

## On the supremum of a quotient of power sums

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## Abstract

We define a function of two real vectors by a certain homogeneous quotient involving power sums, and show that its supremum grows asymptotically linearly w.r.t. the dimension. From this, we deduce a condition under which a parametric set of real matrices satisfies a set of polynomial positivity constraints. This characterization finds an application in mathematical finance, in a recent study on price impact models.

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## 1 Introduction

Reznick [Rez83] has shown, among other results, that the maximum of

$$\left( \sum_{i=1}^n x_i \right) \left( \sum_{i=1}^n x_i^3 \right) \Bigg/ \left( \sum_{i=1}^n x_i^2 \right)^2$$

over  $\mathbb{R}^n$  is of order  $\sqrt{n}$ , as  $n \rightarrow \infty$ . Our contribution is an analytical investigation of a homogeneous quotient involving *differences* of power sums in high dimension, motivated by recent work on price impact models in mathematical finance [HMMK25]. With  $M_p(x) := \sum_{i=1}^n x_i^p$  for  $p > 0$ , we prove the inequality

$$Q(x, y) := \frac{(M_1(x) - M_1(y))(M_2(y) - M_2(x))}{M_3(x) + M_3(y)}$$

$$= \frac{\left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right) \left( \sum_{i=1}^n y_i^2 - \sum_{i=1}^n x_i^2 \right)}{\sum_{i=1}^n x_i^3 + \sum_{i=1}^n y_i^3} < \frac{7\sqrt{7} - 17}{27} n, \quad x, y \in \mathbb{R}_{>0}^n,$$

and show that the constant factor is sharp as  $n \rightarrow \infty$ . Put differently, we maximize the product  $(\|x\|_1 - \|y\|_1)(\|y\|_2^2 - \|x\|_2^2)$  over the  $\ell^3$ -unit ball, with a focus on large dimension. Note that this product is non-positive, if the two vectors  $x, y \in \mathbb{R}_{>0}^n$  are ordered component-wise. It is easy to see, though, and will be shown below, that the supremum is always positive for  $n \geq 2$ . In what follows, we will also consider  $Q(x, y)$  for vectors  $x \in \mathbb{R}_{>0}^{n+1}$  and  $y \in \mathbb{R}_{>0}^n$  of unequal length. In Section 3, we discuss an application: The bound allows to characterize a certain parametric set of matrices that arises in [HMMK25]. This is

also where the constraint of considering only vectors with positive entries comes from. As a side remark, we note that there is a considerable computational literature on optimization of homogeneous polynomials, and refer to [HS14] and the references therein.

## 2 Main result

**Theorem 2.1.** *With*

$$c^* := \frac{7\sqrt{7} - 17}{27} \approx 0.0563,$$

*we have*

$$Q(x, y) < c^*n, \quad x, y \in \mathbb{R}_{>0}^n, \quad n \geq 1, \quad (2.1)$$

*and*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sup \{Q(x, y) : x, y \in \mathbb{R}_{>0}^n\} = c^*. \quad (2.2)$$

*Both assertions remain true if the condition  $x \in \mathbb{R}_{>0}^n$  is replaced by  $x \in \mathbb{R}_{>0}^{n+1}$ .*

First, we verify that  $Q$  has positive values for all  $n \geq 2$ .

**Lemma 2.2.** *For  $n = 1$ , we have  $\max_{x, y > 0} Q(x, y) = 0$ . Furthermore,*

$$\sup \{Q(x, y) : x, y \in \mathbb{R}_{>0}^n\} > 0, \quad n \geq 2,$$

*and*

$$\sup \{Q(x, y) : x \in \mathbb{R}_{>0}^{n+1}, y \in \mathbb{R}_{>0}^n\} > 0, \quad n \geq 1.$$

*Proof.* For  $n = 1$ , note that

$$Q(x, y) = \frac{(x-y)(y^2-x^2)}{x^3+y^3} = -\frac{(x-y)^2(x+y)}{x^3+y^3} \leq 0, \quad x, y > 0.$$

For  $n \geq 2$ , define the vectors

$$\hat{x}^{(n)} := (\underbrace{1, \dots, 1}_{\lceil n/2 \rceil}, \underbrace{n^{-1}, \dots, n^{-1}}_{\lfloor n/2 \rfloor}) \in \mathbb{R}_{>0}^n,$$

$$\hat{y}^{(n)} := \left(\frac{701}{1000}, \dots, \frac{701}{1000}\right) \in \mathbb{R}_{>0}^n,$$

and define  $\tilde{x}^{(n)} \in \mathbb{R}_{>0}^{n+1}$  by adding another component  $n^{-1}$  to  $\hat{x}^{(n)}$ . For even  $n$ ,  $Q(\hat{x}^{(n)}, \hat{y}^{(n)})$  is a rational function of  $n$ . Using a computer algebra system, it is easily verified that this rational function is positive for real  $n \geq 4$ . In particular,  $Q(\hat{x}^{(n)}, \hat{y}^{(n)}) > 0$  for integral  $n \geq 4$ . Analogously, the same holds for odd  $n \geq 5$ , and we have  $Q(\tilde{x}^{(n)}, \hat{y}^{(n)}) > 0$  for  $n \geq 7$ . We can also use computer algebra to find vectors  $x, y$  with  $Q(x, y) > 0$  for the remaining finitely many cases. For instance, we have  $Q(\check{x}, \check{y}) \approx 0.031$  for

$$\begin{aligned} \check{x} &:= \left(\frac{3}{2}, \frac{1}{16777216}, \frac{1}{8388608}, \frac{1}{2097152}, \frac{1}{8192}, \frac{1}{4096}, \frac{3}{4}\right) \in \mathbb{R}_{>0}^7, \\ \check{y} &:= \left(1, \frac{1}{256}, \frac{1}{64}, \frac{1}{16}, \frac{1}{2}, 1\right) \in \mathbb{R}_{>0}^6. \end{aligned}$$

□

*Proof of Theorem 2.1.* We begin with the upper estimate. For  $n = 1$ , we have

$$\sup_{x,y>0} Q(x,y) = 0 < c^*$$

by Lemma 2.2. To prove the upper bound (2.1) for  $n \geq 2$ , it is convenient to work on  $\mathbb{R}_{\geq 0}^n \setminus \{0\}$  instead of  $\mathbb{R}_{>0}^n$ . Suppose that  $Q$  has a local maximum at  $(x,y) = (\xi,\eta) \in \mathbb{R}_{\geq 0}^{2n} \setminus \{(0,0)\}$ . The global maximal value is positive by Lemma 2.2, and so we may consider the logarithmic derivatives

$$\partial_{x_i} \log Q(x,y)|_{(x,y)=(\xi,\eta)} = U(\xi_i), \quad \partial_{y_i} \log Q(x,y)|_{(x,y)=(\xi,\eta)} = V(\eta_i),$$

where

$$U(t) := \frac{1}{M_1(\xi) - M_1(\eta)} - \frac{2t}{M_2(\eta) - M_2(\xi)} - \frac{3t^2}{M_3(\xi) + M_3(\eta)}$$

and

$$V(t) := -\frac{1}{M_1(\xi) - M_1(\eta)} + \frac{2t}{M_2(\eta) - M_2(\xi)} - \frac{3t^2}{M_3(\xi) + M_3(\eta)}.$$

If  $\xi_i > 0$ , then we must have  $U(\xi_i) = 0$ , and if  $\eta_i > 0$ , then  $V(\eta_i) = 0$ . As each of the two quadratic polynomials  $U, V$  has at most two zeros, we conclude that, for a global maximum at  $(\xi,\eta)$ , there exist four numbers  $\alpha, \beta, \gamma, \delta$  with

$$\xi_i \in \{\alpha, \beta, 0\}, \quad \eta_i \in \{\gamma, \delta, 0\}, \quad 1 \leq i \leq n.$$

It thus suffices to consider

$$x = (\underbrace{\alpha, \dots, \alpha}_i, \underbrace{\beta, \dots, \beta}_j, 0, \dots, 0), \quad y = (\underbrace{\gamma, \dots, \gamma}_k, \underbrace{\delta, \dots, \delta}_l, 0, \dots, 0) \quad (2.3)$$

with

$$\begin{aligned} \alpha &> \beta > 0, & \gamma &> \delta > 0, \\ 1 &\leq i \leq n, & 0 &\leq j \leq n-i, \\ 1 &\leq k \leq n, & 0 &\leq l \leq n-k. \end{aligned} \quad (2.4)$$

Then, for  $x, y$  as in (2.3), we have

$$Q(x,y) = \frac{S_1 S_2}{S_3}, \quad (2.5)$$

where

$$\begin{aligned} S_1 &:= i\alpha + j\beta - k\gamma - l\delta, \\ S_2 &:= -i\alpha^2 - j\beta^2 + k\gamma^2 + l\delta^2, \\ S_3 &:= i\alpha^3 + j\beta^3 + k\gamma^3 + l\delta^3. \end{aligned}$$

Thus, we arrive at a rational function of  $(\alpha, \beta, \dots, k, l)$ . By Lemma 2.2, the supremum of this function in the domain defined by (2.4), with real variables

$i, j, k, l$ , is positive. By homogeneity and symmetry, we may assume that  $\alpha$  and  $\gamma$  are bounded. We proceed to determine the maximum of

$$h(\alpha, \beta, \dots, k, l) := \log(S_1 S_2 / S_3),$$

which yields an upper bound for  $Q(x, y)$ . We define  $h := -\infty$  for  $S_1 S_2 \leq 0$ , but note that  $h$  is finitely-valued and smooth in a neighborhood of any global maximum. The partial derivatives of  $h$  are

$$h_\alpha = u(\alpha)i, \quad h_\beta = u(\beta)j, \quad h_\gamma = v(\gamma)k, \quad h_\delta = v(\delta)l,$$

where

$$u(t) := \frac{1}{S_1} - \frac{2t}{S_2} - \frac{3t^2}{S_3}, \quad v(t) := -\frac{1}{S_1} + \frac{2t}{S_2} - \frac{3t^2}{S_3}.$$

Moreover,

$$h_i = U(\alpha), \quad h_j = U(\beta), \quad h_k = V(\gamma), \quad h_l = V(\delta),$$

where

$$U(t) = \frac{t}{S_1} - \frac{t^2}{S_2} - \frac{t^3}{S_3}, \quad V(t) = -\frac{t}{S_1} + \frac{t^2}{S_2} - \frac{t^3}{S_3}.$$

We claim that  $j = l = 0$  holds at any global maximum of  $h$ . The following argument rests on the identities

$$\frac{h_\alpha}{\alpha i} - \frac{h_i}{\alpha^2} - \frac{h_\beta}{\beta j} + \frac{h_j}{\beta^2} = -\frac{2(\alpha - \beta)}{S_3} \quad (2.6)$$

and

$$\frac{h_\gamma}{\gamma k} - \frac{h_k}{\gamma^2} - \frac{h_\delta}{\delta l} + \frac{h_l}{\delta^2} = -\frac{2(\gamma - \delta)}{S_3}. \quad (2.7)$$

They show that no point in the interior of the domain of  $h$  at which  $h$  is  $> -\infty$  satisfies the first order conditions  $h_\alpha = \dots = h_l = 0$ . Furthermore, if there was a maximum at the boundary  $j = n - i$ , then (2.6) would imply

$$\frac{h_j}{\beta^2} = -\frac{2(\alpha - \beta)}{S_3} < 0,$$

but then  $h$  would decrease as we cross the boundary in the  $j$ -direction. The only remaining possibility is a maximum at the boundary  $j = 0$  (equivalently, we could put  $\beta = 0$ ), and similarly we can show from (2.7) that  $l = 0$ . Moreover, by symmetry, it suffices to consider  $i \leq k$ . For  $n = 2$ , (2.1) is now easily verified. By induction, we may assume  $k = n$ , because otherwise the parameters  $\alpha, \gamma, i, k$  would be admissible for the problem with  $n$  replaced by  $k$ , and we can simply note that  $c^*k < c^*n$ . Assuming  $1 = \alpha > \gamma$ , by symmetry and homogeneity, we have formally reduced the problem of optimizing  $Q$  to choosing

$$x = (\underbrace{1, \dots, 1}_i, 0, \dots, 0), \quad y = (\gamma, \dots, \gamma)$$

with  $1 \leq i \leq n$  and  $0 < \gamma < 1$ . We will see, though, that the optimal  $i$  for the function  $h$  is not an integer, which shows that the bound (2.1) is strict. We put  $i = pn$  with unknown  $p \in (0, 1)$ , and so our target becomes

$$\frac{S_1 S_2}{S_3} = \frac{(i - n\gamma)(-i + n\gamma^2)}{i + n\gamma^3} = \frac{(p - \gamma)(-p + \gamma^2)}{p + \gamma^3} n.$$

Maximizing the latter fraction is straightforward, as the system  $\partial_p(\cdot) = 0$ ,  $\partial_\gamma(\cdot) = 0$  can be solved explicitly, with a single positive solution

$$p^* := \frac{16 - 5\sqrt{7}}{27} \approx 0.102, \quad \gamma^* := \frac{\sqrt{7} - 2}{3} \approx 0.215.$$

It is elementary to show that  $(p - \gamma)(-p + \gamma^2)/(p + \gamma^3)$ ,  $(p, \gamma) \in (0, 1)^2$ , has a global maximum at  $(p^*, \gamma^*)$ , with value  $c^*$ . This finishes the proof of (2.1).

Inspired by the upper estimate, we use the vectors

$$x^{(n)} := (\underbrace{1, \dots, 1}_{\lfloor p^* n \rfloor}, n^{-1}, \dots, n^{-1}) \in \mathbb{R}_{>0}^n \quad (2.8)$$

and

$$y^{(n)} := (\gamma^*, \dots, \gamma^*) \in \mathbb{R}_{>0}^n \quad (2.9)$$

to prove the required asymptotic lower bound. From

$$M_\nu(x^{(n)}) = p^* n + O(1), \quad \nu = 1, 2, 3,$$

as  $n \rightarrow \infty$  and

$$M_\nu(y^{(n)}) = (\gamma^*)^\nu n + O(1), \quad \nu = 1, 2, 3,$$

we calculate

$$\begin{aligned} Q(x^{(n)}, y^{(n)}) &= \frac{(p^* n - \gamma^* n + O(1))(-p^* n + (\gamma^*)^2 n + O(1))}{p^* n + (\gamma^*)^3 n + O(1)} \\ &\sim \frac{(p^* - \gamma^*)(-p^* + (\gamma^*)^2)}{p^* + (\gamma^*)^3} n = c^* n, \quad n \rightarrow \infty. \end{aligned}$$

The proof of (2.1) and (2.2) is finished. Proving the remaining assertions is a simple modification: The vector (2.8) from the lower bound receives an additional component  $n^{-1}$ . The range of  $i$  in the upper bound becomes  $1 \leq i \leq n+1$ , which makes no essential difference. For the induction base, we need to compute

$$\sup \{Q(x, y) : x \in \mathbb{R}_{>0}^2, y \in \mathbb{R}_{>0}\}.$$

We may assume  $x_1 = 1$ , and it is then an easy exercise that the maximum of

$$Q(x, y) = \frac{(1 + x_2 - y)(y^2 - 1 - x_2^2)}{1 + x_2^3 + y^3}$$

is located at  $x_2 = 1$ ,  $y = \frac{1}{2}(\sqrt{9 + 4\sqrt{6}} - 1)$ , and has a value that is smaller than  $2c^*$ .  $\square$

### 3 Application

For  $d \geq 2$ , we define the set  $\mathcal{M}_d$  of matrices  $M = (m_{ij}) \in \mathbb{R}^{d \times d}$  such that

$$\Psi_M(z, s) := \sum_{l=1}^d \left( m_{ll} z_l^3 + s_l z_l \sum_{k \neq l} m_{lk} s_k z_k^2 \right) > 0 \quad (3.1)$$

for all  $s \in \{\pm 1\}^d$  and  $z \in \mathbb{R}_{>0}^d$ . In general, this gives  $2^{d-1} - 1$  inequalities which have to be satisfied by all  $z \in \mathbb{R}_{>0}^d$ , as  $s$  and  $-s$  yield the same condition, and  $s = (1, \dots, 1)$  can of course be discarded.

**Lemma 3.1.** *The set  $\mathcal{M}_d$  is a convex cone. Any matrix in  $\mathcal{M}_d$  has non-negative diagonal elements. If  $M$  is diagonally dominant in the sense that*

$$m_{ii} > \sum_{l=1}^d \sum_{k \neq l} |m_{lk}|, \quad 1 \leq i \leq d, \quad (3.2)$$

*i.e. each diagonal element dominates the sum of the absolute values of all off-diagonal elements, then  $M \in \mathcal{M}_d$ . In particular, diagonal matrices with positive diagonal elements are in  $\mathcal{M}_d$ .*

*Proof.* The first statement is obvious. The second one is also clear, by considering large multiples of unit vectors. Now let  $z \in \mathbb{R}_{>0}^d$  and suppose that  $z_i$  is one of the maximal entries of the vector. Then we have for all  $s \in \{\pm 1\}^d$

$$\begin{aligned} \Psi_M(z, s) &= \sum_{l=1}^d \left( m_{ll} z_l^3 + s_l z_l \sum_{k \neq l} m_{lk} s_k z_k^2 \right) \\ &\geq m_{ii} z_i^3 - z_i^3 \sum_{l=1}^d \sum_{k \neq l} |m_{lk}| > 0. \end{aligned} \quad \square$$

The set  $\mathcal{M}_d$  arises naturally in [HMMK25]. As elaborated there, positivity of the term  $h(\mathbf{J}_t)^\top \boldsymbol{\zeta} \mathbf{J}_t$  in (6.1) of [HMMK25], for any vector  $\mathbf{J}_t = (J_t^i)$ , is crucial for the underlying model to make sense. For the standard choice  $h(x) = \text{sgn}(x)\sqrt{|x|}$  (square-root price impact), and with  $z_i = \sqrt{|J_t^i|}$  and  $M = \boldsymbol{\zeta}$ , this amounts to the condition (3.1). It is thus of interest to find conditions that ensure  $M \in \mathcal{M}_d$ , in particular for large dimensions. Presumably, it is hard to give a simple general if-and-only-if condition. Even for  $d = 2$ , where a full characterization is possible, a complicated case distinction involving algebraic expressions of the matrix elements  $m_{ij}$  arises. Beyond the trivial case of diagonal matrices of arbitrary dimension, a reasonable next step is to consider equal off-diagonal elements, and so we define

$$M_d(b) := \begin{pmatrix} 1 & b & \dots & b \\ b & 1 & & \vdots \\ \vdots & & \ddots & b \\ b & \dots & b & 1 \end{pmatrix}, \quad b \in \mathbb{R}_{>0}, \quad (3.3)$$

for which (3.1) reduces to the condition

$$\sum_{l=1}^d z_l^3 + b \sum_{l=1}^d s_l z_l \sum_{k \neq l} s_k z_k^2 > 0, \quad s \in \{\pm 1\}^d, \quad z \in \mathbb{R}_{>0}^d. \quad (3.4)$$

**Lemma 3.2.** *For  $d \geq 2$  and  $b > 0$ , (3.4) is equivalent to*

$$\frac{1}{b} > 1 + Q(x, y), \quad x \in \mathbb{R}_{>0}^{[d/2]}, \quad y \in \mathbb{R}_{>0}^{\lfloor d/2 \rfloor}.$$

*Proof.* We assume that  $d$  is even, as the proof for odd  $d$  is analogous. By symmetry of (3.4), only the number of positive entries in  $s \in \{\pm 1\}^d$  is relevant, not their position. Moreover, it suffices to consider

$$s = (\underbrace{-1, \dots, -1}_d, \underbrace{1, \dots, 1}_d). \quad (3.5)$$

Indeed, the other sign combinations  $s$  lead to a larger number of (+1)-s in the matrix  $(s_i s_j)$ . As the  $z_i$  are positive, this makes the desired inequality (3.4) easier to satisfy. For  $s$  as in (3.5), the second sum in (3.4) becomes

$$\begin{aligned}
\sum_{l=1}^d s_l z_l \sum_{k \neq l} s_k z_k^2 &= \sum_{l=1}^{d/2} s_l z_l \left( \sum_{\substack{k=1 \\ k \neq l}}^{d/2} s_k z_k^2 + \sum_{k=d/2+1}^d s_k z_k^2 \right) \\
&\quad + \sum_{l=d/2+1}^d s_l z_l \left( \sum_{k=1}^{d/2} s_k z_k^2 + \sum_{\substack{k=d/2+1 \\ k \neq l}}^d s_k z_k^2 \right) \\
&= \sum_{l=1}^{d/2} z_l \sum_{\substack{k=1 \\ k \neq l}}^{d/2} z_k^2 - \sum_{l=1}^{d/2} z_l \sum_{k=d/2+1}^d z_k^2 - \sum_{l=d/2+1}^d z_l \sum_{k=1}^{d/2} z_k^2 + \sum_{l=d/2+1}^d z_l \sum_{\substack{k=d/2+1 \\ k \neq l}}^d z_k^2 \\
&= \sum_{l=1}^{d/2} z_l \left( \sum_{k=1}^{d/2} z_k^2 - z_l^2 \right) - \sum_{l=1}^{d/2} z_l \sum_{k=d/2+1}^d z_k^2 \\
&\quad - \sum_{l=d/2+1}^d z_l \sum_{k=1}^{d/2} z_k^2 + \sum_{l=d/2+1}^d z_l \left( \sum_{k=d/2+1}^d z_k^2 - z_l^2 \right) \\
&= -M_3(x) - M_3(y) - (M_1(x) - M_1(y))(M_2(y) - M_2(x)),
\end{aligned}$$

where

$$x := (z_1, \dots, z_{d/2}), \quad y := (z_{d/2+1}, \dots, z_d).$$

As the first sum in (3.4) equals  $M_3(x) + M_3(y)$ , the statement follows.  $\square$

We can now state our main result on membership of (3.3) in the cone  $\mathcal{M}_d$ . While the naive criterion (3.2) shows that  $b \lesssim 1/d^2$  suffices, we can infer from Theorem 2.1 that the true asymptotic order of the maximal value of  $b$  is  $1/d$  for large  $d$ .

**Theorem 3.3.** *For  $d \geq 2$ , the set of admissible  $b$  for the matrices (3.3) is an interval with closure*

$$\text{cl}\{b > 0 : M_d(b) \in \mathcal{M}_d\} = [0, b_d], \quad (3.6)$$

where

$$b_d := \frac{1}{1 + \sup \left\{ Q(x, y) : x \in \mathbb{R}_{>0}^{\lceil d/2 \rceil}, y \in \mathbb{R}_{>0}^{\lfloor d/2 \rfloor} \right\}}. \quad (3.7)$$

The sequence  $b_d > 0$  decreases, satisfies the lower bound

$$b_d \geq \frac{1}{1 + c^* \lfloor d/2 \rfloor}, \quad d \in \mathbb{N}, \quad (3.8)$$

and has the asymptotics

$$b_d \sim \frac{2}{c^* d} = \frac{54}{7\sqrt{7} - 17} \frac{1}{d} \approx \frac{35.52}{d}, \quad d \rightarrow \infty. \quad (3.9)$$

*Proof.* It follows from Lemma 3.1 that the set  $\{b > 0 : M_d(b) \in \mathcal{M}_d\}$  is a non-empty interval. By Lemmas 2.2 and 3.2, we have  $M_d(b) \in \mathcal{M}_d$  if and only if

$$b < \frac{1}{1 + \max\{Q(x, y), 0\}}, \quad x \in \mathbb{R}_{>0}^{[d/2]}, \quad y \in \mathbb{R}_{>0}^{\lfloor d/2 \rfloor}.$$

This proves (3.6), and (3.8) and (3.9) follow from Theorem 2.1. To see that  $b_d$  decreases, let  $d \geq 3$  and  $b, \varepsilon > 0$  be such that  $M_d(b(1 + \varepsilon)) \in \mathcal{M}_d$ . Thus, Lemma 3.2 implies

$$1 > b(1 + \varepsilon)(1 + Q(x, y)), \quad x \in \mathbb{R}_{>0}^{[d/2]}, \quad y \in \mathbb{R}_{>0}^{\lfloor d/2 \rfloor}.$$

Taking  $x_{\lceil d/2 \rceil} \downarrow 0$  for odd  $d$  respectively  $y_{d/2} \downarrow 0$  for even  $d$  yields

$$1 \geq b(1 + \varepsilon)(1 + Q(x, y)) > b(1 + Q(x, y)), \quad x \in \mathbb{R}_{>0}^{\lceil (d-1)/2 \rceil}, \quad y \in \mathbb{R}_{>0}^{\lfloor (d-1)/2 \rfloor}.$$

Therefore,  $M_{d-1}(b) \in \mathcal{M}_{d-1}$ . As  $\varepsilon > 0$  was arbitrary, we have shown that  $b_{d-1} \geq b_d$ .  $\square$

## 4 Numerical values

For  $d = 3$ , the optimization problem in (3.7) can be solved explicitly, as stated at the end of the proof of Theorem 2.1. The number  $b_3$  is the only positive root of the polynomial

$$20x^4 + 60x^3 + 9x^2 - 54x - 27, \quad (4.1)$$

with explicit expression

$$b_3 = \frac{1}{4} \left( \sqrt{\frac{3}{5}(39 + 16\sqrt{6})} - 3 \right) \approx 0.962. \quad (4.2)$$

Alternatively,  $b_d$  can be computed by quantifier elimination [Stu17] from (3.4), which also yields (4.1) and (4.2). As mentioned in the proof of Lemma 3.2, it suffices to consider  $s$  as in (3.5) for even  $d$ . Analogously, for odd  $d$ , the inequality (3.4) needs to be checked only for

$$s = (\underbrace{-1, \dots, -1}_{(d+1)/2}, \underbrace{1, \dots, 1}_{(d-1)/2}).$$

Still, quantifier elimination is computationally hard, and so we could calculate  $b_d$  with this method only until  $d = 4$ . It turns out that  $b_3 = b_4$ . In principle, upper bounds can be computed by using numerical optimization to approximate the supremum in (3.7). It is not trivial, though, to find good starting values. Note that the vectors (2.8) and (2.9) are not useful in that respect for small  $n$ . In fact,  $Q(x^{(n)}, y^{(n)})$  is negative for  $n \leq 9$ . To compute the values  $b_5$  and  $b_6$  in Table 1, we instead used Mathematica's `FindInstance` and `Reduce` commands to show that  $Q$  in (3.7) can be larger than  $\frac{1079}{10000}$ , but not  $\frac{1080}{10000}$ .

Table 2 gives estimates for  $b_d$  for large  $d$ . For the upper estimates, we evaluate  $Q$  at the vectors (2.8) with  $n = \lceil d/2 \rceil$  and (2.9) with  $n = \lfloor d/2 \rfloor$ , which yields an upper bound for the right hand side of (3.7). For large  $d$  our computations show that  $b_d$  is very close to  $b_{d+1}$  for even  $d$ . This is in contrast to the fact that  $b_3 = b_4$ , as mentioned above.

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$d$	2	3	4	5	6
Lower estimate	0.946	0.946	0.898	0.898	0.855
$b_d$	1	0.962	0.962	0.902	0.902

Table 1: The numbers  $b_d$  from (3.7) for small  $d$ , and the lower estimate (3.8).

$d$	50	100	150	200	300	400	500
Lower estimate	0.415	0.262	0.191	0.150	0.105	0.081	0.066
Upper estimate	0.510	0.295	0.210	0.161	0.111	0.084	0.068
Asymptotics	0.710	0.355	0.236	0.177	0.118	0.088	0.071

Table 2: Estimates for  $b_d$  for larger  $d$ . Note that (3.8), which is asymptotically equal to (3.9), gives a better approximation than (3.9).

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