Asian option valuation under price impact

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Abstract

We study the valuation of Asian options in a binomial market with permanent price impact, extending the Cox–Ross–Rubinstein framework under a modified risk–neutral probability. We obtain an exact pathwise representation for geometric Asian options and derive two–sided bounds for arithmetic Asian options. Our analysis identifies the no–arbitrage region in terms of hedging volumes and shows that permanent price impact systematically raises Asian option prices. Numerical examples illustrate the effect of the impact parameter and hedging volumes on the resulting prices.
Keywords: Asian options, price impact, binomial models, derivative pricing, path-dependent derivatives, hedging volumes

1 Introduction

Each time a trader places a buy (sell) order, the transaction pushes the asset price upward (downward). The magnitude of this movement, known as price impact, is of substantial interest because once trading is understood to influence prices, it constitutes an inherent cost to the

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trader. In this work, we examine how such price impact alters option valuation. Our focus is on Asian options, whose payoffs depend on the entire price trajectory. This path dependence makes them particularly sensitive to market impact. The effect of each trade propagates through time and influences the average price, thereby affecting its fair value. We seek to obtain tractable numerical procedures for computing Asian option price under price impact. We study the problem under a binomial option pricing formulation.

The binomial option pricing model is a discrete time approach to pricing options proposed by Cox et al. 1979. This approach is also referred to as the CRR model after the initials of the authors of the paper. It was shown that in the limiting case the CRR model converges to the closed form solution for the price of the European call option as arrived by Black-Scholes-Merton (Black and Scholes 1973; Merton 1973). The CRR model serves as a numerical approach for pricing of contingent claims where analytical solution of the price is not always possible to achieve. Unlike the Black-Scholes-Merton formulation for the European call option, in the CRR model it is assumed trading takes place at discrete time periods. Asian option is an option contract whose payoff depends on the average value of the price of the underlying till the expiry. Unlike the vanilla call or put option whose payoff is dependent only on the terminal value of the asset price, the price of the Asian option depends on the entire price path. The Asian option is therefore a path-dependent option. Formally, the Asian call option has the following payoffs, where V_0 denotes the price of the option at period 0 and K denotes the strike price,

1. Geometric Asian option:

$$V_0 = \max\left(\left(\prod_{i=0}^{n} S_i\right)^{\frac{1}{n+1}} - K, 0\right)$$

2. Arithmetic Asian option:

$$V_0 = \max\left(\left(\frac{\sum_{i=0}^{n} S_i}{n+1}\right) - K, 0\right)$$

The binomial option pricing model has been studied previously in the context of pricing Asian options (Chalasani, Jha, Egriboyun, and

Varikooty 1999; Chalasani, Jha, and Varikooty 1998; Dai 2003; Hsu and Lyuu 2011; K. I. Kim and Qian 2007). In particular the exact analytical pricing of Arithmetic Asian options has been realised to be a difficult problem even under the simplifying Black-Scholes assumptions (Milevsky and Posner 1998). Several bounds for Arithmetic Asian options have been reported (Albrecher et al. 2008; Choe and M. Kim 2021; Chung and Wong 2014a; Fusai et al. 2008; Nielsen and Sandmann 2003). Other analytical and numerical approaches to pricing Arithmetic Asian options include, Choi 2018; Chung and Wong 2014b; Ding et al. 2023; Kemna and Vorst 1990; Kirkby 2016; Linetsky 2004; Mudzimbabwe et al. 2012; Sun et al. 2013; Vecer 2001.

Price impact, as defined by Bouchaud 2009, is the correlation between an incoming order and the subsequent price change. This implies that an increase in buy orders pushes prices upward, while an increase in sell orders pushes prices downward. The empirical presence of price impact is well documented (Hasbrouck and Seppi 2001; Madhavan et al. 1997; Said et al. 2021; Webster 2023), motivating the need to incorporate its effects into option pricing models. Guéant and Pu 2017 formulates the pricing problem under market impact as a stochastic control problem. Loeper 2018 also studies the problem of pricing under a linear market impact. To the best of our knowledge there is no article studying Asian option valuation under price impact.

The paper is structured as follows. In Section 2 we describe the Cox-Ross-Rubinstein (CRR) model and its extension under price impact. We derive the risk-neutral probability under market impact and the no-arbitrage conditions. In Section 3 we use this model to price geometric Asian options and obtain an exact representation for its price. In Section 4 we obtain two sided bounds for the arithmetic Asian options. In Section 5 we check the numerical accuracy of the model against a benchmark approach without impac. We next use the model to illustrate the effect of price impact on the prices. In Section 6 we conclude the article and mention some potential extensions that can be considered.

2 CRR model with price impact

In the standard CRR (binomial) pricing model, time is divided into $n \in \mathbb{N}$ discrete periods. For some $m \in \mathbb{N}$ with $m+1 < n$, let the stock price at period m be S_m. The price evolves according to,

$$S_{m+1} = \begin{cases} uS_m, & \text{with probability } q, \\ dS_m, & \text{with probability } 1-q. \end{cases}$$

where q denotes the probability of an upward movement by factor u, and $1-q$ the probability of a downward movement by factor d. Since the stock can take two possible values at each step, the model is referred to as the binomial option pricing model.

Incorporating price impact into this framework requires modifying the transition dynamics. We consider a linear and permanent impact specification following Bouchaud 2009, which itself is motivated by the seminal model of Kyle 1985. In the Kyle model, trades from informed and noise traders are cleared by a market maker, and the execution of volume v shares produces a predictable shift in the asset price. When hedging an option, the market maker must trade v shares, generating an impact of the form,

$$\Delta S = \lambda v$$

where $\lambda > 0$ is the price-impact coefficient, v_b and v_s denote buy and sell volumes, respectively, and $v = v_b - v_s$ is the net traded volume.

The Kyle specification assumes that impact is linear in trade size and permanent in its effect on the price. Although empirical studies suggest that market impact is typically concave in volume and partially transient (see Bouchaud 2009), the linear and permanent formulation offers analytical tractability and lends itself to incorporation within the CRR framework. For these reasons, we adopt this specification in what follows.

We can rewrite the Kyle impact model as follows,

$$\frac{\Delta p_n}{p_{n-1}} = \frac{\lambda v_n}{p_{n-1}}$$

For small relative changes,

$$\frac{p_n}{p_{n-1}} = 1 + \frac{\lambda v_n}{p_{n-1}}$$

For small impacts where $\frac{\lambda \epsilon_n v_n}{p_{n-1}} \ll 1$,

$$\log\left(1 + \frac{\lambda \epsilon_n v_n}{p_{n-1}}\right) \approx \frac{\lambda \epsilon_n v_n}{p_{n-1}}$$

Define a parameter $\tilde{\lambda}$ such that,

$$\frac{\lambda v_n}{p_{n-1}} = \tilde{\lambda} v_n$$

This gives us,

$$\log(p_n) = \log(p_{n-1}) + \tilde{\lambda} v_n$$

and we obtain,

$$p_n = p_{n-1} \cdot e^{\tilde{\lambda} v_n}$$

We shall use λ instead of $\tilde{\lambda}$ from henceforth. Under price impact, the binomial model therefore becomes,

$$S_{m+1} = \begin{cases} S_u = uS_m e^{\lambda v^u}, & \text{with probability } q, \\ S_d = dS_m e^{-\lambda v^d}, & \text{with probability } 1-q. \end{cases}$$

where v^u and v^d are the net trading volume in the up and down movements respectively. The adjusted up ($\tilde{u}$) and adjusted down ($\tilde{d}$) factors are,

$$\begin{aligned} \tilde{u} &= ue^{\lambda v^u} \\ \tilde{d} &= de^{-\lambda v^d} \end{aligned}$$

The binomial transitions can therefore be rewritten as,

$$S_{m+1} = \begin{cases} S_u = \tilde{u} S_m, & \text{with probability } q, \\ S_d = \tilde{d} S_m, & \text{with probability } 1-q. \end{cases}$$

At any node of the binomial tree, the value of the option may be replicated by constructing a portfolio consisting of Δ shares of the underlying asset and a position B in the risk-free bond. The central replication requirement is that this portfolio must match the option value in both possible successor states. Denoting the option values in the up and down states by V_u and V_d, respectively, and letting r denote the gross risk-free return, the replication conditions are

$$\Delta S_u + Br = V_u, \qquad \Delta S_d + Br = V_d.$$

In the presence of price impact, the underlying transitions are given by $S_u = \tilde{u} S_m$ and $S_d = \tilde{d} S_m$, where $\tilde{u}$ and $\tilde{d}$ denote the impact-adjusted multipliers.

Subtracting the two replication equations eliminates B and yields an expression for the hedge ratio,

$$\Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{V_u - V_d}{S_m(\tilde{u} - \tilde{d})}.$$

This quantity represents the number of shares that must be held in order to replicate the instantaneous change in the option value at the node.

Once Δ is determined, the bond position B follows immediately from either replication equation. Using the down-state relation, we obtain

$$B = \frac{V_d - \Delta S_d}{r} = \frac{V_d - \Delta S\tilde{d}}{r}.$$

An analogous expression follows from the up-state value, and both necessarily agree. The current value of the option is then the cost of establishing this replicating portfolio,

$$V = \Delta S_m + B,$$

which constitutes the unique no-arbitrage price at the node.

The replication relations allow us to rewrite the option value in risk-neutral form. Substituting the expressions for Δ and B into $V = \Delta S + B$ and simplifying yields,

$$V = \frac{1}{r}\left[\frac{r - \tilde{d}}{\tilde{u} - \tilde{d}} V_u + \frac{\tilde{u} - r}{\tilde{u} - \tilde{d}} V_d\right].$$

This motivates the definition of the adjusted risk-neutral probability

$$p^{\mathrm{adj}} = \frac{r - \tilde{d}}{\tilde{u} - \tilde{d}}, \qquad 1 - p^{\mathrm{adj}} = \frac{\tilde{u} - r}{\tilde{u} - \tilde{d}},$$

so that the option value takes the familiar binomial pricing form

$$V = \frac{1}{r}\left[p^{\mathrm{adj}} V_u + (1 - p^{\mathrm{adj}}) V_d\right].$$

Lemma 2.1.

$$\mathbb{E}^{p^{adj}}\left[\frac{S_{m+1}}{S_m}\right] = r,$$

The no-arbitrage condition requires that, under the effective risk-neutral probability, the expected gross return on the underlying equals the risk-free rate. Lemma 2.1 shows the consistency of the pricing measure.

The adjusted risk-neutral probability can be written explicitly as

$$p^{\text{adj}} = \frac{r - de^{-\lambda v^d}}{ue^{\lambda v^u} - de^{-\lambda v^d}}.$$

When the volumes v^u and v^d are constant across time and nodes, the effective risk-neutral probability remains constant throughout the tree. This property greatly simplifies numerical valuation and plays an important role in enabling tractable pricing in the presence of permanent linear price impact.

In the classical CRR model without price impact, the risk-neutral probability,

$$p = \frac{r - d}{u - d}$$

is determined by the requirement that the expected return of the stock under the risk-neutral measure equals the risk-free rate. Importantly, this probability does not represent the real-world likelihood of an upward movement, it is a probability chosen to enforce the no-arbitrage condition that discounted asset prices must evolve as martingales.

When price impact is introduced, the stock no longer moves according to the factors u and d but rather according to the adjusted multipliers $\tilde{u}$ and $\tilde{d}$. Since hedging trades now affect future prices, the replication strategy changes, and so does the probability that ensures the martingale property of the discounted stock price. The adjusted probability

$$p^{\text{adj}} = \frac{r - \tilde{d}}{\tilde{u} - \tilde{d}}$$

therefore incorporates the additional cost and distortion induced by permanent price impact. In the limit $\lambda \to 0$ (no price impact), we recover the standard CRR expression, illustrating that the classical model is a special case of the impact-adjusted framework.

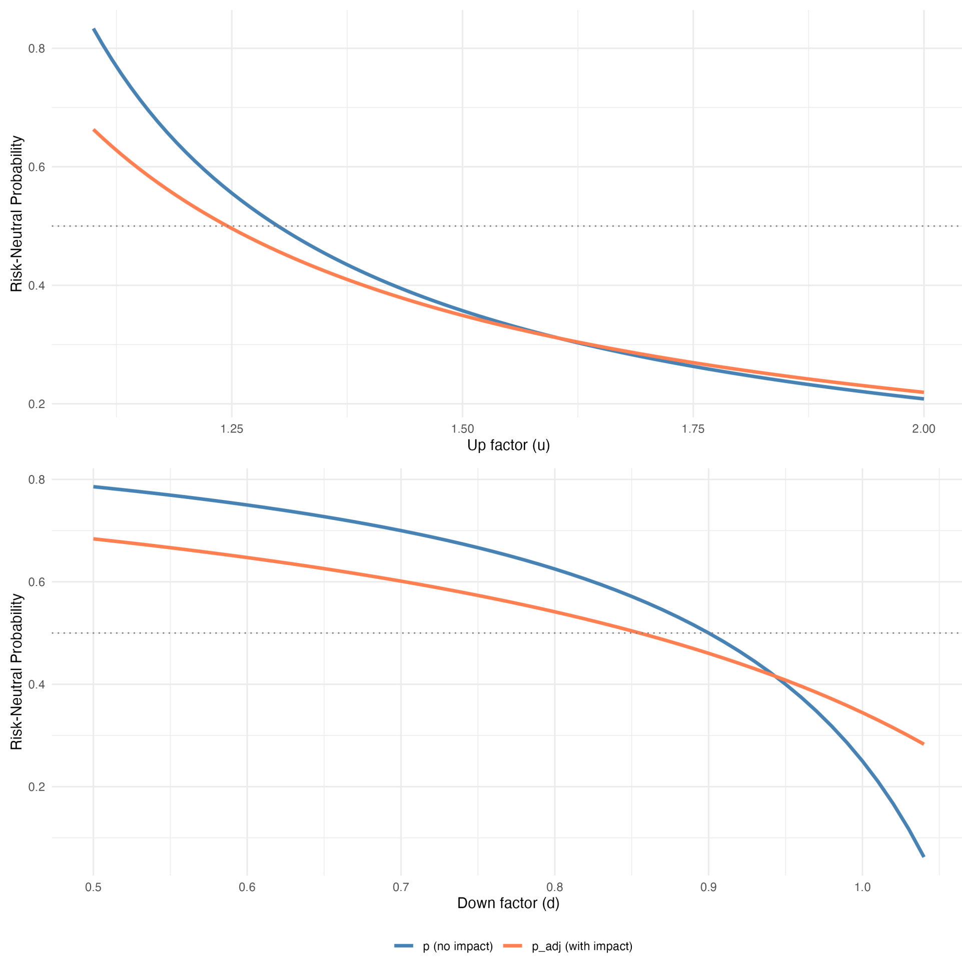


Figure 1: Risk–neutral probabilities in the classical CRR model (blue) and in the impact–adjusted model (orange).

Figure 1 illustrates how permanent linear price impact modifies the risk–neutral measure. The blue curves show the classical CRR probability p, while the orange curves display the impact–adjusted probability p^{adj}.

2.1 No-arbitrage region and admissible hedging volumes

The impact-adjusted CRR model remains arbitrage-free only if the adjusted risk-neutral probability p^{adj} lies in $[0,1]$. From the replicating portfolio construction above, we have,

$$p^{\mathrm{adj}} = \frac{r-\tilde{d}}{\tilde{u}-\tilde{d}},$$

For p^{adj} to define a valid risk-neutral probability, we require,

$$0 \leq p^{\mathrm{adj}} \leq 1.$$

Since we are in a binomial setting, we assume $\tilde{u} > \tilde{d}$, so that the denominator $\tilde{u} - \tilde{d}$ is strictly positive. The lower bound $p^{\mathrm{adj}} \geq 0$ then reduces to

$$\frac{r-\tilde{d}}{\tilde{u}-\tilde{d}} \geq 0 \quad \Longleftrightarrow \quad r-\tilde{d} \geq 0 \quad \Longleftrightarrow \quad r \geq \tilde{d}.$$

Similarly, the upper bound $p^{\mathrm{adj}} \leq 1$ is equivalent to

$$\frac{r-\tilde{d}}{\tilde{u}-\tilde{d}} \leq 1 \quad \Longleftrightarrow \quad r-\tilde{d} \leq \tilde{u}-\tilde{d} \quad \Longleftrightarrow \quad r \leq \tilde{u}.$$

Combining these two inequalities yields the familiar no-arbitrage restriction from the CRR model, now expressed in terms of the impact-adjusted factors:

$$\tilde{d} \leq r \leq \tilde{u}.$$

In words, the risk-free return must lie between the effective down and up multipliers. If $r \leq \tilde{d}$, the stock almost surely dominates the bond, leading to an arbitrage by going long the stock and short the bond. If $r \geq \tilde{u}$, the bond dominates the stock, and an arbitrage arises by shorting the stock and lending at the risk-free rate. The condition $\tilde{d} < r < \tilde{u}$ is therefore the impact-adjusted analogue of the standard CRR no-arbitrage condition $d < r < u$.

Substituting the explicit expressions for $\tilde{u}$ and $\tilde{d}$,

$$\tilde{d} = de^{-\lambda v^d}, \qquad \tilde{u} = ue^{\lambda v^u},$$

we can rewrite the no-arbitrage inequalities as constraints on the hedging volumes:

$$de^{-\lambda v^d} \leq r \leq ue^{\lambda v^u}.$$

These may be rearranged to give minimal admissible volumes,

$$v^d \;\geq\; -\frac{1}{\lambda}\,\ln\!\left(\frac{r}{d}\right) \;=:\; v^d_{\min}, \qquad v^u \;\geq\; \frac{1}{\lambda}\,\ln\!\left(\frac{r}{u}\right) \;=:\; v^u_{\min}.$$

The "no-arbitrage region" in (v^u, v^d)-space is thus described by the set of pairs satisfying

$$v^d \geq v^d_{\min}, \qquad v^u \geq v^u_{\min}.$$

Under the standard CRR assumption $d < r < u$, we have $\ln(r/d) > 0$ and $\ln(r/u) < 0$, so that $v^d_{\min} < 0$ and $v^u_{\min} < 0$. If, in addition, hedging volumes are constrained to be non-negative, $v^u, v^d \geq 0$, these inequalities are automatically satisfied. In this case, every non-negative choice of hedging volumes is compatible with an arbitrage-free, impact-adjusted binomial model, and the condition $0 \leq p^{\text{adj}} \leq 1$ holds without further restriction.

3 Application to Geometric Asian Options

In this section we apply the impact-adjusted binomial framework to the valuation of geometric Asian call options. We consider an underlying asset following the impacted binomial dynamics described above, with impact-adjusted up and down factors $\tilde{u}$ and $\tilde{d}$ and adjusted risk-neutral probability p^{adj}.

A geometric Asian call option with maturity at time n has payoff

$$V_n = \max\left(0, G_n - K\right),$$

where K denotes the strike price and

$$G_n = \left(\prod_{i=0}^{n} S_i\right)^{1/(n+1)}$$

is the geometric average of the underlying prices along the path up to maturity. In contrast to a standard European call option, whose payoff depends only on the terminal value S_n, the geometric Asian payoff depends on the entire price path $(S_0, \ldots, S_n)$ and is therefore path-dependent. In particular, two paths that lead to the same terminal value S_n may nevertheless generate different geometric averages and hence different payoffs.

For a fixed horizon n, a path can be represented by a word in $\{U, D\}^n$, such as UDU, indicating the succession of up and down transitions. For any given path, we write $G(\text{path})$ for the associated geometric average, $V(\text{path})$ for the corresponding payoff

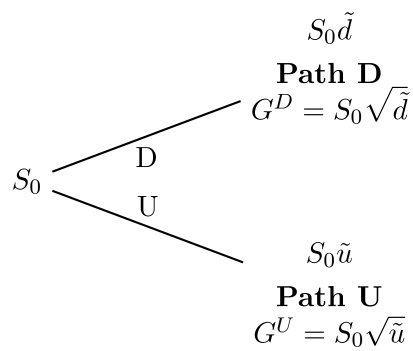
$$V(\text{path}) = \max\big(0, G(\text{path}) - K\big),$$

and $P(\text{path})$ for its risk-neutral probability under the adjusted measure. Since the dynamics are binomial, the probability depends only on the number of up and down moves:

$$P(\text{path}) = \big(p^{\text{adj}}\big)^{\#U(\text{path})}\big(1 - p^{\text{adj}}\big)^{\#D(\text{path})},$$

where $\#U(\text{path})$ and $\#D(\text{path})$ denote the number of up and down moves in the path, with $\#U(\text{path}) + \#D(\text{path}) = n$.

3.1 The case $n = 1$



We begin with the simplest non-trivial case $n = 1$, where the option is defined over two dates $t = 0, 1$. There are exactly two paths, U and D. Starting from S_0, the terminal stock prices are

$$S_1^U = S_0\tilde{u}, \qquad S_1^D = S_0\tilde{d},$$

with corresponding geometric averages

$$G^U = (S_0 \cdot S_0\tilde{u})^{1/2} = S_0\sqrt{\tilde{u}}, \qquad G^D = (S_0 \cdot S_0\tilde{d})^{1/2} = S_0\sqrt{\tilde{d}}.$$

The pathwise payoffs are therefore

$$V^U = \max\left(0, S_0\sqrt{\tilde{u}} - K\right), \qquad V^D = \max\left(0, S_0\sqrt{\tilde{d}} - K\right),$$

and the associated risk-neutral probabilities are

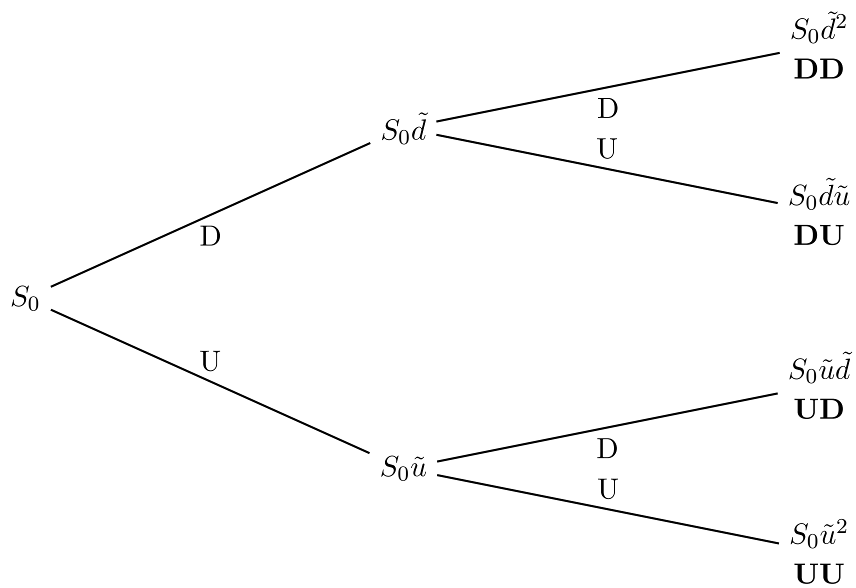
$$P(U) = p^{\text{adj}}, \qquad P(D) = 1 - p^{\text{adj}}.$$

Discounting one period at the risk-free rate r, the option value at time 0 is

$$V_0 = \frac{1}{r}\left[p^{\text{adj}}V^U + (1-p^{\text{adj}})V^D\right] = \frac{1}{r}\left[p^{\text{adj}} \max\left(0, S_0\sqrt{\tilde{u}} - K\right) + (1 - p^{\text{adj}}) \max\left(0, S_0\sqrt{\tilde{d}} - K\right)\right].$$

In this case the geometric feature of the payoff is fully captured by the two effective averages $S_0\sqrt{\tilde{u}}$ and $S_0\sqrt{\tilde{d}}$, so the structure remains close to that of a standard one-period binomial model.

3.2 The case $n = 2$



For $n = 2$ there are four possible paths: UU, UD, DU and DD. The corresponding terminal prices are

$$S_2^{UU} = S_0\tilde{u}^2, \quad S_2^{UD} = S_2^{DU} = S_0\tilde{u}\tilde{d}, \quad S_2^{DD} = S_0\tilde{d}^2.$$

Although UD and DU share the same terminal value, they generate distinct sequences of intermediate prices and hence distinct geometric averages. Denoting the geometric average associated with a given path by G^{path}, a straightforward calculation shows that

$$G^{UU} = S_0\tilde{u}, \qquad G^{UD} = S_0(\tilde{u}^2\tilde{d})^{1/3}, \qquad G^{DU} = S_0(\tilde{d}^2\tilde{u})^{1/3}, \qquad G^{DD} = S_0\tilde{d}.$$

The corresponding payoffs are

$$V^{\text{path}} = \max\Big(0, G^{\text{path}} - K\Big), \qquad \text{for path} \in \{UU, UD, DU, DD\},$$

and the path probabilities under p^{adj} are given by the usual binomial expressions,

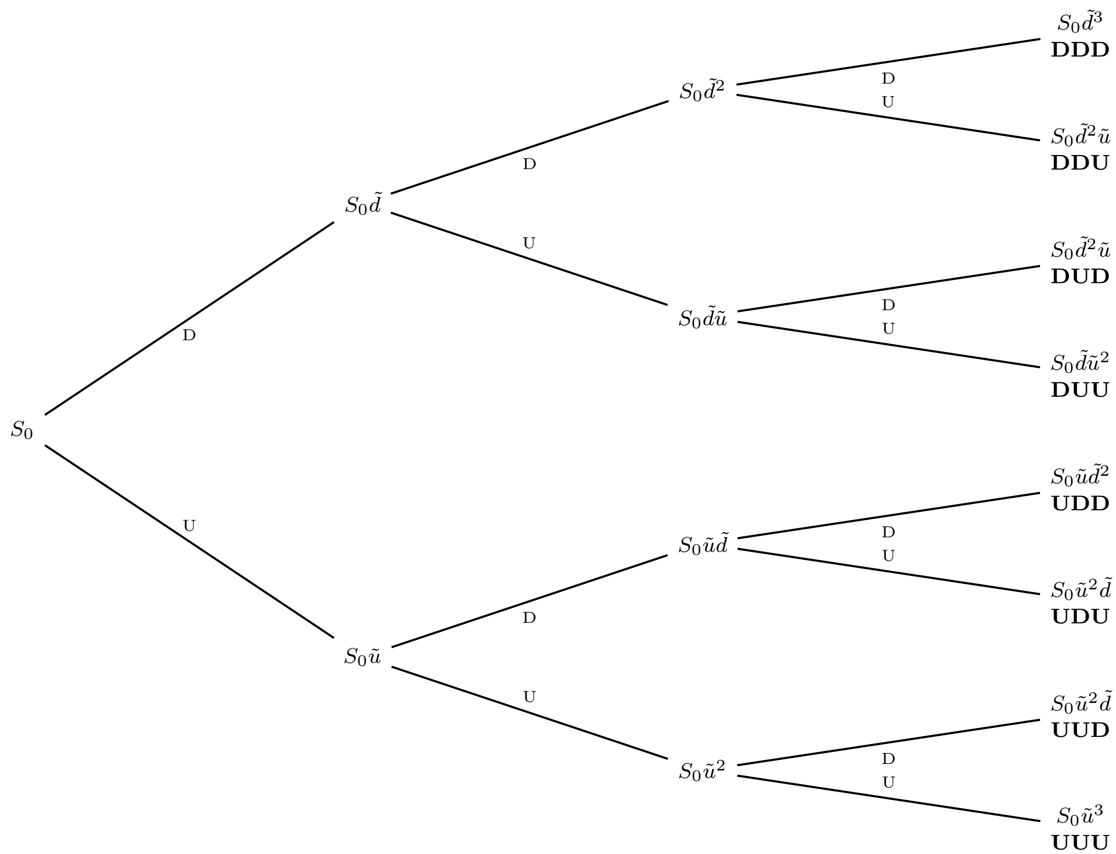
$$P(UU) = (p^{\text{adj}})^2, \quad P(UD) = P(DU) = p^{\text{adj}}(1-p^{\text{adj}}), \quad P(DD) = (1-p^{\text{adj}})^2.$$

Discounting over two periods, the time–0 value is therefore

$$V_0 = \frac{1}{r^2} \sum_{\text{path}\in\{UU,UD,DU,DD\}} P(\text{path})\, V^{\text{path}}.$$

This case already illustrates a key qualitative feature: the stock-price tree itself recombines, but the geometric average does not, so paths that meet at the same terminal stock price can yield different option values.

3.3 The case $n = 3$ and the general pattern



For $n = 3$ the number of possible paths increases to $2^3 = 8$. Each path induces a sequence of stock prices (S_0, S_1, S_2, S_3) and hence a geometric average

$$G(\text{path}) = \left(\prod_{i=0}^{3} S_i \right)^{1/4},$$

with payoff $V(\text{path}) = \max\big(0, G(\text{path}) - K\big)$ and probability $P(\text{path})$ determined by the number of up and down moves. The time–0 value is obtained by discounting the pathwise expectation:

$$V_0 = \frac{1}{r^3} \sum_{\text{all 8 paths}} P(\text{path})\, V(\text{path}).$$

Already at this horizon, explicit enumeration becomes unwieldy, and it is more instructive to describe the general combinatorial structure.

The pattern for general n can be characterised as follows. At time i, the stock price can be written as

$$S_i = S_0 \tilde{u}^{a_i} \tilde{d}^{b_i},$$

where a_i and b_i denote the cumulative numbers of up and down moves up to time i, satisfying $a_i + b_i = i$. For a given path, define

$$A(\text{path}) = \sum_{i=0}^{n} a_i, \qquad B(\text{path}) = \sum_{i=0}^{n} b_i.$$

Since $a_i + b_i = i$ for each i, we have the identity

$$A(\text{path}) + B(\text{path}) = \sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

The geometric average along that path can then be expressed as

$$G(\text{path}) = \left(\prod_{i=0}^{n} S_i \right)^{1/(n+1)} = S_0 \left(\tilde{u}^{A(\text{path})} \tilde{d}^{B(\text{path})} \right)^{1/(n+1)} = S_0 \tilde{u}^{A(\text{path})/(n+1)} \tilde{d}^{B(\text{path})/(n+1)}.$$

The time 0 price of the geometric Asian option can therefore be written in the general form as,

$$V_0 = \frac{1}{r^n} \sum_{\text{all } 2^n \text{ paths}} \left(p^{\text{adj}}\right)^{\#U(\text{path})} \left(1-p^{\text{adj}}\right)^{n-\#U(\text{path})} \max\left(0, S_0 \tilde{u}^{A(\text{path})/(n+1)} \tilde{d}^{B(\text{path})/(n+1)} - K\right).$$

We summarise this representation in the following theorem,

Theorem 3.1 (Geometric Asian option under price impact)**.** *Consider an underlying asset following the impact adjusted binomial stock price model with up and down factors $\tilde{u}$ and $\tilde{d}$ and the impact adjusted risk-neutral probability p^{adj}. Let $n \in \mathbb{N}$ denote the expiry in discrete periods, and let $K > 0$ be the strike price of a geometric Asian call option written on this asset. Then the time 0 price of the option is given by,*

$$V_0 = \frac{1}{r^n} \sum_{\omega \in \{U,D\}^n} (p^{adj})^{\#U(\omega)} (1 - p^{adj})^{\#D(\omega)} \max\left(0, G(\omega) - K\right),$$

where $r > 0$ denotes the gross risk-free rate, and $G(\omega)$ is the geometric average of the underlying prices along the path ω.

More precisely, each path $\omega = (\omega_1, \ldots, \omega_n) \in \{U, D\}^n$ induces a sequence of stock prices $(S_0, S_1, \ldots, S_n)$ with,

$$S_i = S_0 \tilde{u}^{a_i(\omega)} \tilde{d}^{b_i(\omega)}, \qquad i = 0, \ldots, n,$$

where $a_i(\omega)$ and $b_i(\omega)$ denote the cumulative numbers of up and down moves up to time i, satisfying $a_i(\omega) + b_i(\omega) = i$. Define,

$$A(\omega) = \sum_{i=0}^{n} a_i(\omega), \qquad B(\omega) = \sum_{i=0}^{n} b_i(\omega),$$

so that,

$$A(\omega) + B(\omega) = \sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

Then the geometric average along the path ω can be written as,

$$G(\omega) = \left(\prod_{i=0}^{n} S_i \right)^{1/(n+1)} = S_0 \big(\tilde{u}^{A(\omega)} \tilde{d}^{B(\omega)} \big)^{1/(n+1)}.$$

Here $\#U(\omega)$ and $\#D(\omega)$ denote the total numbers of up and down moves in the path ω, with $\#U(\omega) + \#D(\omega) = n$.

4 Bounds for the Arithmetic Asian Option Price

The valuation of an arithmetic Asian option in the CRR framework is well known to be analytically challenging, even in the absence of price impact. The difficulty stems from the fact that the arithmetic average does not preserve the multiplicative structure of the binomial model, and therefore does not admit recombination in the same way as the geometric average. Under price impact, this complexity is exacerbated by the dependence of the effective up and down multipliers on hedging trades. In this section we develop tractable bounds for the arithmetic Asian call price by exploiting convexity properties of the average, namely the classical AM–GM inequality and a reverse AM–GM inequality due to Budimir et al. 2001.

Let $\omega \in \{U, D\}^n$ denote a path of up/down moves and let $\{S_i(\omega)\}_{i=0}^{n}$ be the corresponding sequence of stock prices under the impact-adjusted

CRR dynamics. Define the arithmetic and geometric averages

$$A_n(\omega) = \frac{1}{n+1} \sum_{i=0}^{n} S_i(\omega), \qquad G_n(\omega) = \left(\prod_{i=0}^{n} S_i(\omega) \right)^{1/(n+1)},$$

and the pathwise call payoffs

$$V_A(\omega) = \max\big(0, A_n(\omega) - K\big), \qquad V_G(\omega) = \max\big(0, G_n(\omega) - K\big).$$

We denote by V_0^A and V_0^G the corresponding option prices at time 0 obtained by discounting risk-neutral expectations under the adjusted probability measure Q.

4.1 A lower bound

The first inequality follows immediately from Jensen-type arguments. Because the arithmetic mean always dominates the geometric mean, the arithmetic Asian payoff is pathwise larger than the geometric one, and this relation is preserved under discounting.

Proposition 4.1 (Lower bound via AM–GM)**.** *For every maturity n,*

$$V_0^A \geq V_0^G.$$

Proof. Since $A_n(\omega) \geq G_n(\omega)$ for all paths ω, it follows that $V_A(\omega) \geq V_G(\omega)$. Taking risk-neutral expectations and discounting yields the result. □

This lower bound is tight and requires no structural assumptions on the price-impact parameters. It also illustrates a general phenomenon: price impact affects both the arithmetic and geometric averages, but their ordering remains preserved.

4.2 Upper bounds

While the geometric average provides a natural lower bound, obtaining an upper bound requires more delicate control. A key observation is that the discrepancy between arithmetic and geometric averages can be dominated by the range of stock prices along the path. Let

$$S_m(\omega) = \min_{0 \leq i \leq n} S_i(\omega), \qquad S_M(\omega) = \max_{0 \leq i \leq n} S_i(\omega)$$

be the extremal prices along the path. A reverse AM-GM inequality due to Budimir et al. 2001, Proposition 1, yields a multiplicative bound relating the two averages.

Proposition 4.2 (Upper bound via reverse AM-GM inequality)**.** *For any path $\omega \in \{U, D\}^n$,*

$$A_n(\omega) \leq G_n(\omega)\,\rho(\omega),$$

where

$$\rho(\omega) = \exp\left[\frac{1}{4} \cdot \frac{(S_M(\omega) - S_m(\omega))^2}{S_m(\omega)\,S_M(\omega)}\right].$$

Combining this bound with the structure of the payoff allows us to control the difference between the arithmetic and geometric Asian payoffs. The next result provides a pathwise comparison.

Proposition 4.3 (Pathwise control of payoff differences)**.** *For every path $\omega \in \{U, D\}^n$,*

$$V_A(\omega) - V_G(\omega) \leq G_n(\omega)\,(\rho(\omega) - 1).$$

Proof. The inequality follows from Proposition 4.2 together with a simple case distinction on whether $A_n(\omega)$ and $G_n(\omega)$ lie above or below the strike K. □

Taking expectations yields a pathwise upper bound for the arithmetic Asian price, which is sharp but depends on the path-dependent quantity $\rho(\omega)$.

Proposition 4.4 (Pathwise upper bound)**.** *The arithmetic Asian call price satisfies*

$$V_0^A - V_0^G \leq \frac{1}{r^n} \sum_{\omega \in \{U,D\}^n} P(\omega)\,\big(\rho(\omega) - 1\big)\,G_n(\omega),$$

where $P(\omega)$ denotes the impact-adjusted risk-neutral path probability.

This bound is always tighter than any global bound obtained by replacing $\rho(\omega)$ with a uniform constant, but its computation requires summation over all 2^n paths.

To obtain a more tractable bound, we observe that the extremal prices along any path are themselves bounded by the largest and smallest values attainable in the tree. Under the impact-adjusted CRR dynamics,

$$S_m(\omega) \geq S_0 \tilde{d}^n, \qquad S_M(\omega) \leq S_0 \tilde{u}^n,$$

so that $\rho(\omega)$ is uniformly dominated by

$$\rho^* = \exp\left[\frac{1}{4} \cdot \frac{(\tilde{u}^n - \tilde{d}^n)^2}{\tilde{u}^n \tilde{d}^n}\right].$$

This leads to the following global upper bound.

Proposition 4.5 (Global upper bound)**.** *The arithmetic Asian call price satisfies*

$$V_0^A - V_0^G \le \frac{\rho^* - 1}{r^n}\, \mathbb{E}^Q[G_n],$$

where G_n is the geometric average along a random path under the adjusted risk-neutral measure.

The global bound is weaker but computationally simple, depending only on the impact-adjusted factors $\tilde{u}$ and $\tilde{d}$. As λ increases, the spread between $\tilde{u}^n$ and $\tilde{d}^n$ widens, making ρ^* larger and the bound looser. Likewise, as n grows, extremal path dispersion increases, causing ρ^* to diverge. These effects reflect a fundamental property of arithmetic averaging: the longer the horizon or the larger the impact-induced price variation, the more the arithmetic average can exceed the geometric one.

Theorem 4.1 (Two-sided bounds)**.** *For the arithmetic Asian call option under the impact-adjusted CRR model,*

$$V_0^G \;\le\; V_0^A \;\le\; V_0^G + \frac{\rho^* - 1}{r^n}\, \mathbb{E}^Q[G_n].$$

Finally, we note that the pathwise bound in Proposition 4.4 always improves on the global bound.

Proposition 4.6 (Tightness)**.** *Let*

$$A := \frac{1}{r^n} \sum_{\omega} P(\omega)\, (\rho(\omega) - 1) G_n(\omega), \qquad B := \frac{\rho^* - 1}{r^n}\, \mathbb{E}^Q[G_n].$$

Then $A \le B$.

Proof. Since $\rho(\omega) \le \rho^*$ for every path and $G_n(\omega) \ge 0$, summing against $P(\omega)$ gives the desired inequality. □

Taken together, these results show that the geometric Asian price provides a natural and tight lower bound for the arithmetic Asian price, while a family of tractable upper bounds can be obtained by combining reverse Jensen-type inequalities with extremal path bounds. Pathwise bounds are sharper but computationally expensive, whereas global bounds are coarser but fully explicit in the price-impact parameters.

5 Numerical Illustration

In this section we present numerical illustrations highlighting the approximation quality of the impact–adjusted CRR scheme when applied to Asian options.

5.1 Baseline case with no price impact

Although the model admits no closed-form price in the presence of price impact, the frictionless case provides a natural benchmark via the Kemna–Vorst formula (Kemna and Vorst 1990), which yields an exact price in continuous time for geometric averages under lognormal dynamics. To assess accuracy, we compute the CRR price for increasing numbers of time steps n, keeping the effective up and down factors fixed, and report its deviation from the Kemna–Vorst benchmark.

Table 1 summarizes the results. The CRR approximation converges towards the benchmark as the number of time subdivisions increases. The absolute error decreases steadily, and the relative percentage error stabilizes around 6%–7% for moderate values of n.

n	CRR Price	Kemna-Vorst Price	Absolute Difference	% Error
2	7.14	7.58	0.43	5.73
4	9.15	9.92	0.76	7.68
6	10.74	11.61	0.87	7.49
8	12.03	12.98	0.95	7.29
10	13.13	14.13	0.99	7.03
12	14.08	15.12	1.04	6.89
14	14.92	15.99	1.07	6.71
16	15.67	16.77	1.11	6.60
18	16.34	17.48	1.14	6.50
20	16.96	18.11	1.16	6.40

Table 1: Numerical comparison between the CRR model with no impact and the Kemna-Vorst pricing formula for geometric Asian call options, with CRR parameters as $S_0 = 100$, $K = 100$, $r = 1.05$, $u = 1.2$, $d = 0.8$.

Table 2 summarizes the results for arithmetic Asian call options. For arithmetic Asian options we report the upper bounds given in 4.1. We use the pathwise upper bound from Proposition 4.4. We find that for smaller values of n, the upper bound is closer to the benchmark price, which suggests bounds may be tighter for small values of n.

n	Lower Bound Price	Upper Bound Price	Kemna-Vorst Price
2	7.14	9.71	8.14
4	9.15	16.09	10.87
6	10.74	22.93	12.95
8	12.03	30.43	14.67
10	13.13	38.90	16.20
12	14.08	48.85	17.57
14	14.92	61.26	18.79
16	15.67	78.95	19.91
18	16.34	138.15	21.01
20	16.96	12961.48	21.99

Table 2: Numerical comparison between the CRR model with no impact and the Kemna-Vorst pricing formula for arithmetic Asian call options, with CRR parameters as $S_0 = 100$, $K = 100$, $r = 1.05$, $u = 1.2$, $d = 0.8$.

5.2 Sensitivity to price impact

We will now illustrate the sensitivity of our model to price impact.

Table 3 reports geometric Asian prices together with the lower and upper bounds for the arithmetic Asian call derived. First, the geometric price grows steadily with λ in both regimes, reflecting the convexity of the payoff in response to increasingly dispersed price paths. Second, the lower bound remains tight (as expected), while the upper bound expands extremely rapidly with λ. This behaviour is consistent with the structure of the reverse AM-GM inequality, where the multiplicative factor $\rho(\omega)$ depends exponentially on the spread between the pathwise extrema S_m and S_M. Because price impact widens this spread, the bounds deteriorate sharply for moderate to large λ. Finally, the up and down biased cases track each other closely for small impact levels, but diverge as impact grows.

	Up-biased			**Down-biased**		
λ	Geom.	LB(A)	UB(A)	Geom.	LB(A)	UB(A)
0.00	10.74	10.74	22.93	10.74	10.74	22.93
0.05	12.92	12.92	35.01	12.94	12.94	34.98
0.10	14.94	14.94	52.66	14.91	14.91	52.71
0.15	16.67	16.67	82.17	16.65	16.65	82.62
0.20	18.11	18.11	145.61	18.14	18.14	151.49
0.25	19.30	19.30	358.53	19.40	19.40	484.07
0.30	20.24	20.24	2049.94	20.44	20.44	6693.71
0.35	21.01	21.01	54863.30	21.25	21.25	657809.09

Table 3: Sensitivity of geometric Asian prices (Geom.) and arithmetic Asian bounds (LB(A) denotes the lower bound and UB(A) denotes the upper bound) to the price impact magnitude λ, with parameters $S_0 = 100, K = 100, r = 1.05, n = 6$. We consider two different hedging volume regimes which are up-biased ($v_u = 1.3, v_d = 1$) and down-biased ($v_u = 1, v_d = 1.3$).

Figure 2 illustrates two experiments. In the first (solid red curve), we vary v^d while keeping v^u fixed. In the second (dashed blue curve), we vary v^u while keeping v^d fixed. The behaviour in both cases is similar for small and moderate volumes. Increasing v^d continues to raise the option value, whereas increasing v^u eventually produces a mild decline.

This asymmetry arises because upward and downward effective moves enter the geometric mean multiplicatively but with opposite signs in the price-impact adjustment. Larger v^u boosts $\tilde{u}$, steepening the upside branch of the tree, whereas larger v^d strengthens the contraction in down moves.

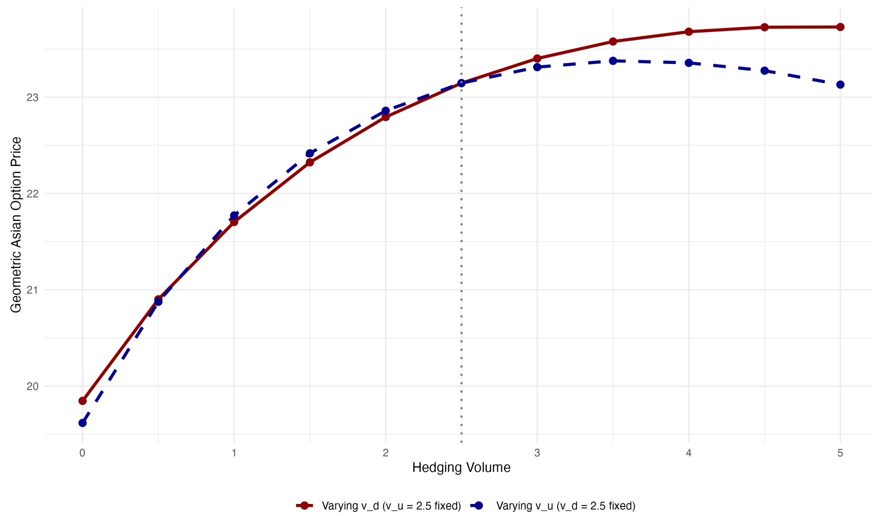


Figure 2: Sensitivity of the geometric Asian call price to hedging volumes. The solid red curve varies v^d while holding $v^u = 2.5$ fixed, and the dashed blue curve varies v^u while holding $v^d = 2.5$ fixed. Parameter values are $S_0 = 100$, $K = 100$, $u = 1.2$, $d = 0.8$, $r = 1.05$, $\lambda = 0.1$, $n = 15$.

We now examine how the geometric Asian option price varies with moneyness (K/S_0) under different hedging regimes and different levels of permanent price impact. Figure 3 reports option values for three representative configurations- down-biased hedging ($v^d > v^u$), symmetric hedging ($v^d = v^u$), and up-biased hedging ($v^u > v^d$).

Across all regimes, option values decrease as the strike increases, reflecting the monotonicity in moneyness. However, the presence of price impact induces a systematic upward shift in the level of option prices.

The magnitude of this effect depends critically on the hedging asymmetry. Under down-biased hedging, where the market maker trades more aggressively in down states, the effective down factor $\tilde{d}$ becomes smaller, producing greater curvature in the lower branch of the tree. This magnifies path variability and results in the largest sensitivity to λ, especially for near-the-money strikes. Symmetric hedging exhibits an intermediate response, while up-biased hedging generates a smaller sensitivity, as the impact predominantly steepens the upper branch of the tree. Interestingly, for high strikes, the three regimes converge, illustrating that extreme out-of-the-money options are less sensitive to the interaction between hedging asymmetry and market impact.

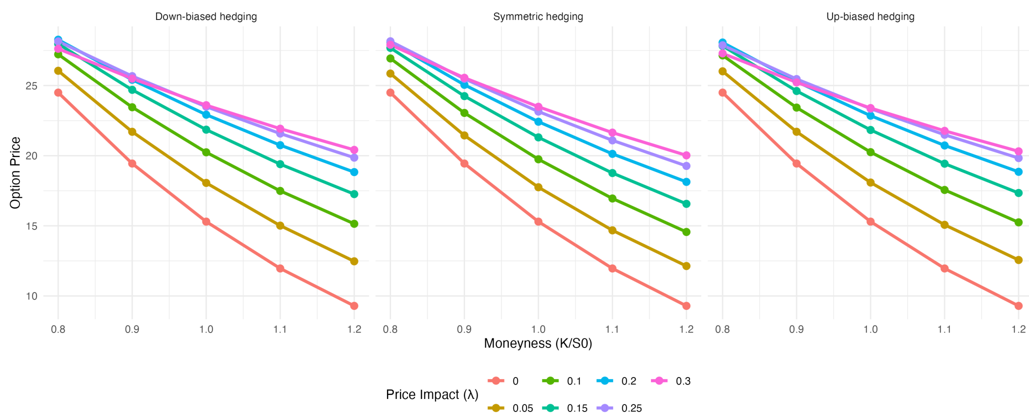


Figure 3: Geometric Asian option prices across moneyness levels (K/S_0) under three hedging regimes- down-biased $(v^d > v^u)$, symmetric $(v^d = v^u)$, and up-biased $(v^u > v^d)$. Each curve corresponds to a different price-impact coefficient λ. The parameters are, up-biased $(v_u = 1.3, v_d = 1)$, down-biased $(v_u = 1, v_d = 1.3)$ and symmetric $(v_u = v_d = 1)$, with $S_0 = 100$, $K = 100$, $u = 1.2$, $d = 0.8$, $r = 1.05$, $n = 15$.

6 Conclusion

This paper develops a tractable framework for valuing Asian options in a binomial market with permanent price impact. By embedding Kyle linear market impact into the Cox–Ross–Rubinstein model, we obtain impact-adjusted up and down factors and a corresponding modified risk–neutral probability. Within this framework, we derived an exact pathwise representation for geometric Asian call options.

For arithmetic Asian options, whose exact valuation is intractable even without impact, we obtain two-sided bounds. The geometric

Asian price yields a tight and model-free lower bound, while an explicit upper bound follows from a reverse AM–GM inequality combined with extremal path analysis. Our analysis also characterises the no-arbitrage region of the impact-adjusted CRR model in terms of admissible hedging volumes.

The numerical illustrations demonstrates several qualitative features of the model. First, geometric Asian prices increase monotonically with the impact parameter, consistent with the convexity of the payoff. Second, the arithmetic bounds behave well for small and moderate maturities but become increasingly loose as either the impact parameter or the horizon grows. Finally, the comparative analysis with respect to hedging volumes shows an asymmetry between up and down biased regimes.

Overall, our results provide a computationally tractable approach to pricing path-dependent options under permanent price impact. Several extensions could be considered including transient or nonlinear impact specifications, multi-asset settings, and optimisation of hedging volumes in the presence of execution costs. The binomial structure developed here could be considered for these extensions.

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