in the heavy-tailed case/rough model considered in [19, 29] the Brownian motions driving the price and volatility process are correlated. The reason for the independence of the Brownian motions in our model is two-fold. First, in the heavy-tailed case the price and volatility processes are scaled differently in space whereas the scaling is the same in light-tailed case considered in [44]. Second, unlike in [40] in our model price changes are conditionally deterministic.

**Remark 2.9** Correlation between the driving Brownian motions can be introduced into our model by correlating price and volatility changes as in [40] and to consider a dynamics of the form

$$\begin{aligned} P_t &= P_0 + \int_0^t \int_{\mathbb{R}^2} u_1 N(ds, du), \\ V_t &= \mu_t + \int_0^t \int_{\mathbb{R}^2} u_2 \cdot \phi(t-s) N(ds, du), \end{aligned}$$

where

$$N(dt, du) := \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau_k \in dt, \xi_k \in du\}}$$

is now a Hawkes random measure on  $[0, \infty) \times \mathbb{R}^2$  with intensity  $V_t dt \nu(du)$ . Here  $\{\tau_k\}$  describes again the arrivals of market orders and the  $\mathbb{R}^2$ -valued sequence  $\{\xi_k\}$  describes the joint changes in prices and volatilities caused by the respective order arrival. Introducing this more general dynamics results in a more cumbersome model analysis as we would have to consider the joint dynamics of prices and volatilities right from the start instead of analyzing the processes separately as we do in what follows. At the same time, introducing correlated movements would not result in any new mathematical challenges. As our focus is on the convergence of the volatility process we prefer to work within the more convenient setting of independent driving noises.

### 3 Proofs

In this section we prove the weak convergence of the rescaled models to a fractional Heston stochastic volatility model. The main challenge is to prove the  $C$ -tightness of the sequence of rescaled volatility processes. The rescaled volatility processes can be expressed as

$$V^{(n)}(t) = V_0^{(n)}(t) + I^{(n)}(t) + J^{(n)}(t), \quad t \geq 0, \quad (3.1)$$

where the processes on the right-hand side of the above equation are defined by

$$\begin{aligned} V_0^{(n)}(t) &:= \frac{1}{n^{2\alpha-1}} (\Lambda_n(nt) + R_n * \Lambda_n(nt)), \\ I^{(n)}(t) &:= \frac{\mu_n}{n^{2\alpha-1}} + \frac{\mu_n}{n^{2\alpha-1}} \cdot \int_0^{nt} R_n(s) ds, \\ J^{(n)}(t) &:= \int_0^t \int_0^{V^{(n)}(s-)} \frac{R_n(n(t-s))}{n^{2\alpha-1}} \tilde{N}_1^{(n)}(ds, dz), \end{aligned} \quad (3.2)$$

and where

$$\tilde{N}_1^{(n)}(ds, dz) := \tilde{N}_{n,1}(n \cdot ds, n^{2\alpha-1} \cdot dz)$$

is a compensated Poisson random measure on  $\mathbb{R}_+^2$  with intensity  $n^{2\alpha} \cdot ds dz$ .

The  $C$ -tightness of the first two processes is obtained in Section 3.1. To obtain the tightness of the rescaled volatility processes, by Corollary 3.33(1) in [42, p.353] it then remains to establish the  $C$ -tightness of the stochastic integral processes, which is more challenging. Two steps are required to overcome the challenge.

- In Section 3.2, we introduce a novel technique to verify the  $C$ -tightness of càdlàg processes based on the classical Kolmogorov-Chentsov tightness criterion for continuous processes.
- In Section 3.3, we establish priori estimates, including uniform estimates for the resolvents  $\{R_n\}_{n \geq 1}$  and their derivatives and a uniform moment estimate of all orders for  $\{J^{(n)}\}_{n \geq 1}$  and  $\{V^{(n)}\}_{n \geq 1}$ .

We then proceed to establish the tightness of the sequence  $\{J^{(n)}\}_{n \geq 1}$  in Section 3.4 and its weak convergence in Section 3.5. Our main theorems are proved in Section 3.6. Section 3.7 contains a counterexample that shows that the weak convergence of the integrated volatility processes do in general not imply the weak convergence of the volatility processes themselves.

### 3.1 $C$ -tightness of $\{V_0^{(n)}\}_{n \geq 1}$ and $\{I^{(n)}\}_{n \geq 1}$

As a preparation, in the next proposition we provide an asymptotic result for the rescaled resolvent sequence  $\{n^{1-\alpha} R_n(n \cdot)\}_{n \geq 1}$  and their integral functions.

**Proposition 3.1** *Recall the constant  $b$  defined in Assumption 2.3(3). As  $n \rightarrow \infty$ , we have*

$$\sup_{t \geq 0} \left| \int_0^t n^{1-\alpha} R_n(ns) ds - \frac{F^{\alpha, \gamma}(t)}{b} \right| \rightarrow 0.$$

*Proof.* Similarly as in the proofs of Lemma 4.3 [45] and Lemma 4.4 in [59], the main idea is to show that the Laplace transform of the function  $R^{(n)}$  converges to the Laplace transform of the Mittag-Leffler density function  $b^{-1} \cdot f^{\alpha, \gamma}$ .

The resolvent  $R_n$  satisfies resolvent equation

$$R_n(t) = \zeta_n \cdot \phi(t) + \zeta_n \cdot \phi * R_n(t), \quad t \geq 0. \quad (3.3)$$

Taking Laplace transform on both sides of this equation shows that

$$\mathcal{L}_{R_n}(\lambda) := \int_0^\infty e^{-\lambda t} R_n(t) dt = \frac{\zeta_n \cdot \mathcal{L}_\phi(\lambda)}{1 - \zeta_n \cdot \mathcal{L}_\phi(\lambda)}, \quad \lambda \geq 0.$$

By a change of variables,

$$\int_0^\infty e^{-\lambda t} n^{1-\alpha} R_n(nt) dt = \frac{\mathcal{L}_{R_n}(\lambda/n)}{n^\alpha} = \frac{\zeta_n \cdot \mathcal{L}_\phi(\lambda/n)}{n^\alpha (1 - \zeta_n) + \zeta_n \cdot n^\alpha (1 - \mathcal{L}_\phi(\lambda/n))}, \quad \lambda \geq 0. \quad (3.4)$$

The monotonicity of  $\mathcal{L}_\phi$  and the assumption that  $\zeta_n \rightarrow 1$  induce that

$$\lim_{n \rightarrow \infty} \zeta_n \cdot \mathcal{L}_\phi(\lambda/n) = 1, \quad \lambda > 0.$$

To analyze the asymptotics of the last denominator in (3.4), we use the fact that the tail-distribution  $\bar{\Phi}$  defined in (2.4) is regularly varying with index  $-\alpha$ . An application of Corollary 8.1.7 in [14, p.334] shows that

$$1 - \mathcal{L}_\phi(z) \sim \Gamma(1 - \alpha)\bar{\Phi}(1/z), \quad \text{as } z \rightarrow 0+.$$

By (2.4), we have  $\bar{\Phi}(n/\lambda) = (1 + \sigma \cdot n/\lambda)^{-\alpha} \sim n^{-\alpha} \cdot \lambda^\alpha / \sigma^\alpha$  as  $n \rightarrow \infty$  and then conclude that

$$\lim_{n \rightarrow \infty} n^\alpha (1 - \mathcal{L}_\phi(\lambda/n)) = \frac{\Gamma(1 - \alpha)}{\sigma^\alpha} \cdot \lambda^\alpha, \quad \lambda > 0.$$

Moreover, by Assumption 2.3(3),  $n^\alpha(1 - \zeta_n) \rightarrow b$ . Plugging this into the denominator in (3.4) yields that

$$\lim_{n \rightarrow \infty} \int_0^\infty e^{-\lambda t} n^{1-\alpha} R_n(nt) dt = \frac{\sigma^\alpha}{b\sigma^\alpha + \Gamma(1 - \alpha)\lambda^\alpha} = \mathcal{L}_{b^{-1}, f^{\alpha, \gamma}}(\lambda), \quad \lambda > 0.$$

The pointwise convergence of the Laplace transform of  $n^{1-\alpha} R_n(n \cdot)$  to that of  $b^{-1} \cdot f^{\alpha, \gamma}$  yields the weak convergence of the finite measure with density function  $n^{1-\alpha} R_n(n \cdot)$  to the finite measure with density function  $b^{-1} \cdot f^{\alpha, \gamma}$ , which, together with the continuity of  $F^{\alpha, \gamma}$ , induces the desired uniform convergence of the integral function of  $n^{1-\alpha} R_n(n \cdot)$  to  $b^{-1} \cdot F^{\alpha, \gamma}$ .  $\square$

**Corollary 3.2** *As  $n \rightarrow \infty$ , we have*

$$\sup_{t \geq 0} \left| V_0^{(n)}(t) - V_*(0)(1 - F^{\alpha, \gamma}(t)) \right| \rightarrow 0 \quad \text{and} \quad \sup_{t \geq 0} \left| I^{(n)}(t) - \frac{a}{b} \cdot F^{\alpha, \gamma}(t) \right| \rightarrow 0.$$

*Proof.* The second uniform convergence result is a direct consequence of Proposition 3.1 and Condition 2.3(3). For the first one, by Assumption 2.3(2) we have uniformly in  $t \geq 0$ ,

$$V_0^{(n)}(t) = \frac{V_{n,0}}{n^{2\alpha-1}} (\bar{\Phi}(nt) + R_n * \bar{\Phi}(nt)) \sim V_*(0) \cdot (\bar{\Phi}(nt) + R_n * \bar{\Phi}(nt)),$$

as  $n \rightarrow \infty$ . It remains to prove that  $\bar{\Phi}(nt) + R_n * \bar{\Phi}(nt)$  converges uniformly to  $1 - F^{\alpha, \gamma}(t)$ . Integrating both sides of (3.3) on  $[t, \infty)$  and then dividing them by  $\zeta_n$ ,

$$\frac{1}{\zeta_n} \int_t^\infty R_n(s) ds = \bar{\Phi}(t) + \int_t^\infty \phi * R_n(s) ds = \bar{\Phi}(t) + \int_0^\infty \phi * R_n(s) ds - \int_0^t \phi * R_n(s) ds. \quad (3.5)$$

By using Fubini's lemma together with the fact that  $\|\phi\|_{L^1} = 1$ , we have

$$\int_0^\infty \phi * R_n(s) ds = \int_0^\infty R_n(s) ds \int_0^\infty \phi(r) dr = \int_0^\infty R_n(s) ds$$

and

$$\int_0^t \phi * R_n(s) ds = \int_0^t R_n(t-s) \int_0^s \phi(r) dr ds = \int_0^t R_n(t-s)(1 - \bar{\Phi}(s)) ds = \int_0^t R_n(s) ds - \bar{\Phi} * R_n(t).$$

Taking these two results back into (3.5), we see that

$$\frac{1}{\zeta_n} \int_t^\infty R_n(s) ds = \bar{\Phi}(t) + \int_t^\infty R_n(s) ds + \bar{\Phi} * R_n(t).$$

Moving the second term on the right side to the left side,

$$\overline{\Phi}(t) + \overline{\Phi} * R_n(t) = \frac{1 - \zeta_n}{\zeta_n} \int_t^\infty R_n(s) ds.$$

By a change of variables, Proposition 3.1 and Assumption 2.3(3),

$$\overline{\Phi}(nt) + \overline{\Phi} * R_n(nt) = \frac{n^\alpha(1 - \zeta_n)}{\zeta_n} \int_t^\infty n^{1-\alpha} R_n(ns) ds = \frac{n^\alpha(1 - \zeta_n)}{\zeta_n} \cdot (\mathcal{I}_{R^{(n)}}(\infty) - \mathcal{I}_{R^{(n)}}(t)),$$

which converges uniformly to  $1 - F^{\alpha,\gamma}(t)$  as  $n \rightarrow \infty$ .  $\square$

### 3.2 A $C$ -tightness criterion for càdlàg processes

In this section we provide a novel  $C$ -tightness criterion for a sequence of stochastic processes  $\{X^{(n)}\}_{n \geq 1}$  in  $\mathbf{D}(\mathbb{R}_+; \mathbb{R})$  by using the well-known Kolmogorov-Chentsov tightness criterion for continuous processes; see Problem 4.11 in [48, p.64].

The Kolmogorov-Chentsov condition states that a sequence of continuous processes  $\{X^{(n)}\}_{n \geq 1}$  is tight if for each  $T \geq 0$ , there exist constants  $C, \beta, p > 0$  and  $\kappa > 1$  such that for any  $0 \leq t_1 < t_2 \leq T$ ,

$$\sup_{n \geq 1} \mathbf{E}[|X^{(n)}(0)|^\beta] < \infty \quad \text{and} \quad \sup_{n \geq 1} \mathbf{E}[|X^{(n)}(t_1) - X^{(n)}(t_2)|^p] \leq C \cdot |t_2 - t_1|^\kappa. \quad (3.6)$$

In what follows we prove that a sequence of càdlàg processes is  $C$ -tight if it can be approximated by a sequence of  $C$ -tight processes. Our notion of approximation is asymptotic indistinguishability as introduced in the following definition.

**Definition 3.3** We say a sequence of stochastic processes  $\{Y^{(n)}\}_{n \geq 1}$  in  $\mathbf{D}(\mathbb{R}_+; \mathbb{R})$  is asymptotically indistinguishable from  $\{X^{(n)}\}_{n \geq 1}$  if for each  $T \geq 0$ , as  $n \rightarrow \infty$ ,

$$\sup_{t \in [0, T]} |X^{(n)}(t) - Y^{(n)}(t)| \xrightarrow{\text{P}} 0. \quad (3.7)$$

As a direct consequence of Theorem 3.1 in [13, p.27], the next proposition shows that tightness and  $C$ -tightness are properties shared by asymptotically indistinguishable sequences of stochastic processes.

**Proposition 3.4** The sequence  $\{X^{(n)}\}_{n \geq 1}$  is tight [resp.  $C$ -tight] if and only if one and hence every asymptotically indistinguishable process sequence  $\{Y^{(n)}\}_{n \geq 1}$  is tight [resp.  $C$ -tight].

For a given sequence of stochastic processes  $\{X_n\}_{n \geq 1}$  we now construct piece-wise linear approximations that allow us to verify the tightness of  $\{X_n\}_{n \geq 1}$  under suitable moment conditions on its jumps, akin to the moment conditions in the Kolmogorov-Chentsov tightness criterion for continuous processes.

In what follows we denote by  $\Delta_h$  the forward difference operator with step size  $h > 0$ , i.e.,

$$\Delta_h f(x) := f(x + h) - f(x), \quad x \in \mathbb{R}.$$

Moreover, we fix some  $\theta > 2$  and define the following piece-wise linear approximations on the grids  $\{\frac{k}{n^\theta} : k = 0, 1, \dots\}$ :

$$X_\theta^{(n)}(t) := X^{(n)}([n^\theta t]/n^\theta) + \Delta_{1/n^\theta} X^{(n)}([n^\theta t]/n^\theta) \cdot (n^\theta t - [n^\theta t]), \quad t \geq 0. \quad (3.8)$$

The next lemma is key to our analysis. It provides sufficient conditions on the sequence  $\{X^{(n)}\}_{n \geq 1}$  that guarantee that the above linear interpolation forms as a tight approximation.

**Lemma 3.5** *If*

$$\sup_{n \geq 1} \mathbf{E}[|X^{(n)}(0)|^\beta] < \infty$$

for some  $\beta > 0$ , then the sequence  $\{X^{(n)}\}_{n \geq 1}$  is  $C$ -tight if for any  $T \geq 0$  and some constant  $\theta > 2$ , the following two conditions hold.

$$(1) \quad \sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h X^{(n)}(k/n^\theta)| \xrightarrow{P} 0 \text{ as } n \rightarrow \infty.$$

(2) There exist some constants  $C > 0$ ,  $p \geq 1$ ,  $m \in \{1, 2, \dots\}$  and pairs  $\{(a_i, b_i)\}_{i=1,\dots,m}$  satisfying

$$a_i \geq 0, \quad b_i > 0, \quad \rho := \min_{1 \leq i \leq m} \{b_i + a_i/\theta\} > 1,$$

such that for all  $n \geq 1$  and  $h \in (0, 1)$ ,

$$\sup_{t \in [0, T]} \mathbf{E}\left[|\Delta_h X^{(n)}(t)|^p\right] \leq C \cdot \sum_{i=1}^m \frac{h^{b_i}}{n^{a_i}}. \quad (3.9)$$

*Proof.* We first prove that condition (1) guarantees that the sequence of linear interpolations  $\{X_\theta^{(n)}\}_{n \geq 1}$  defined in (3.8) is asymptotically indistinguishable from  $\{X^{(n)}\}_{n \geq 1}$ . In fact, for any  $t \in [0, T]$  we have that

$$X^{(n)}(t) - X_\theta^{(n)}(t) = (X^{(n)}(t) - X^{(n)}([n^\theta t]/n^\theta)) - \Delta_{1/n^\theta} X_\theta^{(n)}([n^\theta t]/n^\theta) \cdot (n^\theta t - [n^\theta t])$$

and then by the triangle inequality,

$$|X^{(n)}(t) - X_\theta^{(n)}(t)| \leq 2 \cdot \sup_{h \in [0, 1/n^\theta]} |\Delta_h X^{(n)}([n^\theta t]/n^\theta)|.$$

This shows that

$$\sup_{t \in [0, T]} |X^{(n)}(t) - X_\theta^{(n)}(t)| \leq 2 \cdot \sup_{k=1,\dots,[n^\theta t]} \sup_{h \in [0, 1/n^\theta]} |\Delta_h X^{(n)}(k/n^\theta)|,$$

which goes to 0 in probability as  $n \rightarrow \infty$  because of condition (1).

By Proposition 3.4, it remains to verify that the sequence  $\{X_\theta^{(n)}\}_{n \geq 1}$  satisfies the classical Kolmogorov-Chentsov tightness criterion for continuous processes. The first moment condition in (3.6) holds because

$$\sup_{n \geq 1} \mathbf{E}\left[|X_\theta^{(n)}(0)|^\beta\right] = \sup_{n \geq 1} \mathbf{E}\left[|X^{(n)}(0)|^\beta\right] < \infty.$$

To verify that condition (2) implies the second moment condition in (3.6), it is enough to prove that for some constants  $C > 0$  and  $\kappa \in (1, \rho)$  such that uniformly in  $h \in (0, 1)$  and  $t \in [0, T]$ ,

$$\sup_{n \geq 1} \mathbf{E} \left[ |\Delta_h X_\theta^{(n)}(t)|^p \right] \leq C \cdot h^\kappa.$$

To this end, we fix  $t \in [0, T]$  and  $h \in (0, 1)$  and distinguish the following two cases.

**Case 1.** If  $h \leq n^{-\theta}$ , there exists an integer  $k \geq 0$  such that  $t \in [\frac{k}{n^\theta}, \frac{k+1}{n^\theta}]$  and hence  $t + h \in [\frac{k}{n^\theta}, \frac{k+2}{n^\theta}]$ . Thus,

$$|\Delta_h X_\theta^{(n)}(t)| \leq hn^\theta \cdot \left( |\Delta_{1/n^\theta} X^{(n)}(k/n^\theta)| + |\Delta_{1/n^\theta} X^{(n)}((k+1)/n^\theta)| \right).$$

Raising both sides of this inequality to the  $p$  power, then using the power mean inequality and finally taking expectations, we have

$$\mathbf{E} \left[ |\Delta_h X_\theta^{(n)}(t)|^p \right] \leq 2^p \cdot (hn^\theta)^p \cdot \left( \mathbf{E} \left[ |\Delta_{1/n^\theta} X^{(n)}(k/n^\theta)|^p \right] + \mathbf{E} \left[ |\Delta_{1/n^\theta} X^{(n)}((k+1)/n^\theta)|^p \right] \right).$$

By condition (3.9) and then the two facts that  $h \leq n^{-\theta}$  and  $\theta\kappa - a_i - \theta b_i \leq \theta(\kappa - \rho) < 0$ , there exists a constant  $C > 0$  that is independent of  $t, h$  and  $n$  such that

$$\mathbf{E} \left[ |\Delta_h X_\theta^{(n)}(t)|^p \right] \leq C \sum_{i=1}^m n^{\theta p - a_i - \theta b_i} \cdot h^{p-\kappa} \cdot h^\kappa \leq C \sum_{i=1}^m n^{\theta\kappa - a_i - \theta b_i} \cdot h^\kappa \leq C \cdot h^\kappa.$$

**Case 2.** If  $h > n^{-\theta}$ , there exist two integers  $k < l$  such that  $t \in [\frac{k}{n^\theta}, \frac{k+1}{n^\theta}]$  and  $t + h \in [\frac{l}{n^\theta}, \frac{l+1}{n^\theta}]$ . By the triangle inequality, we can bound  $|\Delta_h X_\theta^{(n)}(t)|$  by

$$|X_\theta^{(n)}(t+h) - X_\theta^{(n)}(l/n^\theta)| + |X_\theta^{(n)}((k+1)/n^\theta) - X_\theta^{(n)}(t)| + |X_\theta^{(n)}(l/n^\theta) - X_\theta^{(n)}((k+1)/n^\theta)|.$$

Applying our result in **Case 1** to the first two terms, there exists a constant  $C > 0$  that is independent of  $t, h$  and  $n$  such that

$$\mathbf{E} \left[ |X_\theta^{(n)}(t+h) - X_\theta^{(n)}(l/n^\theta)|^p \right] + \mathbf{E} \left[ |X_\theta^{(n)}((k+1)/n^\theta) - X_\theta^{(n)}(t)|^p \right] \leq C \cdot h^\kappa.$$

For the third term, by definition of  $X_\theta^{(n)}$  we have

$$|X_\theta^{(n)}(l/n^\theta) - X_\theta^{(n)}((k+1)/n^\theta)| = |X^{(n)}(l/n^\theta) - X^{(n)}((k+1)/n^\theta)|.$$

Applying first the moment estimate (3.9) to the last term, then the inequality  $h > 1/n^\theta$  and finally the fact that  $1 < \kappa < \rho$ , we again obtain that for some constant  $C > 0$  that is independent of  $t, h$  and  $n$  such that

$$\mathbf{E} \left[ |X_\theta^{(n)}(l/n^\theta) - X_\theta^{(n)}((k+1)/n^\theta)|^p \right] \leq C \cdot \sum_{i=1}^m n^{-a_i} \left( \frac{l-k-1}{n^\theta} \right)^{b_i} \leq C \cdot \sum_{i=1}^m h^{b_i + a_i/\theta} \leq C \cdot h^\kappa.$$

which, together with the preceding estimates, induces that  $\mathbf{E} \left[ |\Delta_h X_\theta^{(n)}(t)|^p \right] \leq C \cdot h^\kappa$ .  $\square$

### 3.3 A priori estimates

In this section we provide priori estimates that will be important to our subsequent analysis. We start with a uniform upper bound on the resolvent sequence  $\{R_n\}_{n \geq 1}$ .

**Proposition 3.6** *There exists a constant  $C > 0$  such that for any  $t \geq 0$ ,*

$$\sup_{n \geq 1} R_n(t) \leq C \cdot (1 + t)^{\alpha-1}. \quad (3.10)$$

*Proof.* Let  $R_\alpha$  be the resolvent associated with the kernel  $\phi$ , which is defined as the unique solution of (2.5) with  $\zeta = 1$ . The two functions  $R_n$  and  $R_\alpha$  admit the respective Neumann series expansions

$$R_n = \sum_{k=1}^{\infty} (\zeta_n \cdot \phi)^{*k} \quad \text{and} \quad R_\alpha = \sum_{k=1}^{\infty} \phi^{*k}. \quad (3.11)$$

The convergence of the first series follows from the fact that  $\zeta_n \|\phi\|_{L^1} = \zeta_n < 1$ . For the second series, let  $\phi_\beta^{*k}(t) := e^{-\beta t} \cdot \phi^{*k}(t)$  for  $t \geq 0$  and  $\beta > 0$ . Since  $\|\phi_\beta^{*k}\|_{L^1} < 1$ , we have as  $m \rightarrow \infty$  that

$$\begin{aligned} \sum_{k=1}^m \phi^{*k}(t) &= e^{\beta t} \sum_{k=1}^m e^{-\beta t} \phi^{*k}(t) = e^{\beta t} \sum_{k=1}^m \phi_\beta^{*k}(t) \\ &\rightarrow e^{\beta t} \sum_{k=1}^{\infty} \phi_\beta^{*k}(t) = \sum_{k=1}^{\infty} e^{\beta t} \phi_\beta^{*k}(t) = \sum_{k=1}^{\infty} \phi^{*k}(t), \quad t \geq 0. \end{aligned}$$

The stability condition  $\zeta_n < 1$  induces that  $R_n < R_\alpha$ . It is obvious that  $R_\alpha$  is continuous with  $R_\alpha(0) = \alpha\sigma < \infty$ . Applying Karamata's theorem (see Theorem 8.1.6 in [14, p.333]) to the Laplace transform of  $R_\alpha$  shows that as  $t \rightarrow \infty$ ,

$$R_\alpha(t) \sim C \cdot t^{\alpha-1} \quad (3.12)$$

for some constant  $C > 0$ . Hence, (3.10) holds; see Proposition 4.8 in [59] for more details.  $\square$

Additionally, in the fore-coming tightness proofs we shall also need a uniform upper bound on the derivative sequence  $\{R'_n\}_{n \geq 1}$  of the resolvents  $\{R_n\}_{n \geq 1}$ ; see the next proposition.

**Proposition 3.7** *There exists a constant  $C > 0$  such that uniformly in  $t \geq 0$ ,*

$$\sup_{n \geq 1} |R'_n(t)| \leq C \cdot (1 + t)^{\alpha-2}. \quad (3.13)$$

*Proof.* Recall (2.3). Since the kernel  $\phi$  is completely monotone, that is,

$$(-1)^k \frac{d^k}{dt^k} \phi(t) \geq 0, \quad k \in \{0, 1, 2, \dots\}, \quad t \geq 0,$$

it follows from the well-known Bernstein's theorem on completely monotone functions that it can be uniquely written as the Laplace transform of some non-negative measure  $\mu_1$ , that is,

$$\phi(t) = \int_0^\infty e^{-xt} \mu_1(dx), \quad t \geq 0.$$

Similarly, there exists a unique non-negative measure  $\mu_k$  such that the  $k$ -th order convolution of  $\phi$  can be written as

$$\phi^{*k}(t) = \int_0^\infty e^{-xt} \mu_k(dx), \quad t \geq 0.$$

Plugging these back into the two Neumann series expansions in (3.11),

$$R_n(t) = \sum_{k=1}^{\infty} \int_0^\infty |\zeta_n|^k e^{-xt} \mu_k(dx) \quad \text{and} \quad R_\alpha(t) = \sum_{k=1}^{\infty} \int_0^\infty e^{-xt} \mu_k(dx), \quad t \geq 0. \quad (3.14)$$

Since  $0 < \zeta_n < 1$ , these allow us to bound our derivatives  $R'_n$  by the derivative  $R'_\alpha$  of the resolvent  $R_\alpha$ , i.e., for any  $n \geq 1$  and  $t \geq 0$ ,

$$|R'_n(t)| = \int_0^\infty x \cdot \sum_{k=1}^{\infty} |\zeta_n|^k e^{-xt} \mu_k(dx) \leq \int_0^\infty x \cdot e^{-xt} \sum_{k=1}^{\infty} \mu_k(dx) = |R'_\alpha(t)|.$$

From (3.11), we have  $R'_\alpha = \phi' + \phi' * \sum_{k=1}^{\infty} \phi^{*k}$  and hence  $|R'_\alpha(0)| = |\phi'(0)| = \alpha\sigma(\alpha+1) < \infty$ . Finally, an application of Bernstein's theorem and (3.14) shows that  $R_\alpha$  is completely monotone and hence  $R'_\alpha$  is monotone. By the regular variation of  $R_\alpha$  at infinity (see (3.12)) and Proposition 2.5(b) in [54] we can obtain that as  $t \rightarrow \infty$ ,

$$|R'_\alpha(t)| \sim C \cdot t^{\alpha-2}$$

for some constant  $C > 0$  and hence the desired uniform upper bound follows immediately.  $\square$

The next proposition provides moment estimates for stochastic integrals driven by Poisson random measures when one of the upper integral boundaries is described by stochastic processes. It is a direct corollary of Lemma D.1 in [59] and the proof is omitted.

**Proposition 3.8** *For  $p \geq 1$  and  $T \geq 0$ , assume that  $f$  is a measurable function in  $L^2([0, T]; \mathbb{R})$ , and  $X := \{X(t) : t \geq 0\}$  is a non-negative  $(\mathcal{F}_t)$ -càdlàg process with*

$$L := \sup_{t \in [0, T]} \mathbf{E}[|X(t)|^p] < \infty.$$

*Let  $\tilde{N}_\eta(ds, dz)$  be a compensated Poisson random measure on  $\mathbb{R}_+^2$  with intensity  $\eta \cdot dsdz$  for some constant  $\eta > 0$ . Then there exists a constant  $C > 0$  that depends only on  $p$ ,  $T$  and  $L$  such that*

$$\mathbf{E}\left[\left|\int_0^T \int_0^{X(s-)} f(s) \tilde{N}_\eta(ds, dz)\right|^{2p}\right] \leq C \left(\eta \cdot \int_0^T |f(s)|^2 ds\right)^p + C \cdot \eta \cdot \int_0^T |f(s)|^{2p} ds. \quad (3.15)$$

The above proposition will be repeatedly applied to the stochastic integral in (3.2) to obtain the necessary moment estimates. In our setting the stochastic process  $\omega$  in (3.8) is hence given by the rescaled volatility process(es)  $V^{(n)}$ . To apply this proposition in the sequel, we thus need to verify that the rescaled volatility processes are  $L^p$ -bounded. This is achieved by the next lemma.

**Lemma 3.9** *For any  $p \geq 0$  and  $T \geq 0$ , the stochastic integral term  $J^{(n)}$  defined in (3.2) satisfy*

$$\sup_{n \geq 1} \sup_{t \in [0, T]} \mathbf{E}\left[|J^{(n)}(t)|^{2p}\right] < \infty \quad \text{and} \quad \sup_{n \geq 1} \sup_{t \in [0, T]} \mathbf{E}\left[|V^{(n)}(t)|^{2p}\right] < \infty.$$

*Proof.* By Jensen's inequality it suffices to consider the cases  $2p = 2^k$  with  $k \in \{0, 1, 2, \dots\}$ . We proceed by induction, starting with the case  $k = 0$ .

By Corollary 3.2, the first two functions on the right side of (3.1) are uniformly bounded, i.e.,

$$\sup_{n \geq 1} \sup_{t \geq 0} V_0^{(n)}(t) \leq \sup_{n \geq 1} V_0^{(n)}(0) < \infty \quad \text{and} \quad \sup_{n \geq 1} \sup_{t \geq 0} I^{(n)}(t) \leq \sup_{n \geq 1} I^{(n)}(\infty) < \infty. \quad (3.16)$$

To establish the uniform upper bound for  $\{V^{(n)}\}_{n \geq 1}$ , it hence suffices to consider the stochastic integral processes  $\{J^{(n)}\}_{n \geq 1}$ . We note that these are not (local) martingales because of the time-dependence of the integrands.

Nonetheless, we can still compute their expectations by introducing an auxiliary stochastic process. Specifically, for any  $t \in [0, T]$  and  $n \geq 1$ , we define the process

$$J_t^{(n)}(r) := \int_0^r \int_0^{V^{(n)}(s-)} \frac{\zeta_n}{n^{2\alpha-1}} \cdot R_n(n(t-s)) \tilde{N}_1^{(n)}(ds, dz), \quad r \in [0, t].$$

The process  $J_t^{(n)}$  is a martingale on  $[0, t]$ ; by construction  $J_t^{(n)}(0) = 0$  and  $J_t^{(n)}(t) = J^{(n)}(t)$ . As a result,

$$\mathbf{E}\left[|J^{(n)}(t)|^{2^k}\right] = \mathbf{E}\left[|J_t^{(n)}(t)|^{2^k}\right]. \quad (3.17)$$

When  $k = 0$ , we take expectations on both sides of (2.8) and then use (3.16) to get that

$$\mathbf{E}\left[|V^{(n)}(t)|\right] = \mathbf{E}\left[V^{(n)}(t)\right] = V_0^{(n)}(t) + I^{(n)}(t) \quad \text{and hence} \quad \sup_{n \geq 1} \sup_{t \geq 0} \mathbf{E}\left[|V^{(n)}(t)|\right] \leq C, \quad (3.18)$$

for some constant  $C > 0$  independent of  $n$  and  $t$ . By using the Burkholder-Davis-Gundy inequality and then Jensen's inequality to (3.17), there exists a constant  $C > 0$  that is independent of  $n$  and  $t$  such that

$$\begin{aligned} \mathbf{E}\left[|J^{(n)}(t)|\right] &= \mathbf{E}\left[|J_t^{(n)}(t)|\right] \leq C \mathbf{E}\left[\left|\int_0^t \int_0^{V^{(n)}(s-)} \frac{|R_n(n(t-s))|^2}{n^{4\alpha-2}} N_1^{(n)}(ds, dz)\right|^{1/2}\right] \\ &\leq C \left(\mathbf{E}\left[\int_0^t \int_0^{V^{(n)}(s-)} \frac{|R_n(n(t-s))|^2}{n^{4\alpha-2}} N_1^{(n)}(ds, dz)\right]\right)^{1/2} \\ &= C \left(\int_0^t \mathbf{E}[V^{(n)}(s)] \frac{|R_n(n(t-s))|^2}{n^{2\alpha-2}} ds\right)^{1/2}. \end{aligned}$$

Applying (3.18) and Proposition 3.6 to the last term, we have

$$\mathbf{E}\left[|J^{(n)}(t)|\right] \leq C \left(\int_0^t \frac{(1+ns)^{2\alpha-2}}{n^{2\alpha-2}} ds\right)^{1/2} \leq C \left(\int_0^t s^{2\alpha-2} ds\right)^{1/2} \leq C,$$

uniformly in  $n \geq 1$  and  $t \in [0, T]$ . This proves the desired result for  $k = 0$ . To proceed we assume that the desired inequality holds for  $2p = 2^k$  for some  $k \geq 0$  and prove that it holds for  $2p = 2^{k+1}$ .

First, from the power mean inequality and (3.16), we see that there exists a constant  $C > 0$  such that for all  $t \geq 0$  and  $n \geq 1$ ,

$$\mathbf{E}\left[|V^{(n)}(t)|^{2p}\right] \leq C \cdot \left(1 + \mathbf{E}\left[|J^{(n)}(t)|^{2p}\right]\right).$$

Furthermore, the induction hypotheses allows us to apply Proposition 3.8 to  $\mathbf{E}[|J^{(n)}(t)|^{2p}]$  to obtain that uniformly in  $n \geq 1$  and  $t \in [0, T]$ ,

$$\mathbf{E}\left[|J^{(n)}(t)|^{2p}\right] \leq C \cdot \left(n^{2\alpha} \int_0^t \left(\frac{R_n(ns)}{n^{2\alpha-1}}\right)^2 ds\right)^p + C \cdot n^{2\alpha} \int_0^t \left(\frac{R_n(ns)}{n^{2\alpha-1}}\right)^{2p} ds.$$

- Using that  $\alpha > 1/2$  and the inequality (3.10) the first term on the right side of the above inequality can be estimated as follows:

$$\left(n^{2\alpha} \int_0^t \left(\frac{R_n(ns)}{n^{2\alpha-1}}\right)^2 ds\right)^p \leq C \left(\int_0^t \frac{(1+ns)^{2\alpha-2}}{n^{2\alpha-2}} ds\right)^p \leq C \cdot (1+t)^{2\alpha p}.$$

- Using (3.10) again the second term also can be estimated as follows:

$$\begin{aligned} n^{2\alpha} \int_0^t \left(\frac{R_n(ns)}{n^{2\alpha-1}}\right)^{2p} ds &\leq C \frac{n^{2\alpha+2p(\alpha-1)}}{n^{2p(2\alpha-1)}} \int_0^t (1/n+s)^{2p(\alpha-1)} ds \\ &= \frac{C \cdot n^{2\alpha(1-p)}}{2p(\alpha-1)+1} \left((1/n+t)^{2p(\alpha-1)+1} - (1/n)^{2p(\alpha-1)+1}\right). \end{aligned}$$

We now distinguish two cases:

- If  $2p(\alpha-1) + 1 < 0$ , then using that  $\alpha > 1/2$  and  $p \geq 1$  (induction hypothesis) we see that,

$$n^{2\alpha} \int_0^t \left(\frac{R_n(ns)}{n^{2\alpha-1}}\right)^{2p} ds \leq C \cdot n^{2\alpha(1-p)} \cdot (1/n)^{2p(\alpha-1)+1} = C \cdot n^{-(2p-1)(2\alpha-1)} \leq C.$$

- If  $2p(\alpha-1) + 1 > 0$ , then again using that  $p = 2^k \geq 1$ ,

$$n^{2\alpha} \int_0^t \left(\frac{R_n(ns)}{n^{2\alpha-1}}\right)^{2p} ds \leq C n^{2\alpha(1-p)} \cdot (1/n+t)^{2p(\alpha-1)+1} \leq C(1+t)^{2\alpha p}.$$

Combing the above estimates shows that

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \mathbf{E}\left[|V^{(n)}(t)|^{2p}\right] \leq \sup_{t \in [0, T]} C(1+t)^{2\alpha p} \leq C(1+T)^{2\alpha p} < \infty.$$

□

### 3.4 $C$ -tightness of $\{J^{(n)}\}_{n \geq 1}$

In this section we establish the  $C$ -tightness of the càdlàg process sequence  $\{J^{(n)}\}_{n \geq 1}$ . Because of the existence of jumps, the moments of the increments  $\Delta_h J^{(n)}$  cannot be uniformly bounded by a term of the form  $C \cdot h^{1+\kappa}$ , which prevents us from applying the standard Kolmogorov-Chentsov criterion. Instead, we prove that the sequence meets the conditions presented in Lemma 3.5.

Notice that  $J^{(n)}(0) = 0$  a.s., it suffices to identify that the two conditions in Lemma 3.5 are satisfied. To prove that the first condition holds it will be convenient to represent our integral processes in terms of the derivative of the resolvent function. The resolvents and their first derivatives are smooth and bounded functions, and satisfy

$$R_n(n(t-s)) = R_n(0) + \int_s^t n \cdot R'_n(n(r-s)) dr, \quad n \geq 1, \quad t \geq s \geq 0.$$

As a result of (3.3) and so  $R_n(0) = \alpha\sigma\zeta_n$ , we can write  $J^{(n)}$  as

$$J^{(n)}(t) = J_1^{(n)}(t) + J_2^{(n)}(t), \quad t \geq 0, \quad (3.19)$$

with

$$J_1^{(n)}(t) := \int_0^t \int_0^{V^{(n)}(s-)} \frac{\alpha\sigma\zeta_n}{n^{2\alpha-1}} \tilde{N}_1^{(n)}(ds, dz), \quad (3.20)$$

$$J_2^{(n)}(t) := \int_0^t \int_0^s \int_0^{V^{(n)}(r-)} \frac{R'_n(n(s-r))}{n^{2\alpha-2}} \tilde{N}_1^{(n)}(dr, dz) ds. \quad (3.21)$$

The following lemma proves that condition (1) in Lemma 3.5 is satisfied and hence that the sequence of linear interpolations  $\{J_\theta^{(n)}\}_{n \geq 1}$  defined as in (3.8) is asymptotically indistinguishable from  $\{J^{(n)}\}_{n \geq 1}$ .

**Lemma 3.10** *For any  $\theta > 2$ , the process sequence  $\{J^{(n)}\}_{n \geq 1}$  satisfies as  $n \rightarrow \infty$ ,*

$$\sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h J^{(n)}(k/n^\theta)| \xrightarrow{\text{P}} 0.$$

*Proof.* In terms of the processes  $J_i^{(n)}$ ,  $i = 1, 2$ , introduced in (3.20)-(3.21) it thus suffices to prove that

$$\sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h J_i^{(n)}(k/n^\theta)| \xrightarrow{\text{P}} 0.$$

**Case  $i = 1$ .** We start with the first process that does not involve the derivative of the resolvent. For any  $\eta > 0$ , by Chebyshev's inequality with

$$p > \frac{2\alpha + \theta}{2\alpha - 1} \geq \theta + 2, \quad (3.22)$$

we have for any  $n \geq 1$ ,

$$\mathbf{P}\left(\sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h J_1^{(n)}(k/n^\theta)| \geq \eta\right) \leq \sum_{k=0}^{[Tn^\theta]} \frac{1}{\eta^{4p}} \mathbf{E}\left[\sup_{h \in [0,1/n^\theta]} |\Delta_h J_1^{(n)}(k/n^\theta)|^{4p}\right] \quad (3.23)$$

Applying the Burkholder-Davis-Gundy inequality, together with the fact that  $\zeta_n \sim 1$ , to the last expectation gives that for some constant  $C > 0$  independent of  $n$  and  $k$ ,

$$\mathbf{E}\left[\sup_{h \in [0,1/n^\theta]} |\Delta_h J_1^{(n)}(k/n^\theta)|^{4p}\right] \leq C \cdot \mathbf{E}\left[\left|\int_{k/n^\theta}^{(k+1)/n^\theta} \int_0^{V^{(n)}(s-)} \frac{\tilde{N}_1^{(n)}(ds, dz)}{n^{4\alpha-2}}\right|^{2p}\right].$$

Expressing the Poisson random measure  $\tilde{N}_1^{(n)}(ds, dz)$  as the sum of the compensated measure  $\tilde{N}_1^{(n)}(ds, dz)$  and the compensator  $n^{2\alpha} \cdot ds dz$ , and then using the power mean inequality, the expectation on the right-hand side of the above inequality can be bounded by

$$2^{2p} \cdot \mathbf{E}\left[\left|\int_{k/n^\theta}^{(k+1)/n^\theta} \int_0^{V^{(n)}(s-)} \frac{\tilde{N}_1^{(n)}(ds, dz)}{n^{4\alpha-2}}\right|^{2p}\right] + 2^{2p} \cdot \mathbf{E}\left[\left|\int_{k/n^\theta}^{(k+1)/n^\theta} \frac{V^{(n)}(s)}{n^{2\alpha-2}} ds\right|^{2p}\right].$$

Using Proposition 3.8 and Lemma 3.9 to the first expectation, it can be bounded by

$$C \cdot \int_{k/n^\theta}^{(k+1)/n^\theta} \frac{n^{2\alpha} ds}{n^{4p(2\alpha-1)}} + C \cdot \left( \int_{k/n^\theta}^{(k+1)/n^\theta} \frac{n^{2\alpha} ds}{n^{8\alpha-4}} \right)^p \leq C \cdot \left( n^{2\alpha-\theta-4p(2\alpha-1)} + n^{p(4-6\alpha-\theta)} \right),$$

for some constant  $C > 0$  independent of  $n$  and  $k$ . Applying Hölder's inequality to the second expectation and then using Lemma 3.9, it can be uniformly bounded by

$$n^{-2p(2\alpha-2)-\theta(2p-1)} \int_{k/n^\theta}^{(k+1)/n^\theta} \mathbf{E} \left[ |V^{(n)}(s)|^{2p} \right] ds \leq C \cdot n^{-2p(\theta+2\alpha-2)}.$$

Combining all preceding estimates together and then taking them back into (3.23),

$$\mathbf{P} \left( \sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h J_1^{(n)}(k/n^\theta)| \geq \eta \right) \leq \frac{C}{\eta^{4p}} \left( n^{2\alpha-4p(2\alpha-1)} + n^{p(4-6\alpha-\theta)+\theta} + n^{\theta-2p(\theta+2\alpha-2)} \right).$$

By using the inequality (3.22) as well as the facts that  $\alpha \in (1/2, 1]$  and  $\theta > 2$ , we can show that all powers in the last term are negative, thereby showing that as  $n \rightarrow \infty$ ,

$$\mathbf{P} \left( \sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h J_1^{(n)}(k/n^\theta)| \geq \eta \right) \rightarrow 0.$$

**Case  $i = 2$ .** Let us now focus on the processes  $\{J_2^{(n)}\}_{n \geq 1}$ . For each  $\eta > 0$ , we use Chebyshev's inequality again to get

$$\mathbf{P} \left( \sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h J_2^{(n)}(k/n^\theta)| \geq \eta \right) \leq \frac{1}{\eta} \cdot \mathbf{E} \left[ \sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h J_2^{(n)}(k/n^\theta)| \right] \quad (3.24)$$

Since  $\zeta_n \sim 1$ , by (3.21) we have that

$$|\Delta_h J_2^{(n)}(t)| \leq h \cdot \sup_{r \in [0,T]} \left| \int_0^r \int_0^{V^{(n)}(s-)} \frac{R'_n(n(r-s))}{n^{2\alpha-2}} \tilde{N}_1^{(n)}(ds, dz) \right|,$$

uniformly in  $t \in [0, T]$  and  $n \geq 1$ , which yields that

$$\begin{aligned} \sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h J_2^{(n)}(k/n^\theta)| &\leq \frac{C}{n^\theta} \cdot \sup_{t \in [0,T]} \left| \int_0^t \int_0^{V^{(n)}(s-)} \frac{R'_n(n(t-s))}{n^{2\alpha-2}} \tilde{N}_1^{(n)}(ds, dz) \right| \\ &\leq \frac{C}{n^\theta} \cdot \sup_{t \in [0,T]} \int_0^t \int_0^{V^{(n)}(s-)} \frac{|R'_n(n(t-s))|}{n^{2\alpha-2}} N_1^{(n)}(ds, dz) \\ &\quad + \frac{C}{n^{\theta-2}} \cdot \sup_{t \in [0,T]} \int_0^t V^{(n)}(s-) |R'_n(n(t-s))| ds. \end{aligned}$$

Using Proposition 3.7 and the fact that  $\alpha < 1$  shows that  $\sup_{n \geq 1} \sup_{t \geq 0} |R'_n(nt)| < \infty$  and hence that

$$\sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h J_2^{(n)}(k/n^\theta)| \leq \frac{C}{n^\theta} \int_0^T \int_0^{V^{(n)}(s-)} \frac{N_1^{(n)}(ds, dz)}{n^{2\alpha-2}} + \frac{C}{n^{\theta-2}} \int_0^T V^{(n)}(s) ds.$$

Taking expectations on both sides of this inequality and then using Lemma 3.9,

$$\mathbf{E} \left[ \sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h J_2^{(n)}(k/n^\theta)| \right] \leq C \cdot n^{2-\theta}.$$

Taking this back into (3.24) and using the fact that  $\theta > 2$ , we have as  $n \rightarrow \infty$ ,

$$\mathbf{P}\left(\sup_{k=0,1,\dots,[Tn^\theta]} \sup_{h \in [0,1/n^\theta]} |\Delta_h J_2^{(n)}(k/n^\theta)| \geq \eta\right) \rightarrow 0.$$

□

The next lemma proves that the sequence  $\{J^{(n)}\}_{n \geq 1}$  satisfies condition (2) in Lemma 3.5. For the lemma to hold, it is important to have the locally uniform moment estimates of all orders established in Lemma 3.9.

**Lemma 3.11** *For each  $T \geq 0$ ,  $p > \frac{1}{2}$  and  $\delta \in (0, 1)$  there exists a constant  $C > 0$  such that for any  $h \in (0, 1)$  and  $n \geq 1$  the following moment estimate holds:*

$$\sup_{t \in [0,T]} \mathbf{E}[\Delta_h J^{(n)}(t)]^{2p} \leq C \left( h^{p(2\alpha-1)} + \frac{h^{\frac{\delta}{2-\alpha}}}{n^{2p(2\alpha-1)+\frac{1-\alpha}{2-\alpha}-2\alpha}} \right). \quad (3.25)$$

Moreover, we can choose the two constants  $p$  and  $\delta$  such that condition (2) in Lemma 3.5 is satisfied.

*Proof.* For any  $n \geq 1$  and  $t \in [0, T]$ , we can express the increment  $\Delta_h J^{(n)}(t)$  as the summation of the following two terms

$$\begin{aligned} \epsilon_1^{(n)}(t, h) &:= \int_t^{t+h} \int_0^{V^{(n)}(s-)} \frac{1}{n^{2\alpha-1}} R_n(n(t+h-s)) \tilde{N}_1^{(n)}(ds, dz), \\ \epsilon_2^{(n)}(t, h) &:= \int_0^t \int_0^{V^{(n)}(s-)} \frac{1}{n^{2\alpha-1}} (R_n(n(t+h-s)) - R_n(n(t-s))) \tilde{N}_1^{(n)}(ds, dz). \end{aligned}$$

By the power mean inequality, we have

$$\sup_{t \in [0,T]} \mathbf{E}[\Delta_h J^{(n)}(t)]^{2p} \leq 2^{2p} \cdot \sup_{t \in [0,T]} \mathbf{E}[\epsilon_1^{(n)}(t, h)]^{2p} + 2^{2p} \cdot \sup_{t \in [0,T]} \mathbf{E}[\epsilon_2^{(n)}(t, h)]^{2p}.$$

In the next two steps, we establish the desired uniform upper bound for the two supremums on the right of this inequality, i.e.,

$$\sup_{t \in [0,T]} \mathbf{E}[\epsilon_i^{(n)}(t, h)]^{2p} \leq C \left( h^{p(2\alpha-1)} + \frac{h^{\delta/(2-\alpha)}}{n^{2p(2\alpha-1)+\frac{1-\alpha}{2-\alpha}-2\alpha}} \right), \quad i = 1, 2. \quad (3.26)$$

**Case  $\epsilon_1^{(n)}$ .** The moment estimate of  $\epsilon_1^{(n)}(t, h)$  can be obtained as follows. Applications of Proposition 3.8 together with Lemma 3.9 and the fact that  $\zeta_n \sim 1$  yield that for some constant  $C > 0$  that is independent of  $n$ ,  $t$  and  $h$  such that

$$\begin{aligned} \mathbf{E}[\epsilon_1^{(n)}(t, h)]^{2p} &\leq C \left| \int_0^h n^{2\alpha} \cdot \left( \frac{R_n(ns)}{n^{2\alpha-1}} \right)^2 ds \right|^p + C \left| \int_0^h n^{2\alpha} \cdot \left( \frac{R_n(ns)}{n^{2\alpha-1}} \right)^{2p} ds \right| \\ &\leq C \left| \int_0^h |n^{1-\alpha} R_n(ns)|^2 ds \right|^p + C \cdot n^{2\alpha-2p(2\alpha-1)} \int_0^h (R_n(ns))^{2p} ds. \end{aligned}$$

The upper bound on the resolvent  $R_n$  established in (3.10) and the fact that  $\alpha \in (1/2, 1)$  yields that uniformly in  $n \geq 1$  and  $h \in (0, 1)$ ,

$$\left| \int_0^h |n^{1-\alpha} R_n(ns)|^2 ds \right|^p \leq C \cdot \left| \int_0^h s^{2\alpha-2} ds \right|^p \leq C h^{p(2\alpha-1)}.$$

To bound the second term we use the facts that  $\frac{1}{2-\alpha} < 2p$  and  $\alpha - 1 < 0$  from which we obtain that  $|R_n(ns)|^{2p} \leq C \cdot (1 + ns)^{\frac{\alpha-1}{2-\alpha}} \leq C \cdot (ns)^{\frac{\alpha-1}{2-\alpha}}$  uniformly in  $s \geq 0$  and  $n \geq 1$ , which yields that

$$\frac{1}{n^{2p(2\alpha-1)-2\alpha}} \int_0^h |R_n(ns)|^{2p} ds \leq C \cdot \frac{h^{\frac{1}{2-\alpha}}}{n^{2p(2\alpha-1)-2\alpha+\frac{1-\alpha}{2-\alpha}}}.$$

Therefore, the inequality (3.26) holds uniformly in  $n \geq 1$  and  $h \in (0, 1)$ .

**Case  $\epsilon_1^{(n)}$ .** We now establish the uniform upper-bound (3.26) for  $i = 2$ . The equality

$$R_n(n(t+h-s)) - R_n(n(t-s)) = \int_{t-s}^{t+h-s} nR'_n(nr) dr$$

allows us to write  $\epsilon_2^{(n)}(t, h)$  into

$$\epsilon_2^{(n)}(t, h) = \int_0^t \int_0^{V^{(n)}(s-)} \left( \int_{t-s}^{t+h-s} \frac{R'_n(nr)}{n^{2\alpha-2}} dr \right) \tilde{N}_1^{(n)}(ds, dz).$$

Using Proposition 3.8 and the fact that  $\zeta_n \sim 1$  again to  $\mathbf{E}[|\epsilon_2^{(n)}(t, h)|^{2p}]$ , it can be bounded uniformly in  $n \geq 1$ ,  $t \in [0, T]$  and  $h \in (0, 1)$  by

$$C \left| \int_0^t \left( \int_s^{s+h} n^{2-\alpha} R'_n(nr) dr \right)^2 ds \right|^p + \frac{C}{n^{2p(2\alpha-1)-2\alpha}} \int_0^t \left| \int_s^{s+h} n \cdot R'_n(nr) dr \right|^{2p} ds. \quad (3.27)$$

It follows from (3.13) that  $n^{2-\alpha} R'_n(nr) \leq C \cdot r^{\alpha-2}$  uniformly in  $r > 0$ . Hence, by the power mean inequality the first term can be bounded by

$$\begin{aligned} C \left| \int_0^t \left( \int_s^{s+h} r^{\alpha-2} dr \right)^2 ds \right|^p &\leq C \left| \int_0^h \left( \int_s^{s+h} r^{\alpha-2} dr \right)^2 ds \right|^p + C \left| \int_h^t \left( \int_s^{s+h} r^{\alpha-2} dr \right)^2 ds \right|^p \\ &\leq C \left| \int_0^h s^{2\alpha-2} ds \right|^p + C \left| \int_h^t |s^{\alpha-2} \cdot h|^2 ds \right|^p \leq C \cdot h^{p(2\alpha-1)}. \end{aligned}$$

Here the constant  $C > 0$  is independent of  $t$  and  $h$ . We turn to consider the second term in (3.27). By Proposition 3.6,

$$\sup_{n \geq 1} \sup_{s \geq 0} \left| \int_s^{s+h} n \cdot R'_n(nr) dr \right| \leq \sup_{n \geq 1} R_n(0) < \infty,$$

which, together with the facts that  $\frac{1}{2-\alpha} < 2p$  and  $\alpha - 1 < 0$ , yields that uniformly in  $n \geq 1$ ,  $t \in [0, T]$  and  $h \in (0, 1)$ ,

$$\begin{aligned} \int_0^t \left| \int_s^{s+h} n \cdot R'_n(nr) dr \right|^{2p} ds &\leq C \cdot \int_0^t \left| \int_s^{s+h} n \cdot R'_n(nr) dr \right|^{\frac{1}{2-\alpha}} ds \\ &= C \cdot n^{-\frac{1-\alpha}{2-\alpha}} \int_0^t \left| \int_s^{s+h} n^{2-\alpha} \cdot R'_n(nr) dr \right|^{\frac{1}{2-\alpha}} ds. \end{aligned}$$

By the facts that  $\alpha - 1 < 0$  and  $n^{2-\alpha} R'_n(nr) \leq C \cdot r^{\alpha-2}$  uniformly in  $r > 0$ , the last integral can be bounded uniformly in  $n \geq 1$ ,  $t \in [0, T]$  and  $h \in (0, 1)$  by

$$\int_0^t \left| \int_s^{s+h} r^{\alpha-2} dr \right|^{\frac{1}{2-\alpha}} ds = \int_0^h \left| \int_s^{s+h} r^{\alpha-2} dr \right|^{\frac{1}{2-\alpha}} ds + \int_h^t \left| \int_s^{s+h} r^{\alpha-2} dr \right|^{\frac{1}{2-\alpha}} ds$$

$$\begin{aligned} &\leq C \int_0^h s^{\frac{\alpha-1}{2-\alpha}} ds + C \cdot h^{\frac{1}{2-\alpha}} \int_h^t |s^{\alpha-2}|^{\frac{1}{2-\alpha}} ds \\ &\leq C \cdot h^{\frac{1}{2-\alpha}} (1 + \log h) \leq C \cdot h^{\frac{\delta}{2-\alpha}}, \end{aligned}$$

for any  $\delta \in (0, 1)$ . Taking all preceding estimates back into (3.27) induces that the inequality (3.26) holds uniformly in  $n \geq 1$  and  $h \in (0, 1)$ .

Finally, we show that condition (2) in Lemma 3.5 can be satisfied by choosing

$$p > \frac{1+\theta}{2\alpha-1} \geq \frac{1-\theta(\alpha+\delta-2)}{2(2-\alpha)(2\alpha-1)} + \frac{1}{2}.$$

Indeed, the moment estimate (3.25) is of the form (3.9) with

$$m = 2, \quad a_1 = 0, \quad b_1 = p(2\alpha-1), \quad a_2 = 2p(2\alpha-1) + \frac{1-\alpha}{2-\alpha} - 2\alpha, \quad b_2 = \frac{\delta}{2-\alpha}.$$

It is obvious that  $b_1 + a_1/\theta = b_1 > 1$ . Moreover,

$$\theta b_2 + a_2 - \theta = \frac{\theta \cdot \delta}{2-\alpha} + 2p(2\alpha-1) + \frac{1-\alpha}{2-\alpha} - 2\alpha - \theta = \frac{\theta(\alpha+\delta-2)-1}{2-\alpha} + (2p-1)(2\alpha-1) > 0,$$

which yields that  $b_2 + a_2/\theta > 1$ .  $\square$

**Corollary 3.12** *The sequence  $\{J^{(n)}\}_{n \geq 1}$  is  $C$ -tight.*

*Proof.* The preceding two lemmas show that the sequence  $\{J^{(n)}\}_{n \geq 1}$  in  $\mathbf{D}(\mathbb{R}_+; \mathbb{R})$  satisfies all conditions in Lemma 3.5 and hence is  $C$ -tight.  $\square$

### 3.5 Convergence of $\{J^{(n)}\}_{n \geq 1}$

As the integrands in the definition of the processes  $\{J^{(n)}\}_{n \geq 1}$  depend on the time variable, we cannot establish their weak convergence as in [40] by appealing the standard convergence results established in, e.g. [49]. Instead, we identify the weak limit by identifying the weak limit the sequence of the integrated processes and then differentiate that limit. More precisely, the weak convergence of the sequence  $\{J^{(n)}\}_{n \geq 1}$  in  $\mathbf{D}(\mathbb{R}_+; \mathbb{R})$  implies the weak convergence in  $L^1(\mathbb{R}_+; \mathbb{R})$ , which in turns implies the weak convergence of the integrated processes

$$\mathcal{I}_{J^{(n)}}(t) := \int_0^t J^{(n)}(s) ds, \quad t \geq 0, \quad n \geq 1.$$

That is, as  $n \rightarrow \infty$ ,

$$J^{(n)} \xrightarrow{d} J_* \quad \text{in } \mathbf{D}(\mathbb{R}_+; \mathbb{R}) \quad \text{implies} \quad \mathcal{I}_{J^{(n)}}(t) \xrightarrow{d} \mathcal{I}_{J_*}(t) := \int_0^t J_*(s) ds \quad \text{in } \mathbf{D}(\mathbb{R}_+; \mathbb{R}). \quad (3.28)$$

Since the (unknown) limiting process  $J_*$  is continuous because of the previously established  $C$ -tightness, it can be identified by differentiating the limit of the integrated process. The following proposition confirms our intuition and identifies the weak limit of the integral processes.

**Proposition 3.13** Any accumulation point  $(V_*, J_*) \in \mathbf{C}(\mathbb{R}_+; \mathbb{R}_+ \times \mathbb{R})$  of the sequence  $\{(V^{(n)}, J^{(n)})\}_{n \geq 1}$  is a weak solution to the stochastic equation

$$J_*(t) = \int_0^t \frac{1}{b} f^{\alpha, \gamma}(t-s) \sqrt{V_*(s)} dB(s), \quad t \geq 0. \quad (3.29)$$

*Proof.* The sequences  $\{V_0^{(n)}\}_{n \geq 1}$ ,  $\{I^{(n)}\}_{n \geq 1}$  and  $\{J^{(n)}\}_{n \geq 1}$  are  $C$ -tight. By Corollary 3.33(a) in [42, p.353] and (3.1), the sequence  $\{V^{(n)}\}_{n \geq 1}$  is  $C$ -tight and hence  $\{(V^{(n)}, J^{(n)})\}_{n \geq 1}$  is also  $C$ -tight. Let  $(V_*, J_*) \in \mathbf{C}(\mathbb{R}_+; \mathbb{R}_+ \times \mathbb{R})$  be an accumulation point. W.l.o.g. we may assume that as  $n \rightarrow \infty$ ,

$$(V^{(n)}, J^{(n)}) \xrightarrow{\text{d}} (V_*, J_*) \quad \text{in } \mathbf{D}(\mathbb{R}_+; \mathbb{R}_+ \times \mathbb{R}). \quad (3.30)$$

To prove that the limit process  $(V_*, J_*)$  solves (3.29), we first identify the weak limit of the integrated processes  $\{\mathcal{I}_{J^{(n)}}\}_{n \geq 1}$  and then identify the desired limit of the integrand processes. We proceed in several steps.

**Step 1.** We first bring the integrated process  $\mathcal{I}_{J^{(n)}}$  into a more convenient form. To this end, we introduce the martingale

$$M^{(n)}(t) := \int_0^t \int_0^{V^{(n)}(s-)} n^{-\alpha} \tilde{N}_1^{(n)}(ds, dz), \quad t \geq 0, \quad (3.31)$$

in terms of which we have that

$$J^{(n)}(t) = \int_0^t n^{1-\alpha} R_n(n(t-s)) dM^{(n)}(s), \quad t \geq 0.$$

Integrating both side over  $[0, t]$  and then applying the stochastic Fubini theorem [57, Theorem 2.6],

$$\mathcal{I}_{J^{(n)}}(t) = \int_0^t \int_0^r n^{1-\alpha} R_n(n(r-s)) dM^{(n)}(s) dr = \int_0^t n^{1-\alpha} R_n(n(t-s)) M^{(n)}(s) ds. \quad (3.32)$$

**Step 2.** Next, we establish the weak convergence of the sequence  $\{M^{(n)}\}_{n \geq 1}$ . The martingale  $M^{(n)}$  has predictable quadratic variation

$$\langle M^{(n)} \rangle_t = \int_0^t V^{(n)}(s) ds, \quad t \geq 0.$$

By (3.30) and the continuous mapping theorem, we have as  $n \rightarrow \infty$

$$\langle M^{(n)} \rangle_t \xrightarrow{\text{d}} \int_0^t V_*(s) ds \quad \text{in } \mathbf{C}(\mathbb{R}_+; \mathbb{R}).$$

According to Theorem 4.13 in [42, p.358], the tightness of the predictable quadratic variation process guarantees the tightness of the sequence  $\{M^{(n)}\}_{n \geq 1}$  and hence the weak convergence along a subsequence to a martingale  $M_*$  with predictable quadratic variation

$$\langle M_* \rangle_t = \int_0^t V_*(s) ds, \quad t \geq 0.$$

A standard argument shows that  $M_*$  is a.s. continuous.<sup>4</sup> By the martingale representation theorem; see Theorem 7.1 in [41, p.84], there exists a Brownian motion  $B$  such that the continuous martingale  $M_*$  can be represented as

$$M_*(t) = \int_0^t \sqrt{V_*(s)} dB(s), \quad t \geq 0. \quad (3.33)$$

**Step 3.** We now prove the weak convergence of  $\{\mathcal{I}_{J^{(n)}}\}_{n \geq 1}$ . Based all preceding results, we have

$$(V^{(n)}, J^{(n)}, M^{(n)}) \xrightarrow{d} (V_*, J_*, M_*), \quad \text{in } \mathbf{D}(\mathbb{R}_+; \mathbb{R}_+ \times \mathbb{R}^2).$$

By the Skorokhod representation theorem [41, Theorem 2.7], we may without loss of generality assume that the preceding limit holds almost surely. By the continuity of  $(V_*, J_*, M_*)$ , we have for any  $T \geq 0$ ,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left( |V^{(n)}(t) - V_*(t)| + |J^{(n)}(t) - J_*(t)| + |M^{(n)}(t) - M_*(t)| \right) \xrightarrow{\text{a.s.}} 0. \quad (3.34)$$

With the preceding almost sure convergence, we are ready to prove that  $\mathcal{I}_{J^{(n)}}$  converges almost surely in  $\mathbf{C}(\mathbb{R}_+; \mathbb{R})$  to

$$\xi(t) := \int_0^t \frac{1}{b} f^{\alpha, \gamma}(t-s) M_*(s) ds, \quad t \geq 0. \quad (3.35)$$

By the triangle inequality, we have  $|\mathcal{I}_{J^{(n)}}(t) - \xi(t)| \leq |\epsilon_1^{(n)}(t)| + |\epsilon_2^{(n)}(t)|$  with

$$\begin{aligned} \epsilon_1^{(n)}(t) &:= \int_0^t \left( n^{1-\alpha} R_n(n(t-s)) - \frac{1}{b} f^{\alpha, \gamma}(t-s) \right) M_*(s) ds, \\ \epsilon_2^{(n)}(t) &:= \int_0^t n^{1-\alpha} R_n(n(t-s)) |M^{(n)}(s) - M_*(s)| ds. \end{aligned}$$

By using Corollary 3.2 and then (3.34) as well as the fact that  $\zeta_n \rightarrow 1$  to  $\epsilon_2^{(n)}$ , we have as  $n \rightarrow \infty$ ,

$$\sup_{t \in [0, T]} |\epsilon_2^{(n)}(t)| \leq \int_0^T n^{1-\alpha} R_n(ns) ds \cdot \sup_{t \in [0, T]} |M^{(n)}(t) - M_*(t)| \xrightarrow{\text{a.s.}} 0.$$

We now prove the tightness of  $\{\epsilon_1^{(n)}\}_{n \geq 1}$ . By the change of variables,

$$\epsilon_1^{(n)}(t) = \int_0^t \left( n^{1-\alpha} R_n(ns) - \frac{1}{b} f^{\alpha, \gamma}(s) \right) M_*(t-s) ds.$$

By first using the monotone convergence theorem, then the continuous mapping theorem and finally Lemma 3.9, we obtain that

$$\mathbf{E}[V_*(t)] = \lim_{K \rightarrow \infty} \mathbf{E}[V_*(t) \wedge K] = \lim_{K \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E}[V^{(n)}(t) \wedge K] \leq \lim_{n \rightarrow \infty} \mathbf{E}[V^{(n)}(t)] \leq C,$$

uniformly in  $t \in [0, T+2]$ . This along with the Burkholder-Davis-Gundy inequality yields that

$$\mathbf{E} \left[ \sup_{t \in [0, T+2]} |M_*(t)|^2 \right] \leq C \int_0^{T+2} \mathbf{E}[V^{(n)}(s)] ds < \infty.$$

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<sup>4</sup>We give the complete argument in a slightly more general situation in the proof of Theorem 2.8 below.

For any stopping time  $\tau \leq T$  and  $h \in (0, 1)$ , we have

$$\begin{aligned} \mathbf{E}[|\epsilon_1^{(n)}(\tau + \delta) - \epsilon_1^{(n)}(\tau)|] &\leq \mathbf{E}\left[\int_0^{\tau+h} \left|n^{1-\alpha}R_n(ns) - \frac{1}{b}f^{\alpha,\gamma}(s)\right| |M_*(\tau + h - s) - M_*(\tau - s)| ds\right] \\ &\leq \int_0^{T+1} \left(n^{1-\alpha}R_n(ns) + \frac{1}{b}f^{\alpha,\gamma}(s)\right) \mathbf{E}[|M_*(\tau + h - s) - M_*(\tau - s)|] ds \\ &\leq \int_0^\infty \left(n^{1-\alpha}R_n(ns) + \frac{1}{b}f^{\alpha,\gamma}(s)\right) ds \cdot \mathbf{E}\left[\sup_{t \in [0, T+1]} |M_*(t + h) - M_*(t)|\right] \\ &\leq C \cdot \mathbf{E}\left[\sup_{t \in [0, T+1]} |M_*(t + h) - M_*(t)|\right]. \end{aligned}$$

For some constant  $C > 0$  independent of  $n$  and  $\tau$ . Moreover, the continuity of  $M_*$  induces that as  $h \rightarrow 0+$ ,

$$\sup_{t \in [0, T+1]} |M_*(t + h) - M_*(t)| \xrightarrow{\text{a.s.}} 0 \quad \text{and hence} \quad \mathbf{E}\left[\sup_{t \in [0, T+1]} |M_*(t + h) - M_*(t)|\right] \rightarrow 0.$$

The desired tightness of  $\{\epsilon_1^{(n)}\}_{n \geq 1}$  thus follows from Aldous's criterion; see [3]. To prove that  $\epsilon_1^{(n)}(t) \xrightarrow{\text{a.s.}} 0$  as  $n \rightarrow \infty$  for any  $t \geq 0$ , we introduce two finite measures on  $\mathbb{R}_+$ :

$$m^{(n)}(ds) := n^{1-\alpha}R_n(ns)ds \quad \text{and} \quad m_*(ds) := \frac{1}{b}f^{\alpha,\gamma}(s)ds.$$

By Proposition 3.1, we have  $m^{(n)}(ds) \rightarrow m_*(ds)$  weakly and hence by the continuity of  $M_*$ ,

$$\begin{aligned} \int_0^t n^{1-\alpha}R_n(ns)M_*(t-s)ds &= \int_0^t M_*(t-s)m^{(n)}(ds) \\ &\xrightarrow{\text{a.s.}} \int_0^t M_*(t-s)m_*(ds) = \int_0^t \frac{1}{b}f^{\alpha,\gamma}(s)M_*(t-s)ds, \end{aligned}$$

which immediately yields that  $\epsilon_1^{(n)}(t) \xrightarrow{\text{a.s.}} 0$  as  $n \rightarrow \infty$  and hence  $\epsilon_1^{(n)} \rightarrow 0$  weakly in  $\mathbf{C}(\mathbb{R}_+; \mathbb{R})$ .

Based on preceding results, we conclude that  $\mathcal{I}_{J^{(n)}} \rightarrow \xi$  weakly in  $\mathbf{C}(\mathbb{R}_+; \mathbb{R})$  and also

$$\sup_{t \in [0, T]} |\mathcal{I}_{J^{(n)}}(t) - \xi(t)| \xrightarrow{\text{a.s.}} 0.$$

From this and (3.28), we conclude that  $\mathcal{I}_{J_*} \xrightarrow{\text{a.s.}} \xi$ .

**Step 4.** It remains to show that (3.29) holds. Recall the definition of  $\mathcal{I}_{J^*}$ . Similarly as in (3.32) with  $n^{1-\alpha}R_n$  and  $M^{(n)}$  replaced by  $\frac{1}{b}f^{\alpha,\gamma}$  and  $\int_0^\cdot \sqrt{V_*(s)}dB(s)$ , we also have

$$\begin{aligned} \int_0^t J_*(s)ds = \mathcal{I}_{J^*}(t) &\stackrel{\text{a.s.}}{=} \xi(t) = \int_0^t \frac{1}{b}f^{\alpha,\gamma}(t-r) \int_0^r \sqrt{V_*(s)}dB(s)dr \\ &= \int_0^t \int_0^s \frac{1}{b}f^{\alpha,\gamma}(s-r) \sqrt{V_*(r)}dB(r)ds. \end{aligned}$$

Since the integrands are continuous, the equality (3.29) follows by differentiating both sides of the preceding equality with respect to  $t$ .  $\square$

### 3.6 Proof of the main results

We are now ready to prove the convergence results stated in Section 2. The convergence of the volatility process follows from the arguments given above.

**PROOF OF THEOREM 2.5.** As a conclusion of all preceding results and the Skorokhod representation theorem, we have proved the following almost sure convergence in  $\mathbf{C}(\mathbb{R}_+; \mathbb{R}_+^3 \times \mathbb{R})$ :

$$(V^{(n)}, V_0^{(n)}, I^{(n)}, J^{(n)}) \rightarrow \left( V_*, V_*(0) \cdot (1 - F^{\alpha, \gamma}), \frac{a}{b} \cdot F^{\alpha, \gamma}, \int_0^\cdot \frac{1}{b} \cdot f^{\alpha, \gamma}(\cdot - s) \sqrt{V_*(s)} dB(s) \right). \quad (3.36)$$

Passing both sides of (3.1) to the corresponding limits, we have that

$$V_*(t) = V_*(0) \cdot (1 - F^{\alpha, \gamma}(t)) + \frac{a}{b} \cdot F^{\alpha, \gamma}(t) + \int_0^t \frac{1}{b} \cdot f^{\alpha, \gamma}(t - s) \sqrt{V_*(s)} dB(s).$$

Thus the accumulation point  $V_*$  is a weak solution of (2.10). The weak uniqueness follows directly from Theorem 6.1 in [1].  $\square$

The proof of Theorem 2.6 can be carried out as in [20]. We provide an alternative and simpler proof by solving the corresponding Wiener-Hopf equation.

**PROOF OF THEOREM 2.6.** We first recall the power function  $R^{\alpha, \gamma}$  defined in (A.1), which is the resolvent of the Mittag-leffler probability density function  $f^{\alpha, \gamma}$ ; see (A.2). The stochastic equation (2.11) can be written as

$$V_*(t) = U(t) - R^{\alpha, \gamma} * V_*(t), \quad t \geq 0. \quad (3.37)$$

with

$$U(t) := V_*(0) + \int_0^t \frac{a}{b} R^{\alpha, \gamma}(s) ds + \int_0^t R^{\alpha, \gamma}(t - s) \cdot \frac{1}{b} \sqrt{V_*(s)} dB_s.$$

In view of Proposition A.1, the equation (3.37) is equivalent to

$$V_*(t) = U(t) - f^{\alpha, \gamma} * U(t), \quad t \geq 0. \quad (3.38)$$

Hence it suffices to rewrite this equation into the form (2.10). Indeed, applying Fubini's theorem and the stochastic Fubini theorem to  $f^{\alpha, \gamma} * U(t)$  shows that

$$\begin{aligned} f^{\alpha, \gamma} * U(t) &= V_*(0) \int_0^t f^{\alpha, \gamma}(s) ds + \int_0^t f^{\alpha, \gamma}(t - r) \int_0^r \frac{a}{b} R^{\alpha, \gamma}(s) ds dr \\ &\quad + \int_0^t f^{\alpha, \gamma}(t - r) \int_0^r R^{\alpha, \gamma}(t - s) \cdot \frac{1}{b} \sqrt{V_*(s)} dB_s dr \\ &= V_*(0) F^{\alpha, \gamma} + \int_0^t \frac{a}{b} R^{\alpha, \gamma} * f^{\alpha, \gamma}(s) ds + \int_0^t R^{\alpha, \gamma} * f^{\alpha, \gamma}(t - s) \cdot \frac{1}{b} \sqrt{V_*(s)} dB_s. \end{aligned}$$

Taking this back into (3.38) and then using the equality  $f^{\alpha, \gamma} = R^{\alpha, \gamma} - f^{\alpha, \gamma} * R^{\alpha, \gamma}$  (see (A.2)) yields that

$$\int_0^t R^{\alpha, \gamma}(t - s) \cdot \frac{1}{b} \sqrt{V_*(s)} dB_s - \int_0^t R^{\alpha, \gamma} * f^{\alpha, \gamma}(t - s) \cdot \frac{1}{b} \sqrt{V_*(s)} dB_s$$

$$\begin{aligned}
&= \int_0^t (R^{\alpha,\gamma}(t-s) - R^{\alpha,\gamma} * f^{\alpha,\gamma}(t-s)) \cdot \frac{1}{b} \sqrt{V_*(s)} dB_s \\
&= \int_0^t f^{\alpha,\gamma}(t-s) \cdot \frac{1}{b} \sqrt{V_*(s)} dB_s,
\end{aligned}$$

which allows us to rewrite (3.38) as (2.10).  $\square$

HAVING ESTABLISHED THE CONVERGENCE OF THE VOLATILITY PROCESSES THE CONVERGENCE OF THE PRICE PROCESSES FOLLOWS FROM STANDARD ARGUMENTS.

PROOF OF THEOREM 2.8. Without loss of generality, we assume that the almost sure convergence in (3.36) holds. For convenience, we represent the rescaled price process  $P^{(n)}$  as follows:

$$P^{(n)}(t) = P_0^{(n)} + I_p^{(n)}(t) + J_p^{(n)}(t), \quad t \geq 0, \quad (3.39)$$

where  $P_0^{(n)} := n^{-\alpha} \cdot P_{n,0}$  and

$$\begin{aligned}
I_p^{(n)}(t) &:= n^\alpha \int_{\mathbb{R}} u \nu_n(du) \cdot \int_0^t V^{(n)}(s) ds, \\
J_p^{(n)}(t) &:= \int_0^t \int_{\mathbb{R}} \int_0^{V^{(n)}(s-)} \frac{u}{n^\alpha} \tilde{N}_{n,0}(n \cdot ds, du, n^{2\alpha-1} \cdot dz).
\end{aligned}$$

By Assumption 2.7 and (3.36), the sequence  $\{I_p^{(n)}\}_{n \geq 1}$  converges almost surely in  $\mathbf{C}(\mathbb{R}_+; \mathbb{R})$  to the process

$$b_p \int_0^t V_*(s) ds, \quad t \geq 0.$$

On the other hand, the process  $J_p^{(n)}$  is a martingale with predictable quadratic variation

$$\langle J_p^{(n)} \rangle_t = \int_{\mathbb{R}} |u|^2 \nu_n(du) \cdot \int_0^t V^{(n)}(s) ds,$$

which converges almost surely in  $\mathbf{C}(\mathbb{R}_+; \mathbb{R})$  to

$$\sigma_p^2 \int_0^t V_*(s) ds, \quad t \geq 0.$$

By [42, Theorem 4.13, p.358] this shows that sequence  $\{J_p^{(n)}\}_{n \geq 1}$  is tight. To prove the  $C$ -tightness, let  $J_p^*$  be an accumulation point. By Skorokhod's representation theorem we may without loss of generality assume that

$$J_p^{(n)} \xrightarrow{\text{a.s.}} J_p^* \quad \text{in } \mathbf{D}(\mathbb{R}_+; \mathbb{R}). \quad (3.40)$$

In view of [42, Corollary VI.2.8, p.304] this implies that, as  $n \rightarrow \infty$ ,

$$\sum_{s \leq t} |J_p^{(n)}(s) - J_p^{(n)}(s-)|^{2+\epsilon} \xrightarrow{\text{a.s.}} \sum_{s \leq t} |J_p^*(s) - J_p^*(s-)|^{2+\epsilon} \quad \text{in } \mathbf{D}(\mathbb{R}_+; \mathbb{R}).$$

For each  $t \geq 0$ , it hence follows from Fatou's lemma along with Assumption 2.7, Fubini's theorem and Lemma 3.9 that

$$\begin{aligned}
\mathbf{E} \left[ \sum_{s \leq t} |J_p^*(s) - J_p^*(s-)|^{2+\epsilon} \right] &= \mathbf{E} \left[ \liminf_{n \rightarrow \infty} \sum_{s \leq t} |J_p^{(n)}(s) - J_p^{(n)}(s-)|^{2+\epsilon} \right] \\
&\leq \liminf_{n \rightarrow \infty} \mathbf{E} \left[ \sum_{s \leq t} |J_p^{(n)}(s) - J_p^{(n)}(s-)|^{2+\epsilon} \right] \\
&= \liminf_{n \rightarrow \infty} \mathbf{E} \left[ \int_0^t \int_{\mathbb{R}} \int_0^{V^{(n)}(s-)} \left( \frac{u}{n^\alpha} \right)^{2+\epsilon} N_{n,0}(n \cdot ds, du, n^{2\alpha-1} \cdot dz) \right] \\
&\leq \liminf_{n \rightarrow \infty} \frac{1}{n^{\alpha\epsilon}} \int_{\mathbb{R}} |u|^{2+\epsilon} \nu_n(du) \cdot \int_0^t \mathbf{E}[V^{(n)}(s)] ds \\
&= 0.
\end{aligned}$$

This shows that  $J_p^*$  is almost surely continuous. Using similar arguments as in **Step 2** of the proof of Proposition 3.13, the sequence  $\{J_p^{(n)}\}_{n \geq 1}$  thus converge weakly in  $\mathbf{D}(\mathbb{R}_+; \mathbb{R})$  to the continuous martingale

$$\int_0^t \sigma_p \sqrt{V_*(s)} dW(s), \quad t \geq 0,$$

with  $W$  being a standard Brownian motion.

Hence, the sequence  $\{P^{(n)}\}_{n \geq 1}$  is  $C$ -tight and so is the sequence  $\{(P^{(n)}, P_0^{(n)}, I_p^{(n)}, J_p^{(n)})\}_{n \geq 1}$ . Passing both sides of equation (3.39) to the corresponding limits, we see that any accumulation point  $P_*$  satisfies the stochastic equation (2.13).

It remains to verify that the two Brownian motions  $B$  and  $W$  are independent, i.e. that  $\langle B, W \rangle_t \stackrel{\text{a.s.}}{=} 0$  for any  $t \geq 0$ . To this end, we recall the martingale  $M^{(n)}$  defined in (3.31). Since that  $N_1(ds, dz) = N_0(ds, \mathbb{R}, dz)$ , the covariation of the two martingales  $M^{(n)}$  and  $J_p^{(n)}$  is given by

$$\begin{aligned}
[M^{(n)}, J_p^{(n)}]_t &= \int_0^t \int_{\mathbb{R}} \int_0^{V^{(n)}(s-)} \frac{u}{n^{2\alpha}} N_{n,0}(n \cdot ds, du, n^{2\alpha-1} \cdot dz) \\
&= \int_{\mathbb{R}} u \nu_n(du) \int_0^t V^{(n)}(s) ds \\
&\quad + \int_0^t \int_{\mathbb{R}} \int_0^{V^{(n)}(s-)} \frac{u}{n^{2\alpha}} \tilde{N}_{n,0}(n \cdot ds, du, n^{2\alpha-1} \cdot dz) \\
&= \int_{\mathbb{R}} u \nu_n(du) \int_0^t V^{(n)}(s) ds + \frac{J_p^{(n)}(t)}{n^\alpha},
\end{aligned}$$

which vanishes almost surely as  $n \rightarrow \infty$  because of Assumption 2.7 and (3.40). This along with Theorem 6.26 in [42, p.384] yields that

$$[M_*, J_p^*]_t = \int_0^t V_*(s) d\langle B, W \rangle_t = \lim_{n \rightarrow \infty} [M^{(n)}, J_p^{(n)}]_t \stackrel{\text{a.s.}}{=} 0 \quad \text{and hence} \quad \langle B, W \rangle_t \stackrel{\text{a.s.}}{=} 0, \quad t \geq 0.$$

□

### 3.7 Counterexamples

Our main result states that under mild assumption the sequence of rescaled volatility processes converges to the unique (in law) solution of a stochastic differential equation. [Assumptions 2.3](#) and

2.7 are satisfied if, for instance, we choose  $V_{n,0} = V_*(0) \cdot n^{2\alpha-1}$ ,  $\mu_n = a \cdot n^{\alpha-1}$ ,  $\zeta_n = (1 - b \cdot n^{-\alpha})^+$ ,  $P_{n,0} = P_*(0) \cdot n^\alpha$ , and  $\nu_n$  being normal distributions with mean  $b_p \cdot n^{-\alpha}$  and variance  $\sigma_p^2$ .

We emphasize that our proof requires the  $C$ -tightness of the sequence  $\{J^{(n)}\}_{n \geq 1}$  to identify its limit by identifying the limit of the sequence  $\{\mathcal{I}_{J^{(n)}}\}_{n \geq 1}$ . Without the convergence of the sequence  $\{J^{(n)}\}_{n \geq 1}$  it is not clear to us that the sequence  $\{\mathcal{I}_{J^{(n)}}\}_{n \geq 1}$  converges. Even if the sequence of integrated processes converges, we can in general not expect to identify the weak limit of the sequence  $\{J^{(n)}\}_{n \geq 1}$  by identifying the weak limit of the sequence  $\{\mathcal{I}_{J^{(n)}}\}_{n \geq 1}$  as shown by the following example. The main issue is the convergence of the initial state of the volatility process that cannot always be inferred from the convergence of the integrated processes.

**Example 3.1** Let us assume that

$$\Lambda_n(t) = V_{n,0} \cdot \phi(t), \quad t \geq 0.$$

In this case, we have that  $\Lambda_n(t) + R_n * \Lambda_n(t) = V_{n,0} \cdot R_n(t)/\zeta_n$  and

$$V_n(t) = \frac{V_{n,0}}{\zeta_n} R_n(t) + \mu_n + \mu_n \int_0^t R_n(s) ds + \int_0^t \int_0^{V_n(s-)} \zeta_n R_n(t-s) \tilde{N}_{n,1}(ds, dz).$$

We now distinguish two cases.

(1) If  $\alpha \in (1/2, 1)$ , we choose  $V_{n,0} = V(0) \cdot n^\alpha$  and consider the rescaled processes

$$V^{(n)}(t) := \frac{V_n(nt)}{n^{2\alpha-1}}.$$

In this case, the integrated process satisfies

$$\begin{aligned} \mathcal{I}_{V^{(n)}}(t) &= \frac{V(0)}{\zeta_n} \int_0^t n^{1-\alpha} R_n(ns) ds + \frac{\mu_n \cdot t}{n^{2\alpha-1}} + \frac{\mu_n}{n^{2\alpha-1}} \int_0^t \int_0^{ns} R_n(r) dr ds \\ &\quad + \int_0^t \int_0^s \int_0^{V^{(n)}(r-)} \frac{\zeta_n}{n^{2\alpha-1}} R_n(n(s-r)) \tilde{N}_{n,1}(n \cdot dr, n^{2\alpha-1} \cdot dz) ds. \end{aligned}$$

Based on our preceding analysis, it is not difficult to see that  $\mathcal{I}_{V^{(n)}}$  converges weakly to  $X$  in  $\mathbf{D}(\mathbb{R}_+; \mathbb{R}_+)$  with  $X$  being the unique solution of the ODE

$$X(t) = \frac{V(0)}{b} F^{\alpha,\gamma}(t) + \int_0^t \frac{a}{b} F^{\alpha,\gamma}(s) ds + \int_0^t \frac{1}{b} f^{\alpha,\gamma}(t-s) B(X(s)) ds, \quad t \geq 0.$$

The random variable  $X(t)$  can be written as

$$X(t) = \int_0^t Y(s) ds,$$

where the process  $Y$  satisfies the SDE

$$Y(t) = \frac{V(0)}{b} f^{\alpha,\gamma}(t) + \frac{a}{b} F^{\alpha,\gamma}(t) + \int_0^t \frac{1}{b} f^{\alpha,\gamma}(t-s) \sqrt{Y(s)} dB(s).$$

However, if  $V(0) \neq 0$ , then  $Y(t) \rightarrow \infty$  as  $t \rightarrow 0+$  and hence it is impossible to prove that  $V^{(n)} \rightarrow Y$  weakly in  $\mathbf{D}(\mathbb{R}_+; \mathbb{R}_+)$ .

(2) If  $\alpha > 1$ , then the kernel  $\phi$  is light-tailed. We choose  $\sigma > 0$  such that

$$\vartheta := \int_0^\infty t\phi(t)dt \in (1, \infty).$$

Similarly as in [44, 58], we set  $V_{n,0} = V(0) \cdot n$ ,  $\mu_n \equiv a$  and consider the rescaled processes

$$V^{(n)}(t) := \frac{V_n(nt)}{n}.$$

In this case, the integrated process satisfies

$$\begin{aligned} \mathcal{I}_{V^{(n)}}(t) &= \frac{V(0)}{\zeta_n} \int_0^t R_n(ns)ds + \frac{a \cdot t}{n} + a \int_0^t \int_0^s R_n(nr)dr ds \\ &\quad + \int_0^t \int_0^s \int_0^{V^{(n)}(r-)} \frac{\zeta_n}{n} R_n(n(s-r)) \tilde{N}_{n,1}(n \cdot dr, n \cdot dz)ds. \end{aligned}$$

By Lemma 4.5 in [44], we have  $\int_0^t R_n(ns)ds \rightarrow \frac{1}{b}(1 - e^{-\frac{b}{\vartheta} \cdot t})$  uniformly. Hence it is not difficult to see that  $\mathcal{I}_{V^{(n)}}$  converges weakly to  $X$  in  $\mathbf{D}(\mathbb{R}_+; \mathbb{R}_+)$  with  $X$  being the unique solution of the ODE

$$X(t) = \frac{V(0)}{b} \left(1 - e^{-\frac{b}{\vartheta} \cdot t}\right) + \int_0^t \frac{a}{b} \left(1 - e^{-\frac{b}{\vartheta} \cdot s}\right) ds + \int_0^t \frac{1}{b} e^{-\frac{b}{\vartheta} \cdot (t-s)} B(X(s))ds, \quad t \geq 0.$$

The random variable  $X(t)$  can be written as

$$X(t) = \int_0^t Y(s)ds,$$

where the process  $Y$  satisfies the SDE

$$Y(t) = \frac{V(0)}{\vartheta} e^{-\frac{b}{\vartheta} \cdot t} + \frac{a}{b} \left(1 - e^{-\frac{b}{\vartheta} \cdot t}\right) + \int_0^t \frac{1}{\vartheta} e^{-\frac{b}{\vartheta} \cdot (t-s)} \sqrt{Y(s)} dB(s), \quad t \geq 0.$$

Since

$$V^{(n)}(0) = \frac{\Lambda_n(0) + \mu^{(n)}}{n} \rightarrow V(0) \neq Y(0) = \frac{V(0)}{\vartheta},$$

if  $V(0) \neq 0$ , we see that the sequence  $\{V^{(n)}\}_{n \geq 1}$  does not converge weakly to  $Y$  in  $\mathbf{D}(\mathbb{R}_+; \mathbb{R}_+)$ .

We acknowledge that this example does not apply if the limiting initial volatility equals zero. In our view a vanishing initial volatility fundamentally contradicts the idea of Hawkes processes whose sole purpose is to capture the impact of past events on the arrivals of future ones. Setting the initial volatility to zero means no events occurred prior to time zero, in which case the dynamics of past and future event arrivals is not consistent.

## A Mittag-Leffler function

In this section, we recall some elementary properties of Mittag-Leffler function. The reader may refer to [33, 51] for a detailed discussion of Mittag-Leffler function. For two constants  $\alpha, \kappa > 0$ , the Mittag-Leffler function  $E_{\alpha,\kappa}$  on  $\mathbb{R}$  is given by

$$E_{\alpha,\kappa}(x) := \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \kappa)}, \quad x \in \mathbb{R}.$$

It is locally  $\alpha$ -Hölder continuous. For  $\alpha \in (0, 1)$  and a constant  $\gamma > 0$ , we denote by  $F^{\alpha, \gamma}$  and  $f^{\alpha, \gamma}$  the *Mittag-Leffler distribution* and *density function* on  $\mathbb{R}_+$  with parameters  $(\alpha, \gamma)$ ; they are defined by

$$F^{\alpha, \gamma}(t) := 1 - E_{\alpha, 1}(-\gamma t^\alpha) \quad \text{and} \quad f^{\alpha, \gamma}(t) := \gamma \cdot t^{\alpha-1} E_{\alpha, \alpha}(-\gamma \cdot t^\alpha), \quad t > 0.$$

The Laplace transform of Mittag-Leffler distribution admits the representation

$$\mathcal{L}_{f^{\alpha, \gamma}}(\lambda) := \int_0^\infty e^{-\lambda s} f^{\alpha, \gamma}(s) ds = \frac{\gamma}{\gamma + \lambda^\alpha}, \quad \lambda \geq 0.$$

The asymptotic behavior of the density function  $f^{\alpha, \gamma}$  near zero, respectively infinity is given by

$$f^{\alpha, \gamma}(t) \sim \frac{\gamma \cdot t^{\alpha-1}}{\Gamma(\alpha)} \quad \text{as } t \rightarrow 0+ \quad \text{and} \quad f^{\alpha, \gamma}(t) \sim \frac{\alpha \cdot t^{-\alpha-1}}{\gamma \Gamma(1-\alpha)} \quad \text{as } t \rightarrow \infty.$$

We also define the function

$$R^{\alpha, \gamma}(t) := \frac{\gamma}{\Gamma(\alpha)} \cdot t^{\alpha-1}, \quad t \geq 0, \tag{A.1}$$

whose Laplace transform admits the representation

$$\mathcal{L}_{R^{\alpha, \gamma}}(\lambda) := \int_0^\infty e^{-\lambda s} R^{\alpha, \gamma}(s) ds = \gamma \cdot \lambda^{-\alpha}, \quad \lambda > 0$$

It is easy to identify that  $\mathcal{L}_{R^{\alpha, \gamma}} = \mathcal{L}_{f^{\alpha, \gamma}} + \mathcal{L}_{R^{\alpha, \gamma}} \cdot \mathcal{L}_{f^{\alpha, \gamma}}$ , which yields that  $R^{\alpha, \gamma}$  is the resolvent of  $f^{\alpha, \gamma}$ , i.e.

$$R^{\alpha, \gamma}(t) = f^{\alpha, \gamma}(t) + f^{\alpha, \gamma} * R^{\alpha, \gamma}(t), \quad t > 0. \tag{A.2}$$

The next proposition is a direct consequence of the variation of constants formula for linear Volterra integral equations as given in [31, p.36, Equation (1.2)]. Its proof is omitted.

**Proposition A.1** *For a function  $H \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R})$ , the unique solution to*

$$X(t) = H(t) + f^{\alpha, \gamma} * X(t), \quad t \geq 0,$$

*admits the representation*

$$X(t) = H(t) + R^{\alpha, \gamma} * H(t), \quad t \geq 0.$$

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