

CENTRAL LIMIT THEOREM FOR A PARTIALLY OBSERVED INTERACTING SYSTEM OF HAWKES PROCESSES I: SUBCRITICAL CASE

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ABSTRACT. We consider a system of N Hawkes processes and observe the actions of a subpopulation of size $K \leq N$ up to time t , where K is large. The influence relationships between each pair of individuals are modeled by i.i.d. Bernoulli(p) random variables, where $p \in [0, 1]$ is an unknown parameter. Each individual acts at a *baseline* rate $\mu > 0$ and, additionally, at an *excitation* rate of the form $N^{-1} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) dZ_s^{j,N}$, which depends on the past actions of all individuals that influence it, scaled by N^{-1} (i.e. the mean-field type), with the influence of older actions discounted through a memory kernel $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Here, μ and ϕ are treated as nuisance parameters. The aim of this paper is to establish a central limit theorem for the estimator of p proposed in [21], under the subcritical condition $\Lambda p < 1$.

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2020 *Mathematics Subject Classification.* 62M09, 60J75, 60K35.

Key words and phrases. Multivariate Hawkes processes, Point processes, Statistical inference, Interaction graph, Stochastic interacting particles, Mean field limit.

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1. INTRODUCTION

Hawkes processes, originally introduced by Hawkes [17] in 1971, have been widely applied across various fields such as neuroscience, finance, social network interactions, and criminology, among others (see, e.g., [1, 4, 5, 15, 23, 24, 30, 32, 33, 35] for a non-exhaustive list). From a mathematical perspective, a substantial body of theoretical literature has been devoted to Hawkes processes and their generalizations (see, e.g., [2, 6, 7, 8, 11, 12, 18, 20, 22, 27, 34] for a non-exhaustive list).

Regarding statistical inference for Hawkes processes, most studies have focused on the fixed finite-dimensional case (i.e., fixed N) with the asymptotics $t \rightarrow \infty$. For parametric models, Ogata [26] investigated the maximum likelihood estimator for stationary point processes. In Bacry-Muzzy [3], Delattre et al. [14], Hansen et al. [16], Reynaud-Bouret et al. [29, 30, 31], the non-parametric estimation are considered for the following system: for fixed $N \geq 1$, and $i, j = 1, \dots, N$, the counting process $(Z_s^{i,N})_{i=1\dots N, 0 \leq s \leq t}$ is governed by its intensity process $(\lambda_s^{i,N})_{i=1\dots N, 0 \leq s \leq t}$, defined by

$$(1) \quad \lambda_t^{i,N} := \mu_i + \sum_{j=1}^N \int_0^{t^-} \phi_{ij}(t-s) dZ_s^{j,N},$$

for $\mu_i > 0$ and $\phi_{ij} : [0, \infty) \rightarrow [0, \infty)$ is measurable and locally integrable. They provided estimators for the μ_i and the functions ϕ_{ij} . In [28], Rasmussen considered the Bayesian inference of one dimensional system: the counting process $(Z_s)_{0 \leq s \leq t}$ is determined by its intensity process $(\lambda_s)_{0 \leq s \leq t}$ of the form

$$(2) \quad \lambda_t := \mu_t + \int_0^{t^-} \phi(t-s) dZ_s,$$

where the rate μ_t depends on time t .

In real-world applications, however, it is often necessary to investigate interactions among a large number of measured components within a system. For example, in neuroscience, the number of neurons involved is typically enormous. Therefore, it is natural to consider a double asymptotic case where both $t \rightarrow \infty$ and $N \rightarrow \infty$. Research in this setting remains scarce. In [13], Delattre and Fournier examined a graphical model comprising N Hawkes point processes with pairwise interactions occurring with probability p . They proposed an estimator for p based on observing the entire system $(Z_s^{i,N})_{i=1,\dots,N, 0 \leq s \leq t}$ and gave the explicit rate $N^{-1/2} + N^{1/2}m_t^{-1}$ (up to some arbitrarily small loss), where m_t denotes the mean number of events per point process. Subsequently, Liu [21] studied the same problem of estimating p in the same setting as [13] but using only partial information, specifically, the information obtained from K Hawkes processes where $K \leq N$. The author established that under $(H(q))$ for some $q > 3$, the estimator $\hat{p}_{N,K,t}$ for p with a rate of convergence $K^{-1/2} + N/(K^{1/2}m_t) + N/(Km_t^{1/2})$. More recently, Chevallier, Löcherbach and Ost [9] investigated a system of N interacting $\{0, 1\}$ -valued chains (rather than Hawkes point processes) with binary interactions occurring with unknown probability p on an underlying Erdős-Rényi random graph. By analyzing coalescing random walks that define a backward regeneration representation of the system, they demonstrated that the unknown connection probability p can be estimated by an computationally efficient estimator with a convergence rate $N^{-1/2} + N^{1/2}t + (\log(t)/t)^{1/2}$. Meanwhile, Chevallier and Ost [10] considered the problem of estimating the sets \mathcal{P}_+ and \mathcal{P}_- without prior knowledge of the remaining model parameters, in the same setting as [9].

1.1. Setting. We consider some unknown parameters $p \in [0, 1]$, $\mu > 0$ and a measurable, locally integrable function $\phi : [0, \infty) \rightarrow [0, \infty)$. For $N \geq 1$, let $(\Pi^i(dt, dz))_{i=1,\dots,N}$ be an i.i.d. family of Poisson random measures on $[0, \infty) \times [0, \infty)$ with intensity $dtdz$. Independent of this family, let $(\theta_{ij})_{i,j=1,\dots,N}$ be an i.i.d. family of Bernoulli(p) random variables. We study the following system: for each $i \in \{1, \dots, N\}$ and all $t \geq 0$,

$$(3) \quad Z_t^{i,N} = \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \lambda_s^{i,N}\}} \Pi^i(ds, dz), \text{ where } \lambda_t^{i,N} = \mu + \frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^{t^-} \phi(t-s) dZ_s^{j,N}.$$

The solution $((Z_t^{i,N})_{t \geq 0})_{i=1,\dots,N}$ is a family of counting processes. By [13, Proposition 1], system (3) admits a unique càdlàg solution that is $(\mathcal{F}_t)_{t \geq 0}$ -measurable, provided that ϕ is locally integrable. Here,

$$\mathcal{F}_t = \sigma(\Pi^i(A) : A \in \mathcal{B}([0, t] \times [0, \infty)), i = 1, \dots, N) \vee \sigma(\theta_{ij}, i, j = 1, \dots, N),$$

where $\mathcal{B}([0, t] \times [0, \infty))$ denotes the Borel σ -algebra on the corresponding product space.

Intuitively, the process $Z_t^{i,N}$ counts the actions of individual i in $[0, t]$. We say that individual j influences individual i if and only if $\theta_{ij} = 1$ (allowing for the possibility that $i = j$). At any time t , the i -th individual acts according to the intensity $\lambda_t^{i,N}$. This intensity consists of two components: a constant *autonomous* rate $\mu > 0$, and an *interaction-driven* component of the form

$$N^{-1} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) dZ_s^{j,N},$$

which models imitation. The interaction term depends on the past actions of all individuals that influence i , weighted by N^{-1} , and discounts the influence of older actions through the memory kernel $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

1.2. **Assumptions.** Define $\Lambda := \int_0^\infty \phi(t)dt \in (0, \infty]$. For some $q \geq 1$,

$$(H(q)) \quad \mu \in (0, \infty), \quad \Lambda \in (0, \infty), \quad \Lambda p \in [0, 1), \quad \int_0^\infty s^q \phi(s)ds < \infty, \quad \int_0^\infty (\phi(s))^2 ds < \infty.$$

1.3. **Model.** Consider a system of N individuals. For each individual $j \in \{1, \dots, N\}$, denote by $S_j = \{i \in \{1, \dots, N\} : \theta_{ij} = 1\}$, the set of individuals connected to j . The only action available to individual i is to send a message to every member of S_i . Here $Z_t^{i,N}$ stands for the total number of messages sent by individual i during $[0, t]$. The counting process $(Z_s^{i,N})_{i=1\dots N, 0 \leq s \leq t}$ is governed by its intensity processes $(\lambda_s^{i,N})_{i=1\dots N, 0 \leq s \leq t}$. Informally, the intensity is defined by

$$P\left(Z_t^{i,N} \text{ has a jump in } [t, t+dt] \mid \mathcal{F}_t\right) = \lambda_t^{i,N} dt, \quad i = 1, \dots, N,$$

where \mathcal{F}_t denotes the sigma-field generated by $(Z_s^{i,N})_{i=1\dots N, 0 \leq s \leq t}$ and $(\theta_{ij})_{i,j=1,\dots,N}$. The rate $\lambda_t^{i,N}$ at which i sends messages can be decomposed into the sum of two components:

- **New messages:** new messages generated at rate μ ;
- **Forwarded messages:** messages that i has received and forwards after some delay (possibly infinite) depending on the age of the message, which contributes a sending rate of the form

$$\frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^{t^-} \phi(t-s) dZ_s^{j,N}.$$

If for example $\phi = \mathbf{1}_{[0, K]}$, then $N^{-1} \sum_{j=1}^N \theta_{ij} \int_0^{t^-} \phi(t-s) dZ_s^{j,N}$ is precisely the number of messages that the i -th individual received between the time $t-K$ and t , divided by N .

Remark 1.1. In [25], we see that the Erdős-Rényi graph can be applied to model a social network. And [5] tells us that Hawkes process can model the number of the messages.

1.4. **Main Goal.** In the present work, we follow the same setting as [21]. Specially, we consider a system of N i.i.d. Hawkes point processes $(Z_s^{i,N})_{i=1\dots N, 0 \leq s \leq t}$ and a family of i.i.d. Bernoulli(p) random variables $(\theta_{ij})_{i,j=1,\dots,N}$, where $p \in [0, 1]$ is an unknown parameter. The interactions among the Hawkes processes are binary and encoded by a directed Erdős-Rényi random graph with p . The objective of this paper is to establish the estimation of p through the limit distribution of the corresponding estimator based on partial observations from N Hawkes processes, that is, knowing only the first $1 \ll K \leq N$ processes of $(Z_s^{i,N})_{i=1,\dots,N}$ with t large.

Remark 1.2. Since the family of $(Z_s^{i,N})_{i=1,\dots,N}$ is exchangeable, the observation given by the first K processes is not a restriction.

1.5. **Notations.** Throughout this paper, the conditional expectation given $(\theta_{ij})_{i,j=1,\dots,N}$ is denoted by \mathbb{E}_θ . The corresponding conditional variance and covariance are denoted by Var_θ and Cov_θ , respectively. For $f, g : [0, \infty) \rightarrow \mathbb{R}$, we define their convolution by

$$f * g(t) = \int_0^t f(t-s)g(s)ds, \quad t > 0,$$

and φ^{*n} denotes the n -fold convolution of φ . We adopt the conventions $\phi^{*0}(s)ds = \delta_0(ds)$ and $\phi^{*0}(t-s)ds = \delta_t(ds)$, so that in particular, $\int_0^t s\phi^{*0}(t-s)ds = t$. \xrightarrow{d} and $\xrightarrow{\mathbb{P}}$ refer to the convergence in distribution and convergence in probability, respectively.

We use C to denote a positive constant whose value might change from line to line.

2. MAIN RESULT

2.1. Main result. We assume $(H(q))$ for some $q \geq 1$. The supercritical case ($\Lambda p > 1$) is not addressed in this work. Its treatment would involve different techniques and significantly more technical arguments, and is therefore deferred to a separate, completed paper for independent investigation. We first remind the estimator built in [21]. For $N \geq 1$ and for $((Z_t^{i,N})_{t \geq 0})_{i=1,\dots,N}$ the solution of system (3), we set $\bar{Z}_t^N := N^{-1} \sum_{i=1}^N Z_t^{i,N}$, and $\bar{Z}_t^{N,K} := K^{-1} \sum_{i=1}^K Z_t^{i,N}$. Next, we introduce

$$\varepsilon_t^{N,K} := \frac{1}{t} (\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K}), \quad \mathcal{V}_t^{N,K} := \frac{N}{K} \sum_{i=1}^K \left[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \varepsilon_t^{N,K} \right]^2 - \frac{N}{t} \varepsilon_t^{N,K}.$$

And for $\Delta > 0$ such that $t/(2\Delta) \in \mathbb{N}^*$.

$$\mathcal{X}_{\Delta,t}^{N,K} := \mathcal{W}_{\Delta,t}^{N,K} - \frac{N-K}{K} \varepsilon_t^{N,K},$$

where

$$\mathcal{W}_{\Delta,t}^{N,K} = 2\mathcal{Z}_{2\Delta,t}^{N,K} - \mathcal{Z}_{\Delta,t}^{N,K}, \quad \mathcal{Z}_{\Delta,t}^{N,K} = \frac{N}{t} \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \Delta \varepsilon_t^{N,K})^2.$$

We then introduce the function $\Psi^{(3)}$ defined by

$$\Psi^{(3)}(u, v, w) = \frac{u^2(1 - \sqrt{\frac{u}{w}})^2}{v + u^2(1 - \sqrt{\frac{u}{w}})^2} \quad \text{if } u > 0, v > 0, w > 0 \quad \text{and} \quad \Psi^{(3)}(u, v, w) = 0 \text{ otherwise.}$$

We set

$$\hat{p}_{N,K,t} := \Psi^{(3)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta,t}^{N,K}),$$

with

$$(4) \quad \Delta_t = (2 \lfloor t^{1-4/(q+1)} \rfloor)^{-1} t.$$

The main result of this paper, which is proved in Section 7, is stated below.

Theorem 2.1. *We assume that $p > 0$ and that $(H(q))$ holds for some $q > 3$. Define Δ_t by (4). We set $c_{p,\Lambda} := (1 - \Lambda p)^2 / (2\Lambda^2)$. We always work in the asymptotic $(N, K, t) \rightarrow (\infty, \infty, \infty)$ and $\frac{1}{\sqrt{K}} + \frac{N}{K} \sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda} K} \rightarrow 0$.*

(i) *The dominant term is $\frac{1}{\sqrt{K}}$, i.e. when $[\frac{1}{\sqrt{K}}]/[\frac{N}{K} \sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}}] \rightarrow \infty$, it holds that*

$$\sqrt{K}(\hat{p}_{N,K,t} - p) \xrightarrow{d} \mathcal{N}(0, p^2(1-p)^2).$$

(ii) *The dominant term is $\frac{N}{t\sqrt{K}}$, i.e. when $[\frac{N}{t\sqrt{K}}]/[\frac{1}{\sqrt{K}} + \frac{N}{K} \sqrt{\frac{\Delta_t}{t}}] \rightarrow \infty$, we have*

$$\frac{t\sqrt{K}}{N}(\hat{p}_{N,K,t} - p) \xrightarrow{d} \mathcal{N}\left(0, \frac{2(1-\Lambda p)^2}{\mu^2 \Lambda^4}\right).$$

(iii) *The dominant term is $\frac{N}{K} \sqrt{\frac{\Delta_t}{t}}$, i.e. when $[\frac{N}{K} \sqrt{\frac{\Delta_t}{t}}]/[\frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}}] \rightarrow \infty$, imposing moreover that $\lim_{N,K \rightarrow \infty} \frac{K}{N} = \gamma \in [0, 1]$,*

$$\frac{K}{N} \sqrt{\frac{t}{\Delta_t}}(\hat{p}_{N,K,t} - p) \xrightarrow{d} \mathcal{N}\left(0, \frac{6(1-p)^2}{\Lambda^2} \left((1-\gamma)(1-\Lambda p)^3 + \gamma(1-\Lambda p)\right)^2\right).$$

We will not examine the cases involving two or three dominant terms, as we believe this is not very restrictive in practice. Furthermore, such a study would be much more tedious due to the difficulty of analyzing the correlations between the different terms. An alternative formulation of Theorem 2.1 can also be provided.

Corollary 2.2. *Under the assumption of Theorem 2.1. We also assume $\lim_{N,K \rightarrow \infty} \frac{K}{N} = \gamma \in [0, 1]$ and*

$$\lim \max\left\{\frac{1}{\sqrt{K}}, \frac{N}{K}\sqrt{\frac{\Delta_t}{t}}, \frac{N}{t\sqrt{K}}\right\} / \left(\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} - \max\left\{\frac{1}{\sqrt{K}}, \frac{N}{K}\sqrt{\frac{\Delta_t}{t}}, \frac{N}{t\sqrt{K}}\right\}\right) = \infty.$$

Then we have

$$\begin{aligned} & \left[\max\left\{\frac{p(1-p)}{\sqrt{K}}, \frac{\sqrt{2}(1-\Lambda p)}{\mu\Lambda^2} \frac{N}{t\sqrt{K}}\right\}, \right. \\ & \quad \left. \frac{(1-p)}{\Lambda} \left[(1-\gamma)(1-\Lambda p)^3 + \gamma(1-\Lambda p) \right] \frac{N}{K} \sqrt{\frac{6\Delta_t}{t}} \right]^{-1} \left(\hat{p}_{N,K,t} - p \right) \xrightarrow{d} \mathcal{N}(0, 1). \end{aligned}$$

Remark 2.3. This result allows us to construct an asymptotic confidence interval for p in the subcritical case. We define

$$\hat{\mu}_{N,K,t} := \Psi^{(1)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K}), \quad \hat{\Lambda}_{N,K,t} := \Psi^{(2)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K})$$

where

$$\Psi^{(1)}(u, v, w) := u\sqrt{\frac{u}{w}}, \quad \Psi^{(2)}(u, v, w) := \frac{v + [u - \Psi^{(1)}(u, v, w)]^2}{u[u - \Psi^{(1)}(u, v, w)]},$$

if $u > 0, v > 0, w > u$ and $\Psi^{(1)}(u, v, w) = \Psi^{(2)}(u, v, w) = 0$ otherwise. By [21, Theorem 2.3], we have, when $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \rightarrow 0$,

$$\left(\hat{\mu}_{N,K,t}, \hat{\Lambda}_{N,K,t}, \hat{p}_{N,K,t} \right) \xrightarrow{\mathbb{P}} (\mu, \Lambda, p).$$

Hence by Theorem 2.1, in the cases (i), (ii) or (iii), for $0 < \alpha < 1$,

$$\lim \mathbb{P}\left(|\hat{p}_{N,K,t} - p| \leq I_{N,K,t,\alpha}\right) = 1 - \alpha,$$

where

$$\begin{aligned} I_{N,K,t,\alpha} &= \Phi^{-1}(1 - \frac{\alpha}{2}) \left(\frac{1}{\sqrt{K}} \hat{p}_{N,K,t} (1 - \hat{p}_{N,K,t}) + \frac{N}{t\sqrt{K}} \frac{\sqrt{2}(1 - \hat{\Lambda}_{N,K,t} \hat{p}_{N,K,t})}{\hat{\mu}_{N,K,t} (\hat{\Lambda}_{N,K,t})^2} \right. \\ &\quad \left. + \frac{N}{K} \sqrt{\frac{6\Delta_t}{t}} \frac{(1 - \hat{p}_{N,K,t})}{\hat{\Lambda}_{N,K,t}} \left[(1 - \gamma)(1 - \hat{\Lambda}_{N,K,t} \hat{p}_{N,K,t})^3 + \gamma(1 - \hat{\Lambda}_{N,K,t} \hat{p}_{N,K,t}) \right] \right), \end{aligned}$$

and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{s^2}{2}} ds$.

Concerning the case $p = 0$, the following result shows that $\hat{p}_{N,K,t}$ is not always consistent.

Proposition 2.4. *We assume $p = 0$ and that $(H(q))$ holds for some $q > 3$. We set $c_{p,\Lambda} := (1 - \Lambda p)^2/(2\Lambda^2)$. We always work in the asymptotic $(N, K, t) \rightarrow (\infty, \infty, \infty)$ and in the regime $\frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \rightarrow 0$.*

(i) If $[\frac{N}{t\sqrt{K}}]/[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]^2 \rightarrow \infty$, we have

$$\hat{p}_{N,K,t} \xrightarrow{\mathbb{P}} 0.$$

(ii) If $[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]^2 / [\frac{N}{t\sqrt{K}}] \rightarrow \infty$, we have

$$\hat{p}_{N,K,t} \xrightarrow{d} X$$

where $\mathbb{P}(X = 1) = \mathbb{P}(X = 0) = \frac{1}{2}$.

2.2. Heuristics for the three estimators. The three estimators were initially proposed in [13] and later extended to partially observed settings in [21]. They can essentially be viewed as analogues of the sample mean, the sample variance, and a time-shifted (temporal) empirical variance for stochastic processes. Since our objective is to establish central limit theorem, we focus on the leading term of each estimator. For intuition on their construction, we refer to [13, Section 2.1]. The adaptation to partial observation follows analogously, see [21, Section 3.1]. For the reader's convenience, we also provide a brief explanation below, and for more details, see [13, Section 2.1] or [21, Section 3.1].

Consider the matrix A_N defined by $A_N(i,j) = \theta_{ij}/N$ and set $Q_N = (I - \Lambda A_N)^{-1}$. Under the subcritical condition $\Lambda p < 1$, Q_N exists with high probability and admits the series expansion $\sum_{n \geq 0} \Lambda^n A_N^n$. Let $\ell_N(i) = \sum_{j=1}^N Q_N(i,j)$, $c_N(i) = \sum_{j=1}^N Q_N(j,i)$ and $\bar{\ell}_N^K = \frac{1}{K} \sum_{i=1}^K \ell_N(i)$.

By [13, Section 2.1], it can be informally shown that $\bar{\ell}_N^K \simeq \frac{1}{1-\Lambda p}$ for sufficiently large N . Conditional on $(\theta_{ij})_{i,j=1,\dots,N}$, and for t, N, K large enough, the law of large numbers suggests that $\sum_{i=1}^K Z_t^{i,N} \simeq \sum_{i=1}^K \mathbb{E}_\theta[Z_t^{i,N}]$ (i.e. $(\sum_{i=1}^K Z_t^{i,N}) / (\sum_{i=1}^K \mathbb{E}_\theta[Z_t^{i,N}]) \rightarrow 1$). Assume the limit $\gamma_N(i) := \lim_{t \rightarrow \infty} \mathbb{E}_\theta[Z_t^{i,N}] / t$ exists. Then Definition (3) implies $\gamma_N = \mu \mathbf{1}_N + \Lambda A_N \gamma_N$, so that $\gamma_N = \mu Q_N \mathbf{1}_N$. Consequently, $t^{-1} \bar{Z}_t^{N,K} \simeq (tK)^{-1} \sum_{i=1}^K \mathbb{E}_\theta[Z_t^{i,N}] \simeq \mu K^{-1} \sum_{i=1}^K \ell_N(i) \simeq \frac{\mu}{1-\Lambda p}$.

Next, we explain why we use $\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K}$ rather than $\bar{Z}_t^{N,K}$ itself. Under Assumption $(H(q))$ (see the proof of Lemma 16 in [13]), we have $\mathbb{E}_\theta[Z_t^{i,N}] = \mu \ell_N(i)t + \chi_i^N + \pm t^{1-q}$, where χ_i^N is some finite random variable. Consequently, $t^{-1} \mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]$ converges to $\ell_N(i)$ at rate t^{-q} , which is faster than the rate t^{-1} obtained from $t^{-1} \mathbb{E}_\theta[Z_t^{i,N}]$ alone.

Based on [13, Section 2.1] and proceeding from a similar argument, we have $\mathcal{V}_t^{N,K} \simeq \frac{\mu^2 \Lambda^2 p(1-p)}{(1-\Lambda p)^2}$ and $\mathcal{X}_{\Delta_t,t}^{N,K} \simeq \frac{\mu}{(1-\Lambda p)^3}$. Then we can construct the partial observed estimator for parameters (μ, Λ, p) by arranging $(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K})$.

2.3. Plan of the paper. After some preliminaries stated in Section 3, we study some random matrix in Section 4, some limit theorems for the first and second estimator are established in Section 5, and the limit theorem for the third one is established in Section 6. Finally, we conclude the proof of the main results in Section 7. Moreover, the proofs for some technical Lemmas are presented in Appendix.

3. PRELIMINARIES

3.1. Some notations. For $r \in [1, \infty)$ and $\mathbf{x} \in \mathbb{R}^N$, we set $\|\mathbf{x}\|_r = (\sum_{i=1}^N |x_i|^r)^{\frac{1}{r}}$, and $\|\mathbf{x}\|_\infty = \max_{i=1,\dots,N} |x_i|$. For a $N \times N$ matrix M , we denote by $\|M\|_r$ the operator norm associated to $\|\cdot\|_r$, that is $\|M\|_r = \sup_{\mathbf{x} \in \mathbb{R}^N} \|M\mathbf{x}\|_r / \|\mathbf{x}\|_r$. We have the special cases

$$\|M\|_1 = \sup_{j=1,\dots,N} \sum_{i=1}^N |M_{ij}|, \quad \|M\|_\infty = \sup_{i=1,\dots,N} \sum_{j=1}^N |M_{ij}|.$$

We also have the inequality

$$\|M\|_r \leq \|M\|_1^{\frac{1}{r}} \|M\|_{\infty}^{1-\frac{1}{r}} \quad \text{for any } r \in [1, \infty).$$

We define the $N \times N$ random matrix A_N with $A_N(i, j) := N^{-1}\theta_{ij}$ for $i, j = 1, \dots, N$, and the matrix $Q_N := (I - \Lambda A_N)^{-1}$ on the event on which $I - \Lambda A_N$ is invertible.

For $1 \leq K \leq N$, we introduce the N -dimensional vector $\mathbf{1}_K$ with i -th coordinate $\mathbf{1}_K(i) = \mathbf{1}_{\{1 \leq i \leq K\}}$ for $i = 1, \dots, N$, and the $N \times N$ matrix I_K defined by $I_K(i, j) = \mathbf{1}_{\{i=j \leq K\}}$.

Next, we define $\ell_N := Q_N \mathbf{1}_N$, i.e. $\ell_N(i) := \sum_{j=1}^N Q_N(i, j)$, and $\ell_N^K := I_K \ell_N$, i.e. $\ell_N^K(i) := \ell_N(i) \mathbf{1}_{\{i \leq K\}}$. We also set $\bar{\ell}_N := \frac{1}{N} \sum_{i=1}^N \ell_N(i)$, $\bar{\ell}_N^K := \frac{1}{K} \sum_{i=1}^K \ell_N(i)$, and define the difference vector $\mathbf{x}_N^K := (x_N^K(i))_{i=1, \dots, N} = \ell_N^K - \bar{\ell}_N^K \mathbf{1}_K$ with $x_N^K(i) := (\ell_N(i) - \bar{\ell}_N^K) \mathbf{1}_{\{i \leq K\}}$ and $\mathbf{x}_N := (x_N(i))_{i=1, \dots, N}$, with $x_N(i) := \ell_N(i) - \bar{\ell}_N$.

Recall that $\mathbf{1}_N$ denotes the N -dimensional vector with all coordinates equal 1. Let $\mathbf{L}_N := A_N \mathbf{1}_N$, so that $L_N(i) := \sum_{j=1}^N A_N(i, j)$. We also define $\bar{L}_N := \frac{1}{N} \sum_{i=1}^N L_N(i)$, $\bar{L}_N^K := \frac{1}{K} \sum_{i=1}^K L_N(i)$ and denote the difference vector by $\mathbf{X}_N^K = (X_N^K(i))_{i=1, \dots, N}$, where $X_N^K(i) = (L_N(i) - \bar{L}_N^K) \mathbf{1}_{\{i \leq K\}}$. For convenience, we set $\mathbf{X}_N := \mathbf{X}_N^K$. Let $\mathbf{C}_N := A_N^T \mathbf{1}_N$ (A_N^T is the transpose of A_N), so that $C_N(j) := \sum_{i=1}^N A_N(i, j)$. We also define $\bar{C}_N := \frac{1}{N} \sum_{j=1}^N C_N(j)$, $\bar{C}_N^K := \frac{1}{K} \sum_{j=1}^K C_N(j)$ and consider the event

$$(5) \quad \mathcal{A}_N := \{\|\mathbf{L}_N - p\mathbf{1}_N\|_2 + \|\mathbf{C}_N - p\mathbf{1}_N\|_2 \leq N^{\frac{1}{4}}\}.$$

We assume here that $\Lambda p \in (0, 1)$ and set $a = \frac{1+\Lambda p}{2} \in (\frac{1}{2}, 1)$. We introduce the events

$$\begin{aligned} \Omega_N^1 &:= \left\{ \Lambda \|A_N\|_r \leq a, \text{ for all } r \in [1, \infty] \right\}, \\ \mathcal{F}_N^{K,1} &:= \left\{ \Lambda \|I_K A_N\|_r \leq \left(\frac{K}{N}\right)^{\frac{1}{r}} a, \text{ for all } r \in [1, \infty] \right\}, \\ \mathcal{F}_N^{K,2} &:= \left\{ \Lambda \|A_N I_K\|_r \leq \left(\frac{K}{N}\right)^{\frac{1}{r}} a, \text{ for all } r \in [1, \infty] \right\}, \\ \Omega_{N,K}^1 &:= \Omega_N^1 \cap \mathcal{F}_N^{K,1}, \quad \Omega_{N,K}^2 := \Omega_N^1 \cap \mathcal{F}_N^{K,2}, \quad \Omega_{N,K} = \Omega_{N,K}^1 \cap \Omega_{N,K}^2. \end{aligned}$$

We now review the following lemma established in [21] with $c_{p,\Lambda} = (1 - \Lambda p)^2 / (2\Lambda^2)$.

Lemma 3.1 (Lemma 5.7, [21]). *Assume that $\Lambda p < 1$. It holds that*

$$\mathbb{P}(\Omega_{N,K}) \geq 1 - CN e^{-c_{p,\Lambda} K}$$

for some constants $C > 0$.

Next, we also remind some important result in [13].

Lemma 3.2. *We assume that $\Lambda p < 1$ and recall (5). Then $\Omega_{N,K} \subset \Omega_N^1 \subset \{\|Q_N\|_r \leq C, \text{ for all } r \in [1, \infty]\} \subset \{\sup_{i=1, \dots, N} \ell_N(i) \leq C\}$, where $C = (1 - a)^{-1}$. For any $\alpha > 0$, there exists a constant C_{α} such that*

$$\mathbb{P}(\mathcal{A}_N) \geq 1 - C_{\alpha} N^{-\alpha}.$$

Proof. See [13, Notation 12 and Proposition 14, Step 1]. \square

3.2. Some auxilliary processes. Based on (3), we first introduce a family of martingales: for $i = 1, \dots, N$,

$$M_t^{i,N} = \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \lambda_s^{i,N}\}} \tilde{\pi}^i(ds, dz),$$

where $\tilde{\pi}^i(ds, dz) = \pi^i(ds, dz) - dsdz$. We further define a family of centered processes $U_t^{i,N} = Z_t^{i,N} - \mathbb{E}_\theta[Z_t^{i,N}]$.

Let \mathbf{Z}_t^N (resp. $\mathbf{U}_t^N, \mathbf{M}_t^N$) denote the N -dimensional vector with coordinates $Z_t^{i,N}$ (resp. $U_t^{i,N}, M_t^{i,N}$). Define the vectors

$$\mathbf{Z}_t^{N,K} = I_K \mathbf{Z}_t^N, \quad \mathbf{U}_t^{N,K} = I_K \mathbf{U}_t^N,$$

and the corresponding averages

$$\bar{Z}_t^{N,K} = K^{-1} \sum_{i=1}^K Z_t^{i,N}, \quad \bar{U}_t^{N,K} = K^{-1} \sum_{i=1}^K U_t^{i,N}, \quad \bar{M}_t^{N,K} = K^{-1} \sum_{i=1}^K M_t^{i,N}.$$

From [13, Remark 10 and Lemma 11], we recall the following identities:

$$(6) \quad \mathbb{E}_\theta[\mathbf{Z}_t^{N,K}] = \mu \sum_{n \geq 0} \left[\int_0^t s \phi^{*n}(t-s) ds \right] I_K A_N^n \mathbf{1}_N,$$

$$(7) \quad \mathbf{U}_t^{N,K} = \sum_{n \geq 0} \int_0^t \phi^{*n}(t-s) I_K A_N^n \mathbf{M}_s^N ds,$$

$$(8) \quad [M^{i,N}, M^{j,N}]_t = \mathbf{1}_{\{i=j\}} Z_t^{i,N}.$$

In particular, for $i = 1, \dots, N$,

$$(9) \quad U_t^{i,N} = \sum_{n \geq 0} \int_0^t \phi^{*n}(t-s) \sum_{j=1}^N A_N^n(i, j) M_s^{j,N} ds.$$

Adopting the convention that $\phi^{*0}(s)ds = \delta_0(ds)$ and $\int_0^t s \phi^{*0}(t-s)ds = t$, we establish some prior estimates for the intensity process $\lambda_t^{i,N}$ defined by (3) and the processes introduced in Section 3.2. We first review the following results established in [21, Lemma 6.1].

Lemma 3.3 (Lemma 6.1,[21]). *Assume (H(q)) for some $q \geq 1$.*

(i) *For all r in $[1, \infty]$, all $t \geq 0$, a.s.,*

$$\mathbf{1}_{\Omega_{N,K}} \|\mathbb{E}_\theta[\mathbf{Z}_t^{N,K}]\|_r \leq C t K^{\frac{1}{r}}.$$

(ii) *For any $r \in [1, \infty]$, for all $t \geq s \geq 0$,*

$$\mathbf{1}_{\Omega_{N,K}} \|\mathbb{E}_\theta[\mathbf{Z}_t^{N,K} - \mathbf{Z}_s^{N,K} - \mu(t-s)\ell_N^K]\|_r \leq C(\min\{1, s^{1-q}\}) K^{\frac{1}{r}}.$$

Lemma 3.4. *Assume (H(q)) for some $q \geq 1$. Then the following inequalities holds a.s. on $\Omega_{N,K}$,*

(i) $\sup_{t \in \mathbb{R}_+} \max_{i=1, \dots, N} \mathbb{E}_\theta[\lambda_t^{i,N}] \leq \frac{\mu}{1-a} \quad \text{and} \quad \sup_{t \in \mathbb{R}_+} \max_{i=1, \dots, N} \mathbb{E}_\theta[(\lambda_t^{i,N})^2]^{\frac{1}{2}} \leq C$.

(ii) *For all $t \geq 1$,*

$$\frac{1}{K} \sum_{i=1}^K \mathbb{E}_\theta \left[(\lambda_t^{i,N} - \mu \ell_N(i))^2 \right]^{\frac{1}{2}} \leq \frac{C}{t^q} + \frac{C}{\sqrt{N}}.$$

(iii) For all $t \geq s+1 \geq 1$,

$$\max_{i=1,\dots,N} \mathbb{E}_\theta[(U_t^{i,N} - U_s^{i,N})^4] \leq C(t-s)^2 \quad \text{and} \quad \max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^4] \leq C(t-s)^4.$$

(iv) For all $t \geq s+1 \geq 1$,

$$\mathbb{E}_\theta[(\bar{U}_t^{N,K} - \bar{U}_s^{N,K})^4] \leq \frac{C(t-s)^2}{K^2} \quad \text{and} \quad \mathbb{E}_\theta[(\bar{Z}_t^{N,K} - \bar{Z}_s^{N,K})^4] \leq C(t-s)^4.$$

The proof of Lemma 3.4 is tedious which is deferred to Appendix A.

4. SOME LIMIT THEOREMS FOR THE RANDOM MATRIX

In this section, we prove the asymptotic behavior of the quantities associated with the random matrix Q_N , which determines the asymptotic behavior of the estimators $\varepsilon_t^{N,K}$, $\mathcal{V}_t^{N,K}$ and $\mathcal{X}_{\Delta,t}^{N,K}$, defined in Section 2.1.

4.1. First estimator. Recall from Section 3.1 that the event $\Omega_{N,K}$ and the quantities $\ell_N(i) = \sum_{j=1}^N Q_N(i,j)$ and $\bar{\ell}_N^K = \frac{1}{K} \sum_{i=1}^K \ell_N(i)$. As established in Lemma 5.1, the estimator $\varepsilon_t^{N,K} = (\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K})/t$ is closely related to $\bar{\ell}_N^K$. To establish the limit of $\varepsilon_t^{N,K}$, we therefore require the following inequality for $\bar{\ell}_N^K$ which proved in [21, Lemma 5.9].

Lemma 4.1 (Lemma 5.9, [21]). *If $\Lambda p < 1$, there is $C > 0$ such that for all $1 \leq K \leq N$,*

$$\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}} \left| \bar{\ell}_N^K - \frac{1}{1-\Lambda p} \right|^2\right] \leq \frac{C}{NK}.$$

4.2. Second estimator. Recall the estimator

$$\mathcal{V}_t^{N,K} = \frac{N}{K} \sum_{i=1}^K \left[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \varepsilon_t^{N,K} \right]^2 - \frac{N}{t} \varepsilon_t^{N,K},$$

with $\varepsilon_t^{N,K} = (\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K})/t$, and the definitions from Section 3.1 of the matrices A_N , Q_N , the event $\Omega_{N,K}$, the quantities $\ell_N(i) = \sum_{j=1}^N Q_N(i,j)$ and $\bar{\ell}_N^K = \frac{1}{K} \sum_{i=1}^K \ell_N(i)$. Furthermore, there is a close connection between the second estimator $\mathcal{V}_t^{N,K}$ and $\mathcal{V}_\infty^{N,K} = \frac{N\mu^2}{K} \|\mathbf{x}_N^K\|_2^2$ (see Theorem 5.2), where $\mathbf{x}_N^K = (x_N^K(i))_{i=1,\dots,N} = \boldsymbol{\ell}_N^K - \bar{\ell}_N^K \mathbf{1}_K$ with $x_N^K(i) = (\ell_N(i) - \bar{\ell}_N^K) \mathbf{1}_{\{i \leq K\}}$ defined in Section 3.1. Hence, determining the limit of $\mathcal{V}_t^{N,K}$ is equivalent to finding the limit of $\mathcal{V}_\infty^{N,K}$.

Theorem 4.2. *Assume $\Lambda p < 1$. Then, as $(N, K) \rightarrow (\infty, \infty)$, and $N e^{-c_{p,\Lambda} K} \rightarrow 0$ with $c_{p,\Lambda} = (1 - \Lambda p)^2 / (2\Lambda^2)$,*

$$\mathbf{1}_{\Omega_{N,K}} \sqrt{K} \left(\mathcal{V}_\infty^{N,K} - \frac{\mu^2 \Lambda^2 p (1-p)}{(1-\Lambda p)^2} \right) \xrightarrow{d} \mathcal{N}\left(0, \left(\mu^2 \Lambda^2 \frac{p(1-p)}{(1-\Lambda p)^2}\right)^2\right).$$

We first write the following decomposition

$$\begin{aligned} & \sqrt{K} \left(\mathcal{V}_\infty^{N,K} - \frac{\mu^2 \Lambda^2 p (1-p)}{(1-\Lambda p)^2} \right) \\ &= \frac{N\mu^2}{\sqrt{K}} \left(\|\mathbf{x}_N^K\|_2^2 - (\Lambda \bar{\ell}_N)^2 \|\mathbf{x}_N^K\|_2^2 \right) + \frac{N(\mu \Lambda \bar{\ell}_N)^2}{\sqrt{K}} \|\mathbf{x}_N^K\|_2^2 - \frac{\mu^2 \Lambda^2 p (1-p) \sqrt{K}}{(1-\Lambda p)^2}, \end{aligned}$$

where $\mathbf{X}_N^K = (X_N^K(i))_{i=1,\dots,N} = \boldsymbol{\ell}_N^K - \bar{\ell}_N^K \mathbf{1}_K$ with $X_N^K(i) = (L_N(i) - \bar{L}_N^K) \mathbf{1}_{\{i \leq K\}}$, $L_N(i) = \sum_{j=1}^N A_N(i,j)$, and $\bar{L}_N^K = \frac{1}{K} \sum_{i=1}^K L_N(i)$ defined in Section 3.1. The proof of Theorem 4.2 then

proceeds by analyzing these terms separately. The term $\frac{N(\mu\Lambda\bar{\ell}_N)^2}{\sqrt{K}}\|\mathbf{X}_N^K\|_2^2 - \frac{\mu^2\Lambda^2p(1-p)\sqrt{K}}{(1-\Lambda p)^2}$ constitutes the principal term (see Lemma 4.3-(iv)), whereas the term $\frac{N\mu^2}{\sqrt{K}}(\|\mathbf{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\mathbf{X}_N^K\|_2^2)$ is shown to be negligible (see Lemma 4.3-(iii)).

We now turn to Lemma 4.3, whose proof is deferred to Appendix B.

Lemma 4.3. *Assume $\Lambda p < 1$ and recall \mathcal{A}_N in (5), there is $C > 0$ such that for all $1 \leq K \leq N$,*

$$(i) \mathbb{E}[\|(I_K A_N)^T \mathbf{X}_N^K\|_2^2] \leq \frac{CK^2}{N^3}.$$

$$(ii) \mathbb{E}\left[\left|(I_K A_N \mathbf{X}_N, \mathbf{X}_N^K)\right|\right] \leq \frac{CK}{N^2}, \text{ here } (\cdot, \cdot) \text{ is the inner product between two vectors.}$$

$$(iii) \frac{N}{K} \mathbb{E}\left[\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} \left| \left(\|\mathbf{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\mathbf{X}_N^K\|_2^2 \right) - \|\mathbf{x}_N^K - \bar{\ell}_N \Lambda \mathbf{X}_N^K\|_2^2 \right| \right] \leq \frac{C}{N}.$$

(iv) as $(N, K) \rightarrow (\infty, \infty)$ and $N e^{-c_{p,\Lambda} K} \rightarrow 0$, where $c_{p,\Lambda} = (1 - \Lambda p)^2 / (2\Lambda^2)$,

$$\mathbf{1}_{\Omega_{N,K}} \sqrt{K} \left[\frac{N}{K} (\bar{\ell}_N \|\mathbf{X}_N^K\|_2^2)^2 - \frac{p(1-p)}{(1-\Lambda p)^2} \right] \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{p(1-p)}{(1-\Lambda p)^2}\right)^2\right).$$

Now, we give the proof of Theorem 4.2.

Proof of Theorem 4.2. Recalling that $\mathcal{V}_\infty^{N,K} = \frac{N\mu^2}{K}\|\mathbf{x}_N^K\|_2^2$, we write

$$\begin{aligned} & \sqrt{K} \left(\mathcal{V}_\infty^{N,K} - \frac{\mu^2\Lambda^2p(1-p)}{(1-\Lambda p)^2} \right) \\ &= \frac{N\mu^2}{\sqrt{K}} \left(\|\mathbf{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\mathbf{X}_N^K\|_2^2 \right) + \frac{N\mu^2(\Lambda\bar{\ell}_N)^2}{\sqrt{K}}\|\mathbf{X}_N^K\|_2^2 - \frac{\mu^2\Lambda^2p(1-p)\sqrt{K}}{(1-\Lambda p)^2}. \end{aligned}$$

By Lemma 4.3-(iv), it suffices to check that

$$\zeta_{N,K} := \mathbf{1}_{\Omega_{N,K}} \frac{N}{\sqrt{K}} \left(\|\mathbf{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\mathbf{X}_N^K\|_2^2 \right)$$

converges to 0 in probability. Since $\mathbf{1}_{\mathcal{A}_N} \rightarrow 1$ a.s. by Lemma 3.2, it is enough to verify that $\mathbf{1}_{\mathcal{A}_N} \zeta_{N,K} \rightarrow 0$ in probability. To this end, we write

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{\mathcal{A}_N} \zeta_{N,K}] &\leq \frac{N}{\sqrt{K}} \mathbb{E}[\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} \|\mathbf{x}_N^K - \bar{\ell}_N \Lambda \mathbf{X}_N^K\|_2^2] \\ &\quad + \frac{N}{\sqrt{K}} \mathbb{E}\left[\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} \left| \left(\|\mathbf{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\mathbf{X}_N^K\|_2^2 \right) - \|\mathbf{x}_N^K - \bar{\ell}_N \Lambda \mathbf{X}_N^K\|_2^2 \right| \right]. \end{aligned}$$

By [21, Lemma 5.11], the first term is bounded by C/\sqrt{K} . By Lemma 4.3-(iii), the second term is bounded by $C\sqrt{K}/N \leq C/\sqrt{N}$, which completes the proof. \square

4.3. Third estimator. For $\Delta > 1$ satisfying $t/(2\Delta) \in \mathbb{N}^*$, we recall the definition $\mathcal{X}_{\Delta,t}^{N,K} = \mathcal{W}_{\Delta,t}^{N,K} - \frac{N-K}{K}\varepsilon_t^{N,K}$, where $\mathcal{W}_{\Delta,t}^{N,K} = 2\mathcal{Z}_{2\Delta,t}^{N,K} - \mathcal{Z}_{\Delta,t}^{N,K}$, $\mathcal{Z}_{\Delta,t}^{N,K} = \frac{N}{t} \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \Delta\varepsilon_t^{N,K})^2$ and $\varepsilon_t^{N,K} = (\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K})/t$. Further recall the matrices A_N and Q_N , the event $\Omega_{N,K}$ defined in Section 3.1, as well as $\ell_N(i) = \sum_{j=1}^N Q_N(i,j)$ and $\bar{\ell}_N^K = \frac{1}{K} \sum_{i=1}^K \ell_N(i)$. Now, taking Δ specifically as $\Delta_t = (2\lfloor t^{1-4/(q+1)} \rfloor)^{-1}t$ defined in (4), we will see (Theorem 6.1) that the third estimator $\mathcal{X}_{\Delta_t,t}^{N,K}$ is closely related to

$$\mathcal{X}_{\infty,\infty}^{N,K} = \mathcal{W}_{\infty,\infty}^{N,K} - \frac{(N-K)\mu}{K} \bar{\ell}_N^K,$$

where $\mathcal{W}_{\infty,\infty}^{N,K} = \mu \frac{N}{K^2} A_{\infty,\infty}^{N,K}$, $A_{\infty,\infty}^{N,K} = \sum_{j=1}^N \left(\sum_{i=1}^K Q_N(i,j) \right)^2 \ell_N(j)$. Therefore, establishing the convergence of $\mathcal{X}_{\Delta,t}^{N,K}$ reduces to establishing the convergence of $\mathcal{X}_{\infty,\infty}^{N,K}$. The latter relies on the following two key estimates.

Lemma 4.4 (Lemma 5.19, [21]). *If $\Lambda p < 1$, there is $C > 0$ such that for all $1 \leq K \leq N$,*

$$\mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| \mathcal{X}_{\infty,\infty}^{N,K} - \frac{\mu}{(1-\Lambda p)^3} \right| \right] \leq \frac{C}{K}.$$

The objective of the following lemma is to establish that $\frac{(A_{\infty,\infty}^{N,K})^2}{2K^2}$ is close to $\frac{1}{2} \left(\frac{N-K}{N(1-\Lambda p)} + \frac{K}{N(1-\Lambda p)^3} \right)^2$.

Lemma 4.5. *When (N, K) tends to (∞, ∞) , with $K \leq N$ and in the regime where $\lim_{N,K \rightarrow \infty} \frac{K}{N} = \gamma \in [0, 1]$, we have*

$$\mathbf{1}_{\Omega_{N,K}} \frac{A_{\infty,\infty}^{N,K}}{K} \longrightarrow \frac{1-\gamma}{(1-\Lambda p)} + \frac{\gamma}{(1-\Lambda p)^3},$$

in probability.

Proof. Recalling that $\mathcal{X}_{\infty,\infty}^{N,K} = \mathcal{W}_{\infty,\infty}^{N,K} - \frac{(N-K)\mu}{K} \bar{\ell}_N^K$ and $\mathcal{W}_{\infty,\infty}^{N,K} = (\mu N/K^2) A_{\infty,\infty}^{N,K}$, we obtain

$$\frac{A_{\infty,\infty}^{N,K}}{K} = \frac{K}{\mu N} \mathcal{X}_{\infty,\infty}^{N,K} + \frac{N-K}{N} \bar{\ell}_N^K.$$

The result then follows immediately by combining the convergence $\bar{\ell}_N^K \rightarrow \frac{1}{1-\Lambda p}$ in probability from Lemma 4.1 with $\mathcal{X}_{\infty,\infty}^{N,K} \rightarrow \frac{\mu}{(1-\Lambda p)^3}$ in probability from Lemma 4.4. \square

5. THE LIMIT THEOREMS FOR THE FIRST AND SECOND ESTIMATORS

This section is devoted to establishing the asymptotic behavior of the estimators $\varepsilon_t^{N,K}$ and $\mathcal{V}_t^{N,K}$, defined in Section 2.1.

- For $\varepsilon_t^{N,K}$, its limit follows directly from [21, Lemma 7.3], as stated in Lemma 5.1.
- For $\mathcal{V}_t^{N,K}$, however, a more delicate analysis is required. We begin by decomposing $\frac{t\sqrt{K}}{N} (\mathcal{V}_t^{N,K} - \mathcal{V}_{\infty}^{N,K})$ into several terms, namely, $J_t^{N,K,1}$, $J_t^{N,K,211}$, $J_t^{N,K,212}$, $J_t^{N,K,213}$, $J_t^{N,K,22}$, $J_t^{N,K,23}$, $J_t^{N,K,3}$. We observe that on $\Omega_{N,K}$, the dominant contribution is coming from the term $J_t^{N,K,211}$ (see Lemma 5.3). Further decomposition of this term reveals that the leading-order asymptotic behavior is determined by

$$\frac{2}{t\sqrt{K}} \sum_{i=1}^K \int_t^{2t} (M_s^{i,N} - M_t^{i,N}) dM_s^{i,N},$$

as shown in Step 3 of the proof of Theorem 5.2. This expression converges in distribution to a Gaussian random variable with variance $2\mu^2/(1-\Lambda p)^2$, as established in Lemma 5.4.

Let us remind that $\mathcal{V}_t^{N,K} = \frac{N}{K} \sum_{i=1}^K \left[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \varepsilon_t^{N,K} \right]^2 - \frac{N}{t} \varepsilon_t^{N,K}$ defined in Section 2.1 and that $\mathcal{V}_{\infty}^{N,K} = \frac{N}{K} \mu^2 \|x_N^K\|_2^2$, where $\varepsilon_t^{N,K} = (\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K})/t$, $x_N^K(i) = (\ell_N(i) - \bar{\ell}_N^K) \mathbf{1}_{\{i \leq K\}}$ and $x_N^K = (x_N^K(i))_{i=1,\dots,N}$. The definition of $(\ell_N(i))_{i=1,\dots,N}$ and $\bar{\ell}_N^K$ are introduced in Section 3.1.

Lemma 5.1 (Lemma 7.3, [21]). *Assume $(H(q))$ for some $q \geq 1$, in the regime $\frac{K}{t^{2q}} \rightarrow 0$, we have*

$$\lim_{(N,K,t) \rightarrow (\infty, \infty, \infty)} \mathbf{1}_{\Omega_{N,K}} \sqrt{K} \mathbb{E}_\theta \left[\left| \varepsilon_t^{N,K} - \mu \bar{\ell}_N^K \right| \right] = 0,$$

almost surely.

The main result of this section is the following limit theorem.

Theorem 5.2. *Assume $(H(q))$ for some $q > 1$. When $(N, K, t) \rightarrow (\infty, \infty, \infty)$ and $\frac{t\sqrt{K}}{N}(\frac{N}{t^q} + \sqrt{\frac{N}{Kt}}) + Ne^{-c_{p,\Lambda} K} \rightarrow 0$ with $c_{p,\Lambda} = (1 - \Lambda p)^2/(2\Lambda^2)$, we have*

$$\mathbf{1}_{\Omega_{N,K}} \frac{t\sqrt{K}}{N} (\mathcal{V}_t^{N,K} - \mathcal{V}_\infty^{N,K}) \xrightarrow{d} \mathcal{N} \left(0, \frac{2\mu^2}{(1 - \Lambda p)^2} \right).$$

Prior to the proof, we decompose the difference $\mathcal{V}_t^{N,K} - \mathcal{V}_\infty^{N,K} := J_t^{N,K,1} + J_t^{N,K,2} + J_t^{N,K,3}$, where

$$\begin{aligned} J_t^{N,K,1} &= \frac{N}{K} \left\{ \sum_{i=1}^K \left[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \varepsilon_t^{N,K} \right]^2 - \sum_{i=1}^K \left[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \mu \bar{\ell}_N^K \right]^2 \right\}, \\ J_t^{N,K,2} &= \frac{N}{K} \left\{ \sum_{i=1}^K \left[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \mu \ell_N(i) \right]^2 - \frac{K}{t} \varepsilon_t^{N,K} \right\}, \\ J_t^{N,K,3} &= 2 \frac{N}{K} \sum_{i=1}^K \left[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \mu \ell_N(i) \right] \left[\mu \ell_N(i) - \mu \bar{\ell}_N^K \right]. \end{aligned}$$

We further decompose $J_t^{N,K,2} = J_t^{N,K,21} + J_t^{N,K,22} + J_t^{N,K,23}$, where

$$\begin{aligned} J_t^{N,K,21} &= \frac{N}{K} \left\{ \sum_{i=1}^K \left[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \frac{\mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]}{t} \right]^2 - \frac{K}{t} \varepsilon_t^{N,K} \right\}, \\ J_t^{N,K,22} &= \frac{N}{K} \sum_{i=1}^K \left\{ \frac{\mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]}{t} - \mu \ell_N(i) \right\}^2, \\ J_t^{N,K,23} &= 2 \frac{N}{K} \sum_{i=1}^K \left[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \frac{\mathbb{E}_\theta(Z_{2t}^{i,N} - Z_t^{i,N})}{t} \right] \left[\frac{\mathbb{E}_\theta(Z_{2t}^{i,N} - Z_t^{i,N})}{t} - \mu \ell_N(i) \right]. \end{aligned}$$

Recalling that $U_t^{i,N} = Z_t^{i,N} - \mathbb{E}_\theta[Z_t^{i,N}]$, we further write $J_t^{N,K,21} = J_t^{N,K,211} + J_t^{N,K,212} + J_t^{N,K,213}$, where

$$\begin{aligned} J_t^{N,K,211} &= \frac{N}{K} \sum_{i=1}^K \left\{ \frac{(U_{2t}^{i,N} - U_t^{i,N})^2}{t^2} - \frac{\mathbb{E}_\theta[(U_{2t}^{i,N} - U_t^{i,N})^2]}{t^2} \right\}, \\ J_t^{N,K,212} &= \frac{N}{K} \left\{ \sum_{i=1}^K \frac{\mathbb{E}_\theta[(U_{2t}^{i,N} - U_t^{i,N})^2]}{t^2} - \frac{K}{t} \mathbb{E}_\theta[\varepsilon_t^{N,K}] \right\}, \\ J_t^{N,K,213} &= \frac{N}{K} \left\{ \frac{K}{t} \mathbb{E}_\theta[\varepsilon_t^{N,K}] - \frac{K}{t} \varepsilon_t^{N,K} \right\}. \end{aligned}$$

Finally, we also write $J_t^{N,K,3} := J_t^{N,K,31} + J_t^{N,K,32}$, where

$$\begin{aligned} J_t^{N,K,31} &= 2 \frac{N}{K} \sum_{i=1}^K \left[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \frac{\mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]}{t} \right] \left[\mu\ell_N(i) - \mu\bar{\ell}_N^K \right], \\ J_t^{N,K,32} &= 2 \frac{N}{K} \sum_{i=1}^K \left[\frac{\mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]}{t} - \mu\ell_N(i) \right] \left[\mu\ell_N(i) - \mu\bar{\ell}_N^K \right]. \end{aligned}$$

Although the decomposition above is somewhat involved, the principal term is $J_t^{N,K,211}$, which converges to a Gaussian distribution after normalization. The remaining terms, namely, $J_t^{N,K,1}$, $J_t^{N,K,22}$, $J_t^{N,K,23}$, $J_t^{N,K,213}$, $J_t^{N,K,32}$, $J_t^{N,K,212}$, $J_t^{N,K,31}$ are all suitably bounded as a consequence of Lemma 5.3.

Lemma 5.3. *Assume $(H(q))$ for some $q > 1$. When $(N, K, t) \rightarrow (\infty, \infty, \infty)$ and $\frac{t\sqrt{K}}{N}(\frac{N}{t^q} + \sqrt{\frac{N}{Kt}}) \rightarrow 0$,*

$$\frac{t\sqrt{K}}{N} \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| J_t^{N,K,1} + J_t^{N,K,212} + J_t^{N,K,213} + J_t^{N,K,22} + J_t^{N,K,23} + J_t^{N,K,3} \right| \right] \rightarrow 0.$$

Proof. Bounds for $J_t^{N,K,1}$, $J_t^{N,K,22}$, $J_t^{N,K,23}$, $J_t^{N,K,213}$, $J_t^{N,K,32}$ are provided in [21, Lemma 8.2]. While $J_t^{N,K,212}$ is bounded by [21, Lemma 8.3]. It remains to handle $J_t^{N,K,3} = J_t^{N,K,31} + J_t^{N,K,32}$. For $J_t^{N,K,31}$, [21, Lemma 8.5] implies that

$$\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} \mathbb{E}_\theta [|J_t^{N,K,31}|] \leq C \frac{N}{K\sqrt{t}} \left[\sum_{i=1}^K (\ell_N(i) - \bar{\ell}_N^K)^2 \right]^{1/2} = C \frac{N}{K\sqrt{t}} \|\mathbf{x}_N^K\|_2.$$

Taking expectation and applying the estimate $\mathbb{E}[\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} \|\mathbf{x}_N^K\|_2] \leq CK^{1/2}N^{-1/2}$ from [21, Lemmas 5.14], we obtain $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}} |J_t^{N,K,31}|] \leq C\sqrt{\frac{N}{Kt}}$. The desired result follows by aggregating the individual bounds for all terms. \square

The following tedious lemma will allow us to treat the contribution term $J_t^{N,K,211}$.

Lemma 5.4. *Assume $(H(q))$ for some $q > 1$. For $u \in [0, 1]$, define the process*

$$N_u^{t,i,N} := \int_t^{t+\sqrt{ut}} (M_s^{i,N} - M_t^{i,N}) dM_s^{i,N},$$

where $M^{i,N}$ defined in Section 3.2. When $(N, K, t) \rightarrow (\infty, \infty, \infty)$,

$$(10) \quad \left(\frac{1}{t\sqrt{K}} \sum_{i=1}^K N_u^{t,i,N} \right)_{u \in [0,1]} \xrightarrow{d} \left(\frac{\mu}{\sqrt{2}(1-\Lambda p)} B_u \right)_{u \in [0,1]},$$

where $(B_u)_{u \in [0,1]}$ is a Brownian motion.

Proof. Note that for fixed $t \geq 0$, the process $(N_u^{t,i,N})_{u \in [0,1]}$ is a martingale w.r.t the filtration $\mathcal{F}_{t+\sqrt{ut}}^N$. To prove (10), we apply Jacod-Shiryaev [19, Theorem VIII-3-8], which requires verifying that as $(t, N, K) \rightarrow (\infty, \infty, \infty)$,

- (a) $[\frac{1}{t\sqrt{K}} \sum_{i=1}^K N_u^{t,i,N}, \frac{1}{t\sqrt{K}} \sum_{i=1}^K N_{u-}^{t,i,N}]_u \rightarrow \frac{\mu^2}{2(1-\Lambda p)^2} u$ in probability, for all $u \in [0, 1]$ fixed.
- (b) $\sup_{u \in [0,1]} \frac{1}{t\sqrt{K}} \sum_{i=1}^K |N_u^{t,i,N} - N_{u-}^{t,i,N}| \rightarrow 0$ in probability.

The verification of point (b) is relatively straightforward. Using the independence of the Poisson measures in (3) and the fact that the jumps of $M^{i,N}$ are always of size 1, we obtain

$$\begin{aligned} \frac{1}{t\sqrt{K}}\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\sup_{u\in[0,1]}\sum_{i=1}^K\left|N_u^{t,i,N}-N_{u-}^{t,i,N}\right|\right] &\leq\frac{C}{t\sqrt{K}}\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\sup_{u\in[0,1]}\max_{i=1,\dots,K}\left|M_{t+t\sqrt{u}}^{i,N}-M_t^{i,N}\right|\right] \\ &\leq\frac{C}{t\sqrt{K}}\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\sup_{u\in[0,1]}\left|\sum_{i=1}^K(M_{t+t\sqrt{u}}^{i,N}-M_t^{i,N})^2\right|^{\frac{1}{2}}\right]. \end{aligned}$$

Applying the Cauchy-Schwarz inequality and using (8) yields

$$\begin{aligned} \frac{1}{t\sqrt{K}}\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\sup_{u\in[0,1]}\sum_{i=1}^K\left|N_u^{t,i,N}-N_{u-}^{t,i,N}\right|\right] &\leq\frac{C}{t\sqrt{K}}\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\sup_{u\in[0,1]}\sum_{i=1}^K(M_{t+t\sqrt{u}}^{i,N}-M_t^{i,N})^2\right]^{\frac{1}{2}} \\ &\leq\frac{C}{t\sqrt{K}}\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\left|\sum_{i=1}^K(Z_{2t}^{i,N}-Z_t^{i,N})\right|\right]^{\frac{1}{2}}\leq\frac{C}{\sqrt{t}}. \end{aligned}$$

The last inequality follows from Lemma 3.3-(ii) with $K=N$ and $r=\infty$, which gives us that $\max_{i=1,\dots,N}\mathbb{E}_\theta[Z_t^{i,N}-Z_s^{i,N}]\leq C(t-s)$ on $\Omega_{N,K}\subset\Omega_{N,N}$.

Regarding point (a), recall that $Z_t^{i,N}=M_t^{i,N}+\int_0^t\lambda_s^{i,N}ds$. For fixed u , we write

$$\begin{aligned} \left[\frac{1}{t\sqrt{K}}\sum_{i=1}^KN_{\cdot}^{t,i,N},\frac{1}{t\sqrt{K}}\sum_{i=1}^KN_{\cdot}^{t,i,N}\right]_u &= \frac{1}{t^2K}\sum_{i=1}^K\int_t^{t+\sqrt{ut}}(M_{s-}^{i,N}-M_t^{i,N})^2dZ_s^{i,N} \\ &:= \Upsilon_{t,N,K,u}^1+\Upsilon_{t,N,K,u}^2+\Upsilon_{t,N,K,u}^3, \end{aligned}$$

where,

$$\begin{aligned} \Upsilon_{t,N,K,u}^1 &:= \frac{1}{t^2K}\sum_{i=1}^K\int_t^{t+\sqrt{ut}}(M_{s-}^{i,N}-M_t^{i,N})^2dM_s^{i,N}, \\ \Upsilon_{t,N,K,u}^2 &:= \frac{1}{t^2K}\sum_{i=1}^K\int_t^{t+\sqrt{ut}}(M_s^{i,N}-M_t^{i,N})^2(\lambda_s^{i,N}-\mu\ell_N(i))ds, \\ \Upsilon_{t,N,K,u}^3 &:= \frac{1}{t^2K}\sum_{i=1}^K\mu\ell_N(i)\int_t^{t+\sqrt{ut}}(M_s^{i,N}-M_t^{i,N})^2ds. \end{aligned}$$

Each term will be handled in a separate step.

Step 1. In this step, we verify that $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}}\Upsilon_{t,N,K,u}^1]\rightarrow 0$ as $(N,K,t)\rightarrow(\infty,\infty,\infty)$. Using (8), we obtain

$$\begin{aligned} \mathbb{E}_\theta[(\Upsilon_{t,N,K,u}^1)^2] &= \frac{1}{K^2t^4}\sum_{i=1}^K\mathbb{E}_\theta\left[\int_t^{t+\sqrt{ut}}(M_{s-}^{i,N}-M_t^{i,N})^4dZ_s^{i,N}\right] \\ &= \frac{1}{K^2t^4}\sum_{i=1}^K\mathbb{E}_\theta\left[\int_t^{t+\sqrt{ut}}(M_s^{i,N}-M_t^{i,N})^4\lambda_s^{i,N}ds\right] \\ &\leq \frac{1}{K^2t^4}\sum_{i=1}^K\int_t^{t+\sqrt{ut}}\left\{\mathbb{E}_\theta[(M_s^{i,N}-M_t^{i,N})^4|\lambda_s^{i,N}-\mu\ell_N(i)|]+\mu\mathbb{E}_\theta[(M_s^{i,N}-M_t^{i,N})^4|\ell_N(i)|]\right\}ds. \end{aligned}$$

Applying the Cauchy–Schwarz and Burkholder inequalities, we further obtain

$$\begin{aligned} & \mathbb{E}_\theta[(\Upsilon_{t,N,K,u}^1)^2] \\ & \leq \frac{1}{K^2 t^4} \sum_{i=1}^K \int_t^{t+\sqrt{ut}} \left\{ \mathbb{E}_\theta[(M_s^{i,N} - M_t^{i,N})^8]^{\frac{1}{2}} \mathbb{E}_\theta[|\lambda_s^{i,N} - \mu \ell_N(i)|^2]^{\frac{1}{2}} + C \mu \mathbb{E}_\theta[(Z_s^{i,N} - Z_t^{i,N})^2] |\ell_N(i)| \right\} ds \\ & \leq \frac{C}{K^2 t^4} \sum_{i=1}^K \int_t^{t+\sqrt{ut}} \left\{ \mathbb{E}_\theta[(Z_s^{i,N} - Z_t^{i,N})^4]^{\frac{1}{2}} \mathbb{E}_\theta[|\lambda_s^{i,N} - \mu \ell_N(i)|^2]^{\frac{1}{2}} + \mu \mathbb{E}_\theta[(Z_s^{i,N} - Z_t^{i,N})^4]^{\frac{1}{2}} |\ell_N(i)| \right\} ds. \end{aligned}$$

By Lemma 3.4-(iii), on $\Omega_{N,K}$, we have $\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_s^{i,N} - Z_t^{i,N})^4] \leq C(t-s)^4$ for all $s \geq t$. Moreover, ℓ_N is bounded on $\Omega_{N,K}$. Therefore,

$$\mathbb{E}_\theta[(\Upsilon_{t,N,K,u}^1)^2] \leq \frac{C}{K^2 t^2} \sum_{i=1}^K \int_t^{t+\sqrt{ut}} \left(1 + \mathbb{E}_\theta[|\lambda_s^{i,N} - \mu \ell_N(i)|^2]^{\frac{1}{2}} \right) ds \leq \frac{C}{Kt} \left(1 + \frac{1}{t^q} + \frac{1}{\sqrt{N}} \right),$$

where the last inequality follows from Lemma 3.4-(ii). This completes the step.

Step 2. Similarly, it holds that, on $\Omega_{N,K}$,

$$\begin{aligned} \mathbb{E}_\theta[|\Upsilon_{t,N,K,u}^2|] & \leq \frac{1}{Kt^2} \sum_{i=1}^K \int_t^{t+\sqrt{ut}} \mathbb{E}_\theta[(M_s^{i,N} - M_t^{i,N})^4]^{\frac{1}{2}} \mathbb{E}_\theta[|\lambda_s^{i,N} - \mu \ell_N(i)|^2]^{\frac{1}{2}} \\ & \leq \frac{C}{Kt} \sum_{i=1}^K \int_t^{2t} \mathbb{E}_\theta[|\lambda_s^{i,N} - \mu \ell_N(i)|^2]^{\frac{1}{2}} ds \leq \frac{C}{t^q} + \frac{C}{\sqrt{N}}. \end{aligned}$$

Step 3. Finally, we prove that $\Upsilon_{t,N,K,u}^3 \rightarrow \mu^2 u / [2(1 - \Lambda p)^2]$ in probability as $(N, K, t) \rightarrow (\infty, \infty, \infty)$. Applying Itô's formula and (8), we write

$$\begin{aligned} & (M_s^{i,N} - M_t^{i,N})^2 \\ & = 2 \int_t^s (M_{r-}^{i,N} - M_t^{i,N}) dM_r^{i,N} + Z_s^{i,N} - Z_t^{i,N} \\ & = 2 \int_t^s (M_{r-}^{i,N} - M_t^{i,N}) dM_r^{i,N} + U_s^{i,N} - U_t^{i,N} + \mathbb{E}_\theta[Z_s^{i,N} - Z_t^{i,N} - \mu(s-t)\ell_N(i)] + \mu(s-t)\ell_N(i). \end{aligned}$$

Consequently, we decompose $\Upsilon_{t,N,K,u}^3 := \Upsilon_{t,N,K,u}^{3,1} + \Upsilon_{t,N,K,u}^{3,2} + \Upsilon_{t,N,K,u}^{3,3} + \Upsilon_{t,N,K,u}^{3,4}$, where

$$\begin{aligned} \Upsilon_{t,N,K,u}^{3,1} & := \frac{2}{t^2 K} \sum_{i=1}^K \mu \ell_N(i) \int_t^{t+\sqrt{ut}} \int_t^s (M_{r-}^{i,N} - M_t^{i,N}) dM_r^{i,N} ds, \\ \Upsilon_{t,N,K,u}^{3,2} & := \frac{1}{t^2 K} \sum_{i=1}^K \mu \ell_N(i) \int_t^{t+\sqrt{ut}} (U_s^{i,N} - U_t^{i,N}) ds, \\ \Upsilon_{t,N,K,u}^{3,3} & := \frac{1}{t^2 K} \sum_{i=1}^K \mu \ell_N(i) \int_t^{t+\sqrt{ut}} \mathbb{E}_\theta[Z_s^{i,N} - Z_t^{i,N} - \mu(s-t)\ell_N(i)] ds, \\ \Upsilon_{t,N,K,u}^{3,4} & := \frac{1}{t^2 K} \sum_{i=1}^K \mu^2 (\ell_N(i))^2 \times \frac{ut^2}{2} = \frac{\mu^2 u}{2K} \sum_{i=1}^K (\ell_N(i))^2. \end{aligned}$$

First, noting that $\mathcal{V}_\infty^{N,K} = \frac{N}{K}\mu^2\|\mathbf{x}_N^K\|_2^2$, we have

$$2\Upsilon_{t,N,K,u}^{3,4} = \mu^2 u(\bar{\ell}_N^K)^2 + \frac{\mu^2 u}{K} \sum_{i=1}^K (\ell_N(i) - \bar{\ell}_N^K)^2 = \mu^2 u(\bar{\ell}_N^K)^2 + \frac{\mu^2 u}{K} \|\mathbf{x}_N^K\|_2^2 = \mu^2 u(\bar{\ell}_N^K)^2 + \frac{u}{N} \mathcal{V}_\infty^{N,K}.$$

Then, Lemma 4.1 and Theorem 4.2 implies immediately that $\Upsilon_{t,N,K,u}^{3,4}$ converges to $\mu^2 u/[2(1-\Lambda p)^2]$ in probability.

For the second term, we recall (9) and write for $s \geq t$,

$$U_s^{i,N} - U_t^{i,N} = \sum_{n \geq 0} \int_0^s (\phi^{*n}(s-u) - \phi^{*n}(t-u)) \sum_{j=1}^N A_N^n(i,j) M_u^{j,N} du,$$

so that, by Minkowski's inequality and separating as usual the terms $n = 0$ and $n \geq 1$,

$$\begin{aligned} \mathbb{E}_\theta[|\Upsilon_{t,N,K,u}^{3,2}|^2]^{\frac{1}{2}} &\leq \frac{C}{t^2 K} \int_t^{t+\sqrt{ut}} \mathbb{E}_\theta \left[\left(\sum_{i=1}^K \ell_N(i)(U_s^{i,N} - U_t^{i,N}) \right)^2 \right]^{\frac{1}{2}} ds \\ &\leq \frac{C}{t^2 K} \int_t^{t+\sqrt{ut}} \left\{ \mathbb{E}_\theta \left[\left(\sum_{i=1}^K \ell_N(i)(M_s^{i,N} - M_t^{i,N}) \right)^2 \right]^{\frac{1}{2}} \right. \\ &\quad \left. + \sum_{n \geq 1} \int_0^s (\phi^{*n}(s-r) - \phi^{*n}(t-r)) \mathbb{E}_\theta \left[\left(\sum_{i=1}^K \sum_{j=1}^N \ell_N(i) A_N^n(i,j) M_r^{j,N} \right)^2 \right]^{\frac{1}{2}} dr \right\} ds. \end{aligned}$$

By (8), we see that on $\Omega_{N,K}$, for all $t \leq s \leq 2t$,

$$\begin{aligned} \mathbb{E}_\theta \left[\left(\sum_{i=1}^K \ell_N(i)(M_s^{i,N} - M_t^{i,N}) \right)^2 \right]^{\frac{1}{2}} &= \mathbb{E}_\theta \left[\sum_{i=1}^K (\ell_N(i))^2 (Z_s^{i,N} - Z_t^{i,N}) \right]^{\frac{1}{2}} \\ &= \left\{ \sum_{i=1}^K (\ell_N(i))^2 \mathbb{E}_\theta [Z_s^{i,N} - Z_t^{i,N}] \right\}^{\frac{1}{2}} \\ &\leq C\sqrt{Kt}, \end{aligned}$$

by Lemma 3.3-(i) with $r = \infty$, together with the boundedness of $\ell_N(i)$ on $\Omega_{N,K}$. Next, for $n \geq 1$,

$$\begin{aligned} \mathbb{E}_\theta \left[\left(\sum_{i=1}^K \sum_{j=1}^N \ell_N(i) A_N^n(i,j) M_r^{j,N} \right)^2 \right] &= \sum_{j=1}^N \left(\sum_{i=1}^K \ell_N(i) A_N^n(i,j) \right)^2 \mathbb{E}_\theta [Z_r^{j,N}] \\ &\leq C \sum_{j=1}^N \left(\sum_{i=1}^K A_N^n(i,j) \right)^2 \mathbb{E}_\theta [Z_r^{j,N}] \\ &\leq C \sum_{j=1}^N |||I_K A_N^n|||_1^2 \mathbb{E}_\theta [Z_r^{j,N}] \\ &\leq \frac{CK^2}{N} |||A_N|||_1^{2n-2} r. \end{aligned}$$

The last line follows from $\|I_K A_N\|_1 \leq CK/N$ on $\Omega_{N,K}$ and by another application of Lemma 3.3-(i). Therefore, for any $u \in [0, 1]$ (recalling that $\int_0^\infty \phi^{*n}(u)du = \Lambda^n$),

$$\begin{aligned} \mathbb{E}_\theta[|\mathcal{Y}_{t,N,K,u}^{3,2}|^2]^{\frac{1}{2}} &\leq \frac{C}{t^2 K} \int_t^{t+\sqrt{ut}} \left\{ \sqrt{Kt} + \sum_{n \geq 1} \int_0^s (\phi^{*n}(s-r) - \phi^{*n}(t-r)) \frac{K}{\sqrt{N}} \|A_N\|_1^{n-1} \sqrt{r} dr \right\} ds \\ &\leq \frac{C}{t^2 K} \int_t^{t+\sqrt{ut}} \left\{ \sqrt{Kt} + \sum_{n \geq 1} \sqrt{s} \frac{K}{\sqrt{N}} \Lambda^n \|A_N\|_1^{n-1} \right\} ds \leq \frac{C}{\sqrt{Kt}}. \end{aligned}$$

The last inequality uses the fact that on $\Omega_{N,K}$, we have $\Lambda \|A_N\|_1 \leq a < 1$.

For the third term, since $\ell_N(i)$ is uniformly bounded on $\Omega_{N,K}$, and by Lemma 3.3-(ii) with $r = 1$, we obtain, on $\Omega_{N,K}$,

$$\begin{aligned} \mathbb{E}_\theta[|\mathcal{Y}_{t,N,K,u}^{3,3}|] &\leq \frac{1}{Kt^2} \sum_{i=1}^K \int_t^{t+\sqrt{ut}} |\ell_N(i)| \left| \mathbb{E}_\theta[Z_s^{i,N} - Z_t^{i,N} - \mu(s-t)\ell_N(i)] \right| ds \\ &\leq \frac{C}{Kt^2} \sum_{i=1}^K \int_t^{t+\sqrt{ut}} \left| \mathbb{E}_\theta[Z_s^{i,N} - Z_t^{i,N} - \mu(s-t)\ell_N(i)] \right| ds \leq \frac{C}{t^q}. \end{aligned}$$

Finally, we set $\mathbb{N}_u^{t,i,N} := M_{t+ut}^{i,N} - M_t^{i,N}$. Then $\mathbb{N}_u^{t,i,N}$ is a martingale for the filtration \mathcal{F}_{t+ut}^N with parameter $0 \leq u \leq 1$. Therefore, by (8),

$$[\mathbb{N}_\cdot^{t,i,N}, \mathbb{N}_\cdot^{t,j,N}]_u = \mathbf{1}_{\{i=j\}} (Z_{t+ut}^{i,N} - Z_t^{i,N}).$$

On $\Omega_{N,K}$, using the change of variables $s = t + at$, we obtain

$$\begin{aligned} \mathbb{E}_\theta[(\mathcal{Y}_{t,N,K,u}^{3,1})^2] &= \frac{2}{K^2 t^2} \mathbb{E}_\theta \left[\left(\sum_{i=1}^K \mu \ell_N(i) \int_0^{\sqrt{u}} \int_t^{t+at} (M_{r-}^{i,N} - M_t^{i,N}) dM_r^{i,N} da \right)^2 \right] \\ &= \frac{1}{K^2 t^2} \mathbb{E}_\theta \left[\left(\sum_{i=1}^K \mu \ell_N(i) \int_0^{\sqrt{u}} \int_0^a \mathbb{N}_{b-}^{t,i,N} d\mathbb{N}_b^{t,i,N} da \right)^2 \right] \\ &= \frac{\mu^2}{K^2 t^2} \sum_{i=1}^K \sum_{i'=1}^K \ell_N(i) \ell_N(i') \int_0^{\sqrt{u}} \int_0^{\sqrt{u}} \mathbb{E}_\theta \left[\int_0^a \mathbb{N}_{b-}^{t,i,N} d\mathbb{N}_b^{t,i,N} \int_0^{a'} \mathbb{N}_{b'-}^{t,i',N} d\mathbb{N}_{b'}^{t,i',N} \right] dada' \\ &\leq \frac{C}{K^2 t^2} \sum_{i=1}^K \int_0^1 \int_0^1 \mathbb{E}_\theta \left[\left(\int_0^{a \wedge a'} \mathbb{N}_{b-}^{t,i,N} d\mathbb{N}_b^{t,i,N} \right)^2 \right] dada'. \end{aligned}$$

Using the Burkholder–Davis–Gundy inequality, the above term is bounded by

$$\begin{aligned} &\frac{C}{K^2 t^2} \sum_{i=1}^K \mathbb{E}_\theta \left[\int_0^1 (\mathbb{N}_{b-}^{t,i,N})^2 dZ_{t+bt}^{i,N} \right] \\ &\leq \frac{C}{K^2 t^2} \sum_{i=1}^K \mathbb{E}_\theta \left[\left(\sup_{0 \leq b \leq 1} (\mathbb{N}_b^{t,i,N})^2 \right) Z_{2t}^{i,N} \right]. \end{aligned}$$

Hence, applying the Cauchy–Schwarz inequality and the Burkholder–Davis–Gundy inequality,

$$\mathbb{E}_\theta[(\mathcal{Y}_{t,N,K,u}^{3,1})^2] \leq \frac{C}{K^2 t^2} \sum_{i=1}^K \mathbb{E}_\theta \left[\sup_{0 \leq b \leq 1} (\mathbb{N}_b^{t,i,N})^4 \right]^{\frac{1}{2}} \mathbb{E}_\theta[(Z_{2t}^{i,N})^2]^{\frac{1}{2}} \leq \frac{C}{K^2 t^2} \sum_{i=1}^K \mathbb{E}_\theta[(Z_{2t}^{i,N})^2] \leq \frac{C}{K},$$

where the last inequality follows from Lemma 3.4-(iii). This completes the proof.

□

We are now fully equipped to prove the limit for the second estimator.

Proof of Theorem 5.2. Recall that we operate with $(N, K, t) \rightarrow (\infty, \infty, \infty)$ such that $\frac{t\sqrt{K}}{N}(\frac{N}{t^q} + \sqrt{\frac{N}{Kt}}) + Ne^{-c_{p,\Lambda}K} \rightarrow 0$. At the beginning of the section, we decomposed the difference as

$$\mathcal{V}_t^{N,K} - \mathcal{V}_{\infty}^{N,K} = J_t^{N,K,1} + J_t^{N,K,211} + J_t^{N,K,212} + J_t^{N,K,213} + J_t^{N,K,22} + J_t^{N,K,23} + J_t^{N,K,3}.$$

As shown in Lemma 5.3, all terms except $J_t^{N,K,211}$, when multiplied by $t\sqrt{K}/N$, converge to 0. To complete the proof, it remains to show that under $(t, N, K) \rightarrow (\infty, \infty, \infty)$ and $Ne^{-c_{p,\Lambda}K} \rightarrow 0$,

$$\mathbf{1}_{\Omega_{N,K}} \frac{t\sqrt{K}}{N} J_t^{N,K,211} = \mathbf{1}_{\Omega_{N,K}} \frac{1}{t\sqrt{K}} \sum_{i=1}^K \left\{ (U_{2t}^{i,N} - U_t^{i,N})^2 - \mathbb{E}_{\theta}[(U_{2t}^{i,N} - U_t^{i,N})^2] \right\} \xrightarrow{d} \mathcal{N}\left(0, \frac{2\mu^2}{(1-\Lambda p)^2}\right),$$

which will establish the desired result.

We now work on $\Omega_{N,K}$. Recalling (9), we write

$$(U_{2t}^{i,N} - U_t^{i,N})^2 = (M_{2t}^{i,N} - M_t^{i,N})^2 + 2T_t^{i,N}(M_{2t}^{i,N} - M_t^{i,N}) + (T_t^{i,N})^2,$$

where

$$T_t^{i,N} := \sum_{n \geq 1} \sum_{j=1}^N \int_0^{2t} \phi^{*n}(2t-s) A_N^n(i, j) M_s^{j,N} ds - \sum_{n \geq 1} \sum_{j=1}^N \int_0^t \phi^{*n}(t-s) A_N^n(i, j) M_s^{j,N} ds.$$

These terms will be treated individually in what follows. Here, as usual, we set $\phi(s) = 0$ for $s \leq 0$.

Step 1. In this step, we verify that

$$\lim_{(N, K, t) \rightarrow (\infty, \infty, \infty)} \mathbf{1}_{\Omega_{N,K}} \frac{1}{t\sqrt{K}} \mathbb{E}_{\theta} \left[\sum_{i=1}^K \left| (T_t^{i,N})^2 - \mathbb{E}_{\theta}[(T_t^{i,N})^2] \right| \right] = 0.$$

By the triangle inequality, it suffices to show that for all $i = 1, \dots, K$, $\mathbb{E}_{\theta}[(T_t^{i,N})^2] \leq Ct/N$.

Setting $\beta_n(s, t, r) = \phi^{*n}(t-r) - \phi^{*n}(s-r)$, we rewrite

$$(11) \quad T_t^{i,N} = \sum_{n \geq 1} \int_0^{2t} \beta_n(t, 2t, u) \sum_{j=1}^N A_N^n(i, j) M_u^{j,N} du.$$

Hence,

$$\mathbb{E}_{\theta}[(T_t^{i,N})^2] = \sum_{m, n \geq 1} \int_0^{2t} \int_0^{2t} \beta_m(t, 2t, u) \beta_n(t, 2t, v) \sum_{j, k=1}^N A_N^m(i, j) A_N^n(i, k) \mathbb{E}_{\theta}[M_u^{j,N} M_v^{k,N}] dv du.$$

Note that $\int_0^{2t} \beta_n(t, 2t, u) \leq 2\Lambda^n$ for any $n \geq 0$. Using (8) and Lemma 3.3-(i) with $r = \infty$ on $\Omega_{N,K}$, we obtain $\mathbb{E}_\theta[M_u^{j,N} M_v^{k,N}] = \mathbf{1}_{\{j=k\}} \mathbb{E}_\theta[Z_{u \wedge v}^{j,N}] \leq C(u \wedge v)$. Therefore,

$$\begin{aligned} \mathbb{E}_\theta[(T_t^{i,N})^2] &= \sum_{m,n \geq 1} \int_0^{2t} \int_0^{2t} \beta_m(t, 2t, u) \beta_n(t, 2t, v) \sum_{j,k=1}^N A_N^m(i, j) A_N^n(i, k) \mathbb{E}_\theta[M_u^{j,N} M_v^{k,N}] dv du \\ &\leq Ct \sum_{m,n \geq 1} \int_0^{2t} \int_0^{2t} \beta_m(t, 2t, u) \beta_n(t, 2t, v) dv du \sum_{j=1}^N A_N^m(i, j) A_N^n(i, j) \\ &\leq Ct \sum_{m,n \geq 1} \Lambda^{m+n} \sum_{j=1}^N A_N^m(i, j) A_N^n(i, j) \\ &\leq Ct \sum_{j=1}^N (Q_N(i, j) - \mathbf{1}_{\{i=j\}})^2 \leq \frac{Ct}{N}. \end{aligned}$$

The last step follows from [13, (8)], which states that on $\Omega_{N,K} \subset \Omega_N^1$, $\mathbf{1}_{\{i=j\}} \leq Q_N(i, j) \leq \mathbf{1}_{\{i=j\}} + \Lambda CN^{-1}$.

Step 2. In this step, we verify that

$$\lim_{(N,K,t) \rightarrow (\infty, \infty, \infty)} \mathbf{1}_{\Omega_{N,K}} \frac{1}{t\sqrt{K}} \mathbb{E}_\theta \left[\left| \sum_{i=1}^K \left(T_t^{i,N} (M_{2t}^{i,N} - M_t^{i,N}) - \mathbb{E}_\theta[T_t^{i,N} (M_{2t}^{i,N} - M_t^{i,N})] \right) \right| \right] = 0.$$

Actually, this follows from the variance estimate on $\Omega_{N,K}$:

$$x := \text{Var}_\theta \left[\sum_{i=1}^K (T_t^{i,N} (M_{2t}^{i,N} - M_t^{i,N})) \right] \leq C \frac{Kt^2}{N}.$$

We begin with

$$\begin{aligned} x &= \mathbb{E}_\theta \left[\sum_{i,j=1}^K \left(T_t^{i,N} (M_{2t}^{i,N} - M_t^{i,N}) - \mathbb{E}_\theta[T_t^{i,N} (M_{2t}^{i,N} - M_t^{i,N})] \right) \right. \\ &\quad \left. \left(T_t^{j,N} (M_{2t}^{j,N} - M_t^{j,N}) - \mathbb{E}_\theta[T_t^{j,N} (M_{2t}^{j,N} - M_t^{j,N})] \right) \right]. \end{aligned}$$

Recalling (11) and setting $\alpha_N(u, t, i, j) := \sum_{n \geq 1} \beta_n(t, 2t, u) A_N^n(i, j)$, we obtain

$$\begin{aligned} x &\leq \sum_{i,j=1}^K \int_0^{2t} \int_0^{2t} \sum_{k,m=1}^N |\alpha_N(s, t, i, k) \alpha_N(u, t, j, m)| \\ &\quad |\mathbb{Cov}_\theta[(M_{2t}^{i,N} - M_t^{i,N}) M_s^{k,N}, (M_{2t}^{j,N} - M_t^{j,N}) M_u^{m,N}]| ds du. \end{aligned}$$

Moreover, since $\int_0^{2t} \beta_n(t, 2t, u) \leq 2\Lambda^n$ for any $n \geq 0$, we have

$$\int_0^{2t} |\alpha_N(s, t, i, k)| ds \leq \sum_{n \geq 1} A_N^n(i, k) \int_0^{2t} |\beta_n(t, 2t, s)| ds \leq 2 \sum_{n \geq 1} A_N^n(i, k) \Lambda^n \leq 2(Q_N(i, k) - \mathbf{1}_{\{i=k\}}),$$

which, as noted at the end of Step 1, is bounded by C/N according to [13, (8)]. In addition, by [13, Lemma 22], we have on $\Omega_{N,K}$ that for $s, u \in [0, 2t]$,

$$|\mathbb{Cov}_\theta[(M_{2t}^{i,N} - M_t^{i,N}) M_s^{k,N}, (M_{2t}^{j,N} - M_t^{j,N}) M_u^{m,N}]| \leq C(\mathbf{1}_{\#\{k,i,j,m\}=3} N^{-2} t + \mathbf{1}_{\#\{k,i,j,m\} \leq 2} t^2).$$

Therefore, we conclude that

$$x \leq \frac{C}{N^2} \sum_{i,j=1}^K \sum_{k,m=1}^N (\mathbf{1}_{\#\{k,i,j,m\}=3} N^{-2} t + \mathbf{1}_{\#\{k,i,j,m\}\leq 2} t^2) \leq \frac{C}{N^2} (N^2 K \times N^{-2} t + NK \times t^2),$$

which is bounded by CKt^2/N as desired.

Step 3. It remains to show that

$$(12) \quad \mathbf{1}_{\Omega_{N,K}} \frac{1}{t\sqrt{K}} \left[\sum_{i=1}^K (M_{2t}^{i,N} - M_t^{i,N})^2 - \sum_{i=1}^K \mathbb{E}_\theta[(M_{2t}^{i,N} - M_t^{i,N})^2] \right]$$

converges to a Gaussian random variable with variance $2\mu^2/(1-\Lambda p)^2$. Applying Itô's formula, we obtain

$$(M_{2t}^{i,N} - M_t^{i,N})^2 = 2 \int_t^{2t} (M_{s-}^{i,N} - M_t^{i,N}) dM_s^{i,N} + Z_{2t}^{i,N} - Z_t^{i,N}.$$

Therefore, (12) becomes

$$\mathbf{1}_{\Omega_{N,K}} \frac{1}{t\sqrt{K}} \sum_{i=1}^K \left[2 \int_t^{2t} (M_{s-}^{i,N} - M_t^{i,N}) dM_s^{i,N} + \{(Z_{2t}^{i,N} - Z_t^{i,N}) - \mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]\} \right].$$

From [21, Lemma 7.2-(ii)], we know that $\mathbf{1}_{\Omega_{N,K}} \mathbb{E}_\theta[|\bar{U}_t^{N,K}|^2] \leq \frac{Ct}{K}$. This immediately implies

$$\mathbf{1}_{\Omega_{N,K}} \frac{1}{t\sqrt{K}} \sum_{i=1}^K \{(Z_{2t}^{i,N} - Z_t^{i,N}) - \mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]\} = \mathbf{1}_{\Omega_{N,K}} \frac{\sqrt{K}}{t} [\bar{U}_{2t}^{N,K} - \bar{U}_t^{N,K}] \rightarrow 0.$$

Recalling $N_u^{t,i,N} = \int_t^{t+\sqrt{ut}} (M_{s-}^{i,N} - M_t^{i,N}) dM_s^{i,N}$ defined in Lemma 5.4. Since (10) is established in Lemma 5.4 and $\frac{1}{t\sqrt{K}} \sum_{i=1}^K \int_t^{2t} (M_{s-}^{i,N} - M_t^{i,N}) dM_s^{i,N} = \frac{1}{t\sqrt{K}} \sum_{i=1}^K N_1^{t,i,N}$, we thus complete the proof. \square

6. LIMIT THEOREM FOR THE THIRD ESTIMATOR

In this section, our goal is to establish the asymptotic behavior of the third estimator $\mathcal{X}_{\Delta,t}^{N,K}$, introduced in Section 2.1 (see Theorem 6.1). First, recall that for $\Delta \geq 1$ such that $t/(2\Delta) \in \mathbb{N}^*$,

$$\begin{aligned} \mathcal{X}_{\Delta,t}^{N,K} &= \mathcal{W}_{\Delta,t}^{N,K} - \frac{N-K}{K} \varepsilon_t^{N,K}, \quad \mathcal{W}_{\Delta,t}^{N,K} = 2\mathcal{Z}_{2\Delta,t}^{N,K} - \mathcal{Z}_{\Delta,t}^{N,K}, \\ \mathcal{Z}_{\Delta,t}^{N,K} &= \frac{N}{t} \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \Delta \varepsilon_t^{N,K})^2, \quad \varepsilon_t^{N,K} = (\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K})/t, \end{aligned}$$

where $\bar{Z}_t^N = N^{-1} \sum_{i=1}^N Z_t^{i,N}$ and $\bar{Z}_t^{N,K} = K^{-1} \sum_{i=1}^K t Z_t^{i,N}$. Moreover, Q_N , $(\ell_N(i))_{i=1,\dots,N}$ and $\bar{\ell}_N^K$ are defined in Section 3.1. We also introduce $c_N^K(j) := \sum_{i=1}^K Q_N(i,j)$, $j = 1, \dots, N$.

It was shown in [21] that $\mathcal{X}_{\Delta,t}^{N,K}$ converges to $\mathcal{X}_{\infty,\infty}^{N,K} = \mathcal{W}_{\infty,\infty}^{N,K} - \frac{(N-K)\mu}{K} \bar{\ell}_N^K$, where $\mathcal{W}_{\infty,\infty}^{N,K} = \mu \frac{N}{K^2} A_{\infty,\infty}^{N,K}$, $A_{\infty,\infty}^{N,K} = \sum_{j=1}^N \left(\sum_{i=1}^K Q_N(i,j) \right)^2 \ell_N(j)$. To establish the central limit theorem stated

in Theorem 6.1, we decompose $\mathcal{X}_{\Delta,t}^{N,K} - \mathcal{X}_{\infty,\infty}^{N,K}$ into

$$\begin{aligned} & \mathcal{X}_{\Delta,t}^{N,K} - \mathcal{X}_{\infty,\infty}^{N,K} \\ &= (\mathcal{W}_{\Delta,t}^{N,K} - \mathcal{W}_{\infty,\infty}^{N,K}) - \frac{N-K}{K} (\varepsilon_t^{N,K} - \mu \bar{\ell}_N^K) \\ &= - \underbrace{D_{\Delta,t}^{N,K,1} + 2D_{2\Delta,t}^{N,K,1} - D_{\Delta,t}^{N,K,2} + 2D_{2\Delta,t}^{N,K,2}}_{\text{small error}} - \underbrace{D_{\Delta,t}^{N,K,3} + 2D_{2\Delta,t}^{N,K,3} + D_{\Delta,t}^{N,K,4}}_{\text{principle}} - \underbrace{\frac{N-K}{K} (\varepsilon_t^{N,K} - \mu \bar{\ell}_N^K)}_{\text{small error}}, \end{aligned}$$

where

$$\begin{aligned} D_{\Delta,t}^{N,K,1} &= \frac{N}{t} \left\{ \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left(\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \Delta \varepsilon_t^{N,K} \right)^2 - \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left(\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \Delta \mu \bar{\ell}_N^K \right)^2 \right\}, \\ D_{\Delta,t}^{N,K,2} &= \frac{N}{t} \left\{ \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left(\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \Delta \mu \bar{\ell}_N^K \right)^2 \right. \\ &\quad \left. - \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left(\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \mathbb{E}_\theta[\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K}] \right)^2 \right\}, \\ D_{\Delta,t}^{N,K,3} &= \frac{N}{t} \left\{ \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left(\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \mathbb{E}_\theta[\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K}] \right)^2 \right. \\ &\quad \left. - \mathbb{E}_\theta \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left(\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \mathbb{E}_\theta[\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K}] \right)^2 \right] \right\}, \end{aligned}$$

and finally

$$\begin{aligned} D_{\Delta,t}^{N,K,4} &= \left\{ \frac{2N}{t} \mathbb{E}_\theta \left[\sum_{a=\frac{t}{2\Delta}+1}^{\frac{t}{\Delta}} \left(\bar{Z}_{2a\Delta}^{N,K} - \bar{Z}_{2(a-1)\Delta}^{N,K} - \mathbb{E}_\theta[\bar{Z}_{2a\Delta}^{N,K} - \bar{Z}_{2(a-1)\Delta}^{N,K}] \right)^2 \right] \right. \\ &\quad \left. - \frac{N}{t} \mathbb{E}_\theta \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left(\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \mathbb{E}_\theta[\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K}] \right)^2 \right] - \mathcal{W}_{\infty,\infty}^{N,K} \right\}. \end{aligned}$$

The principle term in this decomposition arises from $D_{\Delta,t}^{N,K,3}$ (see Lemma 6.2), which is approximated by the martingale difference combination $\mathbb{X}_{\Delta,t,v}^{N,K}$ defined in (13) (see Proposition 6.3). We then prove in Proposition 6.7 that $\mathbb{X}_{\Delta,t,v}^{N,K}$ satisfies a central limit theorem, thereby establishing Theorem 6.1.

The proof of Proposition 6.3 relies on Lemmas 6.4, 6.5, and 6.6, which together establish the convergence of each component in the decomposition of $|D_{\Delta,t}^{N,K,3} - \frac{N}{t} \mathbb{X}_{\Delta,t,1}^{N,K}|$. The proof of Proposition 6.7, on the other hand, proceeds in two steps: we first establish Lemma 6.8 and then Proposition 6.9, which itself follows from Lemmas 6.10 and 6.11. We now state Theorem 6.1, which is proved in Section 6.3.

Theorem 6.1. Assume $(H(q))$ for some $q > 3$, $K \leq N$ and $\lim_{(N,K) \rightarrow (\infty,\infty)} \frac{K}{N} = \gamma \leq 1$, $\Delta_t = t/(2\lfloor t^{1-4/(q+1)} \rfloor) \sim t^{4/(q+1)}/2$ (for t large). If $(N, K, t) \rightarrow (\infty, \infty, \infty)$ and $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \rightarrow 0$,

$$\mathbf{1}_{\Omega_{N,K}} \frac{K}{N} \sqrt{\frac{t}{\Delta_t}} (\mathcal{X}_{\Delta_t,t}^{N,K} - \mathcal{X}_{\infty,\infty}^{N,K}) \rightarrow \mathcal{N}\left(0, 6\mu^2 \left(\frac{1-\gamma}{(1-\Lambda p)} + \frac{\gamma}{(1-\Lambda p)^3}\right)^2\right).$$

6.1. Some small terms of the estimator. First, we are going to prove the terms $D_{\Delta,t}^{N,K,1}$, $D_{\Delta,t}^{N,K,2}$, $D_{\Delta,t}^{N,K,4}$ and $\frac{N}{K}|\varepsilon_t^{N,K} - \mu\bar{\ell}_N^K|$ are small.

Lemma 6.2. Assume $(H(q))$ for some $q > 3$. If we choose $\Delta_t = t/(2\lfloor t^{1-4/(q+1)} \rfloor) \sim t^{4/(q+1)}/2$ (for t large), then, If $(N, K, t) \rightarrow (\infty, \infty, \infty)$ and $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \rightarrow 0$, we have the convergence in probability that

$$\mathbf{1}_{\Omega_{N,K}} \frac{K}{N} \sqrt{\frac{t}{\Delta_t}} \left\{ |D_{\Delta,t}^{N,K,1}| + |D_{\Delta,t}^{N,K,2}| + |D_{\Delta,t}^{N,K,4}| + \frac{N}{K} |\varepsilon_t^{N,K} - \mu\bar{\ell}_N^K| \right\} \rightarrow 0.$$

Proof. It is a directly corollary of [21, Lemmas 7.3, 9.2, 9.3 and 9.5]. \square

Next, for $\Delta \geq 1$, we consider the term $D_{\Delta,t}^{N,K,3}$ and prove that it is close to $\mathbb{X}_{\Delta,t,v}^{N,K}$. Recall $c_N^K(j) = \sum_{i=1}^K Q_N(i,j)$, and for $0 \leq v \leq 1$, define

$$(13) \quad \mathbb{X}_{\Delta,t,v}^{N,K} := \sum_{a=\lceil \frac{vt}{\Delta} \rceil + 1}^{\lceil \frac{2vt}{\Delta} \rceil} \left\{ (\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K})^2 - \mathbb{E}_\theta[(\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K})^2] \right\},$$

where

$$(14) \quad \mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K} := \frac{1}{K} \sum_{j=1}^N c_N^K(j) (M_{a\Delta}^{j,N} - M_{(a-1)\Delta}^{j,N}).$$

Proposition 6.3. Assume $(H(q))$ for some $q > 3$, for $\Delta \geq 1$, then we have

$$\frac{K}{N} \sqrt{\frac{t}{\Delta}} \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| D_{\Delta,t}^{N,K,3} - \frac{N}{t} \mathbb{X}_{\Delta,t,1}^{N,K} \right| \right] \leq \frac{CK}{N\Delta} + \frac{C\sqrt{K}}{\sqrt{N\Delta}} + \frac{Ct^{\frac{3}{4}}\sqrt{K}}{\Delta^{1+\frac{q}{2}}}.$$

Before presenting the proof, we require some preparatory steps. Recall $U^{i,N}$ defined in (9). For $a \in \{t/(2\Delta) + 1, \dots, 2t/\Delta\}$, we write for $i = 1, \dots, K$

$$U_{a\Delta}^{i,N} - U_{(a-1)\Delta}^{i,N} = \Gamma_{(a-1)\Delta,a\Delta}^{i,N} + X_{(a-1)\Delta,a\Delta}^{i,N},$$

where

$$(15) \quad \Gamma_{(a-1)\Delta, a\Delta}^{i,N} := \sum_{n \geq 1} \left\{ \int_0^{a\Delta} \phi^{*n}(a\Delta - s) \sum_{j=1}^N A_N^n(i, j) [M_s^{j,N} - M_{a\Delta}^{j,N}] ds \right. \\ \left. - \int_0^{(a-1)\Delta} \phi^{*n}((a-1)\Delta - s) \sum_{j=1}^N A_N^n(i, j) [M_s^{j,N} - M_{(a-1)\Delta}^{j,N}] ds \right\},$$

$$(16) \quad X_{(a-1)\Delta, a\Delta}^{i,N} := \sum_{n \geq 0} \sum_{j=1}^N \left\{ \int_0^{a\Delta} \phi^{*n}(a\Delta - s) ds A_N^n(i, j) M_{a\Delta}^{j,N} \right. \\ \left. - \int_0^{(a-1)\Delta} \phi^{*n}((a-1)\Delta - s) ds A_N^n(i, j) M_{(a-1)\Delta}^{j,N} \right\}.$$

Accordingly, we define $\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K} := \frac{1}{K} \sum_{i=1}^K \Gamma_{(a-1)\Delta, a\Delta}^{i,N}$, $\bar{X}_{(a-1)\Delta, a\Delta}^{N,K} := \frac{1}{K} \sum_{i=1}^K X_{(a-1)\Delta, a\Delta}^{i,N}$. We decompose the difference into three terms:

$$\left| D_{\Delta,t}^{N,K,3} - \frac{N}{t} \mathbb{X}_{\Delta,t,1}^{N,K} \right| = \frac{N}{t} \left| \left\{ \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left(\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \mathbb{E}_\theta[\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K}] \right)^2 \right. \right. \\ \left. \left. - \mathbb{E}_\theta \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left(\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \mathbb{E}_\theta[\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K}] \right)^2 \right] \right\} \right. \\ \left. - \sum_{a=[\frac{vt}{\Delta}]+1}^{[\frac{2vt}{\Delta}]} \left\{ (\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K})^2 - \mathbb{E}_\theta[(\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K})^2] \right\} \right|,$$

which is bounded by

$$\frac{2N}{t} \left[\left(\left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left\{ (\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2 - \mathbb{E}_\theta[(\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2] \right\} \right| \right. \right. \\ \left. \left. + \left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K} \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} - \mathbb{E}_\theta \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K} \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} \right] \right| \right. \right. \\ \left. \left. + \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} \right| \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} \right| + \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} \right|^2 \right) \right] \\ + \frac{2N}{t} \mathbb{E}_\theta \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} \right| \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} \right| + \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} \right|^2 \right].$$

We now prove that all the terms above are negligible. The first term is treated in Lemma 6.5, and the rest are handled in Lemma 6.6.

We begin by recalling $\Gamma_{(a-1)\Delta, a\Delta}^{i,N}$, $i = 1, \dots, K$ defined in (15), which allows us to rewrite their average $\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K} = \frac{1}{K} \sum_{i=1}^K \Gamma_{(a-1)\Delta, a\Delta}^{i,N}$ as $\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K} = C_{a\Delta}^{N,K} + B_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K}$,

where

$$(17) \quad C_{a\Delta}^{N,K} := \frac{1}{K} \sum_{i=1}^K \sum_{j=1}^N \sum_{n \geq 1} \int_0^\Delta \phi^{*n}(s) A_N^n(i, j) (M_{(a\Delta-s)}^{j,N} - M_{a\Delta}^{j,N}) ds,$$

$$(18) \quad B_{a\Delta}^{N,K} := \frac{1}{K} \sum_{i=1}^K \sum_{j=1}^N \sum_{n \geq 1} \int_\Delta^{a\Delta} \phi^{*n}(s) A_N^n(i, j) (M_{(a\Delta-s)}^{j,N} - M_{a\Delta}^{j,N}) ds.$$

We now establish the following bounds for $C_{a\Delta}^{N,K}$ and $B_{a\Delta}^{N,K}$. The proofs are deferred to Appendix D.

Lemma 6.4. *Assume $(H(q))$ for some $q \geq 1$. For $\Delta \geq 1$ and $a, b \in \{\frac{t}{\Delta} + 1, \dots, \frac{2t}{\Delta}\}$. Then a.s. on $\Omega_{N,K}$,*

$$(i) \quad \mathbb{E}_\theta[(B_{a\Delta}^{N,K})^2] \leq \frac{C}{N} \Delta^{1-2q},$$

$$(ii) \quad \mathbb{E}_\theta[(C_{a\Delta}^{N,K})^4] \leq \frac{C}{N^2},$$

$$(iii) \quad \text{Cov}_\theta[(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})^2, (C_{b\Delta}^{N,K} - C_{(b-1)\Delta}^{N,K})^2] \leq \frac{C\sqrt{t}}{N\Delta^{q-1}}, \quad |a - b| \geq 4.$$

We are now in position to give the estimate of $\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left\{ (\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2 - \mathbb{E}_\theta[(\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2] \right\}$.

Lemma 6.5. *Assume $(H(q))$ for some $q \geq 1$, then a.s. on the set $\Omega_{N,K}$, for $\Delta \geq 1$,*

$$\frac{K}{N} \sqrt{\frac{t}{\Delta}} \frac{N}{t} \mathbb{E}_\theta \left[\left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left\{ (\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2 - \mathbb{E}_\theta[(\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2] \right\} \right| \right] \leq \frac{CK\sqrt{t}}{N\Delta^{(q+1)}} + \frac{CK}{N\Delta} + \frac{CKt^{\frac{3}{4}}}{\Delta^{(1+\frac{q}{2})}\sqrt{N}}.$$

Proof. We start from

$$(\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2 = (C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})^2 + 2(B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K})(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K}) + (B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K})^2.$$

Then,

$$\begin{aligned} & \frac{K}{N} \sqrt{\frac{t}{\Delta}} \frac{N}{t} \mathbb{E}_\theta \left[\left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left\{ (\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2 - \mathbb{E}_\theta[(\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2] \right\} \right| \right] \\ & \leq \frac{K}{\sqrt{t\Delta}} \left\{ \mathbb{E}_\theta \left[\left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K})^2 - \mathbb{E}_\theta \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K})^2 \right] \right| \right] \right. \\ & \quad + \mathbb{E}_\theta \left[\left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})^2 - \mathbb{E}_\theta \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})^2 \right] \right| \right] \\ & \quad \left. + 2\mathbb{E}_\theta \left[\left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K})(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K}) \right. \right. \right. \\ & \quad \left. \left. \left. - \mathbb{E}_\theta[(B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K})(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})] \right| \right] \right\}. \end{aligned}$$

Applying Lemma 6.4-(i) gives

$$\begin{aligned} & \mathbb{E}_\theta \left[\left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K})^2 - \mathbb{E}_\theta \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K})^2 \right] \right| \right] \\ & \leq 2 \mathbb{E}_\theta \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K})^2 \right] \\ & \leq \frac{Ct}{N\Delta^{2q}}. \end{aligned}$$

By lemma 6.4-(ii)&(iii), we have

$$\begin{aligned} & \mathbb{E}_\theta \left[\left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})^2 - \mathbb{E}_\theta \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})^2 \right] \right|^2 \right] \\ & = \text{Var}_\theta \left[\sum_{a=\frac{v_t}{\Delta}+1}^{\frac{2v_t}{\Delta}} (C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})^2 \right] \\ & \leq \sum_{\substack{t/\Delta+1 \leq a, b \leq 2t/\Delta \\ |a-b| \leq 3}} \mathbb{E}_\theta [(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})^4]^{\frac{1}{2}} \mathbb{E}_\theta [(C_{b\Delta}^{N,K} - C_{(b-1)\Delta}^{N,K})^4]^{\frac{1}{2}} \\ & \quad + \sum_{\substack{t/\Delta+1 \leq a, b \leq 2t/\Delta \\ |a-b| \geq 4}} \text{Cov}_\theta [(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})^2, (C_{b\Delta}^{N,K} - C_{(b-1)\Delta}^{N,K})^2] \\ & \leq C \left[\frac{t}{\Delta} \frac{1}{N^2} + \frac{t^{\frac{5}{2}}}{\Delta^{q+1} N} \right]. \end{aligned}$$

Moreover, by the Cauchy-Schwarz inequality and Lemmas 6.4-(i)&(ii), we have

$$\begin{aligned} & \mathbb{E}_\theta \left[\left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K})(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K}) - \mathbb{E}_\theta [(B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K})(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})] \right| \right] \\ & \leq 4 \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left\{ \mathbb{E}_\theta \left[\left| B_{a\Delta}^{N,K} C_{a\Delta}^{N,K} \right| \right] + \mathbb{E}_\theta \left[\left| B_{(a-1)\Delta}^{N,K} C_{a\Delta}^{N,K} \right| \right] + \mathbb{E}_\theta \left[\left| B_{a\Delta}^{N,K} C_{(a-1)\Delta}^{N,K} \right| \right] \right. \\ & \quad \left. + \mathbb{E}_\theta \left[\left| B_{(a-1)\Delta}^{N,K} C_{(a-1)\Delta}^{N,K} \right| \right] \right\} \\ & \leq 4 \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left\{ \mathbb{E}_\theta \left[\left| B_{a\Delta}^{N,K} \right|^2 \right]^{\frac{1}{2}} + \left| B_{(a-1)\Delta}^{N,K} \right|^2 \right\} \left\{ \mathbb{E}_\theta \left[\left| C_{a\Delta}^{N,K} \right|^2 \right]^{\frac{1}{2}} + \mathbb{E}_\theta \left[\left| C_{(a-1)\Delta}^{N,K} \right|^2 \right]^{\frac{1}{2}} \right\} \\ & \leq \frac{Ct}{\Delta^{q+\frac{1}{2}}} \frac{1}{N}. \end{aligned}$$

Overall, we have

$$\begin{aligned} & \frac{K}{N} \sqrt{\frac{t}{\Delta}} \frac{N}{t} \mathbb{E}_\theta \left[\left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left\{ (\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2 - \mathbb{E}_\theta[(\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2] \right\} \right| \right] \\ & \leq \frac{K}{\sqrt{t\Delta}} \left\{ \frac{1}{N} \frac{Ct}{\Delta^{2q}} + \left[\frac{t}{\Delta} \frac{1}{N^2} + \frac{t^{\frac{5}{2}}}{\Delta^{q+1} N} \right]^{\frac{1}{2}} + \frac{1}{N} \frac{Ct}{\Delta^{q+\frac{1}{2}}} \right\} \\ & \leq C \left\{ \frac{K}{N\Delta} + \frac{Kt^{\frac{3}{4}}}{\Delta^{(1+\frac{q}{2})}\sqrt{N}} + \frac{K\sqrt{t}}{N\Delta^{(q+1)}} \right\}. \end{aligned}$$

The proof is finished. \square

Recall that $c_N^K(j) = \sum_{i=1}^K Q_N(i, j)$ and that $X_{(a-1)\Delta, a\Delta}^{i,N}$ is defined in (16). Our next step is to show that the term $\bar{X}_{(a-1)\Delta, a\Delta}^{N,K} = \frac{1}{K} \sum_{i=1}^K X_{(a-1)\Delta, a\Delta}^{i,N}$ is close to $\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K}$ defined in (14).

Lemma 6.6. *Assume (H(q)) for some $q \geq 2$. For $\Delta \geq 1$, then we have the following inequalities,*

- (i) $\mathbb{E}_\theta[(\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K})^2] \leq \frac{C}{N} \left[\frac{1}{(a\Delta)^{2q-1}} + \frac{1}{((a-1)\Delta)^{2q-1}} \right]$.
- (ii) $\frac{K}{\sqrt{t\Delta}} \mathbb{E}_\theta \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} \right|^2 \right] \leq \frac{CK}{N\sqrt{t\Delta}^{2q-\frac{1}{2}}}$.
- (iii) $\mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} \right|^4 \right] \leq \frac{C\Delta^2}{K^2}$.
- (iv) $\frac{K}{\sqrt{\Delta t}} \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \sum_{a=\frac{t}{\Delta}}^{\frac{2t}{\Delta}} \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} \right| \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} \right| \right] \leq \frac{C\sqrt{K}}{\Delta^{q-\frac{1}{2}}\sqrt{Nt}}$.
- (v) $\mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \frac{K}{N} \sqrt{\frac{t}{\Delta}} \left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K} \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} - \mathbb{E}_\theta[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K} \bar{X}_{(a-1)\Delta, a\Delta}^{N,K}] \right| \right]$
 $\leq \frac{CK}{N\Delta^q\sqrt{t}} + \frac{C\sqrt{tK}}{\Delta^{q+\frac{1}{2}}\sqrt{N}} + \frac{C\sqrt{K}}{\sqrt{N\Delta}} + \frac{Ct^{\frac{3}{4}}\sqrt{K}}{\Delta^{1+\frac{q}{2}}}$.

We place the proof of Lemma 6.6 in Appendix D. Now, we can give the proof of Proposition 6.3.

Proof of Proposition 6.3. Recalling (13) and (14), as well as Lemmas 6.4, 6.5, 6.6-(i),(ii),(iv)&(v), we have

$$\begin{aligned} & \frac{K}{N} \sqrt{\frac{t}{\Delta}} \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| D_{\Delta,t}^{N,K,3} - \frac{N}{t} \mathbb{X}_{\Delta,t,1}^{N,K} \right| \right] \\ & \leq \frac{CK}{N\Delta} + \frac{CKt^{\frac{3}{4}}}{\Delta^{(1+\frac{q}{2})}\sqrt{N}} + \frac{CK\sqrt{t}}{N\Delta^{(q+1)}} + \frac{CK}{N\Delta^q\sqrt{t}} + \frac{C\sqrt{tK}}{\Delta^{q+\frac{1}{2}}\sqrt{N}} \\ & \quad + \frac{C\sqrt{K}}{\sqrt{N\Delta}} + \frac{Ct^{\frac{3}{4}}\sqrt{K}}{\Delta^{1+\frac{q}{2}}} + \frac{C\sqrt{K}}{\Delta^{q-\frac{1}{2}}\sqrt{Nt}} + \frac{CK}{N\sqrt{t}\Delta^{2q-\frac{1}{2}}} \\ & \leq \frac{CK}{N\Delta} + \frac{CKt^{\frac{3}{4}}}{\Delta^{1+\frac{q}{2}}\sqrt{N}} + \frac{C\sqrt{K}}{\sqrt{N\Delta}} + \frac{Ct^{\frac{3}{4}}\sqrt{K}}{\Delta^{1+\frac{q}{2}}} \\ & \leq \frac{CK}{N\Delta} + \frac{C\sqrt{K}}{\sqrt{N\Delta}} + \frac{Ct^{\frac{3}{4}}\sqrt{K}}{\Delta^{1+\frac{q}{2}}}, \end{aligned}$$

which completes the proof. \square

6.2. The convergence of $\mathbb{X}_{\Delta_t,t,v}^{N,K}$. Recall the process $\mathbb{X}_{\Delta_t,t,v}^{N,K}$ defined in (13). The goal of this subsection is to prove the following proposition, which states that the normalized version of $(\mathbb{X}_{\Delta_t,t,v}^{N,K})_{v \geq 0}$ converges to a Gaussian process.

Proposition 6.7. *Assume $(H(q))$ for some $q > 3$. For $t \geq 1$, set $\Delta_t = t/(2\lfloor t^{1-4/(q+1)} \rfloor) \sim t^{4/(q+1)}/2$ (as $t \rightarrow \infty$). When $(N, K, t) \rightarrow (\infty, \infty, \infty)$ such that $\frac{K}{N} \rightarrow \gamma \leq 1$ and $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_p, \Lambda K} \rightarrow 0$, it holds that*

$$\left(\frac{K}{N} \sqrt{\frac{t}{\Delta_t}} \frac{N}{t} \mathbb{X}_{\Delta_t,t,v}^{N,K} \right)_{v \geq 0} \xrightarrow{d} \sqrt{2}\mu \left(\frac{1-\gamma}{(1-\Lambda p)} + \frac{\gamma}{(1-\Lambda p)^3} \right) (B_{2v} - B_v)_{v \geq 0},$$

for the Skorohod topology, where B is a standard Brownian motion.

Recalling the definition of the martingale summation $\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K}$ defined in (14), we apply Itô's formula to obtain

$$(19) \quad (K\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K})^2 = Q_{a,N,K} + \sum_{j=1}^N \left(c_N^K(j) \right)^2 \left(Z_{a\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N} \right),$$

where

$$Q_{a,N,K} = 2 \int_{(a-1)\Delta}^{a\Delta} \sum_{j=1}^N c_N^K(j) (M_s^{j,N} - M_{(a-1)\Delta}^{j,N}) \sum_{j=1}^N c_N^K(j) dM_s^{j,N}.$$

We can then decompose

$$\frac{K}{\sqrt{\Delta t}} \mathbb{X}_{\Delta_t,t,v}^{N,K} = \mathcal{L}_{N,K}^{t,\Delta}(2v) - \mathcal{L}_{N,K}^{t,\Delta}(v) + rest,$$

where

$$\mathcal{L}_{N,K}^{t,\Delta}(u) := \frac{1}{K\sqrt{\Delta t}} \sum_{a=1}^{\lfloor \frac{t}{\Delta} u \rfloor} Q_{a,N,K}, \quad \text{for } 0 \leq u \leq 2.$$

The proof of Proposition 6.7 proceeds in two steps: we first show that the “rest” term is negligible (see Lemma 6.8), and then prove that $\mathcal{L}_{N,K}^{t,\Delta}$ satisfies a central limit theorem (see Proposition 6.9).

Lemma 6.8. *Assume $(H(q))$ for some $q \geq 1$, then for $0 \leq v \leq 1$, and $\Delta \geq 1$,*

$$\mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| \mathcal{L}_{N,K}^{t,\Delta}(2v) - \mathcal{L}_{N,K}^{t,\Delta}(v) - \frac{K}{\sqrt{\Delta t}} \mathbb{X}_{\Delta_t,t,v}^{N,K} \right| \right] \leq \frac{C}{\sqrt{\Delta t}}.$$

Proof. Noting that $\mathbb{E}[Q_{a,N,K} | \mathcal{F}_{(a-1)\Delta}] = 0$, the process $\mathcal{L}_{N,K}^{t,\Delta}(u)$ is a martingale with respect to the filtration $\mathcal{F}_{[\frac{t}{\Delta} u]\Delta}$. In view of equality (19) and definition of $\mathbb{X}_{\Delta_t,t,v}^{N,K}$ in (13), it remains to verify that

$$\mathbb{E}_\theta \left[\left| \frac{1}{K\sqrt{\Delta t}} \sum_{a=\lfloor \frac{vt}{\Delta} \rfloor + 1}^{\lfloor \frac{2vt}{\Delta} \rfloor} \sum_{j=1}^N \left(c_N^K(j) \right)^2 \left(Z_{a\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N} - \mathbb{E}_\theta [Z_{a\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N}] \right) \right| \right] \leq \frac{C}{\sqrt{\Delta t}}.$$

We first decompose

$$\begin{aligned} & \frac{1}{K\sqrt{\Delta t}} \sum_{a=\lceil \frac{vt}{\Delta} \rceil + 1}^{\lceil \frac{2vt}{\Delta} \rceil} \sum_{j=1}^N \left(c_N^K(j) \right)^2 \left(Z_{a\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N} - \mathbb{E}_\theta[Z_{a\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N}] \right) \\ &= \frac{1}{K\sqrt{\Delta t}} \left\{ \sum_{j=1}^N \left(c_N^K(j) \right)^2 \left(Z_{2vt}^{j,N} - Z_{vt}^{j,N} - \mu vt \ell_N(j) \right) \right. \\ &\quad \left. + \sum_{j=1}^N \mathbb{E}_\theta \left[\left(c_N^K(j) \right)^2 \left(\mu vt \ell_N(j) - Z_{2vt}^{j,N} + Z_{vt}^{j,N} \right) \right] \right\}. \end{aligned}$$

From [13, (8)], on the event $\Omega_{N,K} \subset \Omega_N^1$, we have $\mathbf{1}_{\{i=j\}} \leq Q_N(i,j) \leq \mathbf{1}_{\{i=j\}} + \frac{C}{N}$ for all $i, j = 1, \dots, N$. Recalling that $c_N^K(i) = \sum_{j=1}^K Q_N(j,i)$, $i = 1, \dots, N$, we obtain

$$(20) \quad 1 \leq c_N^K(i) \leq 1 + \frac{CK}{N} \text{ when } 1 \leq i \leq K \text{ and } 0 \leq c_N^K(i) \leq \frac{CK}{N} \text{ when } (K+1) \leq i \leq N.$$

Moreover, by [13, Lemma 16-(ii)], we have

$$\max_{j=1,\dots,N} \mathbb{E}_\theta \left[\left| \left(Z_{2vt}^{j,N} - Z_{vt}^{j,N} - \mu vt \ell_N(j) \right) \right| \right] \leq C.$$

Consequently,

$$\begin{aligned} & \mathbb{E}_\theta \left[\frac{1}{K\sqrt{\Delta t}} \sum_{j=1}^N \left| \left(c_N^K(j) \right)^2 \left(Z_{2vt}^{j,N} - Z_{vt}^{j,N} - \mu vt \ell_N(j) \right) \right| \right] \\ & \leq \frac{C}{K\sqrt{\Delta t}} \left[\sum_{j=1}^K \left(c_N^K(j) \right)^2 + \sum_{j=K}^N \left(c_N^K(j) \right)^2 \right] \leq \frac{C}{\sqrt{\Delta t}}. \end{aligned}$$

Therefore,

$$\mathbb{E}_\theta \left[\left| \frac{1}{K\sqrt{\Delta t}} \sum_{a=\lceil \frac{vt}{\Delta} \rceil + 1}^{\lceil \frac{2vt}{\Delta} \rceil} \sum_{j=1}^N \left(c_N^K(j) \right)^2 \left(Z_{a\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N} - \mathbb{E}_\theta[Z_{a\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N}] \right) \right| \right] \leq \frac{C}{\sqrt{\Delta t}},$$

which ends the proof. \square

We now turn to prove the convergence of $\mathcal{L}_{N,K}^{t,\Delta}(u)$ to a Brownian motion.

Proposition 6.9. *Assume $K \leq N$. For $t \geq 1$, define $\Delta_t := t/(2\lfloor t^{1-4/(q+1)} \rfloor) \sim t^{4/(q+1)}/2$ (for t large). Let $(N, K, t) \rightarrow (\infty, \infty, \infty)$ satisfy $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \rightarrow 0$ and $\frac{K}{N} \rightarrow \gamma \leq 1$. Then, in the Skorokhod topology,*

$$(\mathcal{L}_{N,K}^{t,\Delta_t}(u))_{u \geq 0} \xrightarrow{d} \mu\sqrt{2} \left(\frac{1-\gamma}{(1-\Lambda p)} + \frac{\gamma}{(1-\Lambda p)^3} \right) (B_u)_{u \geq 0},$$

where B is a standard Brownian motion.

According to Lemma 4.5, to prove Proposition 6.9, it suffices, by [19, Theorem VIII.3.8], to verify the following two conditions:

1. The jump size of $\mathcal{L}_{N,K}^{t,\Delta}(u)$ is not large (Lemma 6.10).
2. Its quadratic variation increases linearly in time (Lemma 6.11).

The first condition is addressed by the following lemma.

Lemma 6.10. *Assume $(H(q))$ for some $q \geq 1$, and for $\Delta \geq 1$,*

$$\mathbf{1}_{\Omega_{N,K}} \mathbb{E}_\theta \left[\sup_{0 \leq u \leq 2} \left| \mathcal{L}_{N,K}^{t,\Delta}(u) - \mathcal{L}_{N,K}^{t,\Delta}(u-) \right| \right] \leq C \left(\frac{\Delta}{t} \right)^{\frac{1}{4}}.$$

Proof. First, note that $\mathcal{L}_{N,K}^{t,\Delta}(u)$ is a pure jump process. Hence, at a jump time we have

$$\mathcal{L}_{N,K}^{t,\Delta}(u) - \mathcal{L}_{N,K}^{t,\Delta}(u-) = \frac{1}{K\sqrt{\Delta t}} Q_{[\frac{t}{\Delta}u],N,K}.$$

Next, we are going to show that

$$\mathbb{E}_\theta[(Q_{a,N,K})^4] \leq C(K\Delta)^4.$$

For $0 \leq u \leq 1$, we define

$$q_{a,N,K}(u) := \int_{(a-1)\Delta}^{(a-1)+u]\Delta} \sum_{j=1}^N c_N^K(j)(M_s^{j,N} - M_{(a-1)\Delta}^{j,N}) d\left(\sum_{j=1}^N c_N^K(j) M_s^{j,N}\right).$$

Clearly $q_{a,N,K}(1) = \frac{1}{2}Q_{a,N,K}$ and its quadratic variation

$$[q_{a,N,K}(\cdot), q_{a,N,K}(\cdot)]_u = \int_{(a-1)\Delta}^{(a-1)+u]\Delta} \left(\sum_{j=1}^N c_N^K(j)(M_s^{j,N} - M_{(a-1)\Delta}^{j,N}) \right)^2 \sum_{j=1}^N \left(c_N^K(j) \right)^2 dZ_s^{j,N}.$$

Consequently, recalling $\bar{Z}_t^N = N^{-1} \sum_{i=1}^N Z_t^{i,N}$, $\bar{Z}_t^{N,K} = K^{-1} \sum_{i=1}^K Z_t^{i,N}$ and using (20), we obtain

$$\sum_{j=1}^N \left(c_N^K(j) \right)^2 dZ_s^{j,N} \leq C \left(K d\bar{Z}_s^{N,K} + \frac{K^2}{N} d\bar{Z}_s^N \right).$$

On $\Omega_{N,K}$, using the Burkholder–Davis–Gundy inequality, we obtain

$$\begin{aligned} & \mathbb{E}_\theta[(q_{a,N,K}(v))^4] \leq 4\mathbb{E}_\theta[(q_{a,N,K}(\cdot), q_{a,N,K}(\cdot)]_v]^2 \\ &= 4\mathbb{E}_\theta \left[\left(\int_{(a-1)\Delta}^{(a-1)+v]\Delta} \left(\sum_{j=1}^N c_N^K(j)(M_s^{j,N} - M_{(a-1)\Delta}^{j,N}) \right)^2 d\sum_{j=1}^N \left(c_N^K(j) \right)^2 Z_s^{j,N} \right)^2 \right] \\ &\leq 4\mathbb{E}_\theta \left[\sup_{0 \leq s \leq v\Delta} \left(\sum_{j=1}^N c_N^K(j)(M_{(a-1)\Delta+s}^{j,N} - M_{(a-1)\Delta}^{j,N}) \right)^4 \left(\sum_{j=1}^N \left(c_N^K(j) \right)^2 (Z_{(a-1+v)\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N}) \right)^2 \right] \\ &\leq 8\mathbb{E}_\theta \left[\sup_{0 \leq s \leq v\Delta} \left(\sum_{j=1}^N c_N^K(j)(M_{(a-1)\Delta+s}^{j,N} - M_{(a-1)\Delta}^{j,N}) \right)^8 + \left(\sum_{j=1}^N \left(c_N^K(j) \right)^2 (Z_{(a-1+v)\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N}) \right)^4 \right] \\ &\leq C\mathbb{E}_\theta \left[\left(\sum_{j=1}^N \left(c_N^K(j) \right)^2 (Z_{(a-1+v)\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N}) \right)^4 \right] \\ &\leq C\mathbb{E}_\theta \left[\left(K(\bar{Z}_{(a-1+v)\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K}) + \frac{K^2}{N} (\bar{Z}_{(a-1+v)\Delta}^N - \bar{Z}_{(a-1)\Delta}^N) \right)^4 \right] \\ &\leq C(Kv\Delta)^4. \end{aligned}$$

Here the fourth step uses the Burkholder–Davis–Gundy inequality, while the last bound follows from Lemma 3.4-(iv).

Finally, applying the Cauchy–Schwarz inequality at the third step, we obtain

$$\begin{aligned} & \mathbb{E}_\theta \left[\sup_{0 \leq u \leq 2} \left| \mathcal{L}_{N,K}^{t,\Delta}(u) - \mathcal{L}_{N,K}^{t,\Delta}(u-) \right| \right] \\ &= \frac{1}{K\sqrt{\Delta t}} \mathbb{E}_\theta \left[\sup_{\{i=1, \dots, [\frac{2t}{\Delta}]\}} |Q_{i,N,K}| \right] \leq \frac{1}{K\sqrt{\Delta t}} \mathbb{E}_\theta \left[\left(\sum_{i=1}^{[\frac{2t}{\Delta}]} |Q_{i,N,K}|^4 \right)^{\frac{1}{4}} \right] \\ &\leq \frac{1}{K\sqrt{\Delta t}} \mathbb{E}_\theta \left[\sum_{i=1}^{[\frac{2t}{\Delta}]} |Q_{i,N,K}|^4 \right]^{\frac{1}{4}} \leq C \left(\frac{\Delta}{t} \right)^{\frac{1}{4}}. \end{aligned}$$

This completes the proof. \square

Lemma 6.11. *We assume $(H(q))$ for some $q \geq 1$. For $0 \leq u \leq 2$, and $\Delta \geq 1$,*

$$\mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| [\mathcal{L}_{N,K}^{t,\Delta}(\cdot), \mathcal{L}_{N,K}^{t,\Delta}(\cdot)]_u - \frac{2u(\mu A_{\infty,\infty}^{N,K})^2}{K^2} \right| \right] \leq C \left(\frac{1}{K\Delta} + \frac{1}{\sqrt{N}} + \left(\frac{K\sqrt{t}}{\Delta^{q+1}} \right)^{\frac{1}{2}} + \sqrt{\frac{\Delta}{t}} \right).$$

Proof. For $s \geq 0$, we introduce $\phi_{t,\Delta}(s) = a\Delta$, where a is the unique integer such that $a\Delta \leq s < (a+1)\Delta$. Then we have

$$\mathcal{L}_{N,K}^{t,\Delta}(u) = \frac{2}{K\sqrt{\Delta t}} \int_0^{tu} \sum_{j=1}^N c_N^K(j) (M_s^{j,N} - M_{\phi_{t,\Delta}(s)}^{j,N}) \sum_{i=1}^N c_N^K(i) dM_s^{i,N}.$$

Noting that $Z_t^{i,N} = M_t^{i,N} + \int_0^t \lambda_s^{i,N} ds$ for $i = 1, \dots, N$, we have

$$\begin{aligned} [\mathcal{L}_{N,K}^{t,\Delta}(\cdot), \mathcal{L}_{N,K}^{t,\Delta}(\cdot)]_u &= \frac{4}{K^2 \Delta t} \int_0^{tu} \left(\sum_{j=1}^N c_N^K(j) (M_s^{j,N} - M_{\phi_{t,\Delta}(s)}^{j,N}) \right)^2 \sum_{i=1}^N \left(c_N^K(i) \right)^2 dZ_s^{i,N} \\ &= 4(\mathcal{A}_{N,K}^{u,1} + \mathcal{A}_{N,K}^{u,2} + \mathcal{A}_{N,K}^{u,3}), \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}_{N,K}^{u,1} &:= \frac{1}{K^2 \Delta t} \int_0^{tu} \left(\sum_{j=1}^N c_N^K(j) (M_s^{j,N} - M_{\phi_{t,\Delta}(s)}^{j,N}) \right)^2 \sum_{i=1}^N \left(c_N^K(i) \right)^2 dM_s^{i,N}, \\ \mathcal{A}_{N,K}^{u,2} &:= \frac{1}{K^2 \Delta t} \int_0^{tu} \left(\sum_{j=1}^N c_N^K(j) (M_s^{j,N} - M_{\phi_{t,\Delta}(s)}^{j,N}) \right)^2 \sum_{i=1}^N \left(c_N^K(i) \right)^2 (\lambda_s^{i,N} - \mu \ell_N(i)) ds, \\ \mathcal{A}_{N,K}^{u,3} &:= \left[\mu \sum_{i=1}^N \left(c_N^K(i) \right)^2 \ell_N(i) \right] \frac{1}{K^2 \Delta t} \int_0^{tu} \left(\sum_{j=1}^N c_N^K(j) (M_s^{j,N} - M_{\phi_{t,\Delta}(s)}^{j,N}) \right)^2 ds. \end{aligned}$$

First, we derive an upper-bound for $\mathcal{A}_{N,K}^{u,1}$. Recalling (8), we obtain

$$\begin{aligned} \mathbb{E}_\theta \left[\left(\mathcal{A}_{N,K}^{u,1} \right)^2 \right] &= \frac{1}{K^4 (\Delta t)^2} \mathbb{E}_\theta \left[\int_0^{tu} \left(\sum_{j=1}^N c_N^K(j) (M_s^{j,N} - M_{\phi_{t,\Delta}(s)}^{j,N}) \right)^4 \sum_{i=1}^N \left(c_N^K(i) \right)^4 dZ_s^{i,N} \right] \\ &\leq \frac{C}{K^4 (\Delta t)^2} \mathbb{E}_\theta \left[\max_{a=1, \dots, [\frac{2t}{\Delta}]} \sup_{0 \leq s \leq \Delta} \left(\sum_{j=1}^N c_N^K(j) (M_{(a-1)\Delta+s}^{j,N} - M_{(a-1)\Delta}^{j,N}) \right)^4 \sum_{j=1}^N \left(c_N^K(j) \right)^4 Z_{2t}^{j,N} \right] \\ &\leq \frac{C}{K^4 (\Delta t)^2} \mathbb{E}_\theta \left[\max_{a=1, \dots, [\frac{2t}{\Delta}]} \sup_{0 \leq s \leq \Delta} \left(\sum_{j=1}^N c_N^K(j) (M_{(a-1)\Delta+s}^{j,N} - M_{(a-1)\Delta}^{j,N}) \right)^8 + \left(\sum_{j=1}^N \left(c_N^K(j) \right)^4 Z_{2t}^{j,N} \right)^2 \right]. \end{aligned}$$

Using Doob's inequality, Lemma 3.4-(iii) together with (20), the last expression is bounded by

$$\begin{aligned} & \frac{C}{K^4(\Delta t)^2} \mathbb{E}_\theta \left[\sum_{a=1}^{[\frac{2t}{\Delta}]} \sup_{0 \leq s \leq \Delta} \left(\sum_{j=1}^N c_N^K(j) (M_{(a-1)\Delta+s}^{j,N} - M_{(a-1)\Delta}^{j,N}) \right)^8 \right] + \frac{C}{K^2 \Delta^2} \\ & \leq \frac{C}{K^4(\Delta t)^2} \mathbb{E}_\theta \left[\sum_{a=1}^{[\frac{2t}{\Delta}]} \left(\sum_{j=1}^N (c_N^K(j))^2 (Z_{a\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N}) \right)^4 \right] + \frac{C}{K^2 \Delta^2} \\ & \leq \frac{C\Delta}{t} + \frac{C}{K^2 \Delta^2}. \end{aligned}$$

For the second term, applying the Cauchy-Schwarz inequality, the Burkholder–Davis–Gundy inequality and (20) yields that on $\Omega_{N,K}$,

$$\begin{aligned} & \mathbb{E}_\theta \left[\left| \mathcal{A}_{N,K}^{u,2} \right| \right] \\ & \leq \frac{1}{K^2 \Delta t} \int_0^{tu} \mathbb{E}_\theta \left[\left(\sum_{j=1}^N c_N^K(j) (M_s^{j,N} - M_{\phi_{t,\Delta}(s)}^{j,N}) \right)^4 \right]^{\frac{1}{2}} \mathbb{E}_\theta \left[\left| \sum_{i=1}^N (c_N^K(i))^2 (\lambda_s^{i,N} - \mu \ell_N(i)) \right|^2 \right]^{\frac{1}{2}} ds \\ & \leq \frac{1}{K^2 \Delta t} \int_0^{tu} \mathbb{E}_\theta \left[\left(\sum_{j=1}^N (c_N^K(j))^2 (Z_s^{j,N} - Z_{\phi_{t,\Delta}(s)}^{j,N}) \right)^2 \right]^{\frac{1}{2}} \mathbb{E}_\theta \left[\left| \sum_{i=1}^N (c_N^K(i))^2 (\lambda_s^{i,N} - \mu \ell_N(i)) \right|^2 \right]^{\frac{1}{2}} ds \\ & \leq \frac{C}{K^2 \Delta t} \int_0^{tu} \mathbb{E}_\theta \left[\left(K (\bar{Z}_s^{N,K} - \bar{Z}_{\phi_{t,\Delta}(s)}^{N,K}) + \frac{K^2}{N} (\bar{Z}_s^N - \bar{Z}_{\phi_{t,\Delta}(s)}^N) \right)^2 \right]^{\frac{1}{2}} \\ & \quad \times \mathbb{E}_\theta \left[\left| \sum_{i=1}^N (c_N^K(i))^2 (\lambda_s^{i,N} - \mu \ell_N(i)) \right|^2 \right]^{\frac{1}{2}} ds. \end{aligned}$$

Now, applying Lemma 3.3-(ii), which states that on $\Omega_{N,K}$

$$\max_{i=1,\dots,N} \mathbb{E}_\theta [Z_t^{i,N} - Z_s^{i,N}] \leq C(t-s),$$

together with (20) and Lemma 3.4-(ii), we further obtain

$$\begin{aligned} \mathbb{E}_\theta \left[\left| \mathcal{A}_{N,K}^{u,2} \right| \right] & \leq \frac{C}{Kt} \int_0^{tu} \sum_{i=1}^N (c_N^K(i))^2 \mathbb{E}_\theta \left[\left| (\lambda_s^{i,N} - \mu \ell_N(i)) \right|^2 \right]^{\frac{1}{2}} ds \\ & \leq \frac{C}{Kt} \int_0^{tu} \sum_{i=1}^K \mathbb{E}_\theta \left[\left| \lambda_s^{i,N} - \mu \ell_N(i) \right|^2 \right]^{\frac{1}{2}} ds + \frac{C}{Nt} \int_0^{tu} \sum_{i=1}^N \mathbb{E}_\theta \left[\left| \lambda_s^{i,N} - \mu \ell_N(i) \right|^2 \right]^{\frac{1}{2}} ds \\ & \leq \frac{C}{\sqrt{N}} + \frac{C}{t^q}. \end{aligned}$$

For the third term, we first recall $A_{\infty,\infty}^{N,K} = \sum_{i=1}^N (c_N^K(i))^2 \ell_N(i)$ with $c_N^K(i) = \sum_{j=1}^K Q_N(j,i)$. Then we rewrite

$$\begin{aligned}
& \mathbb{V}ar_\theta(\mathcal{A}_{N,K}^{u,3}) \\
&= \frac{(\mu A_{\infty,\infty}^{N,K})^2}{K^4 \Delta^2 t^2} \mathbb{V}ar_\theta \left[\int_0^{ut} \left(\sum_{j=1}^N c_N^K(j) (M_s^{j,N} - M_{\phi_{t,\Delta}(s)}^{j,N}) \right)^2 ds \right] \\
&= \frac{(\mu A_{\infty,\infty}^{N,K})^2}{K^4 \Delta^2 t^2} \int_0^{ut} \int_0^{ut} \mathbb{C}ov_\theta \left[\left(\sum_{i=1}^N c_N^K(i) (M_s^{i,N} - M_{\phi_{t,\Delta}(s)}^{i,N}) \right)^2, \left(\sum_{j=1}^N c_N^K(j) (M_{s'}^{j,N} - M_{\phi_{t,\Delta}(s')}^{j,N}) \right)^2 \right] ds ds' \\
&= \frac{(\mu A_{\infty,\infty}^{N,K})^2}{K^4 \Delta^2 t^2} \int_0^{ut} \int_0^{ut} \sum_{1 \leq i, i', j, j' \leq N} \mathbb{C}ov_\theta \left[c_N^K(i) c_N^K(i') (M_s^{i,N} - M_{\phi_{t,\Delta}(s)}^{i,N}) (M_s^{i',N} - M_{\phi_{t,\Delta}(s)}^{i',N}), \right. \\
&\quad \left. c_N^K(j) c_N^K(j') (M_{s'}^{j,N} - M_{\phi_{t,\Delta}(s')}^{j,N}) (M_{s'}^{j',N} - M_{\phi_{t,\Delta}(s')}^{j',N}) \right] ds ds' \\
&= \frac{(\mu A_{\infty,\infty}^{N,K})^2}{K^4 \Delta^2 t^2} \int_0^{ut} \int_0^{ut} \left(\mathbf{1}_{\{|s-s'|>3\Delta\}} + \mathbf{1}_{\{|s-s'|\leq 3\Delta\}} \right) \sum_{1 \leq i, i', j, j' \leq N} \\
&\quad \mathbb{C}ov_\theta \left[c_N^K(i) c_N^K(i') (M_s^{i,N} - M_{\phi_{t,\Delta}(s)}^{i,N}) (M_s^{i',N} - M_{\phi_{t,\Delta}(s)}^{i',N}), \right. \\
&\quad \left. c_N^K(j) c_N^K(j') (M_{s'}^{j,N} - M_{\phi_{t,\Delta}(s')}^{j,N}) (M_{s'}^{j',N} - M_{\phi_{t,\Delta}(s')}^{j',N}) \right] ds ds'.
\end{aligned}$$

But on $\Omega_{N,K}$, we have

$$\begin{aligned}
& \sum_{i,i',j,j'=1}^N \int_0^{ut} \int_0^{ut} \mathbf{1}_{\{|s-s'|\leq 3\Delta\}} \mathbb{C}ov_\theta \left[c_N^K(i) c_N^K(i') (M_s^{i,N} - M_{\phi_{t,\Delta}(s)}^{i,N}) (M_s^{i',N} - M_{\phi_{t,\Delta}(s)}^{i',N}), \right. \\
&\quad \left. c_N^K(j) c_N^K(j') (M_{s'}^{j,N} - M_{\phi_{t,\Delta}(s')}^{j,N}) (M_{s'}^{j',N} - M_{\phi_{t,\Delta}(s')}^{j',N}) \right] ds ds' \\
&\leq \int_0^{ut} \int_0^{ut} \mathbf{1}_{\{|s-s'|\leq 3\Delta\}} \mathbb{E}_\theta \left[\left(\sum_{i=1}^N c_N^K(i) (M_s^{i,N} - M_{\phi_{t,\Delta}(s)}^{i,N}) \right)^4 \right]^{\frac{1}{2}} \\
&\quad \times \mathbb{E}_\theta \left[\left(\sum_{i=1}^N c_N^K(i) (M_{s'}^{i,N} - M_{\phi_{t,\Delta}(s')}^{i,N}) \right)^4 \right]^{\frac{1}{2}} ds ds' \\
&\leq \int_0^{ut} \int_0^{ut} \mathbf{1}_{\{|s-s'|\leq 3\Delta\}} \mathbb{E}_\theta \left[\left(\sum_{i=1}^N (c_N^K(i))^2 (Z_s^{i,N} - Z_{\phi_{t,\Delta}(s)}^{i,N}) \right)^2 \right]^{\frac{1}{2}} \\
&\quad \times \mathbb{E}_\theta \left[\left(\sum_{i=1}^N (c_N^K(i))^2 (Z_{s'}^{i,N} - Z_{\phi_{t,\Delta}(s')}^{i,N}) \right)^2 \right]^{\frac{1}{2}} ds ds' \\
&\leq C \int_0^{ut} \int_0^{ut} \mathbf{1}_{\{|s-s'|\leq 3\Delta\}} \mathbb{E}_\theta \left[\left(K (\bar{Z}_s^{N,K} - \bar{Z}_{\phi_{t,\Delta}(s)}^{N,K}) + \frac{K^2}{N} (\bar{Z}_s^N - \bar{Z}_{\phi_{t,\Delta}(s)}^N) \right)^2 \right]^{\frac{1}{2}} \\
&\quad \times \mathbb{E}_\theta \left[\left(K (\bar{Z}_{s'}^{N,K} - \bar{Z}_{\phi_{t,\Delta}(s')}^{N,K}) + \frac{K^2}{N} (\bar{Z}_{s'}^N - \bar{Z}_{\phi_{t,\Delta}(s')}^N) \right)^2 \right]^{\frac{1}{2}} ds ds' \\
&\leq Ct\Delta^3 K^2.
\end{aligned}$$

By [13, Step 6 of the proof of Lemma 30], we already have, when $|s - s'| \geq 3\Delta$, that

$$\begin{aligned} & \mathbb{C}ov_{\theta}[(M_s^{i,N} - M_{\phi_{t,\Delta}(s)}^{i,N})(M_s^{i',N} - M_{\phi_{t,\Delta}(s)}^{i',N}), (M_{s'}^{j,N} - M_{\phi_{t,\Delta}(s')}^{j,N})(M_{s'}^{j',N} - M_{\phi_{t,\Delta}(s')}^{j',N})] \\ & \leq C(\mathbf{1}_{\{i=i'\}} + \mathbf{1}_{\{j=j'\}})t^{1/2}\Delta^{1-q}. \end{aligned}$$

Hence,

$$\begin{aligned} & \mathbf{1}_{\Omega_{N,K}} \sum_{i,i',j,j'=1}^N \int_0^t \int_0^t \mathbf{1}_{\{|s-s'|\geq 3\Delta\}} \mathbb{C}ov_{\theta} \left[c_N^K(i)c_N^K(i') \left(M_s^{i,N} - M_{\phi_{t,\Delta}(s)}^{i,N} \right) \left(M_s^{i',N} - M_{\phi_{t,\Delta}(s)}^{i',N} \right), \right. \\ & \quad \left. c_N^K(j)c_N^K(j') \left(M_{s'}^{j,N} - M_{\phi_{t,\Delta}(s')}^{j,N} \right) \left(M_{s'}^{j',N} - M_{\phi_{t,\Delta}(s')}^{j',N} \right) \right] dsds' \\ & \leq \mathbf{1}_{\Omega_{N,K}} Ct^{5/2}\Delta^{1-q} \left(\sum_{i=1}^N (c_N^K(i))^2 \right) \left(\sum_{i=1}^N c_N^K(i) \right)^2 \leq CK^3 t^{5/2} \Delta^{1-q}. \end{aligned}$$

Overall, we have, on $\Omega_{N,K}$

$$\mathbb{V}ar_{\theta}(\mathcal{A}_{N,K}^{u,3}) \leq \frac{1}{K^4 \Delta^2 t^2} \left(\mu A_{\infty,\infty}^{N,K} \right)^2 \left(\frac{K^3 t^{5/2}}{\Delta^{q-1}} + t \Delta^3 K^2 \right) \leq C \left(\frac{K \sqrt{t}}{\Delta^{q+1}} + \frac{\Delta}{t} \right),$$

due to the fact that on $\Omega_{N,K}$, $|\ell_N(i)| \leq C$ for all $1 \leq i \leq N$, and (20).

Noting that $A_{\infty,\infty}^{N,K} = \sum_{j=1}^N (c_N^K(j))^2 \ell_N(j)$ and that $\int_0^{ut} (s - \phi_{t,\Delta}(s)) ds = \frac{u\Delta t}{2}$, we have on $\Omega_{N,K}$,

$$\begin{aligned} & \left| \mathbb{E}_{\theta}[\mathcal{A}_{N,K}^{u,3}] - \frac{u(\mu A_{\infty,\infty}^{N,K})^2}{2K^2} \right| \\ & = \frac{1}{\Delta t K^2} \left| \mu A_{\infty,\infty}^{N,K} \int_0^{ut} \sum_{j=1}^N \left\{ (c_N^K(j))^2 \mathbb{E}_{\theta} \left[Z_s^{j,N} - Z_{\phi_{t,\Delta}(s)}^{j,N} \right] \right\} ds - \frac{u\Delta t (\mu A_{\infty,\infty}^{N,K})^2}{2} \right| \\ & = \frac{1}{\Delta t K^2} \left| \mu A_{\infty,\infty}^{N,K} \int_0^{ut} \sum_{j=1}^N \left\{ (c_N^K(j))^2 \mathbb{E}_{\theta} \left[Z_s^{j,N} - Z_{\phi_{t,\Delta}(s)}^{j,N} - \mu(s - \phi_{t,\Delta}(s))\ell_N(j) \right] \right\} ds \right|. \end{aligned}$$

By (20) and [13, Lemma 16-(ii)], we obtain

$$\left| \mathbb{E}_{\theta}[\mathcal{A}_{N,K}^{u,3}] - \frac{u(\mu A_{\infty,\infty}^{N,K})^2}{2K^2} \right| \leq \frac{C}{\Delta^q}.$$

Gathering all the previous computations, we obtain

$$\begin{aligned} & \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| [\mathcal{L}_{N,K}^{t,\Delta}(\cdot), \mathcal{L}_{N,K}^{t,\Delta}(\cdot)]_u - \frac{2u(\mu A_{\infty,\infty}^{N,K})^2}{K^2} \right| \right] \\ & \leq 4 \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left\{ \left| \mathcal{A}_{N,K}^{u,1} \right| + \left| \mathcal{A}_{N,K}^{u,2} \right| + \mathbb{V}ar_{\theta}(\mathcal{A}_{N,K}^{u,3})^{\frac{1}{2}} + \left| \mathbb{E}_{\theta}[\mathcal{A}_{N,K}^{u,3}] - \frac{u(\mu A_{\infty,\infty}^{N,K})^2}{2K^2} \right| \right\} \right] \\ & \leq \frac{C}{K\Delta} + C \sqrt{\frac{\Delta}{t}} + \frac{C}{\sqrt{N}} + \frac{C}{t^q} + C \left(\frac{K \sqrt{t}}{\Delta^{q+1}} + \frac{\Delta}{t} \right)^{\frac{1}{2}} + \frac{C}{\Delta^q} \\ & \leq C \left(\frac{1}{K\Delta} + \frac{1}{\sqrt{N}} + \left(\frac{K \sqrt{t}}{\Delta^{q+1}} \right)^{\frac{1}{2}} + \sqrt{\frac{\Delta}{t}} \right). \end{aligned}$$

The proof is finished. \square

Next, we prove Proposition 6.7.

Proof of Proposition 6.7. A direct application of Lemma 6.8, Proposition 6.9 together with Lemma 4.5 gives

$$\frac{K}{\sqrt{t\Delta_t}}(\mathbb{X}_{\Delta_t,t,v}^{N,K})_{v \geq 0} \xrightarrow{d} \mu\sqrt{2}\left(\frac{1-\gamma}{(1-\Lambda p)} + \frac{\gamma}{(1-\Lambda p)^3}\right)(B_{2v} - B_v)_{v \geq 0},$$

as desired. \square

6.3. Proof of Theorem 6.1. We recall that $\mathbb{X}_{\Delta_t,t,v}^{N,K}$ is defined in (13) and note that $\mathbb{X}_{2\Delta_t,t,1}^{N,K} = \mathbb{X}_{\Delta_t,t,\frac{1}{2}}^{N,K}$. By Proposition 6.7, we have

$$\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\frac{N}{t}\left(-\mathbb{X}_{\Delta_t,t,1}^{N,K} + 2\mathbb{X}_{\Delta_t,t,\frac{1}{2}}^{N,K}\right) \xrightarrow{d} \mathcal{N}\left(0, 6\mu^2\left(\frac{1-\gamma}{(1-\Lambda p)} + \frac{\gamma}{(1-\Lambda p)^3}\right)^2\right).$$

By Proposition 6.3, we conclude that

$$\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\left|-D_{\Delta_t,t}^{N,K,3} + 2D_{2\Delta_t,t}^{N,K,3} - 2\frac{N}{t}\mathbb{X}_{2\Delta_t,t,1}^{N,K} + \frac{N}{t}\mathbb{X}_{\Delta_t,t,1}^{N,K}\right|\right] \leq \frac{CK}{N\Delta_t} + \frac{C\sqrt{K}}{\sqrt{N\Delta_t}} + \frac{Ct^{\frac{3}{4}}\sqrt{K}}{\Delta_t^{1+\frac{q}{2}}}.$$

Consequently, by Lemma 6.2, we obtain the following convergence in probability: as $(N, K, t) \rightarrow (\infty, \infty, \infty)$ such that $\frac{K}{N} \rightarrow \gamma \leq 1$ and $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \rightarrow 0$, the limit of

$$\mathbf{1}_{\Omega_{N,K}}\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\left(\mathcal{X}_{\Delta_t,t}^{N,K} - \mathcal{X}_{\infty,\infty}^{N,K}\right)$$

equals the limit of

$$\mathbf{1}_{\Omega_{N,K}}\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\left\{-D_{\Delta_t,t}^{N,K,3} + 2D_{2\Delta_t,t}^{N,K,3}\right\},$$

which in turn equals the limit of

$$\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\frac{N}{t}\left(-\mathbb{X}_{\Delta_t,t,1}^{N,K} + 2\mathbb{X}_{\Delta_t,t,\frac{1}{2}}^{N,K}\right).$$

This finally converges in distribution to $\mathcal{N}\left(0, 6\mu^2\left(\frac{1-\gamma}{(1-\Lambda p)} + \frac{\gamma}{(1-\Lambda p)^3}\right)^2\right)$.

7. PROOF OF THE MAIN RESULT

In this section, we present the proofs of the main results stated in Section 2. First, we recall the estimators $\varepsilon_t^{N,K}$, $\mathcal{V}_t^{N,K}$, and $\mathcal{X}_{\Delta_t,t}^{N,K}$ defined in Section 2, as well as the function

$$\Psi^{(3)}(u, v, w) = \frac{u^2(1 - \sqrt{\frac{u}{w}})^2}{v + u^2(1 - \sqrt{\frac{u}{w}})^2} \quad \text{if } u > 0, v > 0, w > 0 \quad \text{and} \quad \Psi^{(3)}(u, v, w) = 0 \text{ otherwise,}$$

and

$$\hat{p}_{N,K,t} := \Psi^{(3)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K}), \quad \text{with } \Delta_t = (2\lfloor t^{1-4/(q+1)} \rfloor)^{-1}t.$$

We now proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. It can be directly verified that $\Psi^{(3)}\left(\frac{\mu}{1-\Lambda p}, \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2}, \frac{\mu}{(1-\Lambda p)^3}\right) = p$. By the mean value theorem, there exist some vectors $\mathbf{C}_{N,K,t}^i$ for $i = 1, 2, 3$, lying on the segment between

$(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K})$ and $\mathbf{C} := \left(\frac{\mu}{1-\Lambda p}, \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2}, \frac{\mu}{(1-\Lambda p)^3} \right)$, such that

$$\begin{aligned}\hat{p}_{N,K,t} - p &= \Psi^{(3)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K}) - p \\ &= \Psi^{(3)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K}) - \Psi^{(3)}\left(\frac{\mu}{1-\Lambda p}, \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2}, \frac{\mu}{(1-\Lambda p)^3}\right) \\ &= \frac{\partial \Psi^{(3)}}{\partial u}(\mathbf{C}_{N,K,t}^1)\left(\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p}\right) + \frac{\partial \Psi^{(3)}}{\partial v}(\mathbf{C}_{N,K,t}^2)\left(\mathcal{V}_t^{N,K} - \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2}\right) \\ &\quad + \frac{\partial \Psi^{(3)}}{\partial w}(\mathbf{C}_{N,K,t}^3)\left(\mathcal{X}_{\Delta_t,t}^{N,K} - \frac{\mu}{(1-\Lambda p)^3}\right).\end{aligned}$$

From the first paragraph of [21, Section 10], it is established that, when $(N, K, t) \rightarrow (\infty, \infty, \infty)$ and $\frac{1}{\sqrt{K}} + \frac{N}{K} \sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \rightarrow 0$, $(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K})$ converges in probability to \mathbf{C} . Consequently, the three vectors $\mathbf{C}_{N,K,t}^i$, $i = 1, 2, 3$, all converge to $\mathbf{C} := \left(\frac{\mu}{1-\Lambda p}, \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2}, \frac{\mu}{(1-\Lambda p)^3} \right)$ in probability.

We define the following functions from $D' := \{(u, v, w) \in \mathbb{R}^3 : w > u > 0 \text{ and } v > 0\}$ to \mathbb{R}^3 by

$$\Psi^{(1)}(u, v, w) = u\sqrt{\frac{u}{w}}, \quad \Psi^{(2)}(u, v, w) = \frac{v + (u - \Psi^{(1)})^2}{u(u - \Psi^{(1)})}.$$

Then, on D' , we have $\Psi^{(3)}(u, v, w) = \frac{1-u^{-1}\Psi^{(1)}}{\Psi^{(2)}}$.

A series of tedious but straightforward calculations yields

$$\begin{aligned}\frac{\partial \Psi^{(1)}}{\partial v}(\mathbf{C}) &= 0, \quad \frac{\partial \Psi^{(1)}}{\partial w}(\mathbf{C}) = \frac{-(1-\Lambda p)^3}{2}, \quad \frac{\partial \Psi^{(2)}}{\partial v}(\mathbf{C}) = \frac{(1-\Lambda p)^2}{\mu^2 \Lambda p}, \\ \frac{\partial \Psi^{(2)}}{\partial w}(\mathbf{C}) &= \left\{ -2\frac{\partial \Psi^{(1)}}{\partial w} + \frac{\Psi^{(2)} \frac{\partial \Psi^{(1)}}{\partial w}}{(u - \Psi^{(1)})} \right\}(\mathbf{C}) = \frac{(1-\Lambda p)^4(2p-1)}{2\mu p}, \\ \frac{\partial \Psi^{(3)}}{\partial v}(\mathbf{C}) &= -\frac{\frac{\Psi^{(2)} \frac{\partial \Psi^{(1)}}{\partial v}}{u} + (1 - \frac{\Psi^{(1)}}{u}) \frac{\partial \Psi^{(2)}}{\partial v}}{(\Psi^{(2)})^2}(\mathbf{C}) = -\frac{(1-\Lambda p)^2}{(\mu\Lambda)^2}, \\ \frac{\partial \Psi^{(3)}}{\partial w}(\mathbf{C}) &= -\frac{\frac{\Psi^{(2)} \frac{\partial \Psi^{(1)}}{\partial w}}{u} + (1 - \frac{\Psi^{(1)}}{u}) \frac{\partial \Psi^{(2)}}{\partial w}}{(\Psi^{(2)})^2}(\mathbf{C}) = \frac{(1-\Lambda p)^4(1-p)}{\mu\Lambda}.\end{aligned}$$

Case 1. The dominant term is $\frac{1}{\sqrt{K}}$, i.e. when $[\frac{1}{\sqrt{K}}]/[\frac{N}{K} \sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}}] \rightarrow \infty$, we write

$$\begin{aligned}\sqrt{K}(\hat{p}_{N,K,t} - p) &= \sqrt{K} \frac{\partial \Psi^{(3)}}{\partial u}(\mathbf{C}_{N,K,t}^1)\left(\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p}\right) \\ &\quad + \sqrt{K} \frac{\partial \Psi^{(3)}}{\partial v}(\mathbf{C}_{N,K,t}^2)\left(\mathcal{V}_t^{N,K} - \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2}\right) \\ &\quad + \sqrt{K} \frac{\partial \Psi^{(3)}}{\partial w}(\mathbf{C}_{N,K,t}^3)\left(\mathcal{X}_{\Delta_t,t}^{N,K} - \frac{\mu}{(1-\Lambda p)^3}\right).\end{aligned}$$

Based on Lemmas 4.1, 5.1 and 4.4 and Theorem 6.1, we obtain

$$\sqrt{K} \frac{\partial \Psi^{(3)}}{\partial u}(\mathbf{C}_{N,K,t}^1)\left(\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p}\right) + \sqrt{K} \frac{\partial \Psi^{(3)}}{\partial w}(\mathbf{C}_{N,K,t}^3)\left(\mathcal{X}_{\Delta_t,t}^{N,K} - \frac{\mu}{(1-\Lambda p)^3}\right) \xrightarrow{d} 0.$$

Next, we observe that as $(N, K, t) \rightarrow (\infty, \infty, \infty)$,

$$\frac{\partial \Psi^{(3)}}{\partial v}(\mathbf{C}_{N,K,t}^2) \xrightarrow{d} -\frac{(1-\Lambda p)^2}{(\mu\Lambda)^2} \quad \text{in probability.}$$

Therefore, by Theorems 4.2 and 5.2, we conclude that

$$\sqrt{K} \frac{\partial \Psi^{(3)}}{\partial v}(\mathbf{C}_{N,K,t}^2) \left(\mathcal{V}_t^{N,K} - \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2} \right) \xrightarrow{d} \mathcal{N}(0, p^2(1-p)^2),$$

which in turn implies that

$$\sqrt{K} (\hat{p}_{N,K,t} - p) \xrightarrow{d} \mathcal{N}(0, p^2(1-p)^2).$$

Case 2. The dominant term is $\frac{N}{t\sqrt{K}}$, i.e. when $[\frac{N}{t\sqrt{K}}]/[\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}}] \rightarrow \infty$, we write

$$\begin{aligned} \frac{t\sqrt{K}}{N} (\hat{p}_{N,K,t} - p) &= \frac{t\sqrt{K}}{N} \frac{\partial \Psi^{(3)}}{\partial u}(\mathbf{C}_{N,K,t}^1) \left(\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p} \right) \\ &\quad + \frac{t\sqrt{K}}{N} \frac{\partial \Psi^{(3)}}{\partial v}(\mathbf{C}_{N,K,t}^2) \left(\mathcal{V}_t^{N,K} - \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2} \right) \\ &\quad + \frac{t\sqrt{K}}{N} \frac{\partial \Psi^{(3)}}{\partial w}(\mathbf{C}_{N,K,t}^3) \left(\mathcal{X}_{\Delta_t,t}^{N,K} - \frac{\mu}{(1-\Lambda p)^3} \right). \end{aligned}$$

Similarly, according to Lemmas 4.1, 5.1 and 4.4 and Theorem 6.1, we obtain

$$\frac{t\sqrt{K}}{N} \frac{\partial \Psi^{(3)}}{\partial u}(\mathbf{C}_{N,K,t}^1) \left(\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p} \right) + \frac{t\sqrt{K}}{N} \frac{\partial \Psi^{(3)}}{\partial w}(\mathbf{C}_{N,K,t}^3) \left(\mathcal{X}_{\Delta_t,t}^{N,K} - \frac{\mu}{(1-\Lambda p)^3} \right) \xrightarrow{d} 0.$$

Finally, using Theorems 4.2 and 5.2, we find that

$$\frac{t\sqrt{K}}{N} (\hat{p}_{N,K,t} - p) \xrightarrow{d} \mathcal{N}\left(0, \frac{2(1-\Lambda p)^2}{\mu^2 \Lambda^4}\right).$$

Case 3. The dominant term is $\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}$, i.e. when $[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]/[\frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}}] \rightarrow \infty$ and $\frac{K}{N} \rightarrow \gamma \leq 1$. Using Lemmas 4.1, 5.1 and Theorems 4.2 and 5.2, we have

$$\frac{K}{N} \sqrt{\frac{t}{\Delta_t}} \left\{ \frac{\partial \Psi^{(3)}}{\partial u}(\mathbf{C}_{N,K,t}^1) \left(\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p} \right) + \frac{\partial \Psi^{(3)}}{\partial v}(\mathbf{C}_{N,K,t}^2) \left(\mathcal{V}_t^{N,K} - \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2} \right) \right\} \xrightarrow{d} 0.$$

Applying Lemma 4.4 gives

$$\frac{K}{N} \sqrt{\frac{t}{\Delta_t}} \frac{\partial \Psi^{(3)}}{\partial w}(\mathbf{C}_{N,K,t}^3) \left(\mathcal{X}_{\infty,\infty}^{N,K} - \frac{\mu}{(1-\Lambda p)^3} \right) \xrightarrow{d} 0.$$

Consequently, it suffices to analyze

$$\frac{K}{N} \sqrt{\frac{t}{\Delta_t}} \frac{\partial \Psi^{(3)}}{\partial w}(\mathbf{C}_{N,K,t}^3) \left(\mathcal{X}_{\Delta_t,t}^{N,K} - \mathcal{X}_{\infty,\infty}^{N,K} \right).$$

Since

$$\frac{\partial \Psi^{(3)}}{\partial w}(\mathbf{C}_{N,K,t}^3) \xrightarrow{d} \frac{(1-\Lambda p)^4(1-p)}{\mu\Lambda} \quad \text{in probability,}$$

and by Theorem 6.1, we finally conclude that

$$\frac{K}{N} \sqrt{\frac{t}{\Delta_t}} (\hat{p}_{N,K,t} - p) \xrightarrow{d} \mathcal{N}\left(0, \frac{6(1-p)^2}{\Lambda^2} \left((1-\gamma)(1-\Lambda p)^3 + \gamma(1-\Lambda p) \right)^2 \right).$$

□

Next, we move to prove Proposition 2.4.

Proof of Proposition 2.4. We note that for the case $p = 0$, the conclusion of Theorem 6.1 remains valid (and the limit of $\frac{K}{N}$ is no longer required). One can verify directly that $\bar{\ell}_N^K = 1$, $\mathcal{V}_\infty^{N,K} = 0$ and $\mathcal{X}_{\infty,\infty}^{N,K} = \mu$. Define

$$f(u, v, w) := \begin{cases} \frac{u(w-u)}{w+\sqrt{wu}} & \text{when } u > 0, w > 0 \\ 0 & \text{otherwise.} \end{cases}$$

By [21, Lemma 7.3],

$$\mathbf{1}_{\Omega_{N,K}} \frac{K}{N} \sqrt{\frac{t}{\Delta_t}} (\varepsilon_t^{N,K} - \mu) \xrightarrow{\text{in probability.}}$$

Hence, applying Theorem 6.1, we obtain

$$\mathbf{1}_{\Omega_{N,K}} \frac{K}{N} \sqrt{\frac{t}{\Delta_t}} (\mathcal{X}_{\Delta_t,t}^{N,K} - \varepsilon_t^{N,K}) \xrightarrow{d} \mathcal{N}(0, 2\mu^2).$$

From [21, Lemma 7.3, Corollary 9.9], when $(N, K, t) \rightarrow (\infty, \infty, \infty)$ and $\frac{N}{K} \sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_p \Lambda K} \rightarrow 0$, both $\varepsilon_t^{N,K}$ and $\mathcal{X}_{\Delta_t,t}^{N,K}$ converge to μ in probability. Consequently,

$$\mathbf{1}_{\Omega_{N,K}} \frac{K}{N} \sqrt{\frac{t}{\Delta_t}} f(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K}) \xrightarrow{d} \mathcal{N}\left(0, \frac{\mu^2}{2}\right).$$

By Theorem 5.2, we obtain

$$\mathbf{1}_{\Omega_{N,K}} \frac{t\sqrt{K}}{N} \mathcal{V}_t^{N,K} \xrightarrow{d} \mathcal{N}(0, 2\mu^2).$$

Therefore, if $[\frac{N}{t\sqrt{K}}]/[\frac{N}{K} \sqrt{\frac{\Delta_t}{t}}]^2 \rightarrow \infty$, then

$$\left[\frac{N}{K} \sqrt{\frac{\Delta_t}{t}} \right]^{-2} |\mathcal{V}_t^{N,K}| \xrightarrow{\text{in probability.}}$$

Since $\Psi^{(3)}(u, v, w) = \frac{f^2}{v+f^2} \mathbf{1}_{\{v>0\}}$, it follows that

$$\hat{p}_{N,K,t} = \Psi^{(3)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K}) \xrightarrow{d} 0.$$

On the other hand, when $\left[\frac{N}{K} \sqrt{\frac{\Delta_t}{t}} \right]^2 / [\frac{N}{t\sqrt{K}}] \rightarrow \infty$, we have

$$\left[\frac{N}{K} \sqrt{\frac{\Delta_t}{t}} \right]^{-2} |\mathcal{V}_t^{N,K}| \xrightarrow{\text{in probability.}}$$

Thus,

$$\frac{f^2(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K})}{\mathcal{V}_t^{N,K} + f^2(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K})} \xrightarrow{\text{in probability.}}$$

which holds whenever

$$\mathbb{P}(\mathcal{V}_t^{N,K} > 0) \rightarrow \frac{1}{2}.$$

Hence, $\hat{p}_{N,K,t} \xrightarrow{d} X$.

□

APPENDIX A. PROOF OF LEMMA 3.4

A.1. Proof of Lemma 3.4 (i). Observing from $\mathbb{E}_\theta[Z_t^{i,N}] = \int_0^t \mathbb{E}_\theta[\lambda_s^{i,N}] ds$, a direct computation yields that

$$\max_{i=1,\dots,N} \mathbb{E}_\theta[\lambda_t^{i,N}] = \mu + \max_{i=1,\dots,N} \left\{ \sum_{j=1}^N A_N(i,j) \int_0^t \phi(t-s) \mathbb{E}_\theta[\lambda_s^{j,N}] ds \right\}.$$

Define $a_N(t) := \sup_{0 \leq s \leq t} \max_{i=1,\dots,N} \mathbb{E}_\theta[\lambda_s^{i,N}]$. On the event $\Omega_{N,K}$, we have $\Lambda \max_{i=1,\dots,N} \{\sum_{j=1}^N A_N(i,j)\} \leq a < 1$. Since $\Lambda = \int_0^\infty \phi(s) ds$, it follows that

$$a_N(t) \leq \mu + a_N(t)a,$$

which immediately implies the desired result.

For the second part, recalling the definition of $M_t^{i,N}$ in Section 3.2, we have $M_t^{i,N} = Z_t^{i,N} - \int_0^t \lambda_s^{i,N} ds$. We express the intensity process $\lambda_t^{i,N}$ defined in (3) as

$$\lambda_t^{i,N} = \mu + \frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) dM_s^{j,N} + \frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) \lambda_s^{j,N} ds.$$

An application of Minkowski's inequality then yields

$$(21) \quad \begin{aligned} \mathbb{E}_\theta[(\lambda_t^{i,N})^2]^{\frac{1}{2}} &\leq \mu + \mathbb{E}_\theta \left[\left(\frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) dM_s^{j,N} \right)^2 \right]^{\frac{1}{2}} \\ &\quad + \mathbb{E}_\theta \left[\left(\frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) \lambda_s^{j,N} ds \right)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Using (8) and $0 \leq \theta_{ij} \leq 1$, we obtain

$$\begin{aligned} \mathbb{E}_\theta \left[\left(\frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) dM_s^{j,N} \right)^2 \right]^{\frac{1}{2}} &= \frac{1}{N} \mathbb{E}_\theta \left[\sum_{j=1}^N \int_0^t (\theta_{ij} \phi(t-s))^2 dZ_s^{j,N} \right]^{\frac{1}{2}} \\ &= \frac{1}{N} \mathbb{E}_\theta \left[\sum_{j=1}^N \int_0^t (\theta_{ij} \phi(t-s))^2 \lambda_s^{j,N} ds \right]^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{N}} \left[\int_0^t (\phi(t-s))^2 \max_{j=1,\dots,N} \mathbb{E}_\theta[\lambda_s^{j,N}] ds \right]^{\frac{1}{2}}. \end{aligned}$$

From the first part of Lemma 3.4-(i) and assumption $(H(q))$, it follows that

$$(22) \quad \mathbb{E}_\theta \left[\left(\frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) dM_s^{j,N} \right)^2 \right]^{\frac{1}{2}} \leq \frac{C}{\sqrt{N}}.$$

Now, applying Minkowski's inequality to the third term of (21) yields

$$\mathbb{E}_\theta \left[\left(\frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) \lambda_s^{j,N} ds \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) \mathbb{E}_\theta[(\lambda_s^{j,N})^2]^{\frac{1}{2}} ds.$$

Therefore,

$$\max_{i=1,\dots,N} \mathbb{E}_\theta [(\lambda_t^{i,N})^2]^{\frac{1}{2}} \leq \mu + \frac{C}{\sqrt{N}} + \max_{i=1,\dots,N} \left\{ \frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) \mathbb{E}_\theta [(\lambda_s^{j,N})^2]^{\frac{1}{2}} ds \right\}.$$

Define $b_N(t) := \sup_{0 \leq s \leq t} \max_{i=1,\dots,N} \mathbb{E}_\theta [(\lambda_s^{i,N})^2]^{\frac{1}{2}}$, then we have

$$b_N(t) \leq \mu + \frac{C}{\sqrt{N}} + \Lambda \max_{i=1,\dots,N} \left\{ \frac{1}{N} \sum_{j=1}^N \theta_{ij} \right\} b_N(t).$$

Recalling that $A_N(i,j) = \frac{1}{N} \theta_{ij}$, and that on $\Omega_{N,K}$, $\Lambda \max_{i=1,\dots,N} \{\sum_{j=1}^N A_N(i,j)\} \leq a < 1$, we conclude that

$$b_N(t) \leq \mu + C + ab_N(t),$$

which completes the proof.

A.2. Proof of Lemma 3.4 (ii). Starting from the definition $\ell_N = Q_N \mathbf{1}_N = (I - \Lambda A_N)^{-1} \mathbf{1}_N$, we obtain $\ell_N = \mathbf{1}_N + \Lambda A_N \ell_N$. Recalling (3) and writing $\Lambda = \int_0^t \phi(t-s) ds + \int_t^\infty \phi(s) ds$, we obtain

$$\lambda_t^{i,N} - \mu \ell_N(i) = \frac{1}{N} \sum_{j=1}^N \theta_{ij} \left(\int_0^t \phi(t-s) dZ_s^{j,N} - \mu \ell_N(j) \int_0^t \phi(t-s) ds \right) - \frac{\mu}{N} \sum_{j=1}^N \theta_{ij} \ell_N(j) \int_t^\infty \phi(s) ds.$$

Applying Minkowski's inequality, we obtain

$$\begin{aligned} \mathbb{E}_\theta \left[(\lambda_t^{i,N} - \mu \ell_N(i))^2 \right]^{\frac{1}{2}} &\leq \mathbb{E}_\theta \left[\left(\frac{1}{N} \sum_{j=1}^N \theta_{ij} \left(\int_0^t \phi(t-s) dZ_s^{j,N} - \mu \ell_N(j) \int_0^t \phi(t-s) ds \right) \right)^2 \right]^{\frac{1}{2}} \\ &\quad + \mu \mathbb{E}_\theta \left[\left(\frac{1}{N} \sum_{j=1}^N \theta_{ij} \ell_N(j) \int_t^\infty \phi(s) ds \right)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

As in the proof of Lemma 3.4-(i), we reformulate the first right hand side term of the above inequality via the process $M_t^{i,N}$ defined in Section 3.2. In addition, since $\ell_N(j)$ is uniformly bounded on $\Omega_{N,K}$, it follows that

$$\begin{aligned} \mathbb{E}_\theta \left[(\lambda_t^{i,N} - \mu \ell_N(i))^2 \right]^{\frac{1}{2}} &\leq \mathbb{E}_\theta \left[\left(\frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) dM_s^{j,N} \right)^2 \right]^{\frac{1}{2}} \\ &\quad + \mathbb{E}_\theta \left[\left(\frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) |\lambda_s^{j,N} - \mu \ell_N(j)| ds \right)^2 \right]^{\frac{1}{2}} + C \int_t^\infty \phi(s) ds. \end{aligned}$$

Define $F_t^{N,K} := \frac{1}{K} \sum_{i=1}^K \mathbb{E}_\theta [(\lambda_t^{i,N} - \mu \ell_N(i))^2]^{\frac{1}{2}}$. Using (22) and Minkowski's inequality, we obtain

$$\begin{aligned} F_t^{N,K} &\leq \frac{1}{KN} \sum_{j=1}^N \sum_{i=1}^K \theta_{ij} \int_0^t \phi(t-s) \mathbb{E}_\theta \left[|\lambda_s^{j,N} - \mu \ell_N(j)|^2 \right]^{\frac{1}{2}} ds + C \int_t^\infty \phi(s) ds + \frac{C}{\sqrt{N}} \\ &\leq \int_0^t \frac{N}{K} \|I_K A_N\|_1 \phi(t-s) F_s^{N,N} ds + C \int_t^\infty \phi(s) ds + \frac{C}{\sqrt{N}}. \end{aligned}$$

Using $N\|I_K A_N\|_1 = \max_{j=1,\dots,N} \sum_{i=1}^K \theta_{ij}$ and the bound $\frac{N}{K}\|I_K A_N\|_1 \leq a/\Lambda$ on $\Omega_{N,K}$, we obtain

$$F_t^{N,K} \leq \int_0^t \frac{a}{\Lambda} \phi(t-s) F_s^{N,N} ds + C \int_t^\infty \phi(s) ds + \frac{C}{\sqrt{N}}.$$

Defining $g_N(t) := C \int_t^\infty \phi(s) ds + \frac{C}{\sqrt{N}}$, we have on $\Omega_{N,K}$, for all $K = 1, \dots, N$,

$$(23) \quad F_t^{N,K} \leq \int_0^t \frac{a}{\Lambda} \phi(t-s) F_s^{N,N} ds + g_N(t).$$

From assumption $(H(q))$, namely $\int_0^\infty (1+s^q)\phi(s)ds < \infty$, it follows that $g_N(t) \leq C(\frac{1}{t^q} \wedge 1) + CN^{-1/2}$. Moreover, due to Lemma 3.4-(i) and the uniform boundness of $\ell_N(j)$ on $\Omega_{N,K}$, we deduce that $\sup_{t \geq 0} F_t^{N,N} \leq C$, so that $\int_0^t (\frac{a}{\Lambda})^n \phi^{*n}(t-s) F_s^{N,N} ds \leq Ca^n \rightarrow 0$ as $n \rightarrow \infty$. Hence, iterating (23) (using it once with some fixed $K \in \{1, \dots, N\}$ and then always with $K = N$), one concludes that on $\Omega_{N,K}$,

$$\begin{aligned} F_t^{N,N} &\leq \sum_{n \geq 1} \int_0^t \left(\frac{a}{\Lambda}\right)^n \phi^{*n}(t-s) g_N(s) ds + g_N(t) \\ &\leq \sum_{n \geq 1} \int_0^{\frac{t}{2}} \left(\frac{a}{\Lambda}\right)^n \phi^{*n}(t-s) g_N(s) ds + \sum_{n \geq 1} \int_{\frac{t}{2}}^t \left(\frac{a}{\Lambda}\right)^n \phi^{*n}(t-s) g_N(s) ds + g_N(t) \\ &\leq C \sum_{n \geq 1} \int_{\frac{t}{2}}^t \left(\frac{a}{\Lambda}\right)^n \phi^{*n}(s) ds + g_N\left(\frac{t}{2}\right) \sum_{n \geq 1} \int_0^\infty \left(\frac{a}{\Lambda}\right)^n \phi^{*n}(s) ds + g_N(t), \end{aligned}$$

because g_N is non-increasing and bounded. Recalling that $\int_0^\infty \phi^{*n}(s)ds = \Lambda^n$ and, as shown in [13, Proof of Lemma 15-(ii)], that

$$\int_r^\infty \phi^{*n}(u)du \leq Cn^q \Lambda^n r^{-q},$$

we conclude that (since $a \in (0, 1)$)

$$F_t^{N,N} \leq C\left(\frac{t}{2}\right)^{-q} \sum_{n \geq 1} n^q a^n + g_N\left(\frac{t}{2}\right) \frac{a}{1-a} + g_N(t) \leq \frac{C}{t^q} + \frac{C}{\sqrt{N}}.$$

This completes the proof.

A.3. Proof of Lemma 3.4 (iii)&(iv). We restrict our proof to part (iii), since the argument for part (iv) is virtually identical. Recall from (7) that

$$U_t^{i,N} = \sum_{n \geq 0} \int_0^t \phi^{*n}(t-s) \sum_{j=1}^N A_N^n(i,j) M_s^{j,N} ds.$$

We set $\phi(s) = 0$ for $s \leq 0$. Separating the cases $n = 0$ and $n \geq 1$, using $A_N^0(i,j) = \mathbf{1}_{\{i=j\}}$ and Minkowski's inequality implies that on $\Omega_{N,K}$,

$$\begin{aligned} \mathbb{E}_\theta[(U_t^{i,N} - U_s^{i,N})^4]^{\frac{1}{4}} &\leq \mathbb{E}_\theta[(M_t^{i,N} - M_s^{i,N})^4]^{\frac{1}{4}} \\ &\quad + \sum_{n \geq 1} \int_0^t (\phi^{*n}(t-u) - \phi^{*n}(s-u)) \mathbb{E}_\theta \left[\left(\sum_{j=1}^N A_N^n(i,j) M_u^{j,N} \right)^4 \right]^{\frac{1}{4}} du. \end{aligned}$$

For the first term ($n = 0$), an application of (8) and Burkholder's inequality gives

$$\mathbb{E}_\theta[(M_t^{i,N} - M_s^{i,N})^4] \leq C\mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^2].$$

By [13, Lemma 16-(iii)], on $\Omega_{N,K}$, we have, $\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^2] \leq C(t-s)^2$, and therefore

$$(24) \quad \mathbb{E}_\theta[(M_t^{i,N} - M_s^{i,N})^4] \leq C(t-s)^2.$$

For the second term ($n \geq 1$), another application of (8) and Burkholder's inequality yields

$$\begin{aligned} \mathbb{E}_\theta\left[\left(\sum_{j=1}^N A_N^n(i,j)M_u^{j,N}\right)^4\right] &\leq C\mathbb{E}_\theta\left[\left(\left[\sum_{j=1}^N A_N^n(i,j)M^{j,N}, \sum_{j=1}^N A_N^n(i,j)M^{j,N}\right]_u\right)^2\right] \\ &\leq C\mathbb{E}_\theta\left[\left(\sum_{j=1}^N (A_N^n(i,j))^2 Z_u^{j,N}\right)^2\right] \\ &= C \sum_{j,j'=1}^N (A_N^n(i,j))^2 (A_N^n(i,j'))^2 \mathbb{E}_\theta[Z_u^{j,N} Z_u^{j',N}]. \end{aligned}$$

By the Cauchy-Schwarz inequality, $\mathbb{E}_\theta[Z_u^{j,N} Z_u^{j',N}] \leq \sqrt{\mathbb{E}_\theta[(Z_u^{j,N})^2]\mathbb{E}_\theta[(Z_u^{j',N})^2]}$, and from [13, Lemma 16-(iii)], we have $\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_u^{i,N})^2] \leq Cu^2$. Therefore,

$$\mathbb{E}_\theta\left[\left(\sum_{j=1}^N A_N^n(i,j)M_u^{j,N}\right)^4\right] \leq C\left(\sum_{j=1}^N (A_N^n(i,j))^2\right)^2 u^2 \leq C\left(\sum_{j=1}^N A_N^n(i,j)\right)^4 u^2 \leq C|||A_N|||_\infty^{4n} u^2.$$

This implies that

$$\begin{aligned} &\sum_{n \geq 1} \int_0^\infty (\phi^{*n}(t-u) - \phi^{*n}(s-u)) \mathbb{E}_\theta\left[\left(\sum_{i=1}^K \sum_{j=1}^N A_N^n(i,j)M_u^{j,N}\right)^4\right]^{\frac{1}{4}} du \\ &\leq C \sum_{n \geq 1} |||A_N|||_\infty^n \int_0^t \sqrt{u} (\phi^{*n}(t-u) - \phi^{*n}(s-u)) du \\ &\leq C(t-s)^{1/2} \sum_{n \geq 1} \Lambda^n |||A_N|||_\infty^n \leq C(t-s)^{1/2}. \end{aligned}$$

To justify the second inequality, we use the following estimate, which holds for all $n \geq 1$:

$$\begin{aligned} \int_0^t \sqrt{u} (\phi^{*n}(t-u) - \phi^{*n}(s-u)) du &= \int_0^t \sqrt{t-u} \phi^{*n}(u) du - \int_0^s \sqrt{s-u} \phi^{*n}(u) du \\ &\leq \int_0^s [\sqrt{t-u} - \sqrt{s-u}] \phi^{*n}(u) du + \int_s^t \sqrt{t-u} \phi^{*n}(u) du \\ &\leq 2\sqrt{t-s} \int_0^\infty \phi^{*n}(u) du \leq 2\Lambda^n \sqrt{t-s}. \end{aligned}$$

Moreover, on $\Omega_{N,K}$, we have $\Lambda |||A_N|||_\infty \leq a < 1$. This completes the proof of the first part of (iii).

For the second part, we recall from Lemma 3.3-(ii) with $K = N$ and $r = \infty$ that we have $\max_{i=1,\dots,N} \mathbb{E}_\theta[Z_t^{i,N} - Z_s^{i,N}] \leq C(t-s)$ on $\Omega_{N,N} \supset \Omega_{N,K}$, it follows that

$$\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^4] \leq 8 \left\{ \max_{i=1,\dots,N} \mathbb{E}_\theta[Z_t^{i,N} - Z_s^{i,N}]^4 + \max_{i=1,\dots,N} \mathbb{E}_\theta[(U_t^{i,N} - U_s^{i,N})^4] \right\} \leq C(t-s)^4$$

as desired.

APPENDIX B. PROOF OF LEMMA 4.3

Recall that $\mathbf{X}_N^K = (X_N^K(i))_{i=1,\dots,N}$ with $X_N^K(i) = (L_N(i) - \bar{L}_N^K)\mathbf{1}_{\{i \leq K\}}$, where $L_N(i) := \sum_{j=1}^N A_N(i,j) = \frac{1}{N} \sum_{j=1}^N \theta_{ij}$ and that $\bar{L}_N^K := \frac{1}{K} \sum_{i=1}^K L_N(i)$ and $\mathbf{X}_N := \mathbf{X}_N^N$ defined in Section 3.1. Here, $(\theta_{ij})_{i,j=1,\dots,N}$ is a family of i.i.d. Bernoulli(p) random variables, $\mathbf{x}_N^K = (x_N^K(i))_{i=1,\dots,N}$ with $x_N^K(i) = (\ell_N(i) - \bar{\ell}_N^K)\mathbf{1}_{\{i \leq K\}}$, and \mathbf{x}_N are defined in Section 3.1.

B.1. Proof of Lemma 4.3 (i). Since $(\theta_{ij})_{i,j=1,\dots,N}$ are i.i.d. Bernoulli(p) random variables. By symmetry, we have

$$\begin{aligned} \mathbb{E}[\|(I_K A_N)^T \mathbf{X}_N^K\|_2^2] &= \frac{K}{N^2} \mathbb{E}\left[\left(\sum_{j=1}^K \theta_{j1}(L_N(j) - \bar{L}_N^K)\right)^2\right] \\ &\leq \frac{2K}{N^2} \left\{ \mathbb{E}\left[\left(\sum_{j=1}^K \theta_{j1}(L_N(j) - p)\right)^2\right] + \mathbb{E}\left[\left(\sum_{j=1}^K \theta_{j1}(p - \bar{L}_N^K)\right)^2\right] \right\}. \end{aligned}$$

On the one hand, since $\bar{L}_N^K = \frac{1}{NK} \sum_{i=1}^K \sum_{j=1}^N \theta_{ij}$ with $(\theta_{ij})_{i,j=1,\dots,N}$ being i.i.d. Bernoulli(p) random variables and since $\theta_{j1} \leq 1$, it directly follows that

$$\frac{K}{N^2} \mathbb{E}\left[\left(\sum_{j=1}^K \theta_{j1}(p - \bar{L}_N^K)\right)^2\right] \leq \frac{K^3}{N^2} \mathbb{E}[(p - \bar{L}_N^K)^2] \leq \frac{CK^2}{N^3}.$$

On the other hand, writing $L_N(j) = \frac{1}{N} \sum_{i=1}^N \theta_{ji} = \frac{1}{N} (\sum_{i=2}^N \theta_{ji} + \theta_{j1})$, we obtain

$$\begin{aligned} &\frac{K}{N^2} \mathbb{E}\left[\left(\sum_{j=1}^K \theta_{j1}(L_N(j) - p)\right)^2\right] \\ &\leq \frac{2K}{N^4} \left\{ \mathbb{E}\left[\left(\sum_{j=1}^K \sum_{i=2}^N \theta_{j1}(\theta_{ji} - p)\right)^2\right] + \mathbb{E}\left[\left(\sum_{j=1}^K \theta_{j1}(\theta_{j1} - p)\right)^2\right] \right\} \\ &\leq \frac{4K}{N^4} \left\{ \mathbb{E}\left[\left(\sum_{j=1}^K \sum_{i=2}^N (\theta_{j1} - p)(\theta_{ji} - p)\right)^2\right] + p^2 \mathbb{E}\left[\left(\sum_{j=1}^K \sum_{i=2}^N (\theta_{ji} - p)\right)^2\right] + \mathbb{E}\left[\left(\sum_{j=1}^K \theta_{j1}(\theta_{j1} - p)\right)^2\right] \right\}. \end{aligned}$$

Applying the family $\{(\theta_{ji} - p), i = 2, \dots, N, j = 1, \dots, K\}$ is independent and centered, it yields that

$$\begin{aligned} & \mathbb{E}\left[\left(\sum_{j=1}^K \sum_{i=2}^N (\theta_{j1} - p)(\theta_{ji} - p)\right)^2\right] \\ &= \mathbb{E}\left[\sum_{j,j'=1}^K \sum_{i,i'=2}^N (\theta_{j1} - p)(\theta_{j'1} - p)(\theta_{ji} - p)(\theta_{ji'} - p)\right] \\ &= \mathbb{E}\left[\sum_{j=1}^K \sum_{i=2}^N (\theta_{j1} - p)^2 (\theta_{ji} - p)^2\right] \leq CNK. \end{aligned}$$

Similarly, we have $\mathbb{E}[(\sum_{j=1}^K \sum_{i=2}^N (\theta_{ji} - p))^2] \leq CNK$. Furthermore, since $\theta_{j1} \leq 1$ and $|\theta_{j1} - p| \leq 1$, we obtain $\mathbb{E}[(\sum_{j=1}^K \theta_{j1}(\theta_{j1} - p))^2] \leq CK^2$. Consequently, $\frac{K}{N^2} \mathbb{E}\left[\left(\sum_{j=1}^K \theta_{j1}(L_N(j) - p)\right)^2\right] \leq CK^2/N^3$, as desired.

B.2. Proof of Lemma 4.3 (ii). By the definition of \mathbf{X}_N and \mathbf{X}_N^K in Section 3.1, we have

$$(I_K A_N \mathbf{X}_N, \mathbf{X}_N^K) = \sum_{i,j=1}^K A_N(i,j) X_N(j) X_N^K(i).$$

Since $A_N(i,j) = \frac{1}{N} \theta_{ij}$ and $\sum_{i=1}^K X_N^K(i) = \sum_{i=1}^K (L_N(i) - \bar{L}_N^K) = 0$, we have

$$\begin{aligned} (I_K A_N \mathbf{X}_N, \mathbf{X}_N^K) &= \frac{1}{N} \sum_{i,j=1}^K (\theta_{ij} - p) X_N(j) X_N^K(i) \\ &= \frac{1}{N} \left[\sum_{i,j=1}^K (\theta_{ij} - p)(L_N(j) - p) X_N^K(i) + (p - \bar{L}_N) \sum_{i,j=1}^K (\theta_{ij} - p) X_N^K(i) \right] \\ &= \frac{1}{N} \left[\sum_{i,j=1}^K (\theta_{ij} - p)(L_N(j) - p)(L_N(i) - p) + (p - \bar{L}_N^K) \sum_{i,j=1}^K (\theta_{ij} - p)(L_N(j) - p) \right. \\ &\quad \left. + (p - \bar{L}_N) \sum_{i,j=1}^K (\theta_{ij} - p)(L_N(i) - p) + (p - \bar{L}_N)(p - \bar{L}_N^K) \sum_{i,j=1}^K (\theta_{ij} - p) \right]. \end{aligned}$$

We start by analyzing the first term. Recalling that $L_N(i) := \sum_{j=1}^N A_N(i,j) = \frac{1}{N} \sum_{j=1}^N \theta_{ij}$, we obtain

$$\begin{aligned} & \mathbb{E}\left[\left(\sum_{i,j=1}^K (\theta_{ij} - p)(L_N(j) - p)(L_N(i) - p)\right)^2\right] \\ &= \frac{1}{N^4} \mathbb{E}\left[\left(\sum_{i,j=1}^K \sum_{m,n=1}^N (\theta_{ij} - p)(\theta_{jm} - p)(\theta_{in} - p)\right)^2\right] \\ &= \frac{1}{N^4} \mathbb{E}\left[\sum_{i,j,i',j'=1}^K \sum_{m,n,m',n'=1}^N (\theta_{ij} - p)(\theta_{jm} - p)(\theta_{in} - p)(\theta_{i'j'} - p)(\theta_{j'm'} - p)(\theta_{i'n'} - p)\right] \leq \frac{CK^2}{N^2}, \end{aligned}$$

since the family $\{(\theta_{ij} - p), i, j = 1, \dots, N\}$ is i.i.d., centered, and bounded.

For the second term, applying the Cauchy–Schwarz inequality yields

$$\mathbb{E}\left[\left|(p - \bar{L}_N^K) \sum_{i,j=1}^K (\theta_{ij} - p)(L_N(j) - p)\right|\right] \leq \frac{1}{N} \mathbb{E}[(p - \bar{L}_N^K)^2]^{\frac{1}{2}} \mathbb{E}\left[\left(\sum_{i,j=1}^K \sum_{k=1}^N (\theta_{ij} - p)(\theta_{jk} - p)\right)^2\right]^{\frac{1}{2}}.$$

This quantity is bounded by $\frac{\sqrt{K}}{N}$, since on the one hand, we have

$$(25) \quad \mathbb{E}[(p - \bar{L}_N^K)^2] = \frac{1}{N^2 K^2} \mathbb{E}\left[\left(\sum_{i=1}^K \sum_{j=1}^N (\theta_{ij} - p)\right)^2\right] = \frac{\mathbb{E}[(\theta_{11} - p)^2]}{NK} \leq \frac{C}{NK},$$

and on the other hand,

$$\begin{aligned} & \mathbb{E}\left[\left(\sum_{i,j=1}^K \sum_{k=1}^N (\theta_{ij} - p)(\theta_{jk} - p)\right)^2\right] \\ &= \mathbb{E}\left[\sum_{i,j,i',j'=1}^K \sum_{k,k'=1}^N (\theta_{ij} - p)(\theta_{i'j'} - p)(\theta_{jk} - p)(\theta_{j'k'} - p)\right] \leq CNK^2. \end{aligned}$$

For the third term, by (25), we have $\mathbb{E}[(p - \bar{L}_N)^2] = \mathbb{E}[(p - \bar{L}_N^N)^2] \leq \frac{C}{N^2}$. Applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} & \mathbb{E}\left[\left|(p - \bar{L}_N) \sum_{i,j=1}^K (\theta_{ij} - p)(L_N(i) - p)\right|\right] \\ &\leq \frac{1}{N} \mathbb{E}[(p - \bar{L}_N)^2]^{\frac{1}{2}} \mathbb{E}\left[\left(\sum_{i,j=1}^K \sum_{k=1}^N (\theta_{ij} - p)(\theta_{ik} - p)\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{N} \sqrt{\frac{C}{N^2}} \mathbb{E}\left[\sum_{i,j,i',j'=1}^K \sum_{k,k'=1}^N (\theta_{ij} - p)(\theta_{ik} - p)(\theta_{i'j'} - p)(\theta_{i'k'} - p)\right]^{\frac{1}{2}} \\ &\leq \frac{1}{N} \sqrt{\frac{C}{N^2}} \sqrt{K^2 N + K^4} = C \frac{K}{N^{3/2}} + C \frac{K^2}{N^2} \leq C \frac{K}{N}. \end{aligned}$$

Finally, we analyze the last term. Note that $\mathbb{E}[(\sum_{i,j=1}^K (\theta_{ij} - p))^2] = \mathbb{E}[\sum_{i,j=1}^K (\theta_{ij} - p)^2] = CK^2$ and $\mathbb{E}[(p - \bar{L}_N^K)^4] = \frac{1}{N^4 K^4} \mathbb{E}[(\sum_{i=1}^K \sum_{j=1}^N (\theta_{ij} - p))^4] \leq \frac{C}{N^2 K^2}$. Therefore, applying the Cauchy–Schwarz inequality again, we obtain

$$\begin{aligned} & \mathbb{E}\left[\left|(p - \bar{L}_N)(p - \bar{L}_N^K) \sum_{i,j=1}^K (\theta_{ij} - p)\right|\right] \\ &\leq \mathbb{E}[(p - \bar{L}_N)^4]^{\frac{1}{4}} \mathbb{E}[(p - \bar{L}_N^K)^4]^{\frac{1}{4}} \mathbb{E}\left[\left(\sum_{i,j=1}^K (\theta_{ij} - p)\right)^2\right]^{\frac{1}{2}} \\ &\leq C \left(\frac{1}{N^4}\right)^{1/4} \left(\frac{1}{N^2 K^2}\right)^{1/4} \sqrt{K^2} = \frac{\sqrt{K}}{N \sqrt{N}} \leq C \frac{K}{N}. \end{aligned}$$

Together, the preceding arguments complete the proof.

B.3. Proof of Lemma 4.3 (iii). Recall that $\mathbf{x}_N^K = (x_N^K(i))_{i=1,\dots,N}$, where $x_N^K(i) = (\ell_N(i) - \bar{\ell}_N^K)\mathbf{1}_{\{i \leq K\}}$, and that \mathbf{X}_N^K , $\ell_N(i)$, $\bar{\ell}_N^K$ and $\bar{\ell}_N$ are defined in Section 3.1. Here, \mathcal{A}_N is defined in (5). For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, $a > 0$, it's not hard to verify the following elementary equality

$$|\|\mathbf{x}\|_2^2 - a^2\|\mathbf{y}\|_2^2 - \|\mathbf{x} - a\mathbf{y}\|_2^2| = 2|a(\mathbf{x} - a\mathbf{y}, \mathbf{y})|.$$

Then, putting $\mathbf{x} = \mathbf{x}_N^K$, $\mathbf{y} = \mathbf{X}_N^K$ and $a = \Lambda\bar{\ell}_N$, we have

$$\begin{aligned} & \mathbb{E}\left[\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} \left| \left(\|\mathbf{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2 \|\mathbf{X}_N^K\|_2^2 \right) - \|\mathbf{x}_N^K - \Lambda\bar{\ell}_N \mathbf{X}_N^K\|_2^2 \right| \right] \\ &= 2\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} \left| \Lambda\bar{\ell}_N (\mathbf{x}_N^K - \Lambda\bar{\ell}_N \mathbf{X}_N^K, \mathbf{X}_N^K) \right| \right]. \end{aligned}$$

By [21, Lemma 5.11], it holds that

$$\mathbf{x}_N^K - \Lambda\bar{\ell}_N \mathbf{X}_N^K = \Lambda I_K A_N (\mathbf{x}_N - \Lambda\bar{\ell}_N \mathbf{X}_N) - \frac{\Lambda}{K} (I_K A_N \mathbf{x}_N, \mathbf{1}_K) \mathbf{1}_K + \bar{\ell}_N \Lambda^2 I_K A_N \mathbf{X}_N.$$

Since $X_N^K(i) = (L_N(i) - \bar{L}_N^K)\mathbf{1}_{\{i \leq K\}}$ and $\mathbf{1}_K$ is a vector with entries $\mathbf{1}_K(i) = \mathbf{1}_{\{1 \leq i \leq K\}}$, it follows that $(\mathbf{1}_K, \mathbf{X}_N^K) = \sum_{i=1}^K X_N^K(i) = 0$. Consequently,

$$(\mathbf{x}_N^K - \Lambda\bar{\ell}_N \mathbf{X}_N^K, \mathbf{X}_N^K) = \Lambda(I_K A_N (\mathbf{x}_N - \Lambda\bar{\ell}_N \mathbf{X}_N), \mathbf{X}_N^K) + \Lambda^2 \bar{\ell}_N (I_K A_N \mathbf{X}_N, \mathbf{X}_N^K) := e_{N,K,1} + e_{N,K,2}.$$

An application of the Cauchy–Schwarz inequality gives

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} e_{N,K,1}] &= \mathbb{E}\left[\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} \Lambda \left((\mathbf{x}_N - \Lambda\bar{\ell}_N \mathbf{X}_N), (I_K A_N)^T \mathbf{X}_N^K \right) \right] \\ &\leq \Lambda \mathbb{E}\left[\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} \|\mathbf{x}_N - \Lambda\bar{\ell}_N \mathbf{X}_N\|_2^2\right]^{\frac{1}{2}} \mathbb{E}\left[\|(I_K A_N)^T \mathbf{X}_N^K\|_2^2\right]^{\frac{1}{2}}. \end{aligned}$$

From [21, Lemma 5.11], the first term $\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} \|\mathbf{x}_N - \Lambda\bar{\ell}_N \mathbf{X}_N\|_2^2\right]$ is bounded by CN^{-1} , and Lemma 4.3-(i) bounds the second, yielding

$$\mathbb{E}[\mathbf{1}_{\Omega_{N,K} \cap \mathcal{A}_N} e_{N,K,1}] \leq \frac{CK}{N^2}.$$

Furthermore, $\bar{\ell}_N$ is bounded on $\Omega_{N,K}$ by Lemma 3.2, and Lemma 4.3-(ii) implies

$$\mathbb{E}[\mathbf{1}_{\Omega_{N,K}} e_{N,K,2}] \leq \frac{CK}{N^2},$$

which completes the proof.

B.4. Proof of Lemma 4.3 (iv). Recall from [13, Proposition 14] that $\mathbb{E}[\mathbf{1}_{\Omega_N^1} |\bar{\ell}_N - \frac{1}{1-\Lambda p}|^2] \leq \frac{C}{N^2}$. Furthermore, Lemma 3.2 guarantees that $\bar{\ell}_N$ is bounded by a constant C on $\Omega_{N,K}$. In addition, one can verify (see, e.g., [21, Equation (9)]) that $\mathbb{E}[\frac{N^2}{K^2} \|\mathbf{X}_N^K\|_2^4] \leq C$. Together with the Cauchy–Schwarz inequality, these results imply

$$\begin{aligned} & \sqrt{K} \frac{N}{K} \mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}} \left| (\bar{\ell}_N)^2 - \left(\frac{1}{1-\Lambda p}\right)^2 \right| \|\mathbf{X}_N^K\|_2^2 \right] \\ & \leq C \sqrt{K} \mathbb{E}\left[\frac{N^2}{K^2} \|\mathbf{X}_N^K\|_2^4\right]^{\frac{1}{2}} \mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}} \left| \bar{\ell}_N - \frac{1}{1-\Lambda p} \right|^2\right]^{\frac{1}{2}} \leq \frac{C\sqrt{K}}{N} \leq \frac{C}{\sqrt{N}}. \end{aligned}$$

To complete the proof, it suffices to show that $\mathbf{1}_{\Omega_{N,K}} \sqrt{K} [\frac{N}{K} \|\mathbf{X}_N^K\|_2^2 - p(1-p)] \xrightarrow{d} \mathcal{N}(0, p^2(1-p)^2)$. Since $\mathbf{1}_{\Omega_{N,K}}$ tends to 1 in probability, it is enough to verify that $\sqrt{K} [\frac{N}{K} \|\mathbf{X}_N^K\|_2^2 - p(1-p)] \xrightarrow{d}$

$\mathcal{N}(0, p^2(1-p)^2)$. Now observe that

$$\|\mathbf{X}_N^K\|_2^2 = \sum_{i=1}^K (L_N(i) - \bar{L}_N^K)^2 = \sum_{i=1}^K (L_N(i) - p)^2 - K(p - \bar{L}_N^K)^2.$$

As shown in the proof of Lemma 4.3-(ii), we have $\mathbb{E}[(p - \bar{L}_N^K)^2] \leq \frac{C}{NK}$, so that $\sqrt{K}\frac{N}{K}\mathbb{E}[K(p - \bar{L}_N^K)^2] \leq \frac{C}{\sqrt{K}}$. Therefore, it remains to show that

$$\xi_{N,K} := \sqrt{K} \left[\frac{N}{K} \sum_{i=1}^K (L_N(i) - p)^2 - p(1-p) \right] \xrightarrow{d} \mathcal{N}(0, p^2(1-p)^2).$$

Recalling that $L_N(i) = N^{-1} \sum_{j=1}^N \theta_{ij}$, a direct computation shows that

$$\xi_{N,K} = \frac{1}{N\sqrt{K}} \sum_{i=1}^K \sum_{j=1}^N [(\theta_{ij} - p)^2 - p(1-p)] + \frac{1}{N\sqrt{K}} \sum_{i=1}^K \sum_{j=1}^N \sum_{j'=1, j' \neq j}^N (\theta_{ij} - p)(\theta_{ij'} - p).$$

Since the variables $(\theta_{ij} - p)^2 - p(1-p)$ are i.i.d. with mean zero and finite variance, the central limit theorem implies the convergence in distribution of $\frac{1}{\sqrt{NK}} \sum_{i=1}^K \sum_{j=1}^N [(\theta_{ij} - p)^2 - p(1-p)]$. Therefore, the first term tends to 0 in probability. For the second term, applying the central limit theorem again, we obtain

$$\frac{1}{N\sqrt{K}} \sum_{i=1}^K \sum_{j=1}^N \sum_{j'=1, j' \neq j}^N (\theta_{ij} - p)(\theta_{ij'} - p) \xrightarrow{d} \mathcal{N}(0, p^2(1-p)^2),$$

which completes the proof.

APPENDIX C. CONVOLUTION OF ϕ

We first present two lemmas concerning the convolution of the function ϕ introduced in Section 1.5. These will be useful in proving Lemmas 6.4 and 6.6.

Lemma C.1. *We consider $\phi : [0, \infty) \rightarrow [0, \infty)$ such that $\Lambda = \int_0^\infty \phi(s)ds < \infty$ and, for some $q \geq 1$, $\int_0^\infty s^q \phi(s)ds < \infty$. Then, for all $n \geq 1$ and $r \geq 1$,*

$$\int_r^\infty \sqrt{s} \phi^{*n}(s)ds \leq C\Lambda^n n^q r^{\frac{1}{2}-q} \quad \text{and} \quad \int_0^\infty \sqrt{s} \phi^{*n}(s)ds \leq C\sqrt{n}\Lambda^n.$$

Proof. We introduce some i.i.d. random variables X_1, X_2, \dots with density $\Lambda^{-1}\phi$ and set $S_0 = 0$ as well as $S_n = X_1 + \dots + X_n$ for all $n \geq 1$. By the Minkowski inequality and since $\mathbb{E}[X_1^q] = \Lambda^{-1} \int_0^\infty s^q \phi(s)ds < \infty$ by assumption, we obtain $\mathbb{E}[S_n^q] \leq n^q \mathbb{E}[X_1^q] \leq Cn^q$. Consequently,

$$\int_r^\infty \sqrt{s} \phi^{*n}(s)ds = \Lambda^n \mathbb{E}[\sqrt{S_n} \mathbf{1}_{\{S_n \geq r\}}] \leq \Lambda^n r^{\frac{1}{2}-q} \mathbb{E}[S_n^q] \leq \Lambda^n n^q r^{\frac{1}{2}-q} \mathbb{E}[X_1^q] \leq C\Lambda^n n^q r^{\frac{1}{2}-q}.$$

For the second part, we write

$$\int_0^\infty \sqrt{s} \phi^{*n}(s)ds = \Lambda^n \mathbb{E}[\sqrt{S_n}] \leq \sqrt{n}\Lambda^n \sqrt{\mathbb{E}[X_1]} \leq C\sqrt{n}\Lambda^n$$

by the Cauchy-Schwarz inequality. \square

Lemma C.2. *Under the same conditions as in Lemma C.1, we have, for $n \in \mathbb{N}_+$ and $r \geq 1$,*

$$\left| \int_0^t \phi^{*n}(s)ds - \Lambda^n \right| \leq n\Lambda^{n-1} \int_{\frac{t}{n}}^\infty \phi(s)ds.$$

Proof. Consider n i.i.d random variables $\{X_i\}_{i=1,\dots,n}$ with density $\phi(s)/\Lambda$ and write

$$\left| \int_0^t \phi^{*n}(s)ds - \Lambda^n \right| = \Lambda^n \mathbb{P}\left(\sum_{i=1}^n X_i \geq t \right) \leq \Lambda^n \mathbb{P}\left(\max_{i=1,\dots,n} X_i \geq t/n \right) \leq n\Lambda^n \mathbb{P}(X_1 \geq t/n),$$

which completes the proof. \square

APPENDIX D. PROOF OF LEMMAS 6.4 & 6.6

D.1. Proof of Lemma 6.4 (i). We work on the set $\Omega_{N,K}$. Recall (8). By [13, Lemma 16-(iii)] and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathbb{E}_\theta[(M_{(a\Delta-s)}^{j,N} - M_{a\Delta}^{j,N})(M_{(a\Delta-s')}^{j',N} - M_{a\Delta}^{j',N})] &\leq \mathbf{1}_{\{j=j'\}} \mathbb{E}_\theta[Z_{a\Delta}^{j,N} - Z_{(a\Delta-s)}^{j,N}]^{\frac{1}{2}} \mathbb{E}_\theta[Z_{a\Delta}^{j,N} - Z_{(a\Delta-s')}^{j,N}]^{\frac{1}{2}} \\ &\leq \mathbf{1}_{\{j=j'\}} \sqrt{ss'}. \end{aligned}$$

From Lemma C.1, we already have $\int_r^\infty \sqrt{u} \phi^{*n}(u)du \leq C\Lambda^n n^q r^{\frac{1}{2}-q}$. Recalling (18), we obtain

$$\begin{aligned} \mathbb{E}_\theta[(B_{a\Delta}^{N,K})^2] &= \frac{1}{K^2} \sum_{i,i'=1}^K \sum_{j,j'=1}^N \sum_{n,m \geq 1} \int_\Delta^{a\Delta} \int_\Delta^{a\Delta} \phi^{*n}(s) \phi^{*m}(s') A_N^n(i,j) A_N^m(i',j') \\ &\quad \times \mathbb{E}_\theta[(M_{(a\Delta-s)}^{j,N} - M_{a\Delta}^{j,N})(M_{(a\Delta-s')}^{j',N} - M_{a\Delta}^{j',N})] dsds' \\ &\leq \frac{CN}{K^2} \left(\sum_{n,m \geq 1} \|I_K A_N\|_1^2 \|A_N\|_1^{n+m-2} \int_\Delta^\infty \int_\Delta^\infty \sqrt{ss'} \phi^{*n}(s) \phi^{*m}(s') dsds' \right) \\ &\leq \frac{C}{N} \left(\sum_{n \geq 1} n^q \Lambda^n \|A_N\|_1^{n-1} \right)^2 \Delta^{1-2q} \leq \frac{C}{N} \Delta^{1-2q}, \end{aligned}$$

where the last inequality follows because on $\Omega_{N,K}$, $\Lambda \|A_N\|_1 \leq a < 1$ and $\sum_{n \geq 1} n^q a^{n-1} < \infty$.

D.2. Proof of Lemma 6.4 (ii). Recalling (17), we write

$$C_{a\Delta}^{N,K} = \sum_{n \geq 1} \int_0^\Delta \phi^{*n}(s) O_{s,s,a\Delta}^{N,K,n},$$

where for $r \geq 0$ and $0 \leq s \leq a\Delta$,

$$O_{r,s,a\Delta}^{N,K,n} := \frac{1}{K} \sum_{i=1}^K \sum_{j=1}^N A_N^n(i,j) (M_{(a\Delta-s)}^{j,N} - M_{a\Delta-s+r}^{j,N}).$$

For fixed s , $\{O_{r,s,a\Delta}^{N,K,n}\}_{r \geq 0}$ is a family of martingale w.r.t the filtration $(\mathcal{F}_{a\Delta-s+r})_{r \geq 0}$. By (8), we have $[M^{i,N}, M^{j,N}]_t = \mathbf{1}_{\{i=j\}} Z_t^{i,N}$. Hence, for $n \geq 1$, on $\Omega_{N,K}$,

$$\begin{aligned} [O_{.,s,a\Delta}^{N,K,n}, O_{.,s,a\Delta}^{N,K,n}]_r &= \frac{1}{K^2} \sum_{j=1}^N \left(\sum_{i=1}^K A_N^n(i,j) \right)^2 (Z_{a\Delta-s+r}^{j,N} - Z_{a\Delta-s}^{j,N}) \\ &\leq \frac{N}{K^2} \|I_K A_N\|_1^2 (\bar{Z}_{a\Delta-s+r}^N - \bar{Z}_{a\Delta-s}^N) \\ &\leq \frac{N}{K^2} \|I_K A_N\|_1^2 \|A_N\|_1^{2n-2} (\bar{Z}_{a\Delta-s+r}^N - \bar{Z}_{a\Delta-s}^N). \end{aligned}$$

Since $\|I_K A_N\|_1^2 \leq \frac{1}{\Lambda^2} \frac{K^2}{N^2}$ on $\Omega_{N,K}$, we obtain

$$[O_{\cdot,s,a\Delta}^{N,K,n}, O_{\cdot,s,a\Delta}^{N,K,n}]_r \leq \frac{C}{N} \|A_N\|_1^{2n-2} (\bar{Z}_{a\Delta-s+r}^N - \bar{Z}_{a\Delta-s}^N).$$

Applying the Burkholder-Davis-Gundy inequality yields, on $\Omega_{N,K}$,

$$\mathbb{E}_\theta[(O_{r,s,a\Delta}^{N,K,n})^4] \leq 4\mathbb{E}_\theta\left[\left([O_{\cdot,s,a\Delta}^{N,K,n}, O_{\cdot,s,a\Delta}^{N,K,n}]_r\right)^2\right] \leq \frac{C\|A_N\|_1^{4n-4}}{N^2} \mathbb{E}_\theta[(\bar{Z}_{a\Delta-s+r}^N - \bar{Z}_{a\Delta-s}^N)^2].$$

From [13, lemma 16-(iii)], we already have $\sup_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^2] \leq C(t-s)^2$. Using the second part of Lemma C.1 together with the Minkowski inequality, we obtain

$$\begin{aligned} \mathbb{E}_\theta[(C_{a\Delta}^{N,K})^4]^{\frac{1}{4}} &\leq \sum_{n \geq 1} \int_0^\Delta \phi^{*n}(s) \mathbb{E}_\theta[(O_{\cdot,s,a\Delta}^{N,K,n})^4]^{\frac{1}{4}} ds \\ &\leq \frac{1}{\sqrt{N}} \sum_{n \geq 0} \|A_N\|_1^n \int_0^\Delta \sqrt{s} \phi^{*(n+1)}(s) ds \\ &\leq \frac{1}{\sqrt{N}} \sum_{n \geq 0} \sqrt{n+1} \Lambda^{n+1} \|A_N\|_1^n \leq \frac{C}{\sqrt{N}}. \end{aligned}$$

This completes the proof.

D.3. Proof of Lemma 6.4 (iii).

Because

$$\begin{aligned} &\mathbb{C}ov_\theta[(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})^2, (C_{b\Delta}^{N,K} - C_{(b-1)\Delta}^{N,K})^2] \\ &= \frac{1}{K^4} \sum_{i,k,i'=1}^K \sum_{j,l,j',l'=1}^N \sum_{m,n,m',n' \geq 1} \int_0^\Delta \int_0^\Delta \int_0^\Delta \int_0^\Delta \phi^{*n}(s) \phi^{*m}(s) \phi^{*n'}(s') \phi^{*m'}(s') \\ &\quad \times A_N^n(i,j) A_N^m(k,l) A_N^{n'}(i',j') A_N^{m'}(k',l') \\ &\quad \times \mathbb{C}ov_\theta[(M_{(a\Delta-s)}^{j,N} - M_{a\Delta}^{j,N} - M_{((a-1)\Delta-s)}^{j,N} + M_{(a-1)\Delta}^{j,N}) \\ &\quad \quad \times (M_{(a\Delta-s')}^{j',N} - M_{a\Delta}^{j',N} - M_{((a-1)\Delta-s')}^{j',N} + M_{(a-1)\Delta}^{j',N}), \\ &\quad \quad (M_{(b\Delta-r)}^{l,N} - M_{b\Delta}^{l,N} - M_{((b-1)\Delta-r)}^{l,N} + M_{(b-1)\Delta}^{l,N}) \\ &\quad \quad \times (M_{(b\Delta-r')}^{l',N} - M_{b\Delta}^{l',N} + (M_{((b-1)\Delta-r')}^{l',N} - M_{(b-1)\Delta}^{l',N}))] ds dr ds' dr'. \end{aligned}$$

Define $\zeta_{a\Delta,s}^{j,N} := M_{(a\Delta-s)}^{j,N} - M_{a\Delta}^{j,N}$ for $0 \leq s \leq \Delta$. Then we rewrite

$$\begin{aligned} &\mathbb{C}ov_\theta[(M_{(a\Delta-s)}^{j,N} - M_{a\Delta}^{j,N} - M_{((a-1)\Delta-s)}^{j,N} + M_{(a-1)\Delta}^{j,N}) \\ &\quad \times (M_{(a\Delta-s')}^{j',N} - M_{a\Delta}^{j',N} - M_{((a-1)\Delta-s')}^{j',N} + M_{(a-1)\Delta}^{j',N}), \\ &\quad (M_{(b\Delta-r)}^{l,N} - M_{b\Delta}^{l,N} - M_{((b-1)\Delta-r)}^{l,N} + M_{(b-1)\Delta}^{l,N}) \\ &\quad \times (M_{(b\Delta-r')}^{l',N} - M_{b\Delta}^{l',N} + (M_{((b-1)\Delta-r')}^{l',N} - M_{(b-1)\Delta}^{l',N}))] \\ &= \mathbb{C}ov_\theta[(\zeta_{a\Delta,s}^{j,N} - \zeta_{(a-1)\Delta,s}^{j,N})(\zeta_{a\Delta,s'}^{j',N} - \zeta_{(a-1)\Delta,s'}^{j',N}), (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})(\zeta_{b\Delta,r'}^{l',N} - \zeta_{(b-1)\Delta,r'}^{l',N})]. \end{aligned}$$

Since $0 \leq s, s', r, r' \leq \Delta$, we have

$$\mathbb{E}_\theta[\zeta_{(a-1)\Delta,s}^{j,N} \zeta_{a\Delta,s'}^{j',N}] = \mathbb{E}_\theta[\zeta_{a\Delta,s}^{j,N} \zeta_{(a-1)\Delta,s'}^{j',N}] = \mathbb{E}_\theta[\zeta_{(b-1)\Delta,r}^{l,N} \zeta_{b\Delta,r'}^{l',N}] = \mathbb{E}_\theta[\zeta_{b\Delta,r}^{l,N} \zeta_{(b-1)\Delta,r'}^{l',N}] = 0.$$

Without loss of generality, assume $a - b \geq 4$ and $s \leq s'$. First, note that

$$\begin{aligned} & \mathbb{C}ov_{\theta}[\zeta_{a\Delta,s}^{j,N} \zeta_{(a-1)\Delta,s'}^{j',N}, (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})(\zeta_{b\Delta,r'}^{l',N} - \zeta_{(b-1)\Delta,r'}^{l',N})] \\ &= \mathbb{E}_{\theta} \left[\mathbb{E}_{\theta} \left[\zeta_{a\Delta,s}^{j,N} | \mathcal{F}_{(a-1)\Delta} \right] \zeta_{(a-1)\Delta,s'}^{j',N} \right. \\ &\quad \times \left. \left((\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})(\zeta_{b\Delta,r'}^{l',N} - \zeta_{(b-1)\Delta,r'}^{l',N}) - \mathbb{E}_{\theta} \left[(\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})(\zeta_{b\Delta,r'}^{l',N} - \zeta_{(b-1)\Delta,r'}^{l',N}) \right] \right) \right] = 0. \end{aligned}$$

For the same reason, we also obtain

$$\mathbb{C}ov_{\theta}[\zeta_{(a-1)\Delta,s}^{j,N} \zeta_{a\Delta,s'}^{j',N}, (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})(\zeta_{b\Delta,r'}^{l',N} - \zeta_{(b-1)\Delta,r'}^{l',N})] = 0.$$

If $j \neq j'$, the covariance vanishes because

$$\mathbb{E}_{\theta}[(\zeta_{a\Delta,s}^{j,N} - \zeta_{(a-1)\Delta,s}^{j,N})(\zeta_{a\Delta,s'}^{j',N} - \zeta_{(a-1)\Delta,s'}^{j',N}) | \mathcal{F}_{b\Delta}] = 0.$$

Now assume $j = j'$, then

$$\begin{aligned} \mathcal{K} &:= \mathbb{C}ov_{\theta}[(\zeta_{a\Delta,s}^{j,N} - \zeta_{(a-1)\Delta,s}^{j,N})(\zeta_{a\Delta,s'}^{j',N} - \zeta_{(a-1)\Delta,s'}^{j',N}), (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})(\zeta_{b\Delta,r'}^{l',N} - \zeta_{(b-1)\Delta,r'}^{l',N})] \\ &= \mathbb{C}ov_{\theta}[(\zeta_{a\Delta,s}^{j,N} \zeta_{a\Delta,s'}^{j',N} + \zeta_{(a-1)\Delta,s}^{j,N} \zeta_{(a-1)\Delta,s'}^{j',N}), (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})(\zeta_{b\Delta,r'}^{l',N} - \zeta_{(b-1)\Delta,r'}^{l',N})]. \end{aligned}$$

Since $\mathbb{E}_{\theta}[\zeta_{a\Delta,s}^{j,N} \zeta_{a\Delta,s'}^{j,N} | \mathcal{F}_{a\Delta-s'}] = \mathbb{E}_{\theta}[(M_{a\Delta}^{j,N})^2 - (M_{a\Delta-s}^{j,N})^2 | \mathcal{F}_{a\Delta-s'}]$, and as usual $(M_{a\Delta}^{j,N})^2 - (M_{a\Delta-s}^{j,N})^2 = 2 \int_{a\Delta-s}^{a\Delta} M_{\tau-}^{j,N} dM_{\tau}^{j,N} + Z_{a\Delta}^{j,N} - Z_{a\Delta-s}^{j,N}$. We note that $\mathbb{E}_{\theta} \left[\int_{a\Delta-s}^{a\Delta} M_{\tau-}^{j,N} dM_{\tau}^{j,N} | \mathcal{F}_{a\Delta-s} \right] = 0$. Moreover, because $a - b \geq 4$, we obtain

$$\begin{aligned} & \mathbb{C}ov_{\theta}[(M_{a\Delta}^{j,N})^2 - (M_{a\Delta-s}^{j,N})^2, (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})(\zeta_{b\Delta,r'}^{l',N} - \zeta_{(b-1)\Delta,r'}^{l',N})] \\ &= \mathbb{C}ov_{\theta}[Z_{a\Delta}^{j,N} - Z_{a\Delta-s}^{j,N}, (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})(\zeta_{b\Delta,r'}^{l',N} - \zeta_{(b-1)\Delta,r'}^{l',N})]. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathcal{K} &= \mathbb{C}ov_{\theta}[Z_{a\Delta}^{j,N} - Z_{a\Delta-s}^{j,N} + Z_{(a-1)\Delta}^{j,N} - Z_{(a-1)\Delta-s}^{j,N}, (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})(\zeta_{b\Delta,r'}^{l',N} - \zeta_{(b-1)\Delta,r'}^{l',N})] \\ &= \mathbb{C}ov_{\theta}[U_{a\Delta}^{j,N} - U_{a\Delta-s}^{j,N} + U_{(a-1)\Delta}^{j,N} - U_{(a-1)\Delta-s}^{j,N}, (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})(\zeta_{b\Delta,r'}^{l',N} - \zeta_{(b-1)\Delta,r'}^{l',N})]. \end{aligned}$$

Recall that $\beta_n(x, z, r) = \phi^{\star n}(z - r) - \phi^{\star n}(x - r)$. We can write

$$U_{a\Delta}^{i,N} - U_{a\Delta-s}^{i,N} = \sum_{n \geq 0} \int_0^{a\Delta} \beta_n(a\Delta - s, a\Delta, r) \sum_{j=1}^N A_N^n(i, j) M_r^{j,N} dr = R_{a\Delta,a\Delta-s}^{i,N} + T_{a\Delta,a\Delta-s}^{i,N},$$

where

$$\begin{aligned} R_{a\Delta,a\Delta-s}^{i,N} &= \sum_{n \geq 0} \int_{(a-1)\Delta-s}^{a\Delta} \beta_n(x, z, r) \sum_{j=1}^N A_N^n(i, j) (M_r^{j,N} - M_{(a-1)\Delta-s}^{j,N}) dr, \\ T_{a\Delta,a\Delta-s}^{i,N} &= \sum_{n \geq 0} \left(\int_{(a-1)\Delta-s}^{a\Delta} \beta_n(x, z, r) dr \right) \sum_{j=1}^N A_N^n(i, j) M_{(a-1)\Delta-s}^{j,N} \\ &\quad + \sum_{n \geq 0} \int_0^{(a-1)\Delta-s} \beta_n(x, z, r) \sum_{j=1}^N A_N^n(i, j) M_r^{j,N} dr. \end{aligned}$$

The conditional expectation of $R_{a\Delta, a\Delta-s}^{i,N}$ given $\mathcal{F}_{b\Delta}$ is zero. Therefore,

$$\mathcal{K} = \mathbb{C}ov_\theta[T_{a\Delta, a\Delta-s}^{i,N} + T_{(a-1)\Delta, (a-1)\Delta-s}^{i,N}, (\zeta_{b\Delta, r}^{l,N} - \zeta_{(b-1)\Delta, r}^{l,N})(\zeta_{b\Delta, r'}^{l',N} - \zeta_{(b-1)\Delta, r'}^{l',N})].$$

Referring to the proof of Lemma 30, Step 1 in [13] (noting that $T_{a\Delta, a\Delta-s}^{i,N}$ coincides with $X_{a\Delta-s, a\Delta}^{i,N}$ in [13]), we have $\sup_{i=1, \dots, N} \mathbb{E}_\theta[(T_{a\Delta, a\Delta-s}^{i,N})^4] \leq Ct^2\Delta^{-4q}$. Since $r \leq \Delta$ and by [13, Lemma 16-(iii)], we obtain

$$\begin{aligned} \mathbb{E}_\theta[(\zeta_{b\Delta, r}^{l,N} - \zeta_{(b-1)\Delta, r}^{l,N})^4]^{\frac{1}{4}} &\leq \mathbb{E}_\theta[(M_{(b\Delta-r)}^{l,N} - M_{b\Delta}^{l,N})^4]^{\frac{1}{4}} + \mathbb{E}_\theta[(M_{((b-1)\Delta-r)}^{l,N} - M_{(b-1)\Delta}^{l,N})^4]^{\frac{1}{4}} \\ &\leq C\sqrt{\Delta}, \end{aligned}$$

Hence, applying [13, Lemma 16-(iii)] once more,

$$\begin{aligned} |\mathcal{K}| &\leq \{\mathbb{E}_\theta[(T_{a\Delta, a\Delta-s}^{i,N})^2]^{\frac{1}{2}} + \mathbb{E}_\theta[(T_{(a-1)\Delta, (a-1)\Delta-s}^{i,N})^2]^{\frac{1}{2}}\} \\ &\quad \times \mathbb{E}_\theta[(\zeta_{b\Delta, r}^{l,N} - \zeta_{(b-1)\Delta, r}^{l,N})^4]^{\frac{1}{4}} \mathbb{E}_\theta[(\zeta_{b\Delta, r'}^{l',N} - \zeta_{(b-1)\Delta, r'}^{l',N})^4]^{\frac{1}{4}} \\ &\leq Ct^{1/2}\Delta^{-q}\Delta. \end{aligned}$$

Moreover, by symmetry, we conclude that for $|a - b| \geq 4$,

$$\begin{aligned} \mathbb{C}ov_\theta[(\zeta_{a\Delta, s}^{j,N} - \zeta_{(a-1)\Delta, s}^{j,N})(\zeta_{a\Delta, s'}^{j',N} - \zeta_{(a-1)\Delta, s'}^{j',N}), (\zeta_{b\Delta, r}^{l,N} - \zeta_{(b-1)\Delta, r}^{l,N})(\zeta_{b\Delta, r'}^{l',N} - \zeta_{(b-1)\Delta, r'}^{l',N})] \\ \leq C(\mathbf{1}_{\{l=l'\}} + \mathbf{1}_{\{j=j'\}})\sqrt{t}\Delta^{1-q}. \end{aligned}$$

Recalling the definition of $\Omega_{N,K}$, we have

$$|||I_K A_N^n|||_1 \leq |||I_K A_N|||_1 |||A_N|||_1^{n-1} \leq \frac{CK}{N} |||A_N|||_1^{n-1},$$

which implies

$$\begin{aligned} &\mathbb{C}ov_\theta[(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})^2, (C_{b\Delta}^{N,K} - C_{(b-1)\Delta}^{N,K})^2] \\ &\leq \frac{C\sqrt{t}\Delta^{1-q}}{K^4} \sum_{i,k,i',k'=1}^K \sum_{j,l,j',l'=1}^N \sum_{m,n,m',n' \geq 1} \Lambda^{n+m+n'+m'} A_N^n(i,j) A_N^m(k,l) A_N^{n'}(i',j') A_N^{m'}(k',l') \\ &\quad \times (\mathbf{1}_{\{l=l'\}} + \mathbf{1}_{\{j=j'\}}) \\ &\leq \frac{C\sqrt{t}\Delta^{1-q}}{K^4} N^3 \sum_{m,n,m',n' \geq 1} \Lambda^{n+m+n'+m'} |||I_K A_N^n|||_1 |||I_K A_N^m|||_1 |||I_K A_N^{n'}|||_1 |||I_K A_N^{m'}|||_1 \\ &\leq \sum_{n,m,n',m' \geq 1} \frac{1}{K^4} N^3 \left(\frac{K}{N}\right)^4 \Lambda^4 (\Lambda |||A_N|||_1)^{n+m+n'+m'-4} \frac{C\sqrt{t}}{\Delta^{q-1}} \leq \frac{C\sqrt{t}}{N\Delta^{q-1}}. \end{aligned}$$

This completes the proof.

D.4. Proof of Lemma 6.6 (i)&(ii). Recall $c_N^K(j) = \sum_{i=1}^K Q_N(i,j)$ with $Q_N = \sum_{n \geq 0} \Lambda^n A_N^n$. Remind $\bar{X}_{(a-1)\Delta, a\Delta}^{N,K} = \frac{1}{K} \sum_{i=1}^K X_{(a-1)\Delta, a\Delta}^{i,N}$, $\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K}$ and $X_{(a-1)\Delta, a\Delta}^{i,N}$ defined in (14) and (16), respectively. We first rewrite

$$\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} = \frac{1}{K} \sum_{j=1}^N \sum_{i=1}^K \sum_{n \geq 0} \Lambda^n A_N^n(i,j) (M_{a\Delta}^{j,N} - M_{(a-1)\Delta}^{j,N}).$$

Then, using (8) and Lemma C.2, we have, on $\Omega_{N,K}$

$$\begin{aligned}
& \mathbf{1}_{\Omega_{N,K}} \mathbb{E}_\theta [(\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K})^2] \\
& \leq \frac{2}{K^2} \mathbb{E}_\theta \left[\left| \sum_{j=1}^N \left\{ \sum_{n \geq 0} \sum_{i=1}^K \left(\int_0^{a\Delta} \phi^{*n}(a\Delta - s) ds - \Lambda^n \right) A_N^n(i, j) \right\} M_{a\Delta}^{j,N} \right|^2 \right] \\
& \quad + \frac{2}{K^2} \mathbb{E}_\theta \left[\left| \sum_{j=1}^N \left\{ \sum_{n \geq 0} \sum_{i=1}^K \left(\int_0^{(a-1)\Delta} \phi^{*n}((a-1)\Delta - s) ds - \Lambda^n \right) A_N^n(i, j) \right\} M_{(a-1)\Delta}^{j,N} \right|^2 \right] \\
& = \frac{2}{K^2} \sum_{j=1}^N \left\{ \sum_{n \geq 0} \sum_{i=1}^K \left(\int_0^{a\Delta} \phi^{*n}(a\Delta - s) ds - \Lambda^n \right) A_N^n(i, j) \right\}^2 \mathbb{E}_\theta [Z_{a\Delta}^{j,N}] \\
& \quad + \frac{2}{K^2} \sum_{j=1}^N \left\{ \sum_{n \geq 0} \sum_{i=1}^K \left(\int_0^{(a-1)\Delta} \phi^{*n}((a-1)\Delta - s) ds - \Lambda^n \right) A_N^n(i, j) \right\}^2 \mathbb{E}_\theta [Z_{(a-1)\Delta}^{j,N}] \\
& \leq \frac{2}{K^2} \sum_{j=1}^N \left\{ \sum_{n \geq 1} n \int_{(a\Delta)/n}^\infty \phi(s) ds \Lambda^{n-1} \||I_K A_N^n|\|_1 \right\}^2 \mathbb{E}_\theta [Z_{a\Delta}^{j,N}] \\
& \quad + \frac{2}{K^2} \sum_{j=1}^N \left\{ \sum_{n \geq 1} n \int_{(a-1)\Delta/n}^\infty \phi(s) ds \Lambda^{n-1} \||I_K A_N^n|\|_1 \right\}^2 \mathbb{E}_\theta [Z_{(a-1)\Delta}^{j,N}].
\end{aligned}$$

Noting that on $\Omega_{N,K}$, $\||I_K A_N^n|\|_1 \leq \||I_K A_N|\|_1 \||A_N|\|_1^{n-1} \leq \frac{CK}{N} \||A_N|\|_1^{n-1}$, we have

$$\begin{aligned}
& \mathbf{1}_{\Omega_{N,K}} \mathbb{E}_\theta [(\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K})^2] \\
& \leq \frac{C}{N^2} \sum_{j=1}^N \left\{ \sum_{n \geq 1} n \int_{(a\Delta)/n}^\infty \phi(s) ds \Lambda^{n-1} \||A_N|\|_1^{n-1} \right\}^2 \mathbb{E}_\theta [Z_{a\Delta}^{j,N}] \\
& \quad + \frac{C}{N^2} \sum_{j=1}^N \left\{ \sum_{n \geq 1} n \int_{(a-1)\Delta/n}^\infty \phi(s) ds \Lambda^{n-1} \||A_N|\|_1^{n-1} \right\}^2 \mathbb{E}_\theta [Z_{(a-1)\Delta}^{j,N}] \\
& \leq \frac{C}{N^2} \sum_{j=1}^N \left\{ \sum_{n \geq 1} n^{1+q} (a\Delta)^{-q} \int_0^\infty s^q \phi(s) ds \Lambda^{n-1} \||A_N|\|_1^{n-1} \right\}^2 \mathbb{E}_\theta [Z_{a\Delta}^{j,N}] \\
& \quad + \frac{C}{N^2} \sum_{j=1}^N \left\{ \sum_{n \geq 1} n^{1+q} [(a-1)\Delta]^{-q} \int_0^\infty s^q \phi(s) ds \Lambda^{n-1} \||A_N|\|_1^{n-1} \right\}^2 \mathbb{E}_\theta [Z_{(a-1)\Delta}^{j,N}] \\
& \leq \frac{C}{N^2 (a\Delta)^{2q}} \mathbb{E}_\theta \left[\sum_{j=1}^N Z_{a\Delta}^{j,N} \right] + \frac{C}{N^2 ((a-1)\Delta)^{2q}} \mathbb{E}_\theta \left[\sum_{j=1}^N Z_{(a-1)\Delta}^{j,N} \right] \\
& \leq \frac{C}{N} \left[\frac{1}{(a\Delta)^{2q-1}} + \frac{1}{((a-1)\Delta)^{2q-1}} \right].
\end{aligned}$$

The last inequality uses $\max_{j=1,\dots,N} \mathbb{E}_\theta [Z_t^{j,N}] \leq Ct$, proved in Lemma 3.3-(i) with $r = \infty$, which yields exactly (i). For $q \geq 2$, the series $\sum_{a=1}^\infty a^{1-2q} < +\infty$, which concludes (ii).

D.5. Proof of Lemma 6.6 (iii)&(iv). Recalling the definition of $\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K}$ in (14) and (8), and applying the Burkholder-Davis-Gundy inequality, we have on $\Omega_{N,K}$

$$\begin{aligned} & \mathbb{E}\left[\left|\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K}\right|^4\right] \\ & \leq \frac{4}{K^4} \mathbb{E}\left[\mathbb{E}_\theta\left[\sum_{j=1}^N \left(c_N^K(j)\right)^2 (Z_{a\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N})\right]^2\right] \\ & = \frac{4}{K^4} \mathbb{E}\left[\sum_{j,j'=1}^N \left(c_N^K(j)\right)^2 \left(c_N^K(j')\right)^2 \mathbb{E}_\theta\left[(Z_{a\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N})(Z_{a\Delta}^{j',N} - Z_{(a-1)\Delta}^{j',N})\right]\right] \\ & \leq \frac{4}{K^4} \mathbb{E}\left[\sum_{j,j'=1}^N \left(c_N^K(j)\right)^2 \left(c_N^K(j')\right)^2 \mathbb{E}_\theta\left[(Z_{a\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N})^2\right]^{1/2} \mathbb{E}_\theta\left[(Z_{a\Delta}^{j',N} - Z_{(a-1)\Delta}^{j',N})^2\right]^{1/2}\right]. \end{aligned}$$

From [13, lemma 16-(iii)], we already have on $\Omega_{N,K}$, $\sup_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^2] \leq C(t-s)^2$. Moreover, from (20), we have, on $\Omega_{N,K}$,

$$\sum_{j=1}^N \left(c_N^K(j)\right)^2 = \left(\sum_{j=1}^K \left(c_N^K(j)\right)^2 + \sum_{j=K}^N \left(c_N^K(j)\right)^2\right) \leq CK.$$

Consequently,

$$\mathbb{E}\left[1_{\Omega_{N,K}} \left|\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K}\right|^4\right] \leq \frac{C\Delta^2}{K^2},$$

which completes the proof of (iii). From Lemma 6.6-(i)&(iii) and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \frac{K}{\sqrt{\Delta t}} \mathbb{E}\left[1_{\Omega_{N,K}} \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left|\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K}\right| \left|\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta,a\Delta}^{N,K}\right|\right] \\ & \leq \frac{K}{\sqrt{\Delta t}} \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \mathbb{E}\left[1_{\Omega_{N,K}} \left|\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K}\right|^4\right]^{\frac{1}{4}} \mathbb{E}\left[1_{\Omega_{N,K}} \left|\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta,a\Delta}^{N,K}\right|^2\right]^{\frac{1}{2}} \\ & \leq \frac{C\sqrt{K}}{\Delta^{q-\frac{1}{2}}\sqrt{Nt}} \sum_{a=\frac{t}{\Delta}}^{\frac{2t}{\Delta}} a^{\frac{1}{2}-q} \leq \frac{C\sqrt{K}}{\Delta^{q-\frac{1}{2}}\sqrt{Nt}}. \end{aligned}$$

In the last step, we used that for $q \geq 2$, the series $\sum_{a=1}^\infty a^{\frac{1}{2}-q}$ converges.

D.6. Proof of Lemma 6.6 (v). Recalling that $\bar{\Gamma}_{(a-1)\Delta,a\Delta}^{N,K} = C_{a\Delta}^{N,K} + B_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K}$, where $C_{a\Delta}^{N,K}$ and $B_{a\Delta}^{N,K}$ are defined in (17) and (18), respectively. We write

$$\begin{aligned} & \bar{\Gamma}_{(a-1)\Delta,a\Delta}^{N,K} \bar{X}_{(a-1)\Delta,a\Delta}^{N,K} \\ & = \bar{\Gamma}_{(a-1)\Delta,a\Delta}^{N,K} (\bar{X}_{(a-1)\Delta,a\Delta}^{N,K} - \mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K}) + \mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K} \bar{\Gamma}_{(a-1)\Delta,a\Delta}^{N,K} \\ & = \bar{\Gamma}_{(a-1)\Delta,a\Delta}^{N,K} (\bar{X}_{(a-1)\Delta,a\Delta}^{N,K} - \mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K}) + \mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K} (C_{a\Delta}^{N,K} + B_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K}). \end{aligned}$$

Applying the Cauchy–Schwarz inequality together with Lemmas 6.6-(i) and 6.4-(i)&(ii), we obtain

$$\begin{aligned}
& \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| \bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K} \left(\bar{X}_{(a-1)\Delta, a\Delta}^{N,K} - \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} \right) \right|^2 \right]^2 \\
& \leq \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} \right|^2 \right] \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} (\bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K})^2 \right] \\
& = \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} \right|^2 \right] \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left(C_{a\Delta}^{N,K} + B_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K} \right)^2 \right] \\
& \leq 4 \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} - \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} \right|^2 \right] \\
& \quad \times \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \left\{ \left(C_{a\Delta}^{N,K} \right)^2 + \left(B_{a\Delta}^{N,K} \right)^2 + \left(C_{(a-1)\Delta}^{N,K} \right)^2 + \left(B_{(a-1)\Delta}^{N,K} \right)^2 \right\} \right] \\
& \leq \frac{C}{N} \left[(a\Delta)^{1-2q} + \left((a-1)\Delta \right)^{1-2q} \right] \left(\frac{1}{N} + \frac{1}{N} \Delta^{1-2q} \right) \\
& \leq \left[(a\Delta)^{1-2q} + \left((a-1)\Delta \right)^{1-2q} \right] \frac{C}{N^2}.
\end{aligned}$$

Similarly, using the Cauchy–Schwarz inequality again and Lemmas 6.4-(i) and 6.6-(iii), we have

$$\begin{aligned}
& \mathbf{1}_{\Omega_{N,K}} \mathbb{E}_\theta \left[\left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} (B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K}) \right|^2 \right] \\
& \leq \mathbf{1}_{\Omega_{N,K}} \mathbb{E}_\theta \left[\left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} \right|^2 \right] \mathbb{E}_\theta \left[\left(B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K} \right)^2 \right] \\
& \leq \mathbf{1}_{\Omega_{N,K}} \mathbb{E}_\theta \left[\left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} \right|^4 \right]^{1/2} \mathbb{E}_\theta \left[\left(B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K} \right)^2 \right] \leq \frac{C}{NK\Delta^{2q-2}}.
\end{aligned}$$

Next, we consider the term $\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} (C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})$. Recalling $\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K}$ defined in (14) and $C_{a\Delta}^{N,K}$ defined in (17), we write

$$\begin{aligned}
\mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} (C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K}) &= \frac{1}{K^2} \sum_{i=1}^K \sum_{j,j'=1}^N \sum_{n \geq 1} \int_0^\Delta \phi^{*n}(s) A_N^n(i, j) c_N^K(j') \\
&\quad (M_{a\Delta-s}^{j,N} - M_{a\Delta}^{j,N} - M_{(a-1)\Delta-s}^{j,N} + M_{(a-1)\Delta}^{j,N})(M_{a\Delta}^{j',N} - M_{(a-1)\Delta}^{j',N}).
\end{aligned}$$

We set for $1 \leq j, j', l, l' \leq N$ and $a, b \in \{t/(2\Delta) + 1, \dots, 2t/\Delta\}$,

$$\begin{aligned}
\mathcal{T}_{a,b}(j, j', l, l') &:= \mathbb{C}ov_\theta [(M_{a\Delta-s}^{j,N} - M_{a\Delta}^{j,N} - M_{(a-1)\Delta-s}^{j,N} + M_{(a-1)\Delta}^{j,N})(M_{a\Delta}^{j',N} - M_{(a-1)\Delta}^{j',N}), \\
&\quad (M_{b\Delta-s}^{l,N} - M_{b\Delta}^{l,N} - M_{(b-1)\Delta-s}^{l,N} + M_{(b-1)\Delta}^{l,N})(M_{b\Delta}^{l',N} - M_{(b-1)\Delta}^{l',N})].
\end{aligned}$$

Using [13, Lemma 16-(iii)], it is obvious that without any condition on (a, b) , on $\Omega_{N,K}$

$$\begin{aligned}
& |\mathcal{T}_{a,b}(j, j', l, l')| \\
& \leq \left\{ \mathbb{E}_\theta \left[\left(M_{a\Delta}^{j,N} - M_{(a-1)\Delta}^{j,N} \right)^4 \right]^{\frac{1}{4}} + \mathbb{E}_\theta \left[\left(M_{a\Delta-s}^{j,N} - M_{(a-1)\Delta-s}^{j,N} \right)^4 \right]^{\frac{1}{4}} \right\} \mathbb{E}_\theta \left[\left(M_{a\Delta}^{j',N} - M_{(a-1)\Delta}^{j',N} \right)^4 \right]^{\frac{1}{4}} \\
& \quad \left\{ \mathbb{E}_\theta \left[\left(M_{b\Delta}^{l,N} - M_{(b-1)\Delta}^{l,N} \right)^4 \right]^{\frac{1}{4}} + \mathbb{E}_\theta \left[\left(M_{b\Delta-s}^{l,N} - M_{(b-1)\Delta-s}^{l,N} \right)^4 \right]^{\frac{1}{4}} \right\} \mathbb{E}_\theta \left[\left(M_{b\Delta}^{l',N} - M_{(b-1)\Delta}^{l',N} \right)^4 \right]^{\frac{1}{4}} \\
& \leq C\Delta^2,
\end{aligned}$$

and $\mathbf{1}_{\{\#(j, j', l, l')=4\}} |\mathcal{T}_{a,b}(j, j', l, l')| = 0$.

We now consider the case when $a - b \geq 4$. Recall that $\zeta_{a\Delta,s}^{j,N} := M_{(a\Delta-s)}^{j,N} - M_{a\Delta}^{j,N}$ for $0 \leq s \leq \Delta$. Then,

$$\begin{aligned}\mathcal{T}_{a,b}(j,j',l,l') &= \text{Cov}_\theta[(\zeta_{a\Delta,s}^{j,N} - \zeta_{(a-1)\Delta,s}^{j,N})\zeta_{a\Delta,\Delta}^{j',N}, (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})\zeta_{b\Delta,\Delta}^{l',N}] \\ &= \text{Cov}_\theta[\zeta_{a\Delta,s}^{j,N}\zeta_{a\Delta,\Delta}^{j',N}, (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})\zeta_{b\Delta,\Delta}^{l',N}].\end{aligned}$$

Using the same strategy as the proof of Lemma 6.4, we have

$$\begin{aligned}&|\text{Cov}_\theta[\zeta_{a\Delta,s}^{j,N}\zeta_{a\Delta,\Delta}^{j',N}, (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})\zeta_{b\Delta,\Delta}^{l',N}]| \\ &= |\text{Cov}_\theta[T_{a\Delta,(a-1)\Delta}^{j,N}, (\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})\zeta_{b\Delta,\Delta}^{l',N}]| \\ &\leq \{\mathbb{E}_\theta[(T_{a\Delta,(a-1)\Delta}^{j,N})^2]^{\frac{1}{2}}\} \mathbb{E}_\theta[(\zeta_{b\Delta,r}^{l,N} - \zeta_{(b-1)\Delta,r}^{l,N})^4]^{\frac{1}{4}} \mathbb{E}_\theta[(\zeta_{b\Delta,\Delta}^{l',N})^4]^{\frac{1}{4}} \\ &\leq Ct^{1/2}\Delta^{1-q}.\end{aligned}$$

Hence, by symmetry, for $|a - b| \geq 4$, $|\mathcal{T}_{a,b}(j,j',l,l')| \leq C(\mathbf{1}_{\{l=l'\}} + \mathbf{1}_{\{j=j'\}})\sqrt{t}\Delta^{1-q}$. Consequently, still for $|a - b| \geq 4$,

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\left|\text{Cov}_\theta\left[\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K}(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K}), \mathcal{Y}_{(b-1)\Delta,b\Delta}^{N,K}(C_{b\Delta}^{N,K} - C_{(b-1)\Delta}^{N,K})\right]\right|\right] \\ &= \frac{1}{K^4}\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\left|\sum_{i,i'=1}^K \sum_{l,l'=1}^N \sum_{n,n' \geq 1} \int_0^\Delta \int_0^\Delta \phi^{*n}(s)\phi^{*n'}(s')A_N^n(i,j)A_N^{n'}(i',l)\right.\right. \\ &\quad \times c_N^K(j')c_N^K(l')\mathcal{T}_{a,b}(j,j',l,l')dsds'\left|\right|\right] \\ &\leq \frac{t^{1/2}}{K^4\Delta^{q-1}}\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\left|\sum_{j=1}^N (c_N^K(j) - 1)c_N^K(j)\right|\left|\sum_{l=1}^N c_N^K(l)\right|\sum_{n \geq 1} N\Lambda^n||I_K A_N||_1||A_N||_1^{n-1}\right] \\ &\leq \frac{Ct^{1/2}}{K\Delta^{q-1}}.\end{aligned}$$

The last step follows from (20), which implies that on $\Omega_{N,K}$, $\sum_{j=1}^N (c_N^K(j))^2 \leq CK$, together with the facts that on $\Omega_{N,K}$, $||I_K A_N||_1 \leq \frac{K}{N}$, $|\sum_{l=1}^N c_N^K(l)| = K|\bar{\ell}_N^K| \leq CK$.

Next, when $|a - b| \leq 4$,

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\left|\text{Cov}_\theta\left[\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K}(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K}), \mathcal{Y}_{(b-1)\Delta,b\Delta}^{N,K}(C_{b\Delta}^{N,K} - C_{(b-1)\Delta}^{N,K})\right]\right|\right] \\ &\leq \mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\text{Var}_\theta\left[\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K}(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K})\right]\right]^{\frac{1}{2}} \mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\text{Var}_\theta\left[\mathcal{Y}_{(b-1)\Delta,b\Delta}^{N,K}(C_{b\Delta}^{N,K} - C_{(b-1)\Delta}^{N,K})\right]\right]^{\frac{1}{2}} \\ &\leq \mathbb{E}\left[\mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta\left[\left|\mathcal{Y}_{(a-1)\Delta,a\Delta}^{N,K}\right|^4\right]^{\frac{1}{4}} \mathbb{E}_\theta\left[\left(C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K}\right)^4\right]^{\frac{1}{4}} \mathbb{E}_\theta\left[\left|\mathcal{Y}_{(b-1)\Delta,b\Delta}^{N,K}\right|^4\right]^{\frac{1}{4}}\right. \\ &\quad \times \mathbb{E}_\theta\left[\left(C_{b\Delta}^{N,K} - C_{(b-1)\Delta}^{N,K}\right)^4\right]^{\frac{1}{4}}\left.\right] \\ &\leq \frac{C\Delta}{NK}.\end{aligned}$$

Finally,

$$\mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \mathbb{V}ar_{\theta} \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} (C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K}) \right] \right] \leq \frac{Ct}{NK} + \frac{Ct^{5/2}}{K\Delta^{q+1}}.$$

Overall we conclude that

$$\begin{aligned} & \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \frac{K}{N} \sqrt{\frac{t}{\Delta}} \left| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K} \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} - \mathbb{E}_{\theta} \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K} \bar{X}_{(a-1)\Delta, a\Delta}^{N,K} \right] \right| \right] \\ & \leq C \frac{K}{\sqrt{\Delta t}} \left\{ \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \left(\left| \bar{\Gamma}_{(a-1)\Delta, a\Delta}^{N,K} (\bar{X}_{(a-1)\Delta, a\Delta}^{N,K} - \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K}) \right| \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. + \left| \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} (B_{a\Delta}^{N,K} - B_{(a-1)\Delta}^{N,K}) \right| \right) \right] + \mathbb{E} \left[\mathbf{1}_{\Omega_{N,K}} \mathbb{V}ar_{\theta} \left[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \mathcal{Y}_{(a-1)\Delta, a\Delta}^{N,K} (C_{a\Delta}^{N,K} - C_{(a-1)\Delta}^{N,K}) \right] \right]^{\frac{1}{2}} \right\} \\ & \leq \frac{CK}{N\Delta^q \sqrt{t}} + \frac{C\sqrt{tK}}{\Delta^{q+\frac{1}{2}} \sqrt{N}} + \frac{C\sqrt{K}}{\sqrt{N\Delta}} + \frac{Ct^{\frac{3}{4}} \sqrt{K}}{\Delta^{1+\frac{q}{2}}}. \end{aligned}$$

The proof is finished.

ACKNOWLEDGEMENTS

We would like to express our sincere gratitude to N. Fournier and S. Delattre for their invaluable support of this research. This work would not have been possible without their insightful ideas and patient guidance.

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